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Operational properties and matrix representations of quantum measures

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Denoted by $M(A)$, $QM(A)$ and $SQM(A)$ the sets of all measures, quantum measures and subadditive quantum measures on a σ-algebra A, respectively. We observe that these sets are all positive cones in the real vector space $F(A)$ of all real-valued functions on A and prove that $M(A)$ is a face of $SQM(A)$. It is proved that the product of m grade-1 measures is a grade-m measure. By combining a matrix $\boldsymbol{M}_\mu$ to a quantum measure μ on the power set A_n of an n-element set X, it is proved that $\mu \prec\prec \nu$ (resp. $\mu \perp \nu$) if and only if $\boldsymbol{M}_\mu \prec\prec \boldsymbol{M}_\nu$ (resp. $\boldsymbol{M}_\mu\boldsymbol{M}_\nu=0$). Also, it is shown that two nontrivial measures μ and ν are mutually absolutely continuous if and only if $\mu\cdot\nu\in QM(A_n)$. Moreover, the matrices corresponding to quantum measures are characterized. Finally, convergence of a sequence of quantum measures on A_n is introduced and discussed; especially, the Vitali-Hahn-Saks theorem for quantum measures is proved.

quantum measure, absolute continuity, product, matrix representation, convergence

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1 Introduction

Recently, the mathematical properties of quantum theory have attracted much interest, for instance quantum measure theory [1–8], generalized quantum gates [9–20], quantum measurement [21,22], quantum operation [23–25], topology on quantum logic [26,27] and state theory and measure theory on quantum logic [28,29]. These mathematical theory of quantum systems are quite different from those for classical systems. They play important role in quantum information processing and the foundations of quantum mechanics. Quantum measure theory is one of key subjects of quantum theory, which has a lot of deep and difficult problems as well as important applications in functional analysis, probability theory and theoretical physics. Quantum measure spaces were introduced in 1994 by Sorkin in his studies of the histories approach to quantum mechanics and its applications to quantum gravity and cosmology [30,31]. In [1,2] Gudder discussed the theory of finite quantum measure spaces, which can be used to describe coherence between two classical events, and has been generalized to the infinite case in [3]. Subsequently, more literatures on this subject have emerged [4–8].

The aim of this paper is to discuss some operational properties, matrix representations, absolute continuity of quantum measures, discuss the convergence of a sequence of quantum measures on A_n and prove the Vitali-Hahn-Saks theorem for quantum measures.

First, let us recall some concepts and notations used later. Let (Ω,A) be a measurable space consisting of a nonempty set Ω and a σ-algebra A on Ω. If X and Y are disjoint sets, we use the notation $X\hat{\cup}Y$ for their union. Similarly, we write $\hat{\cup}_i X_i$ for the union of a sequence of mutually disjoint sets $\{X_i\}$.

Denote the set of nonnegative real numbers by $\mathbf{R}^+$, a set function $\mu: A\to\mathbf{R}^+$ is said to be additive if

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$$\mu(X \hat{\cup} Y) = \mu(X) + \mu(Y) \tag{1.1}$$

for all disjoint X, Y and μ is said to be countably additive if $\mu(\hat{\cup}_i X_i) = \sum_i \mu(X_i)$ for any sequence of mutually disjoint $X_i \in A$. If μ is countably additive, we call μ a measure and the triple (Ω, A, μ) a measure space.

A quantum measure μ describes the dynamics of a quantum system in the sense that $\mu(X)$ gives the propensity that the event X occurs. In the well-known two-slit experiment, a beam of particles is directed toward a screen containing two slits X_1 and X_2 and the particles that pass through the slits impinge upon a detection screen S. If $\mu(X_i)$ denotes the probability that a particle hits a small region $\Delta \subset S$ after passing through slit X_1 and X_2, then

$$\mu(X_1 \hat{\cup} X_2) \neq \mu(X_1) + \mu(X_2)$$

in general. However, recent experiments involving a three-slit screen indicate that μ satisfies

$$\begin{aligned}\mu(X_1 \hat{\cup} X_2 \hat{\cup} X_3) = \mu(X_1 \hat{\cup} X_2) \\ + \mu(X_2 \hat{\cup} X_3) + \mu(X_1 \hat{\cup} X_3) \\ - \mu(X_1) - \mu(X_2) - \mu(X_3).\end{aligned}$$

We now introduce a generalization of additivity in [3]. A set function $\mu: A \to \mathbf{R}^+$ is said to be grade-2 additive if

$$\begin{aligned}\mu(X \hat{\cup} Y \hat{\cup} Z) = \mu(X \hat{\cup} Y) + \mu(X \hat{\cup} Z) \\ + \mu(Y \hat{\cup} Z) - \mu(X) \\ - \mu(Y) - \mu(Z),\end{aligned} \tag{1.2}$$

and μ is regular if the following two conditions hold:

(1) $\mu(X) = 0 \Rightarrow \mu(X \hat{\cup} Y) = \mu(Y),\ \forall Y \in A$ with $X \cap Y = \varnothing$;

(2) $\mu(X \hat{\cup} Y) = 0 \Rightarrow \mu(X) = \mu(Y)$.

A map $\mu: A \to \mathbf{R}^+$ is said to be grade-1 additive if μ satisfies (1.1). We say that $\mu: A \to \mathbf{R}^+$ is continuous if

$$\lim_{i\to\infty} \mu(X_i) = \mu(\bigcup_{i=1}^{\infty} X_i)$$

for every increasing sequence $\{X_i\}$ in A and

$$\lim_{i\to\infty} \mu(X_i) = \mu(\bigcap_{i=1}^{\infty} X_i)$$

for every decreasing sequence $\{Y_i\}$ in A. A continuous grade-2 additive set function is called a grade-2 measure and a regular grade-2 measure is called a quantum measure (q-measure, for short). If μ is a grade-2 measure (quantum measure), then (Ω, A, μ) is called a grade-2 measure space (a quantum measure space, or a q-measure space). A quantum measure μ on A is said to be subadditive if it satisfies

$$\mu(X \hat{\cup} Y) \leqslant \mu(X) + \mu(Y),\ \forall X, Y \in A \text{ with } X \cap Y = \varnothing.$$

Denote the sets of all measures, quantum measures and subadditive quantum measures on A by $M(A)$, $QM(A)$ and $SQM(A)$, respectively.

It follows from (1.2) that any grade-2 additive function μ satisfies $\mu(\varnothing) = 0$. It is easy to check that if μ is grade-1 additive, then μ is regular and grade-2 additive. A grade-2 measure μ is not necessarily a measure. Thus, Quantum Measure Theory is a genuine generalization of classical measure theory.

In Section 2, we will discuss some operational properties of quantum measures. We shall observe that the sets $M(A)$, $QM(A)$ and $SQM(A)$ are all positive cones in the real vector space $F(A)$ of all real-valued functions on A and prove that $M(A)$ is a face of $SQM(A)$. It is also proved that the product of m grade-1 measures is a grade-m measure. In Section 3, by combining a matrix $\boldsymbol{M}_\mu$ to a quantum measure μ on the power set A_n of an n-element set X, it will be proved that $\mu \prec\prec \nu$ (resp. $\mu \perp \nu$) if and only if $\boldsymbol{M}_\mu \prec\prec \boldsymbol{M}_\nu$ (resp. $\boldsymbol{M}_\mu \boldsymbol{M}_\nu = 0$). Moreover, the matrices corresponding to quantum measures are characterized. Also, a necessary and sufficient condition for the product of two nontrivial measures to be a quantum measure will be given. Finally, the convergence of a sequence of quantum measures on A_n will be introduced and discussed; especially, the Vitali-Hahn-Saks theorem for quantum measures should be proved.

2 The operational properties of quantum measures

For a σ-algebra A on a set Ω, let $F(A)$ be the set of all real-valued functions on A. $\forall \lambda, \mu \in F(A), c \in \mathbf{R}$, and $\forall E \in A$, define

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E), (c\mu)(E) = c\mu(E).$$

Then $F(A)$ becomes a real vector space. A subset E of $F(A)$ is said to be convex if

$$x, y \in E \Rightarrow tx + (1-t)y \in E, \forall 0 \leqslant t \leqslant 1.$$

A convex subset G of a convex set E is said to be a face of E if

$$x, y \in E, \alpha \in (0,1), (1-\alpha)x + \alpha y \in G \Rightarrow x, y \in G.$$

A subset E of $F(A)$ is said to be a positive cone if $x, y \in E \Rightarrow tx, x+y \in E, \forall t \geqslant 0.$

Obviously, $M(A)$, $SQM(A)$ and $QM(A)$ are all positive cones and then convex sets in $F(A)$ with

$$M(A) \subset SQM(A) \subset QM(A) \subset F(A).$$

The following theorem tells us that every measure on A cannot be written as a convex combination of a measure and a subadditive measure that is not a measure on A.

Theorem 2.1 *M*(*A*) is a face of *SQM*(*A*).

Proof

Let $\mu,\nu\in SQM(A)$, $(1-\alpha)\mu+\alpha\nu\in M(A)$ for some $\alpha\in(0,1)$. Then for arbitrary disjoint sets $X,Y\in A$, we have

$$\begin{aligned}&[(1-\alpha)\mu+\alpha\nu](X\hat{\bigcup}Y)\\&=[(1-\alpha)\mu+\alpha\nu](X)+[(1-\alpha)\mu+\alpha\nu](Y)\\&=(1-\alpha)\mu(X)+\alpha\nu(X)+(1-\alpha)\mu(Y)+\alpha\nu(Y)\\&=(1-\alpha)[\mu(X)+\mu(Y)]+\alpha[\nu(X)+\nu(Y)].\end{aligned}$$

So

$$\begin{aligned}&(1-\alpha)[\mu(X\hat{\bigcup}Y)-\mu(X)-\mu(Y)]\\&+\alpha[\nu(X\hat{\bigcup}Y)-\nu(X)-\nu(Y)]=0.\end{aligned}$$

We see that $\mu(X\hat{\bigcup}Y)=\mu(X)+\mu(Y)$, and

$$\nu(X\hat{\bigcup}Y)=\nu(X)+\nu(Y).$$

Since

$$\begin{aligned}&\mu(X\hat{\bigcup}Y)-\mu(X)-\mu(Y)\leqslant 0,\\&\nu(X\hat{\bigcup}Y)-\nu(X)-\nu(Y)\leqslant 0.\end{aligned}$$

This shows that $\mu,\nu\in M(A)$. Consequently, *M*(*A*) is a face of *SQM*(*A*). This completes the proof.

Remark 1 *SQM*(*A*) and *M*(*A*) need not be the faces of *QM*(*A*).

Define a product of two measures as

$$(\mu\cdot\nu)(E)=\mu(E)\nu(E).$$

Recall that [2] a function $\mu:A\to\mathbf{R}^+$ is said to be a grade-*n* measure if it is continuous and grade-*n* additive: $\forall X_1,X_2,\cdots,X_{n+1}\in A$,

$$\begin{aligned}&\mu(X_1\hat{\bigcup}X_2\hat{\bigcup}\cdots\hat{\bigcup}X_{n+1})\\&=\sum_{1\leqslant i_1<i_2<\cdots<i_n\leqslant n+1}\mu(X_{i_1}\hat{\bigcup}X_{i_2}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_n})\\&-\sum_{1\leqslant i_1<i_2<\cdots<i_{n-1}\leqslant n+1}\mu(X_{i_1}\hat{\bigcup}X_{i_2}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{n-1}})\\&+\cdots+(-1)^{n+1}\sum_{1\leqslant i\leqslant n+1}\mu(X_i).\end{aligned}$$

It can be shown by induction that a grade-*n* measure is a grade-(*n*+1) measure, *n*=1, 2, 3···. We denote by $M^n(A)$ the sets of all grade-*n* measures. Especially, it is clear that $M(A)=M^1(A)$. The following two lemmas serve to prove that the product of *m* grade-1 measures is a grade-*m* measure.

Lemma 2.2 $(1\cdot 1=2)$ If $\mu,\nu\in M(A)$, then $\mu\cdot\nu\in M^2(A)$.

Proof

Since $\mu,\nu\in M(A)$, we have $\mu(X\hat{\bigcup}Y)=\mu(X)+\mu(Y)$, $\nu(X\hat{\bigcup}Y)=\nu(X)+\nu(Y)$ for arbitrary disjoint sets $X,Y\in A$. Hence,

$$\begin{aligned}(\mu\cdot\nu)(X\hat{\bigcup}Y)=&(\mu\cdot\nu)(X)+\mu(X)\nu(Y)\\&+\mu(Y)\nu(X)+(\mu\cdot\nu)(Y).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\mu\cdot\nu)(X\hat{\bigcup}Y\hat{\bigcup}Z)\\&=\mu(X\hat{\bigcup}Y\hat{\bigcup}Z)\ \nu(X\hat{\bigcup}Y\hat{\bigcup}Z)\\&=(\mu(X)+\mu(Y)+\mu(Z))(\nu(X)+\nu(Y)+\nu(Z))\\&=(\mu\cdot\nu)(X)+\mu(X)\nu(Y)+\mu(X)\nu(Z)+\mu(Y)\nu(X)\\&\ +(\mu\cdot\nu)(Y)+\mu(Y)\nu(Z)+\mu(Z)\nu(X)+\mu(Z)\nu(Y)\\&\ +(\mu\cdot\nu)(Z)\\&=(\mu\cdot\nu)(X\hat{\bigcup}Y)+(\mu\cdot\nu)(Y\hat{\bigcup}Z)+(\mu\cdot\nu)(X\hat{\bigcup}Z)\\&\ -(\mu\cdot\nu)(X)-(\mu\cdot\nu)(Y)-(\mu\cdot\nu)(Z).\end{aligned}$$

Therefore, $\mu\cdot\nu\in M^2(A)$. This completes the proof.

Remark 2 Lemma 2.2 tells us that $\mu\cdot\nu$ is grade-2 measure. However, $\mu\cdot\nu$ is not necessarily regular. See the following example.

Example 1 Let $\Omega=\{1,2\}$, μ a Dirac measure on the power set $P(\Omega)$ of Ω such that

$$\mu(\varnothing)=0,\mu(\{1\})=0,\mu(\{2\})=1,\mu(\{1,2\})=1,$$

and let ν be the counting measure on $P(\Omega)$. Take

$$X=\{1\},Y=\{2\},$$

then $(\mu\cdot\nu)(X)=0$ and $(\mu\cdot\nu)(Y)=1$, but

$$(\mu\cdot\nu)(X\hat{\bigcup}Y)=\mu(X\hat{\bigcup}Y)\nu(X\hat{\bigcup}Y)=2.$$

This shows that the product $\mu\cdot\nu$ of two measures μ and ν does not satisfy the regularity and then is not a quantum measure on $P(\Omega)$.

However, it is easy to see from Lemma 2.2 that the square μ^2 of a measure μ on *A* is always a quantum measure on *A* (Corollary 2.3 below). Furthermore, a necessary and sufficient condition for the product of two nontrivial measures to be a quantum measure will be given in next section.

Corollary 2.3 If $\mu\in M(A)$, then $\mu^2\in QM(A)$.

Lemma 2.4 $(2\cdot 1=3)$ If $\mu\in M^2(A)$, $\nu\in M(A)$, then $\mu\cdot\nu\in M^3(A)$.

Proof

From the condition, we can get

$$\begin{aligned}&\mu(X\hat{\bigcup}Y\hat{\bigcup}Z)\\&=\mu(X\hat{\bigcup}Y)+\mu(X\hat{\bigcup}Z)+\mu(Y\hat{\bigcup}Z)-\mu(X)-\mu(Y)-\mu(Z),\end{aligned}$$

and

$$\nu(X\hat{\bigcup}Y)=\nu(X)+\nu(Y)$$

for arbitrary disjoint sets $X,Y,Z\in A$. Then for arbitrary disjoint sets $X_1,X_2,X_3,X_4\in A$,

$$\begin{aligned}
&(\mu\cdot\nu)(X_1\hat{\bigcup}X_2\hat{\bigcup}X_3\hat{\bigcup}X_4)\\
&=\mu(X_1\hat{\bigcup}X_2\hat{\bigcup}X_3\hat{\bigcup}X_4)\nu(X_1\hat{\bigcup}X_2\hat{\bigcup}X_3\hat{\bigcup}X_4)\\
&=\left(\sum_{1\leqslant i<j<k\leqslant 4}\mu(X_i\hat{\bigcup}X_j\hat{\bigcup}X_k)-\sum_{1\leqslant i<j\leqslant 4}\mu(X_i\hat{\bigcup}X_j)\right.\\
&\quad\left.+\sum_{1\leqslant i\leqslant 4}\mu(X_i)\right)\left(\sum_{1\leqslant i\leqslant 4}\nu(X_i)\right)\\
&=\left(\sum_{1\leqslant i<j<k\leqslant 4}(\mu(X_i\hat{\bigcup}X_j)+\mu(X_i\hat{\bigcup}X_k)+\mu(X_j\hat{\bigcup}X_k)\right.\\
&\quad-\mu(X_i)-\mu(X_j)-\mu(X_k))-\sum_{1\leqslant i<j\leqslant 4}\mu(X_i\hat{\bigcup}X_j)\\
&\quad\left.+\sum_{1\leqslant i\leqslant 4}\mu(X_i)\right)\left(\sum_{1\leqslant i\leqslant 4}\nu(X_i)\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{1\leqslant i<j<k\leqslant 4}(\mu\cdot\nu)(X_i\hat{\bigcup}X_j\hat{\bigcup}X_k)-\sum_{1\leqslant i<j\leqslant 4}(\mu\cdot\nu)(X_i\hat{\bigcup}X_j)\\
&\quad+\sum_{1\leqslant i\leqslant 4}(\mu\cdot\nu)(X_i)\\
&=\sum_{1\leqslant i<j<k\leqslant 4}\mu(X_i\hat{\bigcup}X_j\hat{\bigcup}X_k)\nu(X_i\hat{\bigcup}X_j\hat{\bigcup}X_k)\\
&\quad-\sum_{1\leqslant i<j\leqslant 4}\mu(X_i\hat{\bigcup}X_j)\nu(X_i\hat{\bigcup}X_j)+\sum_{1\leqslant i\leqslant 4}\mu(X_i)\nu(X_i)\\
&=\sum_{1\leqslant i<j<k\leqslant 4}(\mu(X_i\hat{\bigcup}X_j)+\mu(X_i\hat{\bigcup}X_k)+\mu(X_j\hat{\bigcup}X_k)\\
&\quad-\mu(X_i)-\mu(X_j)-\mu(X_k))\cdot(\nu(X_i)+\nu(X_j)+\nu(X_k))\\
&\quad-\sum_{1\leqslant i<j\leqslant 4}\mu(X_i\hat{\bigcup}X_j)\nu(X_i\hat{\bigcup}X_j)+\sum_{1\leqslant i\leqslant 4}\mu(X_i)\nu(X_i).
\end{aligned}$$

By computation, we have the right-hand sides of above equalities equal. Now we can conclude that

$$\begin{aligned}
&(\mu\cdot\nu)(X_1\hat{\bigcup}X_2\hat{\bigcup}X_3\hat{\bigcup}X_4)\\
&=\sum_{1\leqslant i<j<k\leqslant 4}(\mu\cdot\nu)(X_i\hat{\bigcup}X_j\hat{\bigcup}X_k)-\sum_{1\leqslant i<j\leqslant 4}(\mu\cdot\nu)(X_i\hat{\bigcup}X_j)\\
&\quad+\sum_{1\leqslant i\leqslant 4}(\mu\cdot\nu)(X_i).
\end{aligned}$$

This shows that $\mu\cdot\nu\in M^3(A)$. This completes the proof.

Theorem 2.5 If $\mu_1,\mu_2,\cdots,\mu_m\in M(A)$, then

$$\mu_1\cdot\mu_2\cdot\cdots\cdot\mu_m\in M^m(A).$$

Proof

We prove the result by induction.

Step 1 When m=2, use Lemma 2.2 and when m=3, use Lemma 2.4.

Step 2 Suppose that the conclusion is valid for m−1.

For m+1 disjoint sets $X_1,X_2,\cdots,X_{m+1}$ in A, we have $\mu_1\cdot\mu_2\cdot\cdots\cdot\mu_{m-1}\in M^{m-1}(A)$. Put

$$\mu=\mu_1\cdot\mu_2\cdot\cdots\cdot\mu_{m-1},\nu=\mu_m.$$

Then

$$\begin{aligned}
&(\mu\cdot\nu)(X_1\hat{\bigcup}X_2\hat{\bigcup}\cdots\hat{\bigcup}X_{m+1})\\
&=\mu(X_1\hat{\bigcup}X_2\hat{\bigcup}\cdots\hat{\bigcup}X_{m+1})\nu(X_1\hat{\bigcup}X_2\hat{\bigcup}\cdots\hat{\bigcup}X_{m+1})\\
&=\left(\sum_{1\leqslant i_1<\cdots<i_m\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_m})\right.\\
&\quad-\sum_{1\leqslant i_1<\cdots<i_{m-1}\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{m-1}})+\cdots\\
&\quad\left.+(-1)^{m+1}\sum_{1\leqslant i\leqslant m+1}\mu(X_i)\right)\left(\sum_{1\leqslant i\leqslant m+1}\nu(X_i)\right)\\
&=\sum_{1\leqslant i_1<\cdots<i_m\leqslant m+1}(\mu\cdot\nu)(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_m})\\
&\quad+\sum_{1\leqslant i_1<\cdots<i_m\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_m})\nu(X_{i_{m+1}})\\
&\quad-\sum_{1\leqslant i_1<\cdots<i_{m-1}\leqslant m+1}(\mu\cdot\nu)(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{m-1}})\\
&\quad-\sum_{1\leqslant i_1<\cdots<i_m\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{m-1}})\nu(X_{i_m}\hat{\bigcup}X_{i_{m+1}})\\
&\quad+\cdots+(-1)^{m+1}\sum_{1\leqslant i\leqslant m+1}\mu(X_i)\nu(X_i)\\
&\quad+(-1)^{m+1}\sum_{1\leqslant i\leqslant m+1,j\neq i}\mu(X_i)\nu(X_j).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{1\leqslant i_1<\cdots<i_m\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_m})\nu(X_{i_{m+1}})\\
&=\sum_{1\leqslant i_1<\cdots<i_{m-1}\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{m-1}})\nu(X_{i_{m+1}})\\
&\quad-\sum_{1\leqslant i_1<\cdots<i_{m-2}\leqslant m+1}\mu(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{m-2}})\nu(X_{i_{m+1}})\\
&\quad+\cdots+(-1)^{m+1}\sum_{1\leqslant i\leqslant m+1}\mu(X_i)\nu(X_{i_{m+1}}),
\end{aligned}$$

we have

$$\begin{aligned}
&(\mu\cdot\nu)(X_1\hat{\bigcup}X_2\hat{\bigcup}\cdots\hat{\bigcup}X_{m+1})\\
&=\sum_{1\leqslant i_1<\cdots<i_m\leqslant m+1}(\mu\cdot\nu)(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_m})\\
&\quad-\sum_{1\leqslant i_1<\cdots<i_{m-1}\leqslant m+1}(\mu\cdot\nu)(X_{i_1}\hat{\bigcup}\cdots\hat{\bigcup}X_{i_{m-1}})\\
&\quad+\cdots+(-1)^{m+1}\sum_{1\leqslant i\leqslant m+1}(\mu\cdot\nu)(X_i).
\end{aligned}$$

Therefore, $\mu_1\cdot\mu_2\cdot\cdots\cdot\mu_m\in M^m(A)$. This completes the proof.

Conjecture The product of a grade-p measure and a grade-q measure is a grade-$(p+q)$ measure.

Remark 3 If we can get that for any $\mu \in M^2(A)$, there exist $\mu_1, \mu_2 \in M(A)$ such that $\mu = \mu_1 \cdot \mu_2$, then the Conjecture is equivalent to the proposition that the product of $p+q$ grade-1 additive measures is a grade-$(p+q)$ measure, so by Theorem 2.4, we can prove the Conjecture is valid. However it is unfortunate that not all grade-2 additive measures can be decomposed into the product of two measures. Suppose that $\mu \in M^2(A)$, and there exist $\mu_1, \mu_2 \in M(A)$ satisfy $\mu = \mu_1 \cdot \mu_2$, then for all disjoint X, Y in A, we have

$$\begin{aligned}&\mu(X \hat{\cup} Y)\\ &= \mu_1(X \hat{\cup} Y)\mu_2(X \hat{\cup} Y)\\ &= \mu(X) + \mu(Y) + \mu_1(X)\mu_2(Y) + \mu_1(Y)\mu_2(X),\end{aligned}$$

so $\mu(X \hat{\cup} Y) - \mu(X) - \mu(Y) \geqslant 0$ since

$$\mu_1(X)\mu_2(Y) + \mu_1(Y)\mu_2(X) \geqslant 0.$$

This is not necessarily true. See Example 2.1 below.

Example 2.1 Let $\Omega = (0,1)$, A the Borel σ-algebra on Ω, and let m be the Lebesgue measure. $\forall E \in A$, set $\mu(E) = \frac{1}{2}$ if $\frac{1}{2} \in E$; $\mu(E) = m(E)$ otherwise.

Obviously, μ is a quantum measure. Clearly,

$$\mu((0,\tfrac{1}{2}]) - \mu((0,\tfrac{1}{3}]) - \mu((\tfrac{1}{3},\tfrac{1}{2}]) < 0.$$

3 The matrix representation of a quantum measure

Let $\Omega = \{x_1, x_2, \cdots, x_n\}$ with $x_i \neq x_j (i \neq j)$, A_n denote the power set of Ω and μ a quantum measure on A_n. Then the triple (Ω, A_n, μ) is a quantum measure space. For convenience, we write

$$\mu(x_i) = \mu(\{x_i\}), \mu(x_i, x_j) = \mu(\{x_i, x_j\}),$$

and so on. The following discussion is based on this kind of quantum measure space. First, the Dirac measure δ_i defined by

$\delta_i(X) = 1$ if $x_i \in X$; $\delta_i(X) = 0$ otherwise.

Clearly, δ_i is a quantum measure on A_n. It is proved in [8] that $\hat{\delta}_{ij} := \delta_i \delta_j (i \neq j)$ satisfies grade-2 additivity. However, the following result implies that it does not satisfy the regularity and then not a quantum measure.

Theorem 3.1 When $n>1$, the product $\hat{\delta}_{ij} (i \neq j)$ is not a quantum measure on A_n.

Proof

Let $i \neq j$, $X = \{x_i\}, Y = \{x_j\}$. Then $\hat{\delta}_{ij}(X) = 0$ and $\hat{\delta}_{ij}(Y) = 0$ and $\hat{\delta}_{ij}(X \hat{\cup} Y) = 1$.

This shows that $\hat{\delta}_{ij} = \delta_i \delta_j$ does not satisfy the regularity. This completes the proof.

Lemma 3.2 [8] The set

$$\{\delta_i : 1 \leqslant i \leqslant n\} \cup \{\hat{\delta}_{jk} : j, k = 1, 2, \cdots, n, j < k\}$$

forms a basis for $QM(A_n)$ in such a way that

$$\begin{aligned}\mu &= \sum_{i=1}^{n} \mu(x_i)\delta_i\\ &\quad + \sum_{1 \leqslant i < j \leqslant n} \big(\mu(\{x_i, x_j\}) - \mu(x_i) - \mu(x_j)\big)\hat{\delta}_{ij}\end{aligned}$$

for all $\mu \in QM(A_n)$.

For each $\mu \in QM(A_n)$, we define a matrix $\boldsymbol{M}_\mu$ corresponding to μ:

$$\boldsymbol{M}_\mu = \begin{pmatrix} \mu(x_1) & \mu\{x_1,x_2\} - \mu(x_1) - \mu(x_2) & \cdots & \mu\{x_1,x_n\} - \mu(x_1) - \mu(x_n) \\ 0 & \mu(x_2) & \cdots & \mu\{x_2,x_n\} - \mu(x_2) - \mu(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu(x_n) \end{pmatrix}.$$

Thus, we obtain a mapping $\pi : QM(A_n) \to M_n(\mathbf{R})$, where $M_n(\mathbf{R})$ is the algebra of all n by n matrices over $\mathbf{R}$. It is evident that π is injective. Denote by $S(\boldsymbol{M})$ the sum of all entries of arbitrary matrix $\boldsymbol{M}$.

In the following theorem, $\boldsymbol{M}^{s(k)}$ denotes any k-lever principle sub-matrix of an $n \times n$ matrix $\boldsymbol{M}$ by removing $n-k$ rows and columns from $\boldsymbol{M}$, and write $I_n = \{1, 2, \cdots, n\}$. For example, when $k=1$, $\boldsymbol{M}^{s(1)}$ denotes any one of the 1×1 matrices:

$$[a_{11}], [a_{22}], \cdots, [a_{nn}],$$

where $\boldsymbol{M} = [a_{ij}] \in M_n(\mathbf{R})$.

The following theorem characterizes the matrices corresponding to quantum measures.

Theorem 3.3 Let $\boldsymbol{M} = [a_{ij}] \in M_n(\mathbf{R})$ be an upper triangular matrix. Then there exists a $\mu \in QM(A_n)$, such that $\boldsymbol{M}_\mu = \boldsymbol{M}$ if and only if the following conditions are satisfied.

(1) $S(\boldsymbol{M}^{s(k)}) \geqslant 0$ for every $1 \leqslant k \leqslant n$.

(2) $\sum_{i,j \in I} a_{ij} = 0$ implies $\sum_{i \in I, k \in J} a_{ik} + \sum_{i \in I, k \in J} a_{kj} = 0$ for all $I, J \in I_n$ and $I \cap J = \varnothing$.

(3) $\sum_{i,j \in I} a_{ij} = 0$ implies $\sum_{i,j \in I'} a_{ij} = \sum_{i,j \in I''} a_{ij}$, where $I \subset I_n$, $I = I' \cup I''$ and $I' \cap I'' = \varnothing$.

Proof

Necessity. Suppose that there exists a $\mu \in QM(A_n)$, such that $\boldsymbol{M}_\mu = \boldsymbol{M}$. Then for every $1 \leqslant k \leqslant n$, there exist positive integers

$$1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$$

such that $S(\boldsymbol{M}^{s(k)}) = \mu(x_{i_1}, x_{i_2}, \cdots, x_{i_k}) \geqslant 0$. Therefore, (1) holds. Let $I, J \subset I_n$, $I \cap J = \varnothing$ and $\sum_{i,j \in I} a_{ij} = 0$. Put

$$A = \{x_i : i \in I\}, B = \{x_j : j \in J\}.$$

Then

$$A, B \in A_n, \mu(A) = \sum_{i,j \in I} a_{ij} = 0.$$

Since $A \cap B = \varnothing$, the regularity of μ yields that $\mu(A \cup B) = \mu(B)$. Thus

$$\sum_{i,j \in I \cup J} a_{ij} = \sum_{i,j \in J} a_{ij},$$

so

$$\sum_{i \in I, k \in J} a_{ik} + \sum_{i \in I, k \in J} a_{ik} = 0.$$

This shows that (2) is satisfied. Let

$$I \subset I_n,\ I = I' \cup I'',\ I' \cap I'' = \varnothing \text{ and } \sum_{i,j \in I} a_{ij} = 0.$$

Put $A = \{x_i : i \in I'\}, B = \{x_j : j \in I''\}$. Then

$$A, B \in A_n, A \cap B = \varnothing, \mu(A \cup B) = \sum_{i,j \in I} a_{ij} = 0.$$

From the regularity of μ, we see that $\mu(A) = \mu(B)$, that is, $\sum_{i,j \in I'} a_{ij} = \sum_{i,j \in I''} a_{ij}$. Hence, (3) is also satisfied.

Sufficiency. Suppose that conditions (1)–(3) are satisfied. We define $\mu : A_n \to \mathbf{R}^+$ by

$$\mu(E) := \sum_{x_i, x_j \in E} a_{ij}, \forall E \in A_n, E \neq \varnothing; \mu(\varnothing) = 0. \quad (3.1)$$

For example,

$$\begin{aligned} &\mu(x_i) = a_{ii} (1 \leqslant i \leqslant n), \\ &\mu(x_i, x_j) = a_{ii} + a_{jj} + a_{ij} (1 \leqslant i, j \leqslant n, i \neq j), \\ &\mu(x_i, x_j, x_k) = a_{ii} + a_{jj} + a_{kk} + a_{ij} + a_{jk} + a_{ik}. \end{aligned} \quad (3.2)$$

Let $E \in A_n, E \neq \varnothing$. Then $E = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$ for some positive integers $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Hence, condition (1) implies that

$$\mu(E) = S(\boldsymbol{M}^{s(k)}) \geqslant 0,$$

where $\boldsymbol{M}^{s(k)}$ is a k-lever principle sub-matrix of $\boldsymbol{M}$.

For arbitrary disjoint sets $A, B, C \in A_n$, we have

$$\begin{aligned} \mu(A \hat{\cup} B \hat{\cup} C) &= \sum_{x_i, x_j \in A \hat{\cup} B \hat{\cup} C} a_{ij} \\ &= \sum_{x_i \in A, x_j \in B} a_{ij} + \sum_{x_i \in A, x_j \in C} a_{ij} + \sum_{x_i \in B, x_j \in C} a_{ij} \\ &\quad + \sum_{x_i \in B, x_j \in A} a_{ij} + \sum_{x_i \in C, x_j \in A} a_{ij} + \sum_{x_i \in C, x_j \in B} a_{ij} \\ &\quad + \sum_{x_i, x_j \in A} a_{ij} + \sum_{x_i, x_j \in B} a_{ij} + \sum_{x_i, x_j \in C} a_{ij} \\ &= \mu(A \hat{\cup} B) + \mu(A \hat{\cup} C) + \mu(B \hat{\cup} C) \\ &\quad - \mu(A) - \mu(B) - \mu(C). \end{aligned}$$

It is clear that μ is grade-2 additive. To prove the regularity, we let $A = \{x_i : i \in I\}$ with $\mu(A) = 0$, i.e. $\sum_{i,j \in I} a_{ij} = 0$. Then for every $B = \{x_j : j \in J\} \in A_n$ with $A \cap B = \varnothing$, we have $I \cap J = \varnothing$. Thus, the condition (2) implies that $\sum_{i \in I, k \in J} a_{ik} + \sum_{i \in I, k \in J} a_{ik} = 0$, which implies that $\sum_{i,j \in I \cup J} a_{ij} = \sum_{i,j \in J} a_{ij}$. That is, $\mu(A \cup B) = \mu(B)$.

Let $A = \{x_i : i \in I'\}, B = \{x_j : j \in I''\} \in A_n$ with $A \cap B = \varnothing$, $\mu(A \cup B) = 0$. Then $\sum_{i,j \in I} a_{ij} = 0$, where $I = I' \cup I''$. By (3), we obtain that

$$\sum_{i,j \in I'} a_{ij} = \sum_{i,j \in I''} a_{ij}.$$

That is, $\mu(A)=\mu(B)$. This shows that μ is regular. Thus, μ is a quantum measure. By (3.2), we see that $\boldsymbol{M}_\mu=\boldsymbol{M}$. The proof is completed.

Similarly, we have

Theorem 3.4 If $\mu \in M(A_n)$, then $\boldsymbol{M}_\mu = [a_{ij}]$ is a diagonal matrix with $a_{ii} \geqslant 0$ for each $1 \leqslant i \leqslant n$; Conversely, suppose that $\boldsymbol{M} = [a_{ij}]$ is a diagonal matrix with $a_{ii} \geqslant 0$ for each $1 \leqslant i \leqslant n$, then there exists a unique $\mu \in M(A_n)$ such that $\boldsymbol{M}_\mu = \boldsymbol{M}$.

Proof

If $\mu \in M(A_n)$, then

$$a_{ij} = \mu(\{x_i, x_j\}) - \mu(x_i) - \mu(x_j) = 0$$

when $i<j$ and $a_{ii} \geqslant 0$.

Conversely, let $\boldsymbol{M}=[a_{ij}]$ be a diagonal matrix with $a_{ij} \geqslant 0$ for each $1 \leqslant i \leqslant n$. Similarly to construction (3.1), we can get that there exists a unique μ that satisfies

$$\mu(x_1, x_2, \cdots, x_n) = \mu(x_1) + \mu(x_2) + \cdots + \mu(x_n).$$

Therefore, $\mu \in M(A_n)$ and clearly we obtain $\boldsymbol{M}_\mu = \boldsymbol{M}$. This completes the proof.

Recall that [32] for two measures λ,μ on A, λ is said to be absolutely continuous with respect to μ if $\mu(E) = 0$ implies $\lambda(E) = 0$, written $\lambda \prec\prec \mu$. If there is a set $S \in A$ such that $\lambda(E) = \lambda(S \cap E)$ for every $E \in A$, then we say that λ is concentrated on *S*. Moreover, λ and μ are said to be mutually singular if λ is concentrated on S_1 and μ is concentrated on S_2 and $S_1 \cap S_2 = \varnothing$, written $\lambda \perp \mu$.

Now we generalize these concepts to quantum measures.

Let $\lambda, \mu \in QM(A_n)$. λ is said to be absolutely continuous with respect to μ, written $\lambda \prec\prec \mu$, if

$$\mu(E) = 0 \text{ implies } \lambda(E) = 0.$$

If there is a set $S \in A_n$ such that $\lambda(E) = \lambda(S \cap E)$ for every $E \in A_n$, then we say that λ is concentrated on *S*. Moreover, λ and μ are said to be mutually singular if λ is concentrated on S_1 and μ is concentrated on S_2 and $S_1 \cap S_2 = \varnothing$, written as $\lambda \perp \mu$.

Let $\boldsymbol{M} = [a_{ij}], \boldsymbol{N} = [b_{ij}] \in M_n(\mathbf{R})$. Define

(1) $M \prec\prec N$ if $I \subset I_n, \sum_{i,j \in I} b_{ij} = 0 \Rightarrow \sum_{i,j \in I} a_{ij} = 0$;

(2) $M \perp N$ if $MN = 0$.

Theorem 3.5 Let $\mu, \nu \in QM(A_n)$, $\boldsymbol{M}_\nu = [a_{ij}]$, $\boldsymbol{M}_\mu = [b_{ij}]$. Then

(i) $\nu \prec\prec \mu$ if and only if $\boldsymbol{M}_\nu \prec\prec \boldsymbol{M}_\mu$.

(ii) $\nu \perp \mu$ if and only if $\boldsymbol{M}_\nu \boldsymbol{M}_\mu = 0$.

Proof

(i) By definition, we have $\nu \prec\prec \mu$ if and only if $\mu(E) = 0$ implies $\nu(E) = 0$ if and only if

$$\mu(E) = \sum\nolimits_{x_i, x_j \in E} b_{ij} = 0$$

implies $\nu(E) = \sum\nolimits_{x_i, x_j \in E} a_{ij} = 0$ since (3.1).

(ii) By definition, we have $\nu \perp \mu$ if and only if there exists a pair of disjoint sets S_1, S_2 such that μ is concentrated on S_1 and ν is concentrated on S_2 if and only if $M_\nu M_\mu = 0$. This completes the proof.

Theorem 3.6 Let $\mu, \nu \in M(A_n) \setminus \{0\}$. Then μ and ν are mutually absolutely continuous if and only if $\mu \cdot \nu \in QM(A_n)$.

Proof

Necessity. Let μ and ν be mutually absolutely continuous. Then we can get that

$$\mu(E) = 0 \Leftrightarrow \nu(E) = 0.$$

Therefore, $(\mu \cdot \nu)(E) = 0$ implies $\mu(E) = \nu(E) = 0$ and for all $B \in A_n$ with $E \cap B = \varnothing$,

$$(\mu \cdot \nu)(E \hat{\cup} B) = \mu(E \hat{\cup} B)\nu(E \hat{\cup} B) = (\mu \cdot \nu)(B).$$

Suppose that $(\mu \cdot \nu)(E \hat{\cup} B) = 0$, then $\mu(E \hat{\cup} B) = \nu(E \hat{\cup} B) = 0$. We can see $\mu(E) = \mu(B)$ and $\nu(E) = \nu(B)$ since $\mu, \nu \in M(A_n) \setminus \{0\}$. Thus,

$$(\mu \cdot \nu)(E) = (\mu \cdot \nu)(B).$$

This shows that $\mu \cdot \nu$ is regular. It follows from Lemma 2.2 that $\mu \cdot \nu \in QM(A_n)$.

Sufficiency. Let $\mu \cdot \nu \in QM(A_n)$. Firstly, we will prove that $\nu \prec\prec \mu$. Since $\mu \neq 0$, there exists a set $B \in A_n$ such that $\mu(B) \neq 0$. Suppose that $\mu(E) = 0$. Then $(\mu \cdot \nu)(E) = 0$ and $\mu(B \setminus E) \neq 0$. From the regularity of $\mu \cdot \nu$, we see that

$$(\mu \cdot \nu)(E \hat{\cup} (B \setminus E)) = (\mu \cdot \nu)(B \setminus E)). \tag{3.3}$$

From (3.3), we can compute that

$$\mu(E)\nu(B \setminus E) + \nu(E)\mu(B \setminus E) = 0.$$

Since $\mu(B \setminus E) \neq 0$ and $\mu(E) = 0$, we see that $\nu(E) = 0$. This shows that $\nu \prec\prec \mu$.

Similarly, we can show that $\mu \prec\prec \nu$. Therefore, μ and ν are mutually absolutely continuous. This completes the proof.

Clearly, Theorem 3.6 is also valid for two measures μ and ν on any σ-algebra *A* on a set *X*.

Suppose that $\{\mu_m\}_{m=1}^{\infty} \subset QM(A_n)$. If there exists a $\mu \in QM(A_n)$ such that

$$\| \boldsymbol{M}_{\mu_m} - \boldsymbol{M}_\mu \| \to 0 \ (m \to \infty),$$

then we write $\mu_m \to \mu (m \to \infty)$ and call μ the limit of the sequence.

If we denote

$$\boldsymbol{M}_{\mu_m} = [a_{ij}^{(m)}], \ \boldsymbol{M}_\mu = [a_{ij}],$$

then it is clear that

$$\mu_m \to \mu(m \to \infty) \Leftrightarrow \forall i, j \in I_n, \quad a_{ij} = \lim_{m \to \infty} a_{ij}^{(m)}.$$

From this fact and the definition of $\boldsymbol{M}_{\mu_m}$ and $\boldsymbol{M}_\mu$, we get the following.

Theorem 3.7 Let $\mu_m, \mu \in QM(A_n)$ for all positive integers *m*. Then $\mu_m \to \mu(m \to \infty)$ if and only if $\mu_m(x_i) \to \mu(x_i)$ and $\mu_m(x_i, x_j) \to \mu(x_i, x_j)(i < j)$ for all $x_i \in \Omega$.

A sequence $\{\mu_m\}_{m=1}^{\infty} \subset QM(A_n)$ is said to be a Cauchy sequence if

$$\| \boldsymbol{M}_{\mu_m} - \boldsymbol{M}_{\mu_{m'}} \| \to 0 (\forall m, m' \to \infty).$$

It is easy to see that, every convergent sequence

$\{\mu_m\}_{m=1}^{\infty} \subset QM(A_n)$ is a Cauchy sequence. But the inverse is not valid. In other words, the space $QM(A_n)$ is not complete, see Example 3.1 below. However, the set $M(A_n)$ of all measures is clearly complete.

Example 3.1 Let

$$M_m = \begin{pmatrix} \frac{1}{m} & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

be $n\times n$ matrices for $m=1, 2,\cdots$. Then the conditions (1)–(3) of Theorem 3.3 for $M=M_m$ are satisfied. Thus, $\exists \mu_m \in QM(A_n)$ such that

$$M_m = M_{\mu_m} (m = 1, 2, \cdots) .$$

Since

$$\left\| M_{\mu_m} - M \right\| \to 0 (m \to \infty) ,$$

$\{\mu_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $QM(A_n)$. But does not exist a $\mu \in QM(A_n)$ such that

$$\mu_m \to \mu (m \to \infty)$$

since M does not satisfy the condition (2) of Theorem 3.3.

The Vitali-Hahn-Saks theorem is one of the fundamental results in measure theory [33]. In the quantum measure space, we have the following.

Theorem 3.8 Let $\nu_m, \mu, \nu \in QM(A_n)$ for all positive integers m. If $\nu_m \to \nu (m \to \infty)$ and $\nu_m \prec\prec \mu$ for each $m \in \mathbf{N}$, then $\nu \prec\prec \mu$.

Proof

Denote $\boldsymbol{M}_{\nu_m} = [a_{ij}^{(m)}], \boldsymbol{M}_{\mu} = [b_{ij}],\ \boldsymbol{M}_{\nu} = [a_{ij}]$. Let $\sum_{i,j\in I} b_{ij} = 0$. Since $\nu_m \prec\prec \mu$ for each $m \in \mathbf{N}$, by Theorem 3.5, we have $\boldsymbol{M}_{\nu_m} \prec\prec \boldsymbol{M}_{\mu}$ for each $m \in \mathbf{N}$. Thus, $\forall I \subset I_n$, we have

$$\sum_{i,j\in I} a_{ij}^{(m)} = 0, \quad m=1, 2,\cdots.$$

Since $\nu_m \to \nu$ ($m \to \infty$), $a_{ij}^{(m)} \to a_{ij} (m \to \infty)$ for all i, $j=1,2,\cdots n$. Hence, $\sum_{i,j\in I} a_{ij} = 0$. This shows that $\boldsymbol{M}_{\nu} \prec\prec \boldsymbol{M}_{\mu}$. It follows from Theorem 3.5 that $\nu \prec\prec \mu$. This completes the proof.

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