

UNIQUENESS OF POSITIVE SOLUTIONS OF A CLASS OF ODE WITH NONLINEAR BOUNDARY CONDITIONS

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We study the uniqueness of positive solutions of the boundary value problem $u'' + a(t)u' + f(u) = 0$, $t \in (0, b)$, $B_1(u(0)) - u'(0) = 0$, $B_2(u(b)) + u'(b) = 0$, where $0 < b < \infty$, B_1 and $B_2 \in C^1(\mathbb{R})$, $a \in C[0, \infty)$ with $a \leq 0$ on $[0, \infty)$ and $f \in C[0, \infty) \cap C^1(0, \infty)$ satisfy suitable conditions. The proof of our main result is based upon the shooting method and the Sturm comparison theorem.

1. Introduction

The existence of positive solutions of second order ordinary differential equations (ODEs) with linear boundary conditions has been extensively studied in the literature, see Coffman [1], Henderson and Wang [7], Lan and Webb [8] and the references therein. Also the existence of positive solutions of second order ODEs with nonlinear boundary conditions has been studied by several authors, see Dunninger and Wang [2], Wang [11] and Wang and Jiang [12] for some references along this line. However for the uniqueness problem of second order ODEs, even in the linear boundary conditions case, very little was known, see Ni and Nussbaum [9], Fu and Lin [6] and Peletier and Serrin [10]. To the best of our knowledge, no uniqueness results of positive solutions were established for second order ODEs subject to nonlinear boundary conditions. In this paper, we attempt to prove some uniqueness results in this direction.

More precisely, we consider the uniqueness of positive solutions of the boundary value problem

$$u'' + a(t)u' + f(u) = 0, \quad t \in (0, b) \tag{1.1}$$

$$B_1(u(0)) - u'(0) = 0, \quad B_2(u(b)) + u'(b) = 0, \tag{1.2}$$

where $0 < b < \infty$. We make the following assumptions:

(C1) $f \in C[0, \infty) \cap C^1(0, \infty)$ with $f(0) = 0$,

$$f(u) > 0, \quad uf'(u) < f(u), \quad \text{for } u > 0; \tag{1.3}$$

(C2) $a \in C[0, \infty)$ with $a(t) \leq 0$ for $t \geq 0$;

(C3) $B_i \in C^1[0, \infty)$ satisfies $B_i(0) = 0$, $B_i(x) > 0$ for $x > 0$, $B'_i(x)$ is nondecreasing on $(0, \infty)$ ($i = 1, 2$).

Remark 1.1. Condition (C3) implies that $B'_i(x) \geq 0$ for $x \geq 0$ ($i = 1, 2$).

In fact, we have from $B_i(0) = 0$ and $B_i(x) > 0$ for $x > 0$ that

$$B'_i(0) \geq 0. \tag{1.4}$$

This together with the assumption $B'_i(x)$ is nondecreasing on $(0, \infty)$ implies that $B'_i(x) \geq 0$ for $x \geq 0$.

The main result of this paper is the following.

THEOREM 1.2. *Let (C1)–(C3) hold. Then problem (1.1), (1.2) has at most one positive solution.*

Here we say $u(t)$ is a *positive solution* of (1.1), (1.2), if that $u(t) > 0$ on $[0, b]$ and satisfies the differential equation (1.1) as well as the boundary conditions (1.2).

Remark 1.3. As an application of Theorem 1.2, we consider the nonlinear problem

$$\begin{aligned} u'' + a(t)u' + u^p &= 0, & t \in (0, b), \\ (u(0))^k - u'(0) &= 0, & (u(b))^l + u'(b) = 0, \end{aligned} \tag{1.5}$$

where $p \in (0, 1)$, $k, l \in (1, \infty)$ are given, $a \in C[0, \infty)$ with $a \leq 0$ on $[0, \infty)$. Clearly all of the conditions of Theorem 1.2 are satisfied. Therefore by Theorem 1.2, (1.5) has at most a positive for any $b \in (0, \infty)$.

The proof of the main result is motivated by the work of Erbe and Tang [3, 4, 5] and is based on the shooting method and the Sturm comparison theorem. The rest of the paper is organized as follows. In Section 2, we state and prove some preliminary lemmas. The proof of Theorem 1.2 will be given in Section 3.

2. The preliminary results

To apply the shooting method, we need some properties of the solutions of the initial value problem

$$u'' + \bar{a}(t)u' + \bar{f}(u) = 0, \tag{2.1}$$

$$u(0) = \delta, \quad u'(0) = \lambda. \tag{2.2}$$

LEMMA 2.1. *Let $\bar{a} \in C[0, \infty)$, $\bar{f} \in C[0, \infty) \cap C^1(0, \infty)$ with $\bar{f}(0) = 0$ and $\bar{f}(s) > 0$ for $s > 0$. Let $\delta \in (0, \infty)$ and $\lambda \in \mathbb{R}$ be two given constants. Then (2.1), (2.2) has a unique solution u satisfying either*

(I) $u(t) > 0$ for $t \in [0, \infty)$; or

(II) there exists $\rho \in (0, \infty)$ such that

$$u(t) > 0 \quad \text{on } t \in [0, \rho), \quad u(\rho) = 0, \quad u'(\rho) < 0. \tag{2.3}$$

Proof. For any $r \in (0, \infty)$, let

$$\Omega_r := \{(t, u, p) \mid t \in [0, r], u > 0\}. \tag{2.4}$$

Then the function

$$F(t, u, p) := \bar{a}(t)p + \bar{f}(u) \tag{2.5}$$

satisfies locally Lipschitz condition in Ω_r , and consequently (2.1), (2.2) has a unique solution $u(t)$ such that one of the following cases must occur

- (i) $u > 0$ on $[0, \infty)$;
- (ii) there exists $\rho \in (0, \infty)$ such that $u > 0$ on $[0, \rho)$, and $\lim_{t \rightarrow \rho^-} u(t) = 0$;
- (iii) there exists $T \in (0, \infty)$ such that $u > 0$ on $[0, T)$ and $\limsup_{t \rightarrow T^-} u(t) = \infty$.

We claim that (iii) can not occur.

Assume on the contrary that (iii) occurs, then

$$\limsup_{t \rightarrow T^-} u'(t) = \infty. \tag{2.6}$$

On the other hand, we have from (2.1) that

$$\left(u'(t) \exp \left(\int_0^t \bar{a}(s) ds \right) \right)' + \exp \left(\int_0^t \bar{a}(s) ds \right) \bar{f}(u) = 0, \quad t \in [0, T) \tag{2.7}$$

which together with the condition $\bar{f}(s) > 0$ for $s > 0$ implies that

$$u'(t) \exp \left(\int_0^t \bar{a}(s) ds \right) \text{ is strictly decreasing on } [0, T). \tag{2.8}$$

However this contradicts the fact (2.6).

Therefore either (i) or (ii) must occur.

Suppose on the contrary that (ii) occurs and $u'(\rho) = 0$. Using the similar argument of proving (2.8), we conclude that $u'(t) \exp(\int_0^t \bar{a}(s) ds)$ is strictly decreasing on $[0, \rho)$. Thus $u'(t) \exp(\int_0^t \bar{a}(s) ds) > 0$ on $[0, \rho)$, and accordingly $u'(t) > 0$ on $[0, \rho)$. However this contradicts the fact $\delta = u(0) > u(\rho) = 0$. Therefore $u'(\rho) < 0$ if (ii) occurs. This completes the proof. □

In order to prove Theorem 1.2, we introduce an initial value problem

$$u'' + a(t)u' + f(u) = 0, \tag{2.9}$$

$$u(0) = \alpha > 0, \quad u'(0) = B_1(\alpha). \tag{2.10}$$

For any $\alpha > 0$, we know from Lemma 2.1 that (2.9), (2.10) has a unique solution u such that one of the cases occurs:

- (i) $u > 0$ in $[0, \infty)$;
- (ii) there exists a unique $\rho = \rho(\alpha) \in (0, \infty)$ such that $u(t) > 0$ on $[0, \rho)$, $u(\rho) = 0$ and $u'(\rho) < 0$.

Let

$$T_\alpha = \begin{cases} \infty, & \text{if (i) occurs} \\ \rho(\alpha), & \text{if (ii) occurs.} \end{cases} \quad (2.11)$$

From $\alpha > 0$, we have that $u(0, \alpha) = \alpha > 0$ and $u'(0, \alpha) = B_1(\alpha) > 0$, and consequently

$$B_2(u(0, \alpha)) + u'(0, \alpha) = B_2(\alpha) + B_1(\alpha) > 0. \quad (2.12)$$

Therefore, there exists $\epsilon \in (0, T_\alpha)$ such that

$$B_2(u(t, \alpha)) + u'(t, \alpha) > 0, \quad t \in [0, \epsilon]. \quad (2.13)$$

Denote

$$B(t, \alpha) := B_2(u(t, \alpha)) + u'(t, \alpha). \quad (2.14)$$

When $B(t, \alpha)$ vanishes at some $t_0 \in (0, T_\alpha)$, we define $b(\alpha)$ to be the first zero of $B(t, \alpha)$ in $(0, T_\alpha)$. More precisely, $b(\alpha)$ is a function of α which has the properties

$$B(b(\alpha), \alpha) = 0, \quad B(t, \alpha) > 0, \quad t \in [0, b(\alpha)]. \quad (2.15)$$

If $B(t, \alpha)$ is positive in $[0, T_\alpha)$, then we define $b(\alpha) = T_\alpha$. Let

$$N := \{\alpha \mid \alpha > 0, b(\alpha) < T_\alpha\}. \quad (2.16)$$

It is obvious that (1.1), (1.2) has no positive solution if N is an empty set. (We recall that u is a positive solution means $u(t) > 0$ in $[0, b]$. So in the case $B(T_\alpha, \alpha) = 0$, $u(t, \alpha)$ is not a positive solution of (1.1), (1.2) since $u(T_\alpha, \alpha) = 0$). Hence we suppose $N \neq \emptyset$.

Remark 2.2. It is worth remarking here that if (ii) occurs, and accordingly $u(\rho(\alpha), \alpha) = 0$, then $b(\alpha) \in (0, \rho(\alpha))$,

$$B(b(\alpha), \alpha) = 0, \quad B(t, \alpha) > 0 \quad \text{on } [0, b(\alpha)]. \quad (2.17)$$

In fact, we have from Lemma 2.1 that

$$B(\rho(\alpha), \alpha) = B_2(u(\rho, \alpha)) + u'(\rho(\alpha), \alpha) < 0, \quad (2.18)$$

which together with the fact $B(0, \alpha) > 0$ yields the existence of zero of $B(t, \alpha)$ in $(0, \rho(\alpha))$.

LEMMA 2.3. *Let (C1)–(C3) hold and let $\alpha \in N$. Let $u(t, \alpha)$ be the unique solution of (2.9), (2.10) on $[0, T_\alpha)$. Then*

$$\begin{aligned} u(t, \alpha) &> 0, \quad t \in [0, b(\alpha)], \\ u'(b(\alpha), \alpha) &< 0. \end{aligned} \quad (2.19)$$

Proof. By Remark 2.2, $b(\alpha) \in (0, \rho(\alpha))$. Applying Lemma 2.1, we get that

$$u(t, \alpha) > 0, \quad t \in [0, b(\alpha)]. \quad (2.20)$$

The second inequality in (2.19) can be easily deduced from (2.20) and (C3) and the fact

$$B(b(\alpha), \alpha) = B_2(u(b(\alpha), \alpha)) + u'(b(\alpha), \alpha) = 0. \tag{2.21}$$

□

LEMMA 2.4. Let (C1)–(C3) hold. Let $u(t, \alpha)$ be the unique solution of (2.9), (2.10) on $[0, T_\alpha)$. If $\eta \in (0, T_\alpha)$ is such that

$$B(\eta, \alpha) = 0, \tag{2.22}$$

then

$$B(t, \alpha) > 0, \quad t \in [0, \eta]. \tag{2.23}$$

Proof. From (2.9), we conclude that

$$\left(u' \exp\left(\int_0^t a(s) ds\right)\right)' + \exp\left(\int_0^t a(s) ds\right) f(u) = 0. \tag{2.24}$$

Since $u(t, \alpha) > 0$ for all $t \in [0, \eta]$, we have

$$\left(u'(t, \alpha) \exp\left(\int_0^t a(s) ds\right)\right)' = -\exp\left(\int_0^t a(s) ds\right) f(u(t, \alpha)) < 0, \quad \forall t \in [0, \eta]. \tag{2.25}$$

Suppose on the contrary that there exists $\tau_2 \in [0, \eta)$ such that

$$B(\tau_2, \alpha) = B_2(u(\tau_2, \alpha)) + u'(\tau_2, \alpha) = 0. \tag{2.26}$$

Then we have from condition (C3) and the fact $u(\tau_2, \alpha) > 0$ that

$$u'(\tau_2, \alpha) = -B_2(u(\tau_2, \alpha)) < 0 \tag{2.27}$$

and accordingly

$$u'(\tau_2, \alpha) \exp\left(\int_0^{\tau_2} a(s) ds\right) < 0. \tag{2.28}$$

This together with (2.25) implies that

$$u'(t, \alpha) \exp\left(\int_0^t a(s) ds\right) < 0, \quad t \in [\tau_2, \eta], \tag{2.29}$$

and consequently

$$u'(t, \alpha) < 0, \quad t \in [\tau_2, \eta]. \tag{2.30}$$

This implies

$$u(\tau_2, \alpha) > u(\eta, \alpha). \tag{2.31}$$

By Remark 1.1 and (2.31), we get

$$B_2(u(\tau_2, \alpha)) \geq B_2(u(\eta, \alpha)). \tag{2.32}$$

From (2.30) and (C1)–(C2) and the fact $u''(t, \alpha) = -a(t)u'(t, \alpha) - f(u(t, \alpha))$, it follows that

$$u''(t, \alpha) < 0, \quad t \in [\tau_2, \eta] \tag{2.33}$$

and consequently

$$u'(\tau_2, \alpha) > u'(\eta, \alpha), \tag{2.34}$$

which together with (2.32) implies that

$$B(\tau_2, \alpha) = B_2(u(\tau_2, \alpha)) + u'(\tau_2, \alpha) > B_2(u(\eta, \alpha)) + u'(\eta, \alpha) = 0. \tag{2.35}$$

However this contradicts (2.26). □

Remark 2.5. From Lemmas 2.3 and 2.4, we have that if $\eta \in (0, T_\alpha)$ satisfies

$$B(\eta, \alpha) = 0. \tag{2.36}$$

Then

$$\eta = b(\alpha). \tag{2.37}$$

In other words, if $\alpha \in N$, then $b(\alpha)$ is the unique zero of $B(t, \alpha) = 0$ in $[0, \rho(\alpha))$. Therefore to prove that (1.1), (1.2) has at most one positive solution, it is sufficient to show that for any $l > 0$, there exists at most one $\alpha \in N$ such that $b(\alpha) = l$.

Now we denote the *variation* of $u(t, \alpha)$ by $\phi(t, \alpha) = \partial u(t, \alpha) / \partial \alpha$. Then, $\phi(t, \alpha)$ satisfies

$$\phi'' + a(t)\phi' + f'(u)\phi = 0, \tag{2.38}$$

$$\phi(0, \alpha) = 1, \quad \phi'(0, \alpha) = B'_1(\alpha). \tag{2.39}$$

LEMMA 2.6. *Suppose that*

$$B'_2(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) \neq 0, \quad \alpha \in N. \tag{2.40}$$

Then one of the following cases must occur

- (i) N is an open interval;
- (ii) $N = (0, j_1) \cup (j_2, \infty)$ with $0 < j_1 < j_2 < +\infty$. Moreover, $b'(\alpha) > 0$ for all $(0, j_1)$; $b'(\alpha) < 0$ for all (j_2, ∞) .

Proof. We firstly show that $b(\alpha) \in C^1(N)$ and $b'(\alpha) \neq 0$.

From Lemma 2.3, (C1)–(C2), we conclude that

$$u''(b(\alpha), \alpha) = -a(b(\alpha))u'(b(\alpha), \alpha) - f(u(b(\alpha), \alpha)) < 0. \tag{2.41}$$

This together with

$$B(b(\alpha), \alpha) = 0 \tag{2.42}$$

and (C3) and (2.19) implies that

$$\frac{\partial}{\partial t} B(t, \alpha) \Big|_{t=b(\alpha)} = B'_2(u(b(\alpha), \alpha))u'(b(\alpha), \alpha) + u''(b(\alpha), \alpha) < 0. \tag{2.43}$$

So by Implicit Function theorem, $b(\alpha)$ is well-defined as a function of α in N and $b(\alpha) \in C^1(N)$. Furthermore, it follows from (2.43) that N is an open set.

Differentiating both sides of (2.42) with respect to α , we obtain

$$B'_2(u(b(\alpha), \alpha)) [u'(b(\alpha), \alpha)b'(\alpha) + \phi(b(\alpha), \alpha)] + u''(b(\alpha), \alpha)b'(\alpha) + \phi'(b(\alpha), \alpha) = 0, \tag{2.44}$$

that is,

$$\begin{aligned} & [B'_2(u(b(\alpha), \alpha))u'(b(\alpha), \alpha) + u''(b(\alpha), \alpha)]b'(\alpha) \\ & + B'_2(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) = 0. \end{aligned} \tag{2.45}$$

which together with (2.40) implies that

$$b'(\alpha) \neq 0. \tag{2.46}$$

Next we show that if $\bar{\alpha} \in (0, \infty) \setminus N$ is such that there is a sequence $\{\alpha_n\} \subset N$ and $\alpha_n \rightarrow \bar{\alpha}$ as $n \rightarrow \infty$, then $b(\alpha_n) \rightarrow +\infty$.

Suppose on the contrary that $b(\alpha_n) \not\rightarrow +\infty$, then there exists a subsequence of $\{b(\alpha_n)\}$ which converges to a limit number t^* . Without loss of generality, we may suppose that $b(\alpha_n) \rightarrow t^*$ as $n \rightarrow \infty$, and consequently

$$B(t^*, \bar{\alpha}) = \lim_{n \rightarrow \infty} B(b(\alpha_n), \alpha_n) = 0. \tag{2.47}$$

However this contradicts $\bar{\alpha} \notin N$.

Finally we show that if N is not an open interval, then (ii) must occur.

Suppose $J_1 = (j_0, j_1)$ and $J_2 = (j_2, j_3)$ are two distinct components of N with $0 < j_1 < j_2 < \infty$. Then

$$\lim_{\alpha \rightarrow j_1^-} b(\alpha) = \lim_{\alpha \rightarrow j_2^+} b(\alpha) = +\infty. \tag{2.48}$$

Since $b(\alpha)$ is strictly monotonic in each component of N , we have that $b'(\alpha) > 0$ in J_1 , and $b'(\alpha) < 0$ in J_2 . Meanwhile

$$\lim_{\alpha \rightarrow j_0^+} b(\alpha) < +\infty, \quad \lim_{\alpha \rightarrow j_3^-} b(\alpha) < +\infty. \tag{2.49}$$

It follows that $j_0 = 0$ and $j_3 = +\infty$, and accordingly $N = (0, j_1) \cup (j_2, \infty)$ with $b'(\alpha) > 0$ in $(0, j_1)$, and $b'(\alpha) < 0$ in (j_2, ∞) . □

3. Proof of Theorem 1.2

By Remark 2.5, we only need to show that for any $l > 0$, there exists at most one $\alpha \in N$ such that $b(\alpha) = l$.

Recall that for any given $\alpha \in N$, (2.43), (2.45) hold. If we can show that

$$B'_2(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) > 0, \quad \alpha \in N \tag{3.1}$$

then it follows from (2.43) and (2.45) that

$$b'(\alpha) > 0, \quad \alpha \in N. \tag{3.2}$$

Thus by Lemma 2.6, N must be an open interval. Moreover we know from (3.2) that $b(\alpha)$ is a strictly increasing function on N . Thus, for any given $l > 0$, there is at most one $\alpha \in N$ such that $b(\alpha) = l$, and consequently, (1.1), (1.2) has at most one positive solution.

Proof of Theorem 1.2. Now we prove (3.1).

First we claim that

$$\phi(t, \alpha) > 0, \quad t \in [0, b(\alpha)]. \tag{3.3}$$

Suppose on the contrary that $\phi(t, \alpha)$ has a zero in $(0, b(\alpha)]$. We denote the first zero of $\phi(t, \alpha)$ in $(0, b(\alpha)]$ by t_3 , then $0 < t_3 \leq b(\alpha)$ and

$$u' \phi - u \phi' \Big|_{t=t_3} = -u(t_3, \alpha) \phi'(t_3, \alpha) \geq 0 \tag{3.4}$$

since $\phi(t_3, \alpha) = 0$ and $\phi(t, \alpha) > 0$ on $(0, t_3)$ implies $\phi'(t_3, \alpha) \leq 0$.

Notice that

$$\phi'' + a(t)\phi' + f'(u)\phi = 0 \tag{3.5}$$

so that using (C1) and (1.1) we can compute

$$\left[\exp \left(\int_0^t a(s) ds \right) (u' \phi - u \phi') \right]' = \exp \left(\int_0^t a(s) ds \right) [f'(u)u - f(u)] \phi < 0 \tag{3.6}$$

for $t \in (0, t_3)$. Next we compute from (C3) and (2.39) and (2.10)

$$\exp \left(\int_0^t a(s) ds \right) (u' \phi - u \phi') \Big|_{t=0} = B_1(\alpha) - \alpha B'_1(\alpha) = (B'_1(\xi_1(\alpha)) - B'_1(\alpha)) \alpha \leq 0, \tag{3.7}$$

where $\xi_1(\alpha) \in (0, \alpha)$. This means that

$$\exp \left(\int_0^t a(s) ds \right) (u' \phi - u \phi') \Big|_{t=t_3} < 0 \tag{3.8}$$

and accordingly

$$u' \phi - u \phi' \Big|_{t=t_3} < 0. \tag{3.9}$$

However this contradicts (3.4). Therefore (3.3) is true.

Using (3.5), (1.1), (C1) and (3.3), we can conclude

$$\left[\exp \left(\int_0^t a(s) ds \right) (u' \phi - u \phi') \right]' = \exp \left(\int_0^t a(s) ds \right) [f'(u)u - f(u)] \phi < 0, \quad t \in (0, b(\alpha)] \tag{3.10}$$

which together with (3.7) implies that

$$(u' \phi - u \phi') \Big|_{t=b(\alpha)} < 0. \tag{3.11}$$

Since

$$\begin{aligned} 0 &= B(b(\alpha), \alpha) \\ &= B_2(u(b(\alpha), \alpha)) + u'(b(\alpha), \alpha) \\ &= B'_2(\xi_2(\alpha))u(b(\alpha), \alpha) + u'(b(\alpha), \alpha) \end{aligned} \tag{3.12}$$

for some $\xi_2(\alpha) \in (0, u(b(\alpha), \alpha))$, we have that

$$u'(b(\alpha), \alpha) = -B'_2(\xi_2(\alpha))u(b(\alpha), \alpha). \tag{3.13}$$

This together with (3.11) implies

$$\begin{aligned} &-u(b(\alpha), \alpha) [B'_2(\xi_2(\alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha)] \\ &= -B'_2(\xi_2(\alpha))u(b(\alpha), \alpha)\phi(b(\alpha), \alpha) - u(b(\alpha), \alpha)\phi'(b(\alpha), \alpha) \\ &= u'(b(\alpha), \alpha)\phi(b(\alpha), \alpha) - u(b(\alpha), \alpha)\phi'(b(\alpha), \alpha) \\ &= u' \phi - u \phi' \Big|_{t=b(\alpha)} < 0 \end{aligned} \tag{3.14}$$

and consequently

$$B'_2(\xi_2(\alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) > 0. \tag{3.15}$$

Now we have from (C3) and the facts $\xi_2(\alpha) \leq u(b(\alpha), \alpha)$ and $\phi(b(\alpha), \alpha) > 0$ that

$$B_2(u(b(\alpha), \alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) \geq B'_2(\xi_2(\alpha))\phi(b(\alpha), \alpha) + \phi'(b(\alpha), \alpha) > 0. \tag{3.16}$$

Therefore (3.1) holds. □

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