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Capillary Floating and the Billiard Ball Problem

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Abstract. In a study of capillary floating, Finn (J Math Fluid Mech 11:443–458, 2009) described a procedure for determining cross-sections of non-circular, infinite convex cylinders that float horizontally on a liquid surface in every orientation with contact angle $\pi/2$. Finn's procedure yielded incomplete results for other contact angles; he raised the question as to whether an analogous construction would be feasible in that case. In the note, Finn (J Math Fluid Mech 11:464–465, 2009) pointed out a connection with an independent problem on billiard caustics citing the unpublished work (Gutkin in Proceedings of the Workshop on Dynamics and Related Questions, PennState University, 1993) of the present author. Here we present a solution of the billiard problem in full detail, thus settling Finn's question in a surprising way. In particular, we show that such floating cylinders exist if and only if the contact angle lies in a certain, explicitly described countably dense set. Moreover, for each element γ in this set we exhibit a family of convex, non-circular cylinders that float in every orientation with contact angle γ. Our discussion contains other material of independent interest for the billiard ball problem.

Mathematics Subject Classification (2000). 76B45, 76D45, 52A10, 37A45, 37E10, 37E40.

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Contents

1. Introduction: Floating in Neutral Equilibrium and the Billiard Ball Problem

The mathematical theory of capillarity goes back to 1806. In his famous treatise on celestial mechanics [32] Laplace discussed a broad range of problems related to surface tension at fluid interfaces, among them a theory of capillary floating. One of the major open problems in this subject is to determine

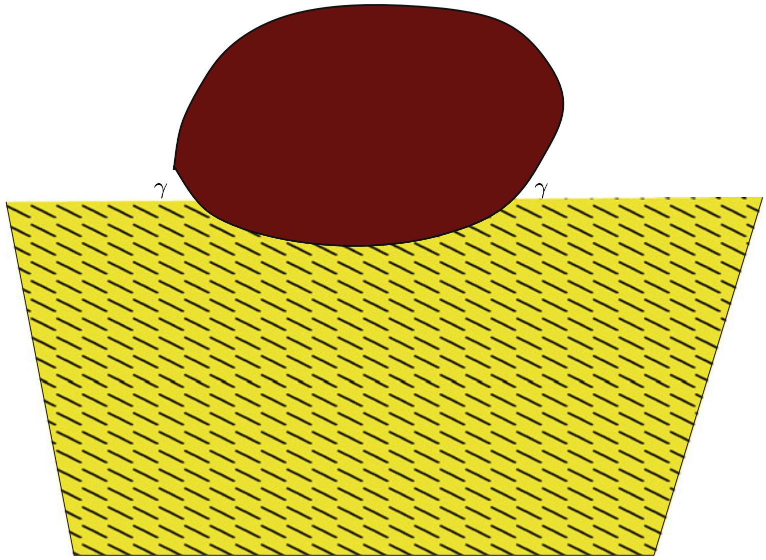

FIG. 1. Floating in neutral equilibrium; γ is the contact angle

configurations at which a particular body will float on a liquid surface. In [32] Laplace characterized some special cases of capillary floating which was an astonishing achievement for his time. There are several physical phenomena that need to be taken into account: The mass distribution in the body, the gravity, the surface tension, etc. This leads to a highly nonlinear free boundary problem.

Among various hypotheses that have been proposed to make specific configurations technically accessible, Finn introduced in [14] the notion of *neutral equilibrium*, essentially assuming that the external fluid free surface for a floating body is ideally flat, but nevertheless allowing for surface forces. This concept thus takes partial account of surface tension. In the model case of two dimensions Finn showed that under reasonable physical assumptions his concept is equivalent to the original theory. As a precautionary note, we observe that there are also other concepts of floating in neutral equilibrium in the literature. The reader should be aware that different concepts lead to different results. See, e.g., Varkonyi [40] for details. In what follows we adopt Finn's approach, and will simply speak of floating (in neutral equilibrium). We will soon reformulate conditions of floating in neutral equilibrium purely geometrically.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex, planar domain. Let $C(\Omega) \subset \mathbb{R}^3$ be an infinite homogeneous cylinder with the cross-section Ω. Then $C(\Omega)$ floats at a constant contact angle in any orientation if and only if the region Ω satisfies a straightforward geometric requirement. Figure 1 illustrates this. In Sect. 2 we interpret this geometric condition from the viewpoint of the billiard on Ω and study this billiard problem in Sect. 3. The first work relating two-dimensional capillary floating with convex geometry appears to be [35]. The main result there says that any convex, bounded, sufficiently regular planar domain will float at a given contact angle in at least four distinct orientations. The proof in [35] is based on the four vertex theorem.

Before describing our results, we will further elaborate on the capillary floating in three dimensions. The conjecture that the round ball is the only body to float in neutral equilibrium at any orientation is usually ascribed to Ulam. See [2,36]. Various authors have mathematically reformulated this question in different, albeit related ways. The interpretation of Finn [14–18] takes into account the surface tension; some other interpretations disregard it [40]. Whatever the interpretation may be, in three dimensions the conditions of floating in neutral equilibrium are much more restrictive than in the model case of two dimensions. Thus, Finn and Sloss [17] show that the only three-dimensional body to float in neutral equilibrium in any orientation at a constant contact angle is the round ball. The problem of floating three

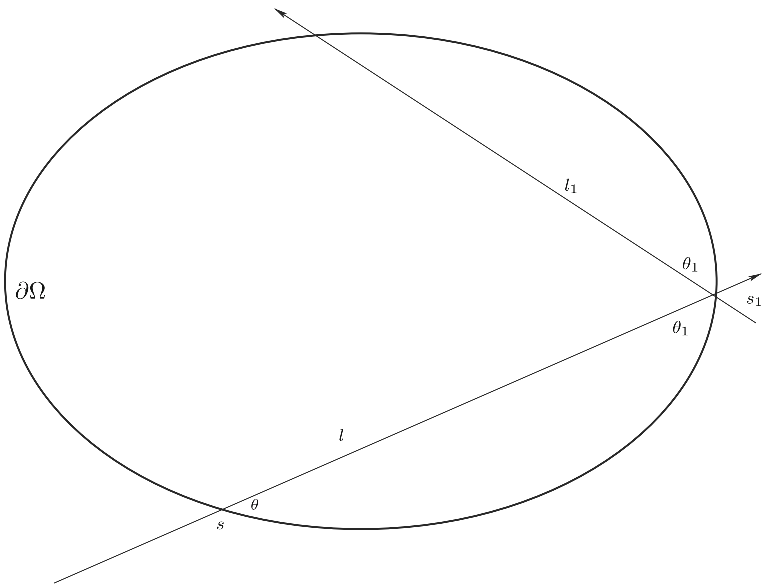

FIG. 2. The billiard phase space as a space of rays intersecting the billiard table

dimensional bodies generates challenging questions about surfaces in $\mathbb{R}^3$. In the present work we relate two-dimensional floating to the geometry of convex curves in $\mathbb{R}^2$.

The geometry of convex planar domains Ω is crucial for the billiard ball problem championed by Birkhoff in the early twentieth century [7]. It can also be viewed as a highly specialized case of a physical situation. The billiard ball is a point that travels with the unit speed inside Ω reflecting at the boundary $\partial\Omega$ according to the law of equal angles. Disregarding the motion of the ball between collisions with the boundary, we reduce the billiard ball problem to the study of the *billiard map* on Ω. Invariant curves of this map provide crucial insights into the billiard dynamics. Let s be the arc length variable on $\partial\Omega$, and let θ be the outgoing angle. See Fig. 3. Beginning with Birkhoff, invariant curves of the form $\theta = h(s)$ played an important role in the study of billiard dynamics.

The functions $\theta = h(s)$ that yield invariant curves have been extensively investigated [13,27,30,33]. The present work is based on a reformulation of the floating problem as a billiard ball problem. Namely, the cylinder with the cross-section Ω floats in neutral equilibrium at any orientation with the contact angle γ if and only if the billiard table Ω admits the invariant curve $\theta = h(s)$ with the constant function $h(s) \equiv \pi - \gamma$. For the reasons that we will explain in Sect. 2, we call these invariant curves the *constant angle caustics* for Ω. The floating problem thus becomes the following billiard problem: Find the regular, convex billiard tables that admit constant angle caustics; determine the corresponding angles. This work provides a fair amount of information on this.

We will now briefly describe the results and the structure of the paper. In Sect. 2 we review the concept of the *billiard map*. Let $\Omega \subset \mathbb{R}^2$ be a bounded, strictly convex domain with the smooth boundary $\partial\Omega$. The *phase space* $Z = Z(\Omega)$ for the billiard map on Ω consists of rays[1] intersecting Ω. Let $0 \leq \theta, \theta_1 \leq \pi$ be the two angles that a ray $l \in Z$ forms at the points of intersection $s, s_1 \in \partial\Omega$. See Fig. 2. Let $F : Z \to Z$ be the billiard map. The ray $l_1 = F(l)$ is obtained by reflecting l at s_1 about $\partial\Omega$, as if $\partial\Omega$ was a perfect mirror. The domain Ω floats in neutral equilibrium at any orientation with a constant contact angle if and only if there exists $0 < \delta < \pi$ such that for any $l \in Z$ satisfying $\theta(l) = \delta$ we have $\theta_1(l) = \delta$.

Further in Sect. 2 we study this geometric condition from the viewpoint of the billiard map. Let $\rho(s)$ be the radius of curvature for $\partial\Omega$; let $c_k, k \in \mathbb{Z}$, be its Fourier coefficients. Note that $\partial\Omega$ is circular if and

[1] I. e., oriented straight lines.

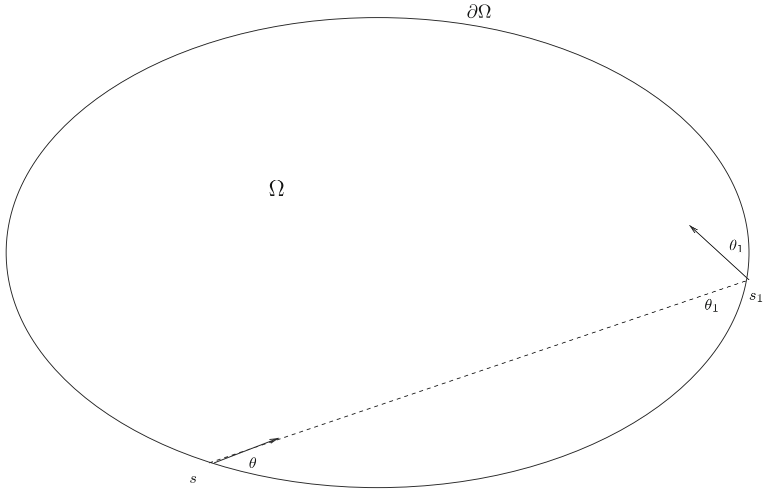

FIG. 3. The billiard map for a regular convex domain

only if $c_k = 0$ for all nonzero k. Theorem 1 says that a noncircular domain Ω has the above property if and only if the following conditions hold: (i) There exist $n > 1$ such that the pairs n, δ satisfy the trigonometric equation (6); (ii) At least one of the corresponding coefficients $c_n \neq 0$; (iii) For all $k \in \mathbb{Z}$ such that the pair k, δ does not satisfy Eq. (6), we have $c_k = 0$.

The value $\delta = \pi/2$ is special since for any odd n the pair $n, \pi/2$ satisfies Eq. (6). The corresponding regions Ω are the domains of constant width; they are well known in geometry. We briefly review this material in Sect. 5.

In Sects. 3 and 4 we study Eq. (6) further, and obtain several applications. The symmetry $\delta' = \pi - \delta$ allows us to reduce the study of Eq. (6) to the range $0 < \delta < \pi/2$. Restricted to this interval, Eq. (6) is equivalent to $\tan nx = n \tan x$. We obtain fairly detailed qualitative information about solutions of these equations. Let $B_n \subset (0, \pi/2)$ denote the set of solutions. We show that B_n has roughly $n/2$ elements; it is (π/n)-dense in the interval $(0, \pi/2)$. For every $n > 3$ we exhibit a one-parameter family $\Omega_{n,\tau}$ of non-circular, real analytic domains that float in neutral equilibrium in every orientation at the contact angles $\pi - \gamma$, where $\gamma \in B_n$.[2] See Theorem 2 and Corollary 2.

A classification of domains that float in neutral equilibrium at constant contact angles hinges on complete analysis of solutions to $\tan nx = n \tan x$. See Question 1. In particular, we need to know whether the sets B_n, B_m are disjoint for $m \neq n$. In Sects. 6 and 7.1 we reduce these questions to a study of roots of an infinite chain of polynomials S_n that are closely related to Chebyshev polynomials. This reveals a number-theoretic aspect of capillary floating.

In Sect. 7.1 we obtain some information about the roots of polynomials S_n. However, the question whether S_m, S_n have nontrivial common roots for $m \neq n$ remains unresolved. This circumstance prevented the author from publishing his findings immediately after the 1993 PennState Dynamics Workshop [20]. The book [37] contains a brief report on some of the results in [20].

In Sect. 7.2 we formulate Conjectures 1 and 2. The former is about the roots of polynomials S_n and the latter is about the sets B_n. They are equivalent, and we will refer to them simply as the *Conjecture*. Further in Sect. 7.2 we present substantial evidence corroborating the Conjecture. In Sects. 8.1 and 8.2, assuming that the Conjecture holds, we derive consequences for the billiard and for the floating respectively. Theorems 4, 5, and Corollary 7 classify billiard tables with constant angle caustics. Theorem 6 gives a full description of regular planar domains that float in neutral equilibrium in any orientation at constant contact angles.

[2] By symmetry, they also float at angle γ.

The present work can be viewed as one of many examples of fruitful relationships between the billiard and other mathematical subjects. We refer the reader to [4,5,12,19,22,24,25,29,31,37,38] for other examples of this nature. The billiard framework offers a variety of open problems that often bear on fundamental and elementary mathematical concepts [23]. The author hopes that the present work will help to advertise the subject in the mathematical fluid mechanics community.

The author is grateful to Bob Finn for bringing the subject of capillary floating to his attention and for pointing out to him its relationship to the billiard framework. Finn encouraged the author to write up the material outlined in [20] and made useful comments on the presentation. The comments of anonymous referee are also gratefully acknowledged. The work was partially supported by the MNiSzW grant N N201 384834.

2. The Birkhoff Billiard: General Caustics Versus Constant Angle Caustics

The billiard in the sense of Birkhoff plays on a compact, convex domain $\Omega \subset \mathbb{R}^2$. We will assume that the boundary $\partial\Omega$ is twice continuously differentiable. Let $0 \leq s \leq |\partial\Omega|$ be an arc length parameter. Then the *curvature* $\kappa(s)$ is a continuous, nonnegative function on $\partial\Omega$. We will assume throughout the paper that Ω is *strictly convex in the sense of differential geometry*: $\kappa > 0$. In what follows we refer to such Ω as *regular billiard tables*, or *regular convex domains*.

The elements of the *phase space of the billiard map* are the inward pointing unit vectors v based on $\partial\Omega$. Let $0 \leq \theta \leq \pi$ be the angle between v and the positively oriented $\partial\Omega$. The coordinates $0 \leq s \leq |\partial\Omega|, 0 \leq \theta \leq \pi$ induce a diffeomorphism of the phase space Z and the cylinder $(\mathbb{R}/|\partial\Omega|\mathbb{Z}) \times [0,\pi]$.

The phase point $(s,\theta) \in Z$ corresponds to the billiard ball located at $s \in \partial\Omega$, which is about to shoot of in the direction that makes angle θ with $\partial\Omega$. This shot lands at $s_1 \in \partial\Omega$. Let $0 \leq \theta_1 \leq \pi$ be the other angle of the chord $[s, s_1]$. The ball bounces elastically at the boundary and is set to shoot of again. The *law of equal angles* yields that the new vector v_1 makes angle θ_1 with $\partial\Omega$. The transformation $F : Z \to Z$ given by $F(s,\theta) = (s_1, \theta_1)$ is the *billiard ball map* for Ω. Figure 3 illustrates the discussion.

By our assumptions on Ω, the billiard map is of class C^1. Let $l(s, s_1)$ denote the length of the chord $[s, s_1]$. The differential of the billiard map is given by the following expressions [27]:

$$\frac{\partial s_1}{\partial s} = \frac{\kappa(s)l(s,s_1) - \sin\theta}{\sin\theta_1}, \qquad \frac{\partial s_1}{\partial \theta} = \frac{l(s,s_1)}{\sin\theta_1}, \qquad \frac{\partial \theta_1}{\partial \theta} = \frac{\kappa(s_1)l(s,s_1) - \sin\theta_1}{\sin\theta_1},$$

and

$$\frac{\partial \theta_1}{\partial s} = \frac{\kappa(s)\kappa(s_1)l(s,s_1) - \kappa(s)\sin\theta_1 - \kappa(s_1)\sin\theta}{\sin\theta_1}.$$

The billiard ball map is an *area preserving twist map*. The classical results of Birkhoff on the dynamics of the billiard map got a “second life” in the subject of area preserving twist maps. See the accounts in [3,34]. We are concerned with a particular aspect of the billiard ball map: *Invariant circles*.

Definition 1. Let Ω be a regular billiard table. An *invariant circle* for the billiard map on Ω is a closed curve $\Gamma \subset Z$ which is homotopic to a boundary component of Z and is invariant under the billiard map.

By a theorem of Birkhoff, any invariant circle Γ is the graph of a lipshitz function: $\theta = h_\Gamma(s)$. Thus, for every base point $s \in \partial\Omega$ there is a unique angle $\theta = h_\Gamma(s)$ such that the ball shooting from s in the direction θ will “stay” on the invariant circle Γ. For a typical Γ the function h_Γ is not constant. See [3,21,27,28,30,33]. We will study invariant circles such that h_Γ is constant. Both boundary components of Z are trivial invariant circles of that type. We will consider only nontrivial invariant circles in what follows. To simplify the terminology, we will often call them the *invariant curves*. This is justified, since we will not study other invariant curves.

Definition 2. Let $\Gamma \subset Z$ be an invariant circle, and let $\theta = h_\Gamma(s)$ the corresponding lipshitz function. If $h_\Gamma = \text{const}$, we will say that Γ is a *constant angle invariant circle*. A constant angle invariant circle is determined by that angle, say $0 < \delta < \pi$. We will denote it by Γ_δ. See Fig. 4.

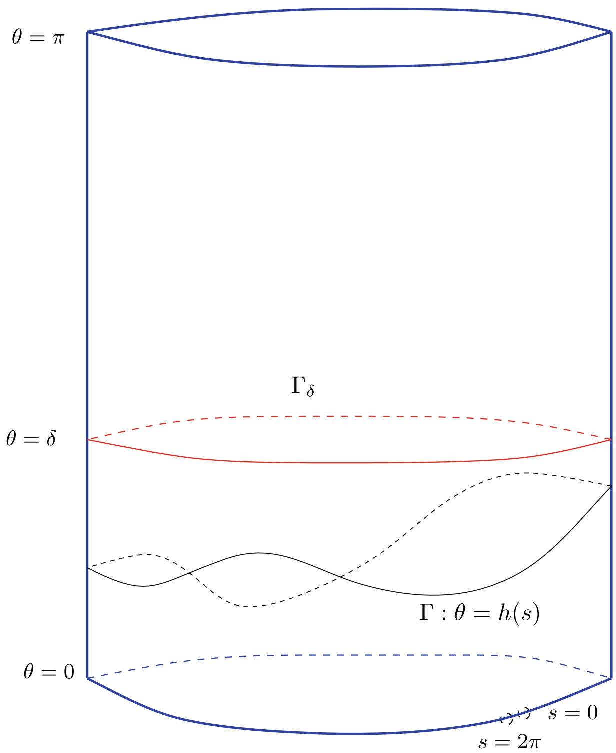

FIG. 4. Billiard map phase space with a general invariant circle and a constant angle invariant circle

It is instructive to think of Z as the space of oriented lines (i. e., *rays*) intersecting Ω, or, alternatively, as the space of directed chords in Ω. In this representation, an invariant circle Γ is a one-parameter family of rays. Its *envelope* $\gamma \subset \mathbb{R}^2$ is the *caustic of* Ω corresponding to the invariant circle Γ.[3]

Let Γ' be the family obtained from Γ by reversing the directions of rays. Then Γ' is an invariant circle as well. This is a consequence of the well known fact that the direction reversing involution $\sigma : Z \to Z$ conjugates the billiard map with its inverse: $\sigma F \sigma = F^{-1}$. Note that Γ and Γ' have the same envelope; hence, the correspondence between invariant circles and the caustics is 2-to-1. The geometry of billiard caustics offers challenging open questions. See [13,27,28] for this material. Since invariant circles are determined by their caustics essentially uniquely, in what follows we identify them; in particular, we will speak of *general caustics* and of *constant angle caustics*.

Remark 1. The reader should keep in mind that the two invariant circles, say Γ and $\Gamma' = \sigma(\Gamma)$ corresponding to the caustic γ are distinct subsets of the phase space Z. Let $0 < r(\Gamma) < 1$ be the *rotation number* of the invariant circle. Then $r(\Gamma') = 1 - r(\Gamma)$.

Since $\partial\Omega$ is strictly convex, we parameterize it by the direction $0 \leq \alpha \leq 2\pi$ of the tangent ray to $s \in \partial\Omega$. Thus, $s = s(\alpha)$. The derivative $\rho(\alpha) = ds/d\alpha$ is the *radius of curvature* function for Ω. Set $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Then the billiard map is a diffeomorphism of $\mathbb{T} \times [0, \pi]$; we will use the notation $F(\alpha, \theta) = (\alpha_1, \theta_1)$.

Proposition 1. *Let $\Omega \subset \mathbb{R}^2$ be a billiard table, and let $\rho(\alpha), 0 \leq \alpha \leq 2\pi$, be its radius of curvature. Then Ω has the constant angle caustic Γ_δ iff the function $\rho(\cdot)$ satisfies the identity*

$$\int_{\alpha-\delta}^{\alpha+\delta} \rho(\xi) \sin(\alpha - \xi) d\xi = 0. \tag{1}$$

[3]The term "caustic" is widely used in geometric optics, mechanics, and the geometric theory of singularities; in these contexts it means different, although related things. We refer the reader to [1,6,11] for more information.

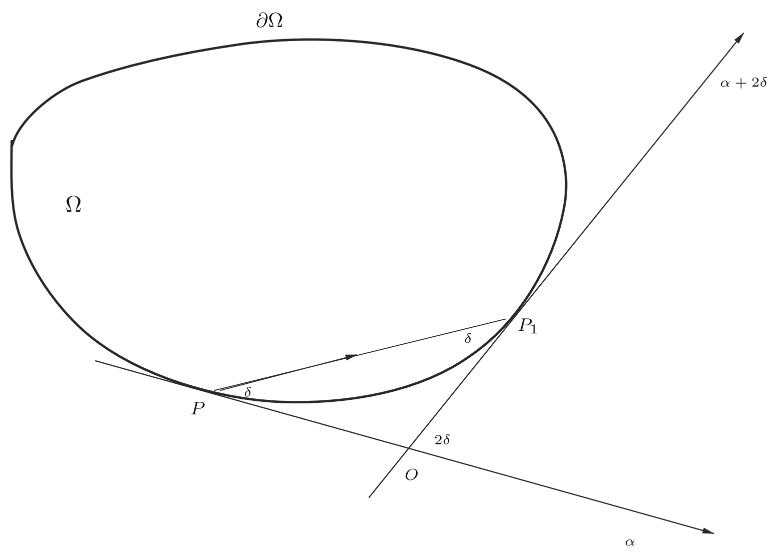

FIG. 5. The billiard map restricted to a constant angle invariant circle

Proof. Set $P = P(\alpha) = (x(\alpha), y(\alpha))$ and let $P_1 = P(\alpha_1)$. Let O be the intersection point of the tangent lines at α and α_1. From the triangle POP_1 we have $\alpha_1 = \alpha + 2\delta$. See Fig. 5. As is well known

$$x'(\alpha) = \rho(\alpha)\cos\alpha,\ \ y'(\alpha) = \rho(\alpha)\sin\alpha. \tag{2}$$

Thus

$$x(\alpha + 2\delta) - x(\alpha) = \int\limits_{\alpha}^{\alpha+2\delta} \rho(\xi)\cos\xi d\xi,$$

$$y(\alpha + 2\delta) - y(\alpha) = \int\limits_{\alpha}^{\alpha+2\delta} \rho(\xi)\sin\xi d\xi.$$

The direction of the chord $[PP_1]$ is $\alpha + \delta$. We introduce the new variable $\beta = \alpha + \delta$. Thus, the slope of $[PP_1]$ is $\tan\beta$. Computing the slope from the coordinates of points P and P_1, we obtain

$$\frac{\int_{\beta-\delta}^{\beta+\delta} \rho(\xi)\sin\xi d\xi}{\int_{\beta-\delta}^{\beta+\delta} \rho(\xi)\cos\xi d\xi} = \tan\beta. \tag{3}$$

Equation (3) is an identity that holds for any $\beta \in \mathbb{T}$. Performing elementary trigonometric manipulations in Eq. (3), and renaming the independent variable by α again, we obtain the claim. □

We will briefly review basic facts from harmonic analysis on the circle. The reader may find proofs of the statements below in most analysis textbooks.

If g is a distribution on $\mathbb{T}$, its Fourier transform is defined by $\hat{g}(n) = \int\limits_{\mathbb{T}} g(\alpha)e^{-in\alpha}d\alpha$ for $n \in \mathbb{Z}$. The radius of curvature has a Fourier expansion

$$\rho(\alpha) = \sum_{n\in\mathbb{Z}} c_n e^{in\alpha} \tag{4}$$

where $c_n = \hat{\rho}(n)/2\pi$ are the Fourier coefficients. The Fourier coefficients of a real function satisfy $c_{-n} = \bar{c}_n$. Equation (4) is equivalent to the trigonometric expansion

$$\rho(\alpha) = a_0 + \sum_{n\geq 1} a_n \cos n\alpha + b_n \sin n\alpha$$

whose coefficients are real. The coefficients in these equations are related by $a_0 = c_0$ and $a_n = 2\Re(c_n), b_n = -2\Im(c_n)$ for $n > 0$.

Denote by $x + y$ the group operation on $\mathbb{T}$. Let $k(\cdot)$ be a function or a distribution on $\mathbb{T}$. The operator of convolution with k is defined by

$$(K\rho)(x) = \int_{\mathbb{T}} \rho(x-\xi)k(\xi)d\xi = \int_{\mathbb{T}} \rho(\xi)k(x-\xi)d\xi. \tag{5}$$

The standard notation for convolution operators is $K(\rho) = \rho * k = k * \rho$.

Let F_n be the complex line in the space of functions on $\mathbb{T}$ spanned by e^{inx}. We view Eq. (4) as the orthogonal decomposition by the subspaces F_n, $n \in \mathbb{Z}$. Convolution operators preserve this decomposition. The restriction $K|_{F_n}$ is the operator of multiplication by $\hat{k}(n)$. The above discussion yields the following statement which is crucial for Theorem 1.

Lemma 1. *Let $k(\cdot)$ be a distribution on $\mathbb{T}$, and let $\hat{k}(n)$, $n \in \mathbb{Z}$, be its Fourier transform. Let K be the operator of convolution with the distribution $k(\cdot)$. Let $\rho(\cdot)$ be a function on $\mathbb{T}$, and let c_n, $n \in \mathbb{Z}$, be its Fourier coefficients.*

Then $K\rho = 0$ iff $\hat{k}(n)c_n = 0$ for all $n \in \mathbb{Z}$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a regular, noncircular billiard table. Let $\rho(\cdot)$ be the radius of curvature of $\partial\Omega$, and let c_n, $n = 1, 2 \dots$ be its Fourier coefficients. Then Ω has the constant angle caustic Γ_δ if and only if the following conditions hold*:

(i) *There exist $n > 1$ such that*

$$\frac{\sin(n-1)\delta}{n-1} = \frac{\sin(n+1)\delta}{n+1}; \tag{6}$$

(ii) *We have $c_k = 0$ for all $k > 1$ such that Eq. (6) is not satisfied.*

(iii) *We have $c_n \neq 0$ for at least one $n > 1$ such that Eq. (6) is satisfied.*

Proof. By Proposition 1, Γ_δ is a caustic for Ω iff $\rho(\cdot)$ belongs to the zero space of the convolution with the function

$$k(x) = (\sin x)\, 1_{[-\delta,\delta]}. \tag{7}$$

The function $k(\cdot)$ is odd, hence $\hat{k}(0) = 0$. By a straightforward computation, for $n > 1$ we have

$$i \cdot \hat{k}(n) = \frac{\sin(n-1)\delta}{n-1} - \frac{\sin(n+1)\delta}{n+1}.$$

It is well known that for any billiard table Ω we have $c_1 = 0$. By Lemma 1, Γ_δ is a caustic for Ω iff $c_m\hat{k}(m) = 0$ for all m. By the preceding discussion, Γ_δ is a caustic iff

$$c_m \left[\frac{\sin(m-1)\delta}{m-1} - \frac{\sin(m+1)\delta}{m+1}\right] = 0$$

for $m > 1$. Since Ω is not circular, at least one coefficient, say c_n, does not vanish. But Γ_δ being a caustic, Eq. (6) holds for all $n > 1$ such that $c_n \neq 0$. □

3. Constant Angle Caustics and a Chain of Trigonometric Equations

By Theorem 1, the description of billiard tables with constant angle caustics hinges on solving Eq. (6). In this section we will reduce Eq. (6) to a chain of trigonometric equations involving the function $\tan(\cdot)$.

Let $A_n \subset (0,\pi)$ be the set of δ such that the pair δ, n satisfies Eq. (6). Set

$$A = \cup_{n=2}^{\infty} A_n.$$

Lemma 2. *Let $n > 1$. Then the following claims hold.*

(i) *We have $\frac{\pi}{2} \in A_n$ iff n is odd.*
(ii) *Set $\tilde{A}_n = A_n \setminus \{\frac{\pi}{2}\}$. Then $\tilde{A}_n$ is the set of solutions in $(0,\pi)$ of the equation $\tan n\delta = n \tan\delta$.*

Proof. Set $\delta = \frac{\pi}{2}$. If n is odd, then both sides in Eq. (6) vanish, hence $\frac{\pi}{2} \in A_n$. If n is even, then the numerators in Eq. (6) are ± 1, and their signs are opposite. Thus, $\frac{\pi}{2} \notin A_n$, proving claim (i).

Let $\delta \in A_n$. Arguing as above, we establish that $\sin(n+1)\delta = 0$ iff n is odd and $\delta = \frac{\pi}{2}$. Hence for $\delta \in \tilde{A}_n$ we have $\sin(n-1)\delta, \sin(n+1)\delta \neq 0$. Therefore, $\tilde{A}_n$ is the set of $\delta \in (0,\pi)$ satisfying

$$\frac{\sin(n-1)\delta}{\sin(n+1)\delta} = \frac{n-1}{n+1}.$$

We rewrite this as

$$\frac{\sin n\delta \cos\delta - \cos n\delta \sin\delta}{\sin n\delta \cos\delta + \cos n\delta \sin\delta} = \frac{n-1}{n+1}. \tag{8}$$

If $\cos n\delta = 0$, then the left hand side in Eq. (8) is 1, which is impossible. Thus, $\cos n\delta \neq 0$. Dividing the numerators and the denominators in Eq. (8) by $\cos\delta \cos n\delta$, we obtain

$$\frac{\tan n\delta - \tan\delta}{\tan n\delta + \tan\delta} = \frac{n-1}{n+1}.$$

Claim 2 follows. □

Remark 2. We point out that equations similar to $\tan nx = n \tan x$ arise in several contexts. Some of these contexts are directly related to floating [10,40,41], while others are formally independent, but intrinsically related to it [39].

For $X \subset \mathbb{R}$ and $a \in \mathbb{R}$ let $\{a - X\} = \{a - x : x \in X\}$. Set $\tilde{A} = A \setminus \{\frac{\pi}{2}\}$. Then

$$\tilde{A} = \cup_{n=2}^{\infty} \tilde{A}_n. \tag{9}$$

Set $B_n = A_n \cap (0,\pi/2)$ and $B = A \cap (0,\pi/2)$. Lemma 2 and the preceding discussion imply the following.

Proposition 2. *Let $n > 1$. Then for n even, $A_n = B_n \cup \{\pi - B_n\}$ and for n odd, $A_n = B_n \cup \{\pi - B_n\} \cup \{\frac{\pi}{2}\}$. Moreover, B_n is the set of solutions in $(0,\pi/2)$ of the equation*

$$\tan nx = n \tan x. \tag{10}$$

We proceed to analyze the chain of Eq. (10) in the interval $(0,\pi/2)$.

Proposition 3. 1. *Let $n > 1$ be even. Then B_n consists of $\frac{n}{2} - 1$ points $\xi_k^{(n)}$, where*

$$\frac{2k}{2n}\pi < \xi_k^{(n)} < \frac{(2k+1)}{2n}\pi : k = 1, \dots, \frac{n}{2} - 1.$$

2. *Let $n > 1$ be odd. Then B_n consists of $\frac{n-1}{2} - 1$ points $\xi_k^{(n)}$, where*

$$\frac{2k}{2n}\pi < \xi_k^{(n)} < \frac{(2k+1)}{2n}\pi : k = 1, \dots, \frac{n-1}{2} - 1.$$

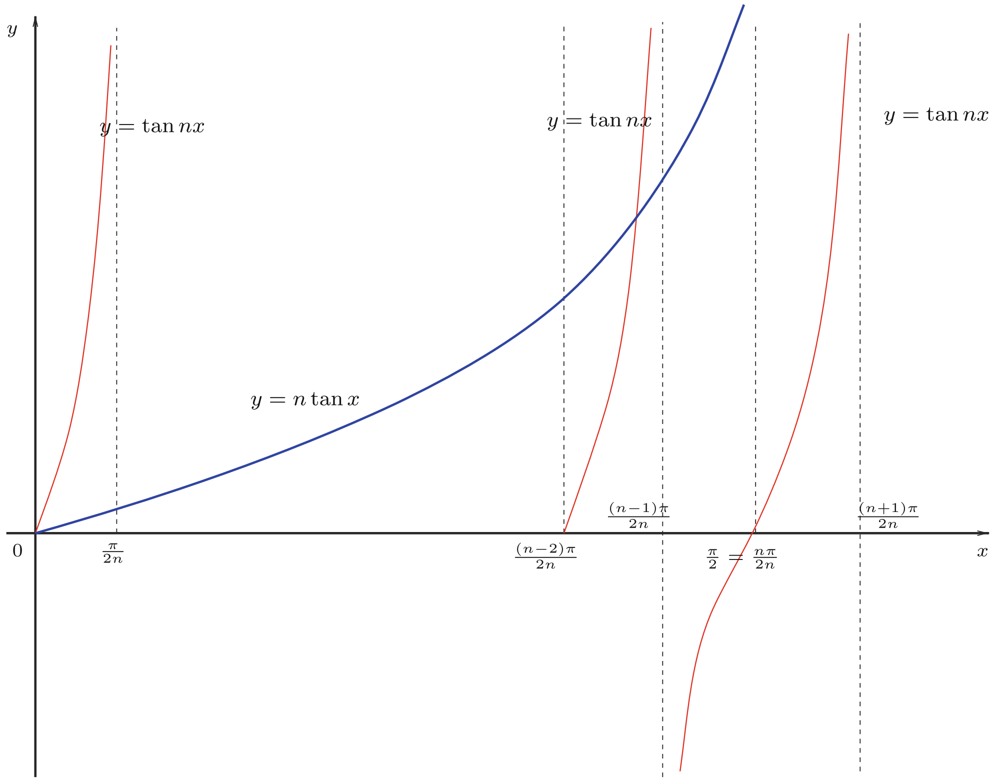

FIG. 6. The graphs of functions $y = \tan nx$ and $y = n \tan x$ for n even

Proof. The graph of the function $y = \tan nx$ on $(0, \pi/2)$ is the disjoint union of n connected curves; we will call them *branches*. A branch is defined on the interval $\frac{k}{2n}\pi < x < \frac{k+1}{2n}\pi : 0 \le k \le n-1$. Set $I_k^{(n)} = (\frac{k}{2n}\pi, \frac{k+1}{2n}\pi)$. Each branch extends by continuity to one of the endpoints of $I_k^{(n)}$. These endpoints don't enter in our analysis, and we ignore them in what follows. We say that a branch is *positive* (resp. *negative*) if it belongs the the upper (resp. lower) halfplane.

Positive branches correspond to $I_k^{(n)}$ with k even. Thus, there are $n/2$ (resp. $(n-1)/2$) positive branches if n is even (resp. odd). We observe that each point in B_n belongs to the intersection of the graph of $y = n \tan x$ on $(0, \pi/2)$ with a positive branch; this intersection contains at most one point. See Figs. 6 and 7.

Comparing the asymptotics of $n \tan x$ and $\tan nx$ as $x \to 0+$, we see that the first branch, which corresponds to $k = 0$, does not yield an intersection point. When n is even, all other positive branches intersect the graph $n \tan x$. This proves claim 1. Let now n be odd. Then both the last branch and the graph of $y = n \tan x$ are asymptotic to the vertical line $x = \pi/2$. Comparing the asymptotics of $n \tan x$ and $\tan nx$ as $x \to \frac{\pi}{2}-$, we see that the curves do not intersect. This proves claim 2. We leave details to the reader. □

We will state an immediate consequence of Proposition 3.

Corollary 1. *We have* $|A_n| = n - 2$. *The sets* A_n *are* $2\pi/n$ *dense in* $(0, \pi)$.

Proof. By Lemma 2 and Proposition 2, $|A_n| = 2|B_n|$ if n is even and $|A_n| = 2|B_n| + 1$ if n is odd. The first claim now follows from Proposition 3. The above propositions imply that the distances between consecutive points of A_n are at most π/n. Besides, the distances from A_n and the endpoints of $(0, \pi)$ are at most $2\pi/n$. □

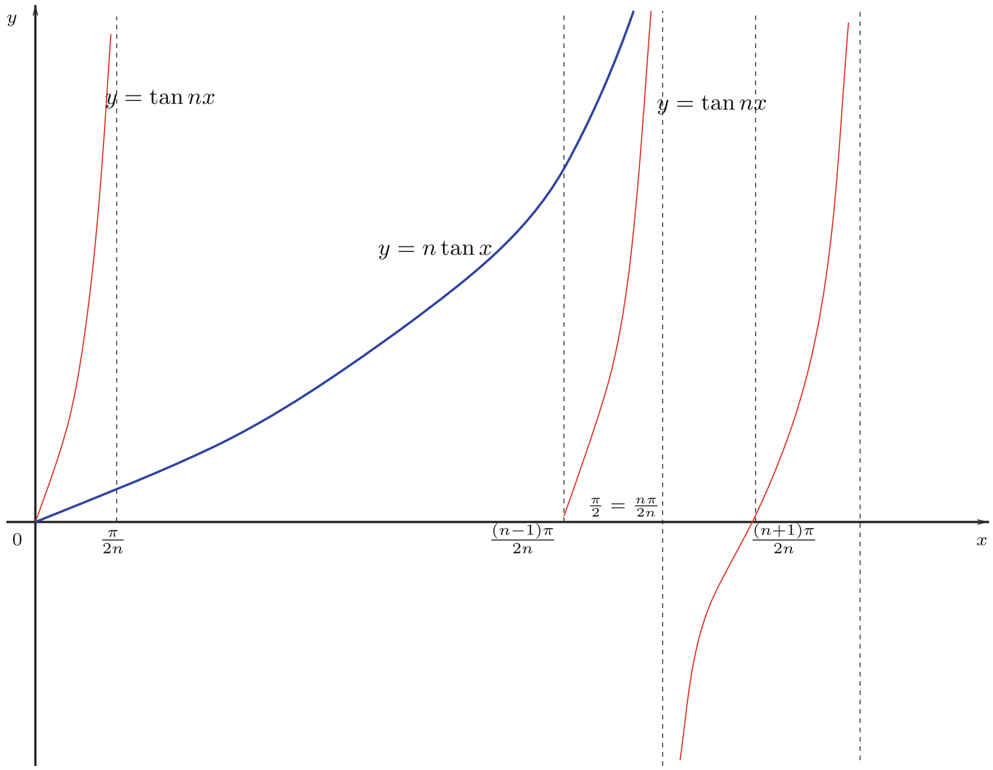

FIG. 7. The graphs of functions $y = \tan nx$ and $y = n \tan x$ for n odd

4. Immediate Applications to the Billiard and Floating

The above results have immediate consequences for the billiard and the floating. We begin with the former. We will say that the billiard tables Ω_1, Ω_2 are *equivalent* if there is a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ which is an isometry up to a homothety, and such that $F(\Omega_1) = \Omega_2$. For instance, all discs in $\mathbb{R}^2$ are equivalent. All squares are also equivalent.

Theorem 2. *There is a dense countable set $\tilde{A} \subset (0, \pi) \setminus \{\pi/2\}$ such that the following holds.*

1. *For any $\delta \in \tilde{A}$ there is $n > 1$ and a real analytic 1-parameter family $\Omega_{n,\tau}$, $0 \leq \tau < 1$, of pairwise inequivalent, regular billiard tables having the constant angle caustic Γ_δ. The curves $\partial\Omega_{n,\tau}$ are real analytic; $\partial\Omega_{n,0}$ is the unit circle.*
2. *A regular billiard table Ω has the caustic $\Gamma_{\pi/2}$ iff $\partial\Omega$ is a curve of constant width.*
3. *Let $0 < \delta < \pi$ belong to the complement of $\tilde{A} \cup \{\pi/2\}$ in $(0, \pi)$. If a regular billiard table Ω has the constant angle caustic Γ_δ, then Ω is circular.*

Proof. Let $\tilde{A}_n$ and $\tilde{A}$ be as in Eq. (9). Then $\delta \in \tilde{A}$ iff there exists $n > 1$ such that $\tan n\delta = n \tan \delta$. Let $a, b \in \mathbb{R}$ be arbitrary. Set

$$\rho(\alpha) = 1 + a \cos n\alpha + b \sin n\alpha.$$

By elementary trigonometry, there exists α_0 depending on a, b, n such that $\rho(\alpha) = 1 + \sqrt{a^2 + b^2} \sin n(\alpha + \alpha_0)$. This is the radius of curvature of a regular billiard table iff $a^2 + b^2 < 1$. Different values of α_0 correspond to isometric billiard tables. Set $\rho_{n,\tau}(\alpha) = 1 + \tau \sin n\alpha$.

Integrating Eq. (2), we obtain

$$x_{n,\tau}(\alpha) = \xi_0 + \sin\alpha + \frac{\tau}{2(n-1)}\cos(n-1)\alpha - \frac{\tau}{2(n+1)}\cos(n+1)\alpha, \tag{11}$$

$$y_{n,\tau}(\alpha) = \eta_0 - \cos\alpha + \frac{\tau}{2(n-1)}\sin(n-1)\alpha - \frac{\tau}{2(n+1)}\sin(n+1)\alpha. \tag{12}$$

Fixing the constants ξ_0, η_0, we obtain a real analytic family $\Omega_{n,\tau}$. This proves claim 1. Claim 2 follows from the material in Sect. 5. Claim 3 is immediate from Theorem 1. □

Set $\rho(\alpha) = c + a\cos n\alpha + b\sin n\alpha$. This formula, provided $0 \le \sqrt{a^2+b^2} < c$, yields a 3-parameter family of functions that serve as radii of curvature for billiard tables Ω having the caustic Γ_δ. Equations (11), (12) yield a 5-parameter family of these domains. However, the equivalence eats up 4 of the parameters. We view Eqs. (11), (12) as a deformation $\Omega_{n,\tau}$ of the circular table.

The following is the counterpart of Corollary 2 for the floating in neutral equilibrium. Its claims are the reformulations of the corresponding claims in Corollary 2; we do not repeat the proofs.

Corollary 2. *There is a dense countable set $\tilde{A} \subset (0,\pi) \setminus \{\pi/2\}$ such that the following holds.*

1. *For any $\delta \in \tilde{A}$ there is $n > 1$ and a real analytic 1-parameter family $\Omega_{n,\tau}$, $0 < \tau < 1$, of pairwise inequivalent planar domains with real analytic boundaries that float in neutral equilibrium at any orientation with the contact angle $\pi - \delta$. The domain $\Omega_{n,0}$ is the unit circle.*
2. *A regular convex domain floats in neutral equilibrium at any orientation with the contact angle $\pi/2$ if and only if its boundary is a curve of constant width.*
3. *Let $0 < \delta < \pi$ belong to the complement of $\tilde{A} \cup \{\pi/2\}$ in $(0,\pi)$. If a regular convex domain floats in neutral equilibrium at any orientation with the contact angle δ, then it is a disc.*

For each $\delta \in B$ Corollary 2 exhibited a 1-parameter family of billiard tables with a constant angle caustic Γ_δ. Are there more?

Question 1. Let $\delta \in B$. Describe the set of billiard tables Ω such that Γ_δ is a caustic. Equivalently, describe the set of cross-sections of cylinders that float in neutral equilibrium with the contact angle $\pi - \delta$ at any orientation.

In order to make progress on Question 1, we need to investigate the intersection sets $B_n \cap B_m$ for $1 < m < n$. This leads to number theoretic questions which we will study in Sect. 7. The contact angle $\delta = \pi/2$ is special. Geometers investigated the corresponding planar domains from an independent viewpoint. We briefly review this material in the next section.

5. The Caustics $\Gamma_{\pi/2}$ and Curves of Constant Width

To illustrate the preceding discussion, we will now study the question: Which billiard tables have the caustic $\Gamma_{\pi/2}$? Let Ω be a regular billiard table. Then $\Gamma_{\pi/2}$ is a caustic if and only if any chord which is perpendicular to $\partial\Omega$ at one of its ends, is also perpendicular to $\partial\Omega$ at the other end. The values of the angle parameter at these points are $\alpha, \alpha+\pi$. Since the chord $[P(\alpha)P(\alpha+\pi)]$ is perpendicular to $\partial\Omega$ at both end points, its length $d(\alpha)$ is the *width of Ω in direction α*. Moreover, the orthogonality of $[P(\alpha)P(\alpha+\pi)]$ and $\partial\Omega$ implies that $d(\alpha) = \text{const}$. These curves and the domains that they bound are known in geometry as the *curves and domains of constant width* [8]. Thus, a regular billiard table Ω has the caustic $\Gamma_{\pi/2}$ iff Ω is a domain of constant width.

We point out that our analysis assumes that $\partial\Omega$ is twice continuously differentiable. In particular, it is not valid for domains of constant width with corners. The boundary of a famous example of such a domain, the Reuleaux triangle [8], consists of three circular arcs of the same radius; it has corners at the endpoints of the arcs. See Fig. 8. The Reuleaux triangle is not a regular billiard table.

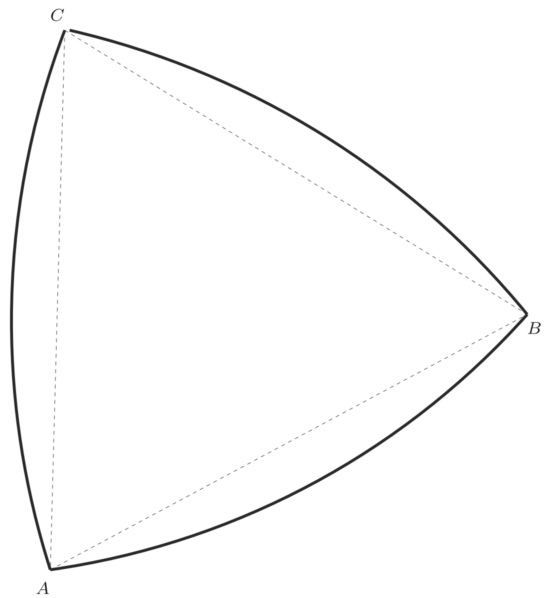

Fig. 8. Reuleaux triangle: a domain of constant width with corners

Corollary 3. *Let Ω be a regular billiard table, and let $\rho(\cdot)$ be its radius of curvature. Then $\Gamma_{\pi/2}$ is a caustic for Ω iff we have the identity*

$$\rho(\alpha) + \rho(\alpha + \pi) = const. \tag{13}$$

Proof. Let $c_m, m \in \mathbb{Z}$, be the Fourier coefficients of ρ. By the proof of Theorem 1, $\Gamma_{\pi/2}$ is a caustic iff

$$c_m \left[\frac{\sin \frac{(m-1)\pi}{2}}{m-1} - \frac{\sin \frac{(m+1)\pi}{2}}{m+1} \right] = 0$$

for all $m > 1$. For $m = 2k$ this means $4kc_{2k}/(4k^2 - 1) = 0$, yielding $c_{2k} = 0$. For odd m the equation holds for any c_m. Thus, Ω has the caustic $\Gamma_{\pi/2}$ iff the radius of curvature has the Fourier expansion of the form

$$\rho(\alpha) = c_0 + \sum_{m\,\mathrm{odd}} c_m e^{im\alpha}. \tag{14}$$

Equation (14) is equivalent to the identity $\rho(\alpha + \pi) + \rho(\alpha) = 2c_0$. □

Remark 3. We point out that the identity Eq. (14) characterizes all billiard tables Ω with the caustic $\Gamma_{\pi/2}$, including the circular billiard table. By the discussion preceding Corollary 3, the width of any such Ω is constant, and is equal to $2c_0$. *Let* $|\partial\Omega|$ be the perimeter of Ω. If Ω has constant width, we denote it by $w(\Omega)$. By the above argument, for a curve of constant width we have

$$\rho(\alpha + \pi) + \rho(\alpha) = w(\Omega).$$

Integrating this equation and using that $\int_{\mathbb{T}} \rho(\alpha)d\alpha = |\partial\Omega|$, *we obtain the identity*

$$\pi \cdot w(\Omega) = |\partial\Omega|. \tag{15}$$

Note that we have used the regularity of $\partial\Omega$ to derive Eq. (15). *In fact, it is valid for arbitrary curves of constant width; it is called Barbier's theorem. Another amusing fact about domains of constant width is the Blaschke–Lebesgue theorem* [8]. *It says that amongst the domains of a fixed constant width the*

Reuleaux triangle has the smallest area. By the isoperimetric theorem, the disc has the biggest area. Let Ω be any domain of constant width w; let $|\Omega|$ be the area of Ω. By an elementary calculation

$$\frac{\pi-\sqrt{3}}{2}w^2 \leq |\Omega| \leq \frac{\pi}{4}w^2.$$

The equalities take place only for the Reuleaux triangle and the disc.

6. Trigonometric Equations and a Family of Polynomials

We will now obtain quantitative information about the solutions of Eq. (10).

Lemma 3. *Let $n \geq 1$. There are polynomials P_n, Q_n such that*

$$\tan nx = \frac{P_n(\tan x)}{Q_n(\tan x)}. \tag{16}$$

Polynomials P_n, Q_n are uniquely determined by the recurrence relations

$$P_{n+1}(z) = P_n(z) + zQ_n(z), \quad Q_{n+1}(z) = Q_n(z) - zP_n(z) \tag{17}$$

and the initial data $P_1(z) = z,\ Q_1(z) = 1$. The polynomial P_n (resp. Q_n) is odd (resp. even). The degree of each of the two polynomials is either n or $n-1$, depending on the parity of n.

Proof. The formula

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

in the special case $y = nx$ yields

$$\tan(n+1)x = \frac{\tan nx + \tan x}{1 - \tan nx \tan x}.$$

The claims follow by induction on n. □

Remark 4. Polynomials P_n, Q_n can be expressed in terms of the Chebyshev polynomials of the first and the second kind. We will not pursue this approach here.

Proposition 4. *The polynomials in Eq. (16) satisfy*

$$-2P_n(z) = i^{n+1}(z-i)^n + (-i)^{n+1}(z+i)^n \tag{18}$$

and

$$2Q_n(z) = i^n(z-i)^n + (-i)^n(z+i)^n. \tag{19}$$

Proof. We rewrite Eq. (17) as

$$\begin{bmatrix} P_{n+1}(z) \\ Q_{n+1}(z) \end{bmatrix} = \begin{bmatrix} 1 & z \\ -z & 1 \end{bmatrix} \begin{bmatrix} P_n(z) \\ Q_n(z) \end{bmatrix}.$$

The claims follow by the elementary algebra. □

Corollary 4. *For $n > 1$ set*

$$R_n(z) = -\frac{1}{2}\, i^n \left[(nz+i)(z-i)^n + (-1)^n(nz-i)(z+i)^n\right]. \tag{20}$$

Let $0 < x < \pi/2$, and set $z = \tan x$. Then $x \in B_n$ iff z is a positive root of the polynomial R_n.

Proof. By Proposition 2, Eq. (10), and Lemma 3, $x \in B_n$ iff $\tan x = z > 0$ satisfies $P_n(z) - nzQ_n(z) = 0$. By Eqs. (18) and (19), $P_n(z) - nzQ_n(z) = R_n(z)$. □

7. Floating and Number Theory

We have reduced our investigation of Eq. (10) to a study of roots of the polynomials R_n. We now continue to study these polynomials, and bring in some number theory.

7.1. Polynomials and Fractional Linear Transformations

The following lemma summarizes the immediate properties of polynomials R_n.

Lemma 4. *Let $n \geq 1$. Then the following holds:*

(i) *The polynomials R_n are real, odd polynomials;*
(ii) *The degree of R_n is equal to $n+1$ for n even, and to n for n odd;*
(iii) *The highest coefficient of R_n is $2n$ for n even and ± 1 for n odd;*
(iv) *The roots of R_n are real and simple, except for the zero root, which has multiplicity three.*

Proof. Claims (i)–(iii) follow either from $R_n(z) = P_n(z) - nzQ_n(z)$ or directly from Eq. (20). We will prove claim (iv). Suppose n is even; set $n = 2k$. Then $\deg(R_n) = 2k+1$. By Eq. (20), R_n has at most $2k-2$ nonzero roots, counted with multiplicities. By claim 1 in Proposition 3 and Corollary 4, R_n has $k-1$ distinct positive roots. By (i), R_n has $k-1$ distinct negative roots, hence the claim. The case of odd n is similar, and we leave it to the reader. □

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a nondegenerate matrix. We will use the notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z = \frac{az+b}{cz+d}.$$

Let A, A' be nondegenerate matrices. We will write $A \sim A'$ to mean that $A'A^{-1}$ is a scalar matrix. Then $A \sim A'$ holds iff $A \circ z \equiv A' \circ z$.

Proposition 5. *Let $n > 1$. There is a 1-to-1 correspondence, preserving the multiplicities, between the nonzero roots of R_n and the roots of the equation*

$$\zeta^n = (-1)^{n+1} \begin{bmatrix} n+1 & n-1 \\ n-1 & n+1 \end{bmatrix} \circ \zeta, \tag{21}$$

other than $\zeta = \pm 1$.

Proof. By Eq. (20), we have $R_n(z) = 0$ iff

$$\left(\frac{z-i}{z+i}\right)^n = (-1)^{n+1} \frac{nz-i}{nz+i}. \tag{22}$$

We recall a few well known facts. The fractional linear transformations

$$\zeta = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \circ z, \;\; z = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \circ \zeta.$$

are inverse to each other; they induce a diffeomorphism of $\mathbb{R} \cup \infty$ onto the unit circle which sends the natural orientation of the real axis to the counter clockwise orientation of the unit circle.

Setting $F_n(z) = \frac{nz-i}{nz+i}$, we rewrite Eq. (22) as

$$(F_1(z))^n = (-1)^{n+1} F_n(z).$$

Setting $F_1(z) = \zeta, z = F_1^{-1}(\zeta)$, and using that

$$\begin{bmatrix} n & -i \\ n & i \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} ni+i & ni-i \\ ni-i & ni+i \end{bmatrix} \sim \begin{bmatrix} n+1 & n-1 \\ n-1 & n+1 \end{bmatrix},$$

we obtain Eq. (21).

We have proved that the transformation $\zeta = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \circ z$ induces a multiplicity preserving isomorphism between the roots z of R_n such that $F_1(z) \neq \infty$ and the solutions $\zeta \neq F_1(\infty)$ of Eq. (21). Using that $F_1(0) = -1, F_1(\infty) = 1$, and the information about the roots of R_n contained in Lemma 4, we obtain the claim. □

Remark 5. Proposition 5 singles out the roots $\zeta = \pm 1$ of Eq. (21). Observe that -1 is always a root of multiplicity three for this equation, while 1 is a (simple) root iff n is odd. To explain this, we note that $-1 = F_1(0)$, while $1 = F_1(\infty)$. Observe that 0 is a multiplicity three root of R_n; the appearance of ∞ as a "root" of R_n is due to the circumstance that in the beginning of the proof of Proposition 5 we have put the equation $R_n(z) = 0$ in the form

$$(nz + i)(z - i)^n = (-1)^{n+1}(nz - i)(z + i)^n. \tag{23}$$

If n is odd, the leading terms in both sides of Eq. (23) have the same sign when $z \to \infty$; if n is even, the signs are opposite.

Equation (21) involves a rational function whose denominator is $(n-1)\zeta + (n+1)$. Getting rid of the denominator and using the variable $x = -\zeta$, we obtain an equivalent polynomial equation:

$$-(n-1)x^{n+1} + (n+1)x^n - (n+1)x + (n-1) = 0.$$

The corollaries below are immediate from Proposition 5 and the preceding discussion.

Corollary 5. *Let $n > 1$. Set*

$$S_n(x) = (n-1)\left[x^{n+1} - 1\right] - (n+1)\left[x^n - x\right]. \tag{24}$$

Then all roots of the polynomials S_n belong to the unit circle $\{|x| = 1\}$. The number 1 is a root of multiplicity three. The number -1 is a simple root of S_n if n is odd, and $S_n(-1) \neq 0$ if n is even. The remaining roots of R_n are simple.

In what follows we will refer to the roots $x \neq \pm 1$ of S_n as the *complex roots*.

Corollary 6. *Let $n > 1$. The transformation $z \mapsto x$ given by*

$$x = -\frac{z - i}{z + i}$$

induces a 1-to-1 correspondence between the nonzero roots of the polynomial R_n and the complex roots of the polynomial S_n. Moreover, this transformation sends the positive (resp. negative) roots of R_n to the roots of S_n such that $\Im x > 0$ (resp. $\Im x < 0$).

7.2. Characterizing Billiard Tables with a Particular Constant Angle Caustic

Let $0 < \delta < \pi$, $\delta \neq \pi/2$, be an element in A. We will analyze the set of billiard tables with the constant angle caustic Γ_δ. Our results hinge on certain number theoretic claims, which for the time being, we cannot prove. In view of overwhelming evidence in favor of these claims, we will state them as conjectures.

Conjecture 1. *Let $m, n > 1$ be distinct integers; let S_m, S_n be the corresponding polynomials in Eq. (24). Then their sets of complex roots are disjoint.*

Conjecture 2. *Let $m, n > 1$ be distinct integers. Then equations $\tan mx = m \tan x$, $\tan nx = n \tan x$ have no common solutions in $(0, \pi/2)$.*

By Sects. 6 and 7.1, these conjectures are equivalent. For reader's convenience, we outline a proof. Recall that B_k denotes the set of roots of the equation $\tan kx = k \tan x$ in $(0, \pi/2)$. By Lemma 3, Proposition 4, and Corollary 4, the set $\{\tan x : x \in B_k\}$ is the set of positive roots of the polynomial R_k. See Eq. (20). Corollary 6 provides a fractional linear transformation that sends the positive roots of

$R_k, k > 1$, to the roots of S_k in the semi-circle $\{|z| = 1, \Im z > 0\}$. Now the information about the roots of S_k contained in Corollary 5 implies the claim.

In view of the equivalence of Conjectures 1 and 2, from now on we will simply refer to them as *the Conjecture*.

We will say that a solution x is nontrivial if $\tan x \neq 0$. The following proposition lends support to the Conjecture.

Proposition 6. *Let $n > 1$. Then the following holds.*

1. *The system*

$$\tan nx = n \tan x,\ \ \tan(n+k)x = (n+k)\tan x \tag{25}$$

has no nontrivial solutions for $k = 1, 2$.

2. *The system*

$$\tan nx = n \tan x,\ \ \tan knx = kn \tan x \tag{26}$$

has no nontrivial solutions for $k = 2, 3$.

3. *The systems*

$$\tan nx = n \tan x, \quad \tan(2n \pm 1)x = (2n \pm 1)\tan x \tag{27}$$

have no nontrivial solutions.

Proof. We have

$$\tan(n+k)x = \frac{\tan nx + \tan kx}{1 - \tan nx \tan kx}.$$

Substituting this into Eq. (25) and using that $\tan x \neq 0$, we obtain $1 + n(n+1)\tan^2 x = 0$ in the case $k = 1$, and $(n+1)^2 \tan^2 x = 0$ in the case $k = 2$. This proves claim 1.

We have

$$\tan 2nx = \frac{2\tan nx}{1 - \tan^2 nx}, \quad \tan 3nx = \frac{3\tan nx - \tan^3 nx}{1 - 3\tan^2 nx}.$$

Substituting these identities into Eq. (26), and assuming $\tan x \neq 0$, we obtain $1 - n^2 \tan x = 1$, $8n^2 \tan x = 0$ if $k = 2$, $k = 3$ respectively. This proves claim 2.

Equation (27) and the identity

$$\tan(2n \pm 1)x = \frac{\tan 2nx \pm \tan x}{1 \mp \tan 2nx \tan x}$$

yield the relationship $n^3 \pm 2n^2 + n = 0$ which has no solutions $n > 1$. □

Remark 6. A refinement of the above argument yields that the systems

$$\tan nx = n \tan x,\ \ \tan(3n \pm 1)x = (3n \pm 1)\tan x$$

do not have nontrivial solutions as well. The proof is rather long and we do not reproduce it here.

Proposition 6 and Remark 6 yield particular families of pairs of integers $m \neq n$ such that the system $\tan mx = m \tan x$, $\tan nx = n \tan x$ has no nontrivial solutions. This provides some evidence supporting our conjecture. Other supporting evidence, which we will now describe, comes from the work [9].

Let $n \geq 4$. Set $\tilde{S}_n(x) = S_n(x)/(x-1)^3(x+1)$ if n is odd and $\tilde{S}_n(x) = S_n(x)/(x-1)^3$ if n is even. By Corollary 5, $\tilde{S}_n$ are polynomials with integer coefficients; their roots are simple and belong to the unit circle. Let X be a property of natural numbers. Let $\mathbb{N}(X) \subset \mathbb{N}$ be the set of numbers having this property. We say that property X *holds for almost all positive integers* if $\mathbb{N}(X) \subset \mathbb{N}$ is a subset of *density one*. A property that holds for *almost all pairs of positive integers* is defined analogously.

The work [9] puts forward several conjectures about irreducibility of polynomials over $\mathbb{Q}$. It conjectures, in particular, that polynomials $\tilde{S}_n$ are irreducible. See Conjecture 3 in [9]. Let $m \neq n$ be natural numbers. We will say that the pair m, n satisfies the Conjecture if the sets of complex roots of the polynomials S_m, S_n are disjoint.

Proposition 7. *The Conjecture holds for almost all pairs of positive integers.*

Proof. By the preceding discussion, it suffices to show that for almost all pairs $m \neq n$ the root sets of $\tilde{S}_m, \tilde{S}_n$ are disjoint. Let $\mathbb{I} \subset \mathbb{N}$ be the set of integers k such that $\tilde{S}_k$ is irreducible. Let $\mathbb{J} \subset \mathbb{I} \times \mathbb{I}$ be the set of distinct pairs. By Theorem 4 in [9], $\mathbb{I} \subset \mathbb{N}$ is a set of density one. Thus, the sets $\mathbb{J} \subset \mathbb{I} \times \mathbb{I} \subset \mathbb{N} \times \mathbb{N}$ have density one. But for pairs $(m, n) \in \mathbb{J}$ the polynomials $\tilde{S}_m, \tilde{S}_n$ have disjoint roots. □

We will now briefly digress into the general Birkhoff billiard. Let Ω be any Birkhoff billiard table; let $F : Z \to Z$ be the billiard map on Ω. Let $\Gamma \subset Z$ be an invariant curve of the form $\theta = h(s)$. Thus, Γ is homeomorphic to the standard circle $T = \{0 \leq \alpha \leq 2\pi\}$. The circle T carries the classical family of rotations $R_\rho : \alpha \mapsto \alpha + 2\pi\rho \mod 2\pi$, $0 \leq \rho < 1$. If $F|_\Gamma$ is conjugate to R_ρ, we say that Γ is a *rotational invariant circle* with the *rotation number* ρ.

By Poincaré, the transformation $F|_\Gamma$ always has a rotation number $\rho(\Gamma)$. The general Γ, however, is not a rotational invariant circle. If $\rho(\Gamma)$ is irrational, then, under mild regularity assumptions, Γ is an invariant circle with the rotation number $\rho(\Gamma)$. Thus, there is a diffeomorphism of Γ onto the standard circle that conjugates $F|_\Gamma$ and $R_{\rho(\Gamma)}$. By [13,33] every sufficiently regular Ω has a large family of rotational invariant circles Γ with irrational $\rho(\Gamma)$. On the other hand, there is no theorem that guarantees the existence of invariant circles Γ with rational $\rho(\Gamma)$. Several results in the billiard literature suggest that such invariant curves are extremely rare [28,30].[4]

Let $0 < x < \pi/2$ satisfy Eq. (10). Then, by Theorem 2 and the proof of Proposition 1, there is a continuous family of billiard tables Ω_r[5] with invariant circles Γ such that $\rho(\Gamma) = x/\pi$. In view of the above, the author conjectured that the numbers x/π are irrational [20,26].[6] The work of Van Cyr [12] corroborated the conjecture.

Theorem 3. Van Cyr [12].[7] *Let $x \in (0, \pi/2)$ satisfy $\tan nx = n \tan x$ for some $n > 1$. Then x/π is irrational.*

8. Further Applications to the Billiard and Floating

We will now derive applications of the preceding material to the billiard problem and to the capillary floating. Throughout this section we assume the validity of the Conjecture.

8.1. Billiard Tables with Constant Angle Caustics

In Sect. 4 we called billiard tables $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ equivalent if there is a homothety $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Omega_1 = h(\Omega_2)$. To simplify the terminology, we will now say that Ω_1, Ω_2 are *equal up to a homothety*.

Proposition 8. *Let $A \subset (0, \pi)$ be the set defined in Sect. 4. Let $\delta \in A \setminus \{\pi/2\}$, and let $\Omega \subset \mathbb{R}^2$ be a non-circular, regular billiard table with the caustic Γ_δ. Then there is a unique integer $n \geq 4$ and a unique parameter $0 < \tau < 1$ such that up to a homothety $\Omega = \Omega_{n,\tau}$, where $\Omega_{n,\tau}$ is given by Eqs. (11), (12).*

Proof. Let $\rho(\alpha)$ be the radius of curvature function for $\partial\Omega$. By Theorem 1, there are unique constants a, b, c satisfying $0 < a^2 + b^2 < c$ such that $\rho(\alpha) = c + a \cos n\alpha + b \sin n\alpha$. The claim now follows from Theorem 2. □

Theorem 4. *Let $\Omega \subset \mathbb{R}^2$ be a noncircular, regular billiard table. The following statements are equivalent.*

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[4] See, however, [5] for a different viewpoint.

[5] In fact, the one-parameter family $\Omega_{n,\tau}, 0 \leq \tau < 1$, if the Conjecture holds.

[6] The work [26] is a preliminary version of the present paper.

[7] Van Cyr proved this theorem a few years ago; he was then an undergraduate student at SUNY Buffalo.

1. *The table* Ω *has a caustic* Γ_δ, $\delta \neq \pi/2$.
2. *There is* $n > 3$ *such that the Fourier coefficients of the radius of curvature* $\rho(\cdot)$ *of* $\partial\Omega$ *satisfy (i)* $c_n \neq 0$*; (ii)* $c_k = 0$ *for all positive* $k \neq n$.
3. *There is a unique* $n > 3$ *and a unique* $0 < \tau < 1$ *such that up to a homothety* $\Omega = \Omega_{n,\tau}$.

Proof. Proposition 8 proves the implication $1 \Rightarrow 3$, while $2 \Rightarrow 1$ is a byproduct of Theorem 2. The implication $3 \Rightarrow 2$ is obvious. □

Corollary 7. *Let* $\Omega \subset \mathbb{R}^2$ *be a regular billiard table. Suppose that* Ω *has a constant angle caustic* Γ_δ *where* $\delta \neq \pi/2$ *and* δ/π *is rational. Then* Ω *is a disc.*

Proof. The claim is immediate from Theorems 3 and 2, claim 3. □

Theorem 5. *There is a dense countable set* $R \subset (0, 1)$ *of irrational numbers such that the following claims hold.*

1. *For every* $\rho \in R$ *there is a one-parameter family* $\{\Omega_\tau : 0 \leq \tau < 1\}$ *of real analytic billiard tables having a constant angle caustic with the rotation number* ρ*. Every regular billiard table having a constant angle caustic with the rotation number* ρ *is homothetic to a unique table* Ω_τ.
2. *Let* $\rho \in (0, 1) \setminus R$*. Suppose that a regular billiard table* Ω *has a constant angle caustic with the rotation number* ρ. (i) *If* $\rho = 1/2$ *then* Ω *has constant width.* (ii) *If* $\rho \neq 1/2$ *then* Ω *is a disc.*

Proof. Claim 1 is immediate from Theorem 4 and Corollary 7. Claim 2 follows by combining these statements with Theorem 1. □

8.2. Two-Dimensional Capillary Floating in Neutral Equilibrium

We will now apply the preceding material to the capillary floating.

Theorem 6. *Let* $\Omega \subset \mathbb{R}^2$ *be a regular, compact, convex domain. Then the following holds.*

1. *Suppose that* Ω *is not a disc. Then* Ω *floats in neutral equilibrium at any orientation with the contact angle* $\gamma \neq \pi/2$ *if and only if there is* $n > 3$ *and* $0 < \tau < 1$ *such that* Ω *is homothetic to the domain* $\Omega_{n,\tau}$ *given by Eqs.* (11), (12)*. The numbers* n, τ *are uniquely determined by* Ω.
2. *If* Ω *floats in neutral equilibrium at any orientation with the contact angle* $\gamma \neq \pi/2$ *then* γ/π *is irrational.*
3. *There is a countable dense set* $A \subset (0, \pi)$ *containing* $\pi/2$ *and symmetric about this point such that the following holds:*

If Ω *floats in neutral equilibrium at any orientation with the contact angle* $\gamma \in (0, \pi) \setminus A$ *then* Ω *is a disc.*

Proof. The claims are the counterparts of statements in Theorems 4, 5, and Corollary 7. □

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