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Generalized diamond-α dynamic opial inequalities

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Abstract

We establish some new dynamic Opial-type diamond alpha inequalities in time scales. Our results in special cases yield some of the recent results on Opial's inequality and also provide new estimates on inequalities of this type. Also, we introduce an example to illustrate our result.

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1. Introduction and Preliminaries

In 1960, the Polish Mathematician Zdzidlaw Opial [1] published an inequality involving integrals of functions and their derivatives;

$$\int_0^h |x(t)x'(t)|dt \le \frac{h}{4}\int_0^h |x'(t)|^2 dt \tag{1.1}$$

where $x \in C^1[0, h]$, $x(0) = x(h) = 0$ and $x(t) > 0$ in $(0, h)$, and the constant $h/4$ is the best possible.

Inequalities which involve integrals of functions and their derivatives are of great importance in mathematics with applications in the theory of differential equations, approximations and probability. It has been shown that inequalities of the form (1.1) can be deduced from those of Wirtinger and Hardy type, but the importance of Opial's result is in the establishment of the best possible constant. The monograph [2] is the first book dedicated to the theory of Opial type inequalities.

The positivity requirement of $x(t)$ in the original proof of Opial was shown to be unnecessary later by Olech [3] where he proved that the inequality (1.1) holds even for functions $x(t)$ which are only absolutely continuous in [0, h]. Moreover, Olech's proof is simpler than that of Opial.

Theorem 1.1. (Olech): Let $x(t)$ be absolutely continuous in $[0, h]$ and $x(0) = x(h) = 0$. Then the following inequality holds;

$$\int_0^h |x(t)x'(t)|dt \le \frac{h}{4}\int_0^h (x'(t))^2 dt \tag{1.2}$$

Works containing discrete analogues of Opial-type inequalities started in 1967-69 with the articles of Lasota [4], Wong [5], and Beesack [6] which provided discrete versions of the inequality (1.1).

The next result is the discrete analogue of the above theorem.

Theorem 1.2. (Lasota's Inequality): Let $\{x_i\}_{i=0}^{N}$ be a sequence of numbers, and $x_0 = x_N = 0$. Then the following inequality holds

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \leq \frac{1}{2} \left[\left[\frac{N+1}{2}\right]\right] \sum_{i=0}^{N-1} |\Delta x_i|^2 \tag{1.3}$$

where Δ is the forward difference operator, and $[[\cdot]]$ is the greatest integer function.

Now, if we consider Olech's result under weaker conditions, we get the following estimate with a bound which is less sharp.

Theorem 1.3. Let $x(t)$ be absolutely continuous in $[0, a]$ and $x(0) = 0$. Then the following inequality holds.

$$\int_0^a |x(t)x'(t)|dt \leq \frac{a}{2} \int_0^a (x'(t))^2 dt \tag{1.4}$$

In (1.4) equality holds if and only if $x(t) = ct$.

The following theorem is a non-trivial generalization of Theorem 1.3 and is given in Hua [7].

Theorem 1.4. (Hua's generalization) Let $x(t)$ be absolutely continuous on $[0, a]$, and $x(0) = 0$. Further let ϵ be a positive integer. Then the following inequality holds

$$|x^{\epsilon}(t)x'(t)|dt \leq \frac{a^{\epsilon}}{\epsilon + 1} |x'(t)|^{\epsilon+1} dt \tag{1.5}$$

with equality being valid in (1.5) if and only if $x(t) = ct$.

Finally we give a discrete analogue of Theorem 1.4 due to Wong [5].

Theorem 1.5. (Wong's inequality) Let $\{x_i\}_{i=0}^{\tau}$ be a non-decreasing sequence of non-negative numbers, and $x_0 = 0$. Then for $\epsilon \geq 1$, the following inequality holds

$$\sum_{i=1}^{\tau} x_i^{\epsilon} \nabla x_i \leq \frac{(\tau+1)^{\epsilon}}{\epsilon+1} \sum_{i=1}^{\tau} (\nabla x_i)^{\epsilon+1}. \tag{1.6}$$

where ∇ is the backward difference operator, that is $\nabla x_i = x_i - x_{i-1}$.

Remark 1.6. *In terms of the forward difference operator,* Δx_i, *above Wong's inequality* (1.6) *can be restated as follows;*

$\{x_i\}_{i=0}^{\tau}$ *is a non-decreasing sequence of non-negative numbers with* $x_0 = 0$, *for* $\epsilon \geq 1$, *the inequality*

$$\sum_{i=0}^{\tau-1} x_{i+1}^{\epsilon} \Delta x_i \leq \frac{(\sigma(\tau))^{\epsilon}}{\epsilon+1} \sum_{i=0}^{\tau-1} (\Delta x_i)^{\epsilon+1}. \tag{1.7}$$

holds where Δ *is the forward difference operator.*

1.1. Time-scale set-up of basic Opial type inequality

For convenience we now recall the following easiest versions of Opial's inequality.

Theorem 1.7. [Continuous Opial inequality, [[2], Theorem 1.4.1]] For absolutely continuous x: $[0, h] \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h |xx'|(t)\mathrm{d}t \le \frac{h}{2} \int_0^h |x'|^2(t)\mathrm{d}t,$$

with equality when $x(t) = ct$.

Theorem 1.8. [Discrete Opial inequality, [[2], Theorem 5.2.2]] For $x_0 = 0$ and a sequence $\{x_i\}_{0\le i\le h} \subset \mathbb{R}$, we have

$$\sum_{i=1}^{h-1} |x_i(x_{i+1} - x_i)| \le \frac{h-1}{2} \sum_{i=0}^{h-1} |x_{i+1} - x_i|^2,$$

with equality when $x_i = ci$.

In [8], a dynamic Opial inequality is proven that contains both Theorems 1.7 and 1.8 as special cases. For details of time-scale calculus we refer to [9,10]. We now give this simplest version of Opial's inequality on time scales as presented in [8].

2. Main results

Theorem 2.1. [Delta Dynamic Opial inequality] Let $\mathbb{T}$ be a time scale. For delta differentiable $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \le h \int_0^h |x^\Delta|^2(t)\Delta t,$$

with equality when $x(t) = ct$, provided all Δ-anti derivatives exist.

We refer to [8] for the proof of Theorem 2.1.

Next a generalization of Theorem 2.1 is offered where $x(0)$ does not need to be equal to 0. This result is not found in the book [2] (neither a continuous nor a discrete version of it), but both a weaker version of Theorem 2.1 (with $x(0) = 0$) and the subsequent Corollary 2.3 (with $x(0) = x(h) = 0$) are corollaries of Theorem 2.2, and continuous [[2], Theorem 1.3.1] and discrete [2, Theorem 5.2.1, "Lasota's inequality'] versions of Theorem 2.3 can be found in the book by Agarwal and Pang [2].

Theorem 2.2. Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be Δ-differentiable and rd-continuous function. Then

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \le \alpha \int_0^h |x^\Delta(t)|^2\Delta t + 2\beta \int_0^h |x^\Delta(t)|\,\Delta t,$$

where

$$\alpha \in \mathbb{T}, \quad \mathrm{dist}(h/2,\ \alpha) = \mathrm{dist}(h/2,\ \mathbb{T}) \tag{2.1}$$

$\beta = \max\{|x(0)|, |x(h)|\}$.

Corollary 2.3. Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be Δ-differentiable and rd-continuous function with $x(0) = x(h) = 0$. Then

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \le \alpha \int_0^h |x^\Delta(t)|^2\Delta t,$$

where α is given in (2.1).

Proof. This follows easily from Theorem 2.2 since in this case we have $\beta = 0$. □

2.1. Improved diamond-alpha Opial inequalities

The diamond-alpha derivative of $f : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t),$$

where $\alpha \in [0, 1]$. A function f: $[0, h] \to \mathbb{R}$ is said to be in $C^1_{\diamond_\alpha}$ if f is $\diamond_\alpha$-differentiable such that αf^{Δ} is rd-continuous, $(1 - \alpha)f^{\nabla}$ is ld-continuous, and $\alpha(1-\alpha)f^{\diamond_\alpha}$ is continuous. Note that $C^1_{\diamond_\alpha}$ is, for $\alpha \in (0, 1)$, the class of functions that are Δ-differentiable and ∇-differentiable such that f^{Δ} is rd-continuous, f^{∇} is ld-continuous, and $f^{\diamond\alpha}$ is continuous. Moreover, $C^1_{\diamond_0}$ coincides with the class of functions that are ∇-differentiable such that f^{∇} is ld-continuous, and likewise $C^1_{\diamond_1}$ is equal the class of functions that are Δ-differentiable such that f^{Δ} is rd-continuous.

First we introduce a set of Opial type Diamond-alpha Inequalities obtained by Bohner-Duman [11]

Theorem 2.4. Let $\alpha \in [0, 1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\diamond_\alpha}$ with $f(0) = 0$ and $\alpha(1 - \alpha)f^{\Delta}f^{\nabla} \geq 0$, we have

$$\alpha^3 \int_0^h |(f + f^{\sigma})f^{\Delta}|(t)\Delta t + (1-\alpha)^3 \int_0^h |(f + f^{\rho})f^{\nabla}|(t)\nabla t \leq h \int_0^h (f^{\Delta})^2(t) \diamond_\alpha t. \quad (2.2)$$

Theorem 2.5. Let $\alpha \in [0, 1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\diamond_\alpha}$ with $\alpha f^{\Delta} \geq 0$ and $(1 - \alpha)f^{\nabla} \geq 0$, we have

$$\alpha^3 \int_0^h |(f + f^{\sigma})f^{\Delta}|(t)\Delta t + (1-\alpha)^3 \int_0^h |(f + f^{\rho})f^{\nabla}|(t)\nabla t \leq h\beta \int_0^h (f^{\diamond_\alpha})^2(t)\diamond_\alpha t + 2\gamma(1 - 3\alpha + 3\alpha^2)(f(h) - f(0)),$$

where $\beta := \min_{u\in[0,h]\cap\mathbb{T}} \max\{u, h-u\}$ and $\gamma := \max\{|f(0)|, |f(h)|\}$.

Corollary 2.6. Let $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\diamond_\alpha}$ with $\alpha f^{\Delta} \geq 0$, $(1 - \alpha)f^{\nabla} \geq 0$, and $f(0) = f(h) = 0$, we have

$$\alpha^3 \int_0^h |(f + f^{\sigma})f^{\Delta}|(t)\Delta t + (1-\alpha)^3 \int_0^h |(f + f^{\rho})f^{\nabla}|(t)\nabla t \leq \beta \int_0^h (f^{\phi_\alpha})^2(t)\diamond_\alpha t,$$

where β is as in Theorem 2.5.

Theorem 2.7. Let $\alpha \in [0, 1]$ and $h \in \mathbb{T}$ with $h > 0$. Assume that g: $[0, h] \to \mathbb{R}^+$ is a non increasing continuous function. For any $f \in C^1_{\diamond_\alpha}$ with $\alpha(1 - \alpha)f^{\Delta}f^{\nabla} \geq 0$, we have

$$\alpha^3 \int_0^h [g^{\sigma}|(f + f^{\sigma})f^{\Delta}|](t)\Delta t + (1-\alpha)^3 \int_0^h [g^{\rho}|(f + f^{\rho})f^{\nabla}|](t)\nabla t \leq h \int_0^h g(t)(f^{\diamond_\alpha})^2(t)\diamond_\alpha t.$$

In this article, the above given Theorem 2.4 is improved in the sense of removing the restriction given by the condition $\alpha(1-\alpha)f^{\Delta}f^{\nabla} \geq 0$ as well as the left hand side of inequality (2.2) refined to a compact form, being composed of a single diamond-alpha integral. In this sense, the next theorem and its consequences extend and unify the previously obtained delta and nabla Opial dynamic inequalities in a more accurate way than that given by Theorem 2.4.

Theorem 2.8. Let $\mathbb{T}$ be a time scale. For $\Diamond_\alpha$ differentiable $x : [\rho(0), \sigma(h)]_{\mathbb{T}} \to \mathbb{R}$ with $x \in C^1$ and $x(0) = 0$ we have

$$\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t \leq h \int_0^h |x^{\Diamond_\alpha}|^2(t)\Diamond_\alpha t, \tag{2.3}$$

with equality when $x(t) = ct$.

Proof. Starting with the left side of (2.3)and using the fact that

$$xx^{\Diamond_\alpha} = \alpha xx^{\Delta} + (1-\alpha)xx^{\nabla}$$

we get,

$$\begin{aligned}
\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t &= \int_0^h |xx^{\Diamond_\alpha} + \alpha x^\sigma x^\Delta + (1-\alpha)x^\rho x^\nabla|(t)\Diamond_\alpha t \\
&= \alpha \int_0^h |xx^{\Diamond_\alpha}| + \alpha x^\sigma x^\Delta + (1-\alpha)x^\rho x^\nabla|(t)\Delta t \\
&+ (1-\alpha)\int_0^h |xx^{\Diamond_\alpha}| + \alpha x^\sigma x^\Delta + (1-\alpha)x^\rho x^\nabla|(t)\nabla t \\
&\leq \alpha \int_0^h |xx^{\Diamond_\alpha}|(t)\Delta t + \alpha^2 \int_0^h |x^\sigma x^\Delta|(t)\Delta t + \alpha(1-\alpha)\int_0^h |x^\rho x^\nabla|(t)\Delta t \\
&+ (1-\alpha)\int_0^h |xx^{\Diamond_\alpha}|(t)\nabla t + \alpha(1-\alpha)\int_0^h |x^\sigma x^\Delta|(t)\nabla t \\
&+ (1-\alpha)^2 \int_0^h |x^\rho x^\nabla|(t)\nabla t \\
&= \alpha \int_0^h |\alpha xx^\Delta + (1-\alpha)xx^\nabla|(t)\Delta t + \alpha^2 \int_0^h |x^\sigma x^\Delta|(t)\Delta t \\
&+ \alpha(1-\alpha)\int_0^h |x^\rho x^\nabla|(t)\Delta t + (1-\alpha)\int_0^h |\alpha xx^\Delta| + (1-\alpha)xx^\nabla|(t)\nabla t \\
&+ \alpha(1-\alpha)\int_0^h |x^\sigma x^\Delta|(t)\nabla t + (1-\alpha)^2 \int_0^h |x^\rho x^\nabla|(t)\nabla t \\
&= \alpha^2 \int_0^h (|x| + |x^\sigma|)|x^\Delta|(t)\Delta t + \alpha(1-\alpha)\int_0^h (|x| + |x^\rho|)|x^\nabla|(t)\Delta t \\
&+ \alpha(1-\alpha)\int_0^h (|x| + |x^\sigma|)x^\Delta(t)\Delta t + (1-\alpha)^2 \int_0^h (|x| + |x^\rho|)x^\nabla|(t)\Delta t \\
&= \alpha^2 \int_0^h [(|x| + |x^\sigma|)|x^\Delta|](t)\Delta t + (1-\alpha)^2 \int_0^h [(|x| + |x^\rho|)|x^\nabla|](t)\nabla t \\
&+ \alpha(1-\alpha)\int_0^h [(|x| + |x^\sigma|)|x^\Delta|](t)\nabla t + \alpha(1-\alpha)\int_0^h [(|x| + |x^\rho|)|x^\nabla|](t)\Delta t
\end{aligned}$$

Letting $y(t) = \int_0^t |x^{\diamond_\alpha}(s)| \diamond_\alpha s$, we then have $y^\Delta = |x^\Delta|$, $y^\nabla = |x^\nabla|$ and since

$$y(t) = \int_0^t |x^{\diamond_\alpha}(s)| \diamond_\alpha s \geq |\int_0^t |x^{\diamond_\alpha}(s)| \diamond_\alpha s| = |x(t) - x(0)| = |x(t)|$$

yielding $|x| \leq y$ as long as $\diamond_\alpha$-anti derivative of x exists. So

$$\begin{aligned}
&\leq \alpha^2 \int_0^h [(y + y^\sigma)y^\Delta](t)\Delta t + (1-\alpha)^2 \int_0^h [(y + y^\rho)y^\nabla](t)\nabla t \\
&+ \alpha(1-\alpha) \int_0^h [(y + y^\sigma)y^\Delta](t)\nabla t + \alpha(1-\alpha) \int_0^h [(y + y^\rho)y^\nabla](t)\Delta t \\
&= \int_0^h (y^2)^{\diamond_\alpha}(t)\diamond_\alpha t \\
&= y^2(h) - y^2(0) \\
&= \left[\int_0^h |x^{\diamond_\alpha}(s)| \diamond_\alpha s\right]^2 \\
&\leq h \int_0^h |x^{\diamond_\alpha}(s)|^2 \diamond_\alpha s.
\end{aligned}$$

where we have used Hölder's inequality for $h(x) = 1$ and $p = 2$. □

We illustrate the result obtained in Theorem 2.8 with an example.

Example 2.9. Let $\mathbb{T} = [-1, 4] \cap \mathbb{Z}$ be the time scale and $x : \mathbb{T} \to \mathbb{R}$ with $\alpha = \frac{1}{2}$,

$$x(t) = \begin{cases} -1 & \text{if } t = \pm 1 \\ 0 & \text{if } t = 0 \\ 5 & \text{if } t \geq 2. \end{cases}$$

Then,

$$\begin{aligned}
\int_0^h |(x^2)^{\diamond_\alpha}|(t)\diamond_\alpha t &= \frac{1}{2}\int_0^3 |(x^2)^\Delta(t) + (x^2)\nabla(t)|\diamond_\alpha t \\
&= \frac{1}{4}\left[\int_0^3 |(x^2)^\Delta(t) + (x^2)\nabla(t)|\Delta(t) + \int_0^3 |(x^2)^\Delta(t) + (x^2)\nabla(t)|\nabla(t)\right] \\
&= \frac{1}{4}\left[\sum_{t=0}^2 |(x(t+1) + x(t))(x(t+1) - x(t)) + (x(t) + x(t-1))(x(t) - x(t-1))|\right. \\
&\left. + \sum_{t=1}^3 |(x(t+1) + x(t))(x(t+1) - x(t)) + (x(t) + x(t-1))(x(t) - x(t-1))|\right] \\
&= \frac{1}{4}\left[\sum_{t=0}^2 |(x^2(t+1) - x^2(t-1))| + \sum_{t=1}^3 |(x^2(t+1) - x^2(t-1))|\right] = \frac{49}{2}
\end{aligned}$$

and similarly,

$$\begin{aligned}\int_0^h |(x^{\Diamond_\alpha})^2|(t)\Diamond_\alpha t &= \frac{1}{4}\int_0^3 (x^\Delta(t) + (x^\nabla(t)))^2\Diamond_\alpha t\\ &= \frac{1}{8}\left[\int_0^3 (x^\Delta(t) + (x^\nabla(t)))^2\Delta t + \int_0^3 (x^\Delta(t) + (x^\nabla(t)))^2\nabla t\right]\\ &= \frac{1}{8}\left[\sum_{t=0}^{2}(x(t+1) - (x(t-1)))^2 + \sum_{t=1}^{3}(x(t+1) - (x(t-1)))^2\right] = \frac{63}{4}\end{aligned}$$

and so we obtain $3.\frac{63}{4} = \frac{189}{4}$.

Therefore the inequality in Theorem 2.8 holds.

The next two theorems are extensions of Theorem 2.8.

Theorem 2.10. Let $x : [p(0), \sigma(h)]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-differentiable. Then

$$\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t \le \gamma \int_0^h |x^{\Diamond_\alpha}|^2 \Diamond_\alpha + 2\beta \int_0^h |x^{\Diamond_\alpha}|\Diamond_\alpha t,$$

where $\beta = \max\{|x(0), |x(h)|\}$ $\gamma = \min_{u\in[0,h]\cap\mathbb{T}} \max\{u, h-u\}$

Proof. We consider $y(t) = \int_0^t |x^{\Diamond_\alpha}(s)|\Diamond_\alpha s$ and $z(t) = \int_t^h |x^{\Diamond_\alpha}(s)|\Diamond_\alpha s$. Then we have, $z^{\Diamond_\alpha} = -|x^{\Diamond_\alpha}|$, $z^{\Diamond_\alpha} = -|x^{\Diamond_\alpha}|$, $y^\Delta = |x^\Delta|$, $y^\Delta = |x^\Delta|$, $y^\nabla = |x^\nabla|$, $y^\nabla = |x^\nabla|$ yielding

$$\begin{aligned}|x(t)| &\le |x(t) - x(0)| + |x(0)|\\ &= \left|\int_0^t x^{\Diamond_\alpha}(s)\Diamond_\alpha s\right| + |x(0)|\\ &\le \int_0^t |x^{\Diamond_\alpha}(s)|\Diamond_\alpha s + |x(0)|\\ &= y(t) + |x(0)|\end{aligned}$$

and similarly, $|x(t)| \le z(t) + |x(h)|$.

Let $u \in [\rho(0), \sigma(h)]_{\mathbb{T}}$. Then

$$\begin{aligned}\int_0^h |(x^2)\Diamond_\alpha|(t)\Diamond_\alpha t &\le \int_0^u [(y + |x(0)|)y^{\Diamond_\alpha} + \alpha(y^\sigma + |x(0)|)y^\Delta\\ &\quad + (1-\alpha)(y^\rho + |x(0)|)y^\nabla]\Diamond_\alpha t\\ &= y^2(u) - y^2(0) + 2|x(0)|y(u)\\ &\le u\int_0^u |x^{\Diamond_\alpha}|^2\Diamond_\alpha t + 2|x(0)|\int_0^u |x^{\Diamond_\alpha}(t)|\Diamond_\alpha t,\end{aligned}$$

where we have used Hölder's inequality for $h(x) = 1$ and $p = 2$. Therefore we get

$$\begin{aligned}\int_u^h |(x^2)\Diamond_\alpha|(t)\Diamond_\alpha t &\le z^2(u) - z^2(h) + 2|x(h)|z(u)\\ &\le (h-u)\int_u^h |x^{\Diamond_\alpha}|^2\Diamond_\alpha t + 2|x(h)|\int_u^h |x^{\Diamond_\alpha}|\Diamond_\alpha t.\end{aligned}$$

By putting $\gamma = \min_{u\in[0,h]\cap\mathbb{T}} \max\{u, h-u\}$, $\beta = \max\{|x(0), |x(h)|\}$ and adding the above two inequalities, we find

$$\int_0^h |(x^2)\diamondsuit_\alpha|(t)\diamondsuit_\alpha t \le \gamma \int_0^h |x^{\diamondsuit_\alpha}|^2\diamondsuit_\alpha + 2\beta\int_0^h |x^{\diamondsuit_\alpha}|\diamondsuit_\alpha t.$$

□

Theorem 2.11. Let $x : [p(0), \sigma(h)]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-differentiable with $x(0) = x(h) = 0$. Then

$$\int_0^h |(x^2)^{\diamondsuit_\alpha} \le \gamma \int_0^h |x^{\diamondsuit_\alpha}|^2\diamondsuit_\alpha, \tag{2.4}$$

where γ is given in Theorem 2.10.

Proof. We consider $y(t) = \int_0^t |x^{\diamondsuit_\alpha}(s)|\diamondsuit_\alpha s$ and $z(t) = \int_t^h |x^{\diamondsuit_\alpha}(s)|\diamondsuit_\alpha s$. Then, $y^{\diamondsuit_\alpha} = |x^{\diamondsuit_\alpha}|$, $z^{\diamondsuit\alpha} = -|x^{\diamondsuit_\alpha}|$, $y^{\Delta} = |x^{\Delta}|$, $y^{\nabla} = |x^{\nabla}|$ and $|x(t)| \le |x(t) - x(0)| + |x(0)| = |\int_0^t x^{\diamondsuit_\alpha}(s)|\diamondsuit_\alpha s + |x(0)| \le \int_0^t |x^{\diamondsuit_\alpha}(s)|\diamondsuit_\alpha s + |x(0)| = y(t) + |x(0)|$ and similarly $|x(t)| \le z(t) + |x(h)|$.

Let $u \in [\rho(0), \sigma(h)]_{\mathbb{T}}$, then

$$\begin{aligned}\int_0^h |(x^2)^{\diamondsuit_\alpha}|(t)\diamondsuit_\alpha t &\le \int_0^u [(y + |x(0)|)y^{\diamondsuit_\alpha} + \alpha(y^\sigma + |x(0)|)y^\Delta \\ &+ (1-\alpha)(y^\rho + |x(0)|)y^\nabla]\diamondsuit_\alpha \\ &= y^2(u) - y^2(0) + 2|x(0)|y(u) \\ &\le u\int_0^u |x^{\diamondsuit_\alpha}|^2\diamondsuit_\alpha t + 2|x(0)|\int_0^u |x^{\diamondsuit_\alpha}(t)|\diamondsuit_\alpha t\end{aligned}$$

where we have used Hölder's inequality for $h(x) = 1$ and $p = 2$.

Therefore we get

$$\begin{aligned}\int_u^h |(x^2)^{\diamondsuit_\alpha}|(t)\diamondsuit_\alpha t &\le z^2(u) - z^2(h) + 2|x(h)|z(u) \\ &\le (h-u)\int_u^h |x^{\diamondsuit_\alpha}|^2\diamondsuit_\alpha t + 2|x(h)|\int_u^h |x^{\diamondsuit_\alpha}|\diamondsuit_\alpha t.\end{aligned}$$

By putting $\gamma = \min_{u\in[0,h]\cap\mathbb{T}} \max\{u, h-u\}$ and adding the above two inequalities, we find

$$\int_0^h |(x^2)^{\diamondsuit_\alpha}|(t)\diamondsuit_\alpha t \le \gamma \int_0^h |x^{\diamondsuit_\alpha}|^2\diamondsuit_\alpha$$

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Authors' contributions
All authors have contributed evenly to the manuscript and read and approved the final version.

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