Published for SISSA by Springer
Received: July 19, 2012
Accepted: September 8, 2012
Published: October 4, 2012

Conformal field theory of Painlevé VI

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Abstract: Generic Painlevé VI tau function $\tau(t)$ can be interpreted as four-point correlator of primary fields of arbitrary dimensions in 2D CFT with $c = 1$. Using AGT combinatorial representation of conformal blocks and determining the corresponding structure constants, we obtain full and completely explicit expansion of $\tau(t)$ near the singular points. After a check of this expansion, we discuss examples of conformal blocks arising from Riccati, Picard, Chazy and algebraic solutions of Painlevé VI.

Keywords: Integrable Equations in Physics, Integrable Field Theories

ArXiv ePrint: 1207.0787

doi:10.1007/JHEP10(2012)038

JHEP10(2012)038

Contents

1 Introduction

Starting from the early seventies, Painlevé equations have been playing an increasingly important role in mathematical physics, especially in the applications to classical and quantum integrable systems and random matrix theory [12, 19, 27, 35–39, 41]–[44, 46]. A considerable progress has been made since then in the study of various analytic, asymptotic and geometric properties of Painlevé transcendents. The interested reader is referred to [7, 9, 13] for details and further references.

The sixth Painlevé equation (PVI)

$$w'' = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)(w')^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)w' + \frac{2w(w-1)(w-t)}{t^2(t-1)^2}\left(\left(\theta_\infty-\frac{1}{2}\right)^2-\frac{\theta_0^2 t}{w^2}+\frac{\theta_1^2(t-1)}{(w-1)^2}-\frac{\left(\theta_t^2-\frac{1}{4}\right)t(t-1)}{(w-t)^2}\right), \tag{1.1}$$

is on the top of the classification of 2nd order ODEs without movable critical points. The latter property means that $w(t)$ is a meromorphic function on the universal cover of $\mathbb{P}^1\backslash\{0,1,\infty\}$. Four complex parameters $\boldsymbol{\theta}=(\theta_0,\theta_t,\theta_1,\theta_\infty)$ and two integration constants form a six-dimensional PVI parameter space $\mathcal{M}$.

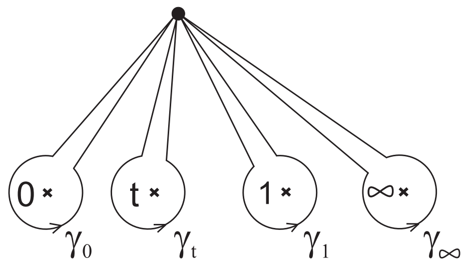

Figure 1. Homotopy basis for 4-punctured sphere.

The most natural mathematical framework for Painlevé equations is the theory of monodromy preserving deformations. For example, equation (1.1) is associated to rank 2 system

$$\frac{d\Phi}{dz} = \left(\frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1} \right) \Phi, \tag{1.2}$$

with four regular singular points $0, t, 1, \infty$ on $\mathbb{P}^1$. Traceless 2×2 matrices $\mathcal{A}_\nu$ $(\nu = 0, t, 1)$ are independent of z and have eigenvalues $\pm\theta_\nu$ that coincide with PVI parameters. Also, $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 \stackrel{def}{=} -\mathcal{A}_\infty = \operatorname{diag}\{-\theta_\infty, \theta_\infty\}$.

The fundamental matrix solution $\Phi(z)$ is multivalued on $\mathbb{P}^1 \backslash \{0, 1, t, \infty\}$, as its analytic continuation along non-contractible closed loops produces nontrivial monodromy. Full monodromy group is generated by three matrices $\mathcal{M}_{0,t,1} \in G = \mathrm{SL}(2, \mathbb{C})$ which correspond to the loops $\gamma_{0,t,1}$ in figure 1 (note that $\mathcal{M}_\infty \mathcal{M}_1 \mathcal{M}_t \mathcal{M}_0 = \mathbf{1}$). Right multiplication of Φ by a constant matrix gives another solution, and therefore $\{\mathcal{M}_\nu\}$ are fixed by (1.2) only up to overall conjugation.

As is well-known, Schlesinger equations of isomonodromic deformation of (1.2)

$$\frac{d\mathcal{A}_0}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_0]}{t}, \qquad \frac{d\mathcal{A}_1}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_1]}{t-1}$$

are equivalent to (1.1). Parameter space $\mathcal{M}$ may be identified with G^3/G and many questions on Painlevé VI can be recast in terms of monodromy. This approach, characteristic for classical integrable systems in general, turns out to be quite successful. In particular, it represents the key element of the solution of PVI connection problem [18], as well as of the construction [5, 11] and classification [25] of algebraic solutions.

Logarithmic derivative of Painlevé VI tau function

$$\sigma(t) = t(t-1)\frac{d}{dt}\ln\tau = (t-1)\operatorname{tr}\mathcal{A}_0\mathcal{A}_t + t\operatorname{tr}\mathcal{A}_t\mathcal{A}_1 \tag{1.3}$$

can be expressed in terms of $t, w(t)$ and $w'(t)$ (see e.g. eq. (2.9) in [26]). It solves a nonlinear 2nd order ODE called σ-form of Painlevé VI (σPVI), which may be written as

$$\Big(t(t-1)\sigma''\Big)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}. \tag{1.4}$$

It is much more natural to work with $\tau(t)$ than with $w(t)$ for a number of reasons:

- First, it is the tau function which typically shows up in the applications of Painlevé equations, representing gap probabilities in random matrix theory, correlation functions of Ising model and sine-Gordon field theory at the free-fermion point etc.

- The notion of tau function extends to the general isomonodromic setting [20] and its mathematical meaning is rather clear: it gives the determinant of a Cauchy-Riemann operator whose domain consists of multivalued functions with appropriate monodromy [34].

- Thirdly and (arguably) most importantly, the tau function has an intimate connection with quantum field theory.

The last point was discovered by Sato, Miwa and Jimbo in the first two papers of the series [35–39]. There it was shown that the Riemann-Hilbert problem for rank r linear systems with an arbitrary number of regular singularities on $\mathbb{P}^1$ admits a formal solution in terms of correlation functions in the theory describing r free massless chiral fermion copies. Besides fermions, the correlators involve local fields of another type (below we use the term "monodromy fields" instead of SMJ's "holonomic") which can be seen in the operator formalism as Bogoliubov transformations of the fermion algebra ensuring the required monodromy properties. General isomonodromic tau function was originally *defined* as the correlator of monodromy fields. In particular, for Painlevé VI it is given by a four-point correlator

$$\tau(t) = \langle \mathcal{O}_{\mathcal{M}_0}(0)\mathcal{O}_{\mathcal{M}_t}(t)\mathcal{O}_{\mathcal{M}_1}(1)\mathcal{O}_{\mathcal{M}_\infty}(\infty)\rangle. \tag{1.5}$$

We would like to put isomonodromic deformations into the context of subsequent developments in conformal field theory [4, 10, 48]. To our knowledge, no such attempt has been made so far except for a short general discussion in [31]. The present paper mainly deals with Painlevé VI which represents the simplest nontrivial example of isomonodromy equations.

It will be argued below that the relevant chiral CFT has central charge $c = 1$ and monodromy fields in (1.5) are Virasoro primaries with conformal dimensions $\Delta_\nu = \frac{1}{2}\operatorname{tr}\mathcal{A}_\nu^2 = \theta_\nu^2$, where $\nu = 0, 1, t, \infty$. Therefore the structure of the expansion of $\tau(t)$ near, say, $t = 0$, is strongly constrained by conformal invariance. In fact, the chiral correlator of four primary fields for any c is given by

$$\langle \phi_0(0)\phi_t(t)\phi_1(1)\phi_\infty(\infty)\rangle = \sum_p C_{0t}^p C_{p\infty}^1 t^{\Delta_p-\Delta_0-\Delta_t}\mathcal{F}(\Delta, \Delta_p, c; t), \tag{1.6}$$

where the sum runs over conformal families appearing in the OPE of ϕ_0 and ϕ_t, Δ_p denotes the dimension of the corresponding intermediate primary field ϕ_p and $\Delta = (\Delta_0, \Delta_t, \Delta_1, \Delta_\infty)$ is the set of external dimensions. Conformal block $\mathcal{F}(\Delta, \Delta_p, c; t) = \sum_{k=0}^{\infty} \mathcal{F}_k(\Delta, \Delta_p, c)\, t^k$ associated to the channel p is a power series normalized as $\mathcal{F}_0(\Delta, \Delta_p, c) = 1$. It is completely fixed by conformal symmetry [4]. The structure constants C_{0t}^p, $C_{p\infty}^1$ combine conformal blocks into correlation functions of specific theories

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and should be obtained from another source. CFT correlators usually contain contributions of holomorphic and antiholomorphic conformal blocks subject to the requirement of invariance under the action of braid group on the positions of fields. The chiral correlators (1.5), (1.6) are not invariant under this action but transform in a natural way, induced by the Hurwitz action on monodromy matrices.

Direct computation of the coefficients $\mathcal{F}_k(\Delta, \Delta_p, c)$ becomes quite laborious with the growth of k. However, very recently this problem was completely solved in the framework of AGT conjecture [3] relating Liouville CFT and $\mathcal{N} = 2$ 4D supersymmetric gauge theories. The latter correspondence produces conjectural combinatorial evaluations of conformal blocks, subsequently proven by Alba, Fateev, Litvinov and Tarnopolsky [2]. For $c = 1$, which is the only case of interest for PVI, another derivation was given by Mironov, Morozov and Shakirov in [30]. In particular, AGT representation expresses the contribution of fixed level of descendants of ϕ_p to 4-point conformal block $\mathcal{F}(\Delta, \Delta_p, c; t)$ in terms of sums of simple explicit functions of Δ, Δ_p and c over bipartitions with a fixed number of boxes.

Hence, to obtain full expansion of PVI tau function it suffices to determine the dimension spectrum of primaries present in the OPEs of monodromy fields and the associated structure constants. To formulate the final result, we introduce monodromy exponents $\boldsymbol{\sigma} = (\sigma_{0t}, \sigma_{1t}, \sigma_{01})$ by

$$p_{\mu\nu} = \operatorname{tr} \mathcal{M}_\mu \mathcal{M}_\nu = 2\cos 2\pi\sigma_{\mu\nu}, \qquad \mu, \nu = 0, t, 1.$$

Together with $\boldsymbol{\theta}$, these parameters define seven invariant functions on the space $\mathcal{M}$ of monodromy data subject to a relation [18]

$$\begin{aligned} p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_{0t}p_{1t}p_{01} + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0 p_t p_1 p_\infty \\ = (p_0 p_t + p_1 p_\infty) p_{0t} + (p_1 p_t + p_0 p_\infty) p_{1t} + (p_0 p_1 + p_t p_\infty) p_{01} + 4, \end{aligned} \tag{1.7}$$

where $p_\nu = \operatorname{tr} \mathcal{M}_\nu = 2\cos 2\pi\theta_\nu$ $(\nu = 0, t, 1, \infty)$. This of course agrees with the dimension of $\mathcal{M}$, and allows to interpret the triple $\boldsymbol{\sigma}$ as a pair of PVI integration constants. Below we assume that $\boldsymbol{\theta}$, $\boldsymbol{\sigma}$ are generic complex numbers verifying Jimbo-Fricke relation (1.7).

Let $\mathbb{Y}$ be the set of all partitions identified with Young diagrams. Given $\lambda \in \mathbb{Y}$, we write λ_i and λ'_j for the number of boxes in the ith row and the jth column of λ, and denote by $|\lambda|$ the total number of boxes in λ. The quantity $h_\lambda(i, j) = \lambda'_j - i + \lambda_i - j + 1$ is called the hook length of the box $(i, j) \in \lambda$.

Our main statement is the following

Claim. *Complete expansion of Painlevé VI tau function near $t = 0$ can be written as*

$$\tau(t) = \text{const} \cdot \sum_{n\in\mathbb{Z}} C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})\, t^{(\sigma_{0t}+n)^2 - \theta_0^2 - \theta_t^2} \mathcal{B}(\boldsymbol{\theta}, \sigma_{0t} + n; t). \tag{1.8}$$

The function $\mathcal{B}(\boldsymbol{\theta}, \sigma; t)$ is a power series in t which coincides with the general $c = 1$ con-

formal block $\mathcal{F}\left(\theta_0^2,\theta_t^2,\theta_1^2,\theta_\infty^2,\sigma^2,1;t\right)$ *and is explicitly given by*

$$\mathcal{B}\left(\boldsymbol{\theta},\sigma;t\right) =(1-t)^{2\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right)t^{|\lambda|+|\mu|}, \tag{1.9}$$

$$\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right) = \prod_{(i,j)\in\lambda}\frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2}\times \tag{1.10}$$
$$\times\prod_{(i,j)\in\mu}\frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2}.$$

The structure constants $\{C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)\}_{n\in\mathbb{Z}}$ *can be written in terms of Barnes G-function,*

$$C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) = s^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\big(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\big)\,G\big(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\big)}{G\big(1+2(\sigma_{0t}+n)\big)G\big(1-2(\sigma_{0t}+n)\big)}, \tag{1.11}$$

with s given by

$$\begin{aligned} s^{\pm1}\left(\cos 2\pi(\theta_t\mp\sigma_{0t})-\cos 2\pi\theta_0\right)\left(\cos 2\pi(\theta_1\mp\sigma_{0t})-\cos 2\pi\theta_\infty\right)& \\ =\left(\cos 2\pi\theta_t\cos 2\pi\theta_1+\cos 2\pi\theta_0\cos 2\pi\theta_\infty\pm i\sin 2\pi\sigma_{0t}\cos 2\pi\sigma_{01}\right)& \\ -\left(\cos 2\pi\theta_0\cos 2\pi\theta_1+\cos 2\pi\theta_t\cos 2\pi\theta_\infty\mp i\sin 2\pi\sigma_{0t}\cos 2\pi\sigma_{1t}\right)e^{\pm 2\pi i\sigma_{0t}}&. \end{aligned} \tag{1.12}$$

Analogous expansions of $\tau(t)$ at $t=1,\infty$ are obtained from (1.8)–(1.12) by applying a suitable transformation of parameters.

Painlevé transcendents are generally believed to be rather complicated special functions. In our opinion, this reputation is somewhat undeserved. Painlevé VI, for instance, enjoys most of the basic properties of the Gauss hypergeometric equation. In particular, it has many elementary solutions (see [25] and references therein), Bäcklund transformations [33], quadratic Landen-type transformations [21], and its connection problem is solved [18]. The present work adds to this list a link to representation theory of the Virasoro algebra and the series representation of PVI solutions, which can also be seen as an efficient tool for their numerical computation.

The outline of the paper is as follows. The next section starts with a brief survey of the isomonodromy problem. After exhibiting global conformal symmetry of the tau function, in subsection 2.3 we introduce monodromy fields and explain how various mathematical objects of the theory of monodromy preserving deformations can be written in terms of correlation functions in 2D CFT. Subsection 2.4 deals more specifically with Painlevé VI. Here we present the arguments leading to (1.8)–(1.12), and discuss analytic continuation of PVI solutions and their Bäcklund transformations from CFT point of view. Section 3 is devoted to direct analytic verification of the above expansion. This task is further pursued in section 4, where the known special PVI solutions are examined from the field-theoretic perspective. We conclude with a list of open questions and directions for future work.

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2 CFT approach to isomonodromy

2.1 General

It is instructive to start with the general case of rank N linear system with n regular singular points $a = \{a_1, \ldots, a_n\}$ on $\mathbb{P}^1$. Instead of (1.2) one has

$$\partial_z \Phi = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \sum_{\nu=1}^{n} \frac{\mathcal{A}_\nu}{z - a_\nu}. \tag{2.1}$$

The absence of singularity at infinity implies that $N \times N$ constant matrices $\{\mathcal{A}_\nu\}$ satisfy the constraint $\sum_{\nu=1}^{n} \mathcal{A}_\nu = 0$. They are assumed to be diagonalizable so that $\mathcal{A}_\nu = \mathcal{G}_\nu \mathcal{T}_\nu \mathcal{G}_\nu^{-1}$ with some $\mathcal{T}_\nu = \operatorname{diag}\{\lambda_{\nu,1}, \ldots, \lambda_{\nu,N}\}$. The fundamental solution will be normalized by $\Phi(z_0) = \mathbf{1}_N$. It is useful to introduce the matrix

$$\mathcal{J}(z) = \Phi^{-1}\partial_z \Phi = \Phi^{-1}\mathcal{A}(z)\Phi.$$

The coefficients of the Taylor series of $\Phi(z)$ around $z = z_0$ can be expressed in terms of $\mathcal{J}$ and its derivatives. In particular,

$$\Phi(z \to z_0) = \mathbf{1}_N + \mathcal{J}(z_0)(z - z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right)\frac{(z - z_0)^2}{2} + \ldots \tag{2.2}$$

Near the singular points, the fundamental solution has the following expansions (under additional non-resonancy assumption $\lambda_{\nu,j} - \lambda_{\nu,k} \notin \mathbb{Z}$ for $j \neq k$):

$$\Phi(z \to a_\nu) = \mathcal{G}_\nu(z)\,(z - a_\nu)^{\mathcal{T}_\nu}\,\mathcal{C}_\nu. \tag{2.3}$$

Here $\mathcal{G}_\nu(z)$ is holomorphic and invertible in a neighborhood of $z = a_\nu$ and satisfies $\mathcal{G}_\nu(a_\nu) = \mathcal{G}_\nu$. The connection matrix $\mathcal{C}_\nu$ is independent of z. Counterclockwise continuation of $\Phi(z)$ around a_ν leads to monodromy matrix $\mathcal{M}_\nu = \mathcal{C}_\nu^{-1} e^{2\pi i \mathcal{T}_\nu}\mathcal{C}_\nu$.

Let us now vary the positions of singularities and normalization point, simultaneously evolving $\mathcal{A}_\nu$'s in such a way that the monodromy is preserved. A classical result translates this requirement into a system of PDEs

$$\partial_{a_\nu}\Phi = -\frac{z_0 - z}{z_0 - a_\nu}\frac{\mathcal{A}_\nu}{z - a_\nu}\Phi, \tag{2.4}$$

$$\partial_{z_0}\Phi = -\mathcal{A}(z_0)\,\Phi. \tag{2.5}$$

It is important to note that the matrix $\mathcal{J}(z)$ remains invariant under isomonodromic variation of z_0. Schlesinger deformation equations are obtained as compatibility conditions of (2.1), (2.4) and (2.5). Explicitly,

$$\partial_{a_\mu}\mathcal{A}_\nu = \frac{z_0 - a_\nu}{z_0 - a_\mu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \mu \neq \nu, \tag{2.6}$$

$$\partial_{a_\nu}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \partial_{z_0}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{z_0 - a_\mu}. \tag{2.7}$$

Lax form of the Schlesinger system (2.6)–(2.7) implies that the eigenvalues of $\mathcal{A}_\nu$'s are conserved under deformation. This is of course expected due to the obvious relation between the spectra of $\mathcal{A}_\nu$'s and monodromy matrices.

Isomonodromic tau function $\tau(a)$ is defined by

$$d\ln\tau = \sum_{\mu<\nu} \operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu\, d\ln\left(a_\mu - a_\nu\right). \tag{2.8}$$

It is a nontrivial consequence of the deformation equations that the 1-form in the r.h.s. is closed. To show that it does not depend on z_0, one can rewrite (2.8) as

$$\partial_{a_\mu}\ln\tau = \sum_{\nu\neq\mu} \frac{\operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu}{a_\mu - a_\nu} = \frac{1}{2}\operatorname{res}_{z=a_\mu}\operatorname{tr}\mathcal{J}^2(z). \tag{2.9}$$

Here the first equality follows from (2.8) and the second one from the fact that $\mathcal{J}(z)$ is conjugate to $\mathcal{A}(z)$.

Finally, let us decompose all $\mathcal{A}_\nu$'s into the sum of scalar and traceless part as $\mathcal{A}_\nu = \frac{\operatorname{tr}\mathcal{A}_\nu}{N}\mathbf{1}_N + \hat{\mathcal{A}}_\nu$. It can then be easily checked that

$$\Phi_{\mathcal{A}}(z) = \prod_\nu \left(\frac{z-a_\nu}{z_0-a_\nu}\right)^{\frac{\operatorname{tr}\mathcal{A}_\nu}{N}} \Phi_{\hat{\mathcal{A}}}(z), \tag{2.10}$$

$$\mathcal{J}_{\mathcal{A}}(z) = \frac{1}{N}\sum_\nu \frac{\operatorname{tr}\mathcal{A}_\nu}{z-a_\nu}\,\mathbf{1}_N + \mathcal{J}_{\hat{\mathcal{A}}}(z), \tag{2.11}$$

$$\tau_{\mathcal{A}}(a) = \prod_{\mu<\nu}\left(a_\mu - a_\nu\right)^{\frac{\operatorname{tr}\mathcal{A}_\mu\operatorname{tr}\mathcal{A}_\nu}{N}} \tau_{\hat{\mathcal{A}}}(a). \tag{2.12}$$

This allows to assume without any loss of generality that $\mathcal{A}_\nu$'s are traceless, but we deliberately postpone the imposition of this condition.

2.2 Global conformal symmetry

Fractional linear maps $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ form the automorphism group of the Riemann sphere. It is clear that under the action of these transformations on z, z_0 and a the quantities $\Phi(z)$ and $\{\mathcal{A}_\nu\}$ transform as functions and $\mathcal{J}(z)$ as a vector field. Our task in this subsection is to understand the effect of global conformal mappings on the tau function.

Let us first compute $\tau(a)$ explicitly in the case $n=3$. Since $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 = 0$, the coefficients $\operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu$ in (2.8) are conserved quantities. Hence, integrating (2.8), we find

$$\tau(a_1,a_2,a_3) = \text{const}\cdot (a_1-a_2)^{\Delta_3-\Delta_2-\Delta_1}(a_1-a_3)^{\Delta_2-\Delta_1-\Delta_3}(a_2-a_3)^{\Delta_1-\Delta_2-\Delta_3},$$

where $\Delta_\nu = \frac{1}{2}\operatorname{tr}\mathcal{A}_\nu^2$ with $\nu = 1,2,3$. One recognizes here the general expression for the three-point correlation function of (quasi)primary fields of dimensions $\Delta_{1,2,3}$ in the two-dimensional conformal field theory.

Given the above example, it is natural to assume that for general n the tau function transforms as the n-point function of primaries with appropriate dimensions:

$$\tau\left(f(a)\right) = \prod_{\nu=1}^n \left[f'\left(a_\nu\right)\right]^{-\Delta_\nu}\tau(a). \tag{2.13}$$

To prove the last formula, it is sufficient to consider infinitesimal transformations generated by the vector field $Z = \left(A + Bz + Cz^2\right)\partial_z$, which amounts to checking three differential constraints:

$$\sum_\nu \partial_{a_\nu} \ln \tau = 0,$$
$$\sum_\nu \left(a_\nu \partial_{a_\nu} \ln \tau + \Delta_\nu\right) = 0,$$
$$\sum_\nu \left(a_\nu^2 \partial_{a_\nu} \ln \tau + 2\Delta_\nu a_\nu\right) = 0.$$

These relations can indeed be straightforwardly demonstrated using the first equality in (2.9) and the condition $\sum_\nu \mathcal{A}_\nu = 0$.

2.3 Field content

The fundamental solution Φ is completely fixed by its monodromy, normalization and singular behaviour (2.3). The last point is particularly important: indeed, adding an integer to any diagonal element of $\mathcal{T}_\nu$ modifies the asymptotics of Φ as $z \to a_\nu$ without changing monodromy matrices. Therefore, in what follows, the notion of monodromy will include not only the set of $\mathcal{M}_\nu$'s but also the choice of their logarithm branches $\mathcal{L}_\nu = \mathcal{C}_\nu^{-1}\mathcal{T}_\nu\mathcal{C}_\nu$.

Let us now try to construct a formal QFT solution of the isomonodromic deformation problem, extending the ideas of [31, 35, 36]. The starting point will be the following ansatz for Φ:

$$\Phi_{jk}(z) = (z - z_0)^{2\Delta} \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \bar{\varphi}_j(z_0)\varphi_k(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}, \qquad j, k = 1, \ldots, N. \tag{2.14}$$

Here it is assumed that $\{\mathcal{O}_{\mathcal{L}_\nu}\}$, $\{\bar{\varphi}_j\}$, $\{\varphi_k\}$ are primary fields in a 2D CFT characterized by some central charge c. Further, we want the OPEs of $\bar{\varphi}$'s with φ's to contain the identity operator. This forces them to have equal dimensions, to be denoted by Δ. Normalization of these fields is fixed by the normalization of Φ; the leading OPE term should be equal to

$$\bar{\varphi}_j(z_0)\varphi_k(z) \sim (z - z_0)^{-2\Delta}\, \delta_{jk}. \tag{2.15}$$

Since $\Phi(z)$ may be represented by an entire series near $z = z_0$, the dimensions of all other primaries appearing in this OPE should be given by strictly positive integers. Monodromy fields $\{\mathcal{O}_{\mathcal{L}_\nu}\}$ are defined by the condition that their complete OPEs with $\{\varphi_k\}$ have the form

$$\mathcal{O}_{\mathcal{L}_\nu}(a_\nu)\varphi_k(z) = \sum_{j=1}^{n} \left((z - a_\nu)^{\mathcal{L}_\nu}\right)_{jk} \sum_{\ell=0}^{\infty} \mathcal{O}_{\mathcal{L}_\nu, j, \ell}(a_\nu)\,(z - a_\nu)^{\ell},$$

where $\{\mathcal{O}_{\mathcal{L}_\nu, j, \ell}\}$ are some local fields. In particular, the raw vector $(\varphi_1 \ldots \varphi_n)$ should be multiplied by $\mathcal{M}_\nu$ when continued around $\mathcal{O}_{\mathcal{L}_\nu}$. If one succeeds in finding a set of fields with all mentioned properties, the correlator ratio (2.14) will automatically give the solution of the linear system (2.1).

The definition (2.8) of the tau function arises very naturally in the CFT framework. To illustrate this, let us compute two more orders in the OPE (2.15). The identity field has no level 1 descendants, therefore the leading correction is given by a new primary field J_{jk} of dimension 1. The next-to-leading order correction comes from three sources: i) nonvanishing level 2 descendant of the identity operator given by the energy-momentum tensor T, ii) level 1 current descendant ∂J_{jk} and iii) new primaries of dimension 2 which can be combined into a single field S_{jk}. Thus

$$\begin{aligned}\bar{\varphi}_j(z_0)\varphi_k(z) = (z-z_0)^{-2\Delta}\Big[&\delta_{jk} + J_{jk}(z_0)\,(z-z_0) \\ &+\left(\frac{4\Delta}{c}\,T(z_0)\delta_{jk} + (\partial J_{jk})(z_0) + S_{jk}(z_0)\right)\frac{(z-z_0)^2}{2} + O\left((z-z_0)^3\right)\Big].\end{aligned} \tag{2.16}$$

We will make a further assumption of tracelessness of S, which is essentially motivated by the examples considered below. Now, substituting (2.16) into (2.14) and matching the result with (2.2), one finds that

$$\mathcal{J}(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}, \tag{2.17}$$

$$\operatorname{tr}\mathcal{J}^2(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)T(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}\,\frac{4N\Delta}{c}. \tag{2.18}$$

Standard CFT arguments allow to rewrite the r.h.s. of the last formula as

$$\frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)T(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle} = \sum_{\nu=1}^{n}\left\{\frac{\tilde{\Delta}_\nu}{(z-a_\nu)^2} + \frac{1}{z-a_\nu}\,\partial_{a_\nu}\ln\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle\right\},$$

where $\tilde{\Delta}_\nu$ denotes conformal dimension of $\mathcal{O}_{\mathcal{L}_\nu}$. Comparison of (2.18) with (2.9) then shows that the tau function can be identified with a power of the correlator of monodromy fields,

$$\tau(a) = \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle^{\frac{2N\Delta}{c}}. \tag{2.19}$$

In what follows, we will be exclusively interested in the case when

$$c = 2N\Delta. \tag{2.20}$$

Such a condition implies, in particular, that the dimensions $\tilde{\Delta}_\nu$ of monodromy fields coincide with the quantities $\Delta_\nu = \frac{1}{2}\operatorname{tr}\mathcal{A}_\nu^2$ from the previous subsection.

One possible realization of the above conditions is provided by the theory of N free complex fermions. Its central charge $c = N$ agrees with the conformal dimension $\Delta = \frac{1}{2}$ of fermionic fields $\{\bar{\psi}_j\}$, $\{\psi_k\}$ which play the role of $\bar{\varphi}$'s and φ's. The currents are by definition given by $J_{jk} = (\bar{\psi}_j\,\psi_k)$, while the energy-momentum tensor T and the fields $\{S_{jk}\}$ may be expressed as

$$T = \frac{1}{2}\sum_k\left[(\bar{\psi}_k\,\partial\psi_k) - (\partial\bar{\psi}_k\,\psi_k)\right],$$

$$S_{jk} = (\bar{\psi}_j\,\partial\psi_k) - (\partial\bar{\psi}_j\,\psi_k) - \frac{2}{N}\,T\,\delta_{jk}.$$

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To represent monodromy fields, recall the usual bosononization formulas

$$\begin{aligned} \bar{\psi}_k &= :e^{-i\phi_k}:, \qquad \psi_k =: e^{i\phi_k}:, \\ J_{jk} &= \begin{cases} :e^{i(\phi_k-\phi_j)}:, & j\neq k, \\ i\,\partial\phi_k, & j=k, \end{cases} \\ T &= -\frac{1}{2}\sum_k \left(\partial\phi_k\,\partial\phi_k\right), \end{aligned}$$

where $\{\phi_k\}_{k=1,\ldots,N}$ are free complex bosonic fields with the propagator $\langle\phi_k(w)\phi_k(z)\rangle \sim -\ln(z-w)$. Also note that for $\mathcal{C}\in GL(N,\mathbb{C})$, monodromy matrices for the linearly transformed fermions

$$\bar{\psi}'_j = \sum_k \mathcal{C}_{jk}\bar{\psi}_k, \qquad \psi'_j = \sum_k \mathcal{C}^{-1}_{kj}\psi_k,$$

are obtained from $\mathcal{M}_\nu$'s by conjugation by $\mathcal{C}$. In particular, setting $\mathcal{C}=\mathcal{C}_\nu$, one obtains fermions $\{\bar{\psi}^{(\nu)}_k\}$, $\{\psi^{(\nu)}_k\}$ with diagonal monodromy around a_ν. Denote by $\{\phi^{(\nu)}_k\}$ bosonic fields associated to this "diagonal" fermionic basis, then monodromy field $\mathcal{O}_{\mathcal{L}_\nu}$ can be written as

$$\mathcal{O}_{\mathcal{L}_\nu} =: e^{i\sum_k \lambda_{\nu,k}\phi^{(\nu)}_k}: .$$

We thus need to deal with n different bosonization schemes of the same theory, each of them being adapted for representing one of the monodromy fields. The corresponding N-tuples of bosons are related by complicated nonlocal transformations.

The formulas (2.10)–(2.12) are a signature of the well-known decomposition of fermionic CFT into the direct sum $\hat{u}(1)\oplus\hat{su}(N)_1$ of two WZW theories. Fermion and monodromy fields are given by products of fields from the two summands:

$$\begin{aligned} \bar{\psi}_k &= :e^{-i\phi_0/\sqrt{N}}:\otimes\hat{\bar{\varphi}}_k, \qquad \psi_k =: e^{i\phi_0/\sqrt{N}}:\otimes\hat{\varphi}_k, \\ \mathcal{O}_{\mathcal{L}_\nu} &= :e^{\frac{i\,\mathrm{tr}\,\mathcal{A}_\nu}{\sqrt{N}}\phi_0}:\otimes\mathcal{O}_{\hat{\mathcal{L}}_\nu}. \end{aligned}$$

Bosonic field ϕ_0 in the $\hat{u}(1)$ factors is expressed in terms of fields introduced before as $\phi_0=\frac{1}{\sqrt{N}}\sum_{k=1}^N\phi_k$. The fields $\{\hat{\bar{\varphi}}_k\}$, $\{\hat{\varphi}_k\}$ and $\{\mathcal{O}_{\hat{\mathcal{L}}_\nu}\}$ live in the $\hat{su}(N)_1$ WZW theory and can be formally written as ordered exponentials of integrated linear combinations of $\hat{su}(N)_1$-currents. It should be emphasized that they are Virasoro primaries but not necessarily WZW primaries. The fields $\{\hat{\bar{\varphi}}_k\}$ and $\{\hat{\varphi}_k\}$ have the same dimension $\Delta=\frac{N-1}{2N}$, in accordance with the central charge $c_{\hat{su}(N)_1}=N-1$ [23]. The dimension of $\mathcal{O}_{\hat{\mathcal{L}}_\nu}$ is equal to $\frac{1}{2}\,\mathrm{tr}\,\hat{\mathcal{A}}_\nu^2$, where as above, $\hat{\mathcal{A}}_\nu=\mathcal{A}_\nu-\frac{\mathrm{tr}\,\mathcal{A}_\nu}{N}\mathbf{1}_N$ stands for the traceless part of $\mathcal{A}_\nu$.

Now it becomes clear that imposing the tracelessness of $\mathcal{A}(z)$ corresponds to factoring out the $\hat{u}(1)$ piece from the fermionic theory. This innocently looking procedure is in fact crucial, as it drastically reduces the number of primary fields in the OPEs and thus makes the computation of correlation functions much more efficient as compared to fermionic realization. Therefore, in what follows we set $\mathrm{tr}\,\mathcal{A}(z)=0$, remove the hats from $\hat{\mathcal{A}}_\nu$'s, $\hat{\mathcal{L}}_\nu$'s, $\hat{\bar{\varphi}}$'s and $\hat{\varphi}$'s to lighten the notation, and interpret the isomonodromic tau function as a correlation function of primaries with dimensions Δ_ν in a CFT with $c=N-1$.

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We close this subsection with an example of application of field-theoretic machinery in the case $N = 2$. It is somewhat distinguished from CFT point of view, since for $c = 1$ the dimension $\Delta = \frac{1}{4}$ of $\bar{\varphi}$'s and φ's corresponds to level 2 degenerate states, and the dimension 1 of $\{J_{jk}\}$ is degenerate at level 3. Hence the correlation functions

$$\begin{aligned}\mathcal{P}_{jk} &= \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle,\\ \mathcal{Q}_{jk} &= \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J_{jk}(z)\rangle,\end{aligned}$$

have to satisfy linear PDEs of order 2 and 3, fixed by Virasoro symmetry. This results into the following statement (cf observations made in [32]):

Proposition 1. *Under assumption* $\operatorname{tr}\mathcal{A}(z) = 0$*, the matrices*

$$\mathcal{P} = (z-z_0)^{-\frac{1}{2}}\tau\Phi, \qquad \mathcal{Q} = \tau\Phi^{-1}\partial_z\Phi,$$

satisfy the differential equations

$$\begin{aligned}\partial_{zz}\mathcal{P} &= \left\{\frac{1}{z-z_0}\,\partial_{z_0} + \frac{1}{4\,(z-z_0)^2} + \sum_\nu\left(\frac{1}{z-a_\nu}\,\partial_{a_\nu} + \frac{\Delta_\nu}{(z-a_\nu)^2}\right)\right\}\mathcal{P},\\ \partial_{zzz}\mathcal{Q} &= \left\{4\sum_\nu\left(\frac{1}{z-a_\nu}\,\partial_{a_\nu z} + \frac{\Delta_\nu}{(z-a_\nu)^2}\,\partial_z\right) + 2\sum_\nu\left(\frac{1}{(z-a_\nu)^2}\,\partial_{a_\nu} + \frac{2\Delta_\nu}{(z-a_\nu)^3}\right)\right\}\mathcal{Q}.\end{aligned}$$

Proof. Straightforward but tedious calculation using the relations (2.1), (2.4), (2.5), (2.9) and the identity $A^2 = \frac{1}{2}\operatorname{tr}A^2\,\mathbf{1}_2$ verified by any traceless 2×2 matrix A. □

2.4 Painlevé VI

Recall that global conformal symmetry allows to fix the positions of three singular points. Painlevé VI equation corresponds to setting $N = 2$, $n = 4$ and sending these three points to 0, 1 and ∞. The remaining singular point, $z = t$, represents the cross-ratio of singularities, which is preserved by Möbius transformations.

For $\nu = 0, t, 1, \infty$, let us denote by $\pm\theta_\nu$ the eigenvalues of $\mathcal{A}_\nu$. Preceding arguments show that PVI tau function $\tau(t)$ defined by (1.3) is nothing but the four-point correlator of monodromy fields,

$$\tau(t) = \langle \mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)\mathcal{O}_{\mathcal{L}_1}(1)\mathcal{O}_{\mathcal{L}_\infty}(\infty)\rangle, \tag{2.21}$$

and that these fields are Virasoro primaries with dimensions $\Delta_\nu = \theta_\nu^2$ in a $c = 1$ conformal field theory. The field at infinity should be understood according to the usual CFT prescription

$$\langle\ldots\mathcal{O}\,(\infty)\rangle \stackrel{def}{=} \lim_{R\to\infty} R^{2\Delta_{\mathcal{O}}}\langle\ldots\mathcal{O}\,(R)\rangle.$$

It is clear that auxiliary fields $\{\varphi_k\}$ should have monodromy $\mathcal{M}_t\mathcal{M}_0$ around all fields in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$. Let $e^{\pm 2\pi i\sigma_{0t}}$ denote the eigenvalues of $\mathcal{M}_t\mathcal{M}_0$ and $\mathcal{C}_{0t}$ be its diagonalizing transformation. Since σ_{0t} is defined only up to an integer, it is natural to

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expect that the set of primaries present in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$ consists of an infinite number of monodromy fields $\mathcal{O}_{\mathcal{L}_{0t}^{(n)}}$ with $n \in \mathbb{Z}$ and

$$\mathcal{L}_{0t}^{(n)} = \mathcal{C}_{0t}^{-1} \begin{pmatrix} \sigma_{0t} + n & 0 \\ 0 & -\sigma_{0t} - n \end{pmatrix} \mathcal{C}_{0t},$$

i.e. of all possible monodromy fields associated to the monodromy matrix $\mathcal{M}_t\mathcal{M}_0$. Taking into account that conformal dimension of $\mathcal{O}_{\mathcal{L}_{0t}^{(n)}}$ is equal to $(\sigma_{0t} + n)^2$, the first part of our main statement (formulas (1.8)–(1.10)) now follows from the general formula (1.6) and AGT combinatorial representations of conformal blocks [3].

The structure constants $C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})$ of the expansion (1.8) can be determined from the so-called Jimbo asymptotic formula [18], expressing the asymptotics of PVI tau function as $t \to 0$ in terms of monodromy. In fact we have already obtained the "easier half" of this formula. E.g. if $-\frac{1}{2} < \operatorname{Re}\sigma_{0t} < \frac{1}{2}$, then (1.8) implies that the leading behaviour of $\tau(t)$ is given by

$$\tau(t \to 0) \sim \text{const} \cdot t^{\sigma_{0t}^2 - \theta_0^2 - \theta_t^2}.$$

Subleading asymptotics, fixing the second PVI integration constant, can be rewritten in the form of a recursion relation on the coefficients $C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})$. Namely,

$$\begin{aligned}\frac{C_{n\pm1}}{C_n} =& \frac{\Gamma^2\left(1 \mp 2(\sigma_{0t} + n)\right)}{\Gamma^2\left(1 \pm 2(\sigma_{0t} + n)\right)} \prod_{\epsilon=\pm} \frac{\Gamma\left(1+\epsilon\theta_0+\theta_t \pm (\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1 \pm (\sigma_{0t}+n)\right)}{\Gamma\left(1+\epsilon\theta_0+\theta_t \mp (\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1 \mp (\sigma_{0t}+n)\right)} \times \\ &\times \frac{\left(\theta_0^2 - (\theta_t \mp (\sigma_{0t} + n))^2\right)\left(\theta_\infty^2 - (\theta_1 \mp (\sigma_{0t} + n))^2\right)}{4\left(\sigma_{0t} + n\right)^2\left(1 \pm 2(\sigma_{0t} + n)\right)^2} (-s)^{\pm 1},\end{aligned}$$

where s is defined by (1.12). This relation can be easily solved in terms of Barnes functions, with the answer given by (1.11). It is interesting to note that, up to a common multiplier and appropriately symmetrized s^n factors, C_n's essentially coincide with the chiral parts [40] of the corresponding structure constants in the time-like Liouville theory [16, 47].

Remark 2. *The structure constants (1.11) can not be completely factorized into the products of three-point functions due to the presence of the parameter s. This is an artifact of non-trivial braid group action on the correlation functions of monodromy fields.*

To illustrate what we have in mind, consider the analytic continuation of $\tau(t)$ along a closed counterclockwise contour around the branch point $t = 0$. In general, such a continuation induces an action of the 3-braid group (more precisely, of the modular group $\Gamma(2)$) on monodromy [11]. In the case at hand, new monodromy matrices are given by

$$\mathcal{M}_0' = \mathcal{M}_t\mathcal{M}_0\mathcal{M}_t^{-1}, \qquad \mathcal{M}_t' = \left(\mathcal{M}_t\mathcal{M}_0\right)\mathcal{M}_t\left(\mathcal{M}_t\mathcal{M}_0\right)^{-1}, \qquad \mathcal{M}_1' = \mathcal{M}_1,$$

so that $\sigma_{0t}' = \sigma_{0t}$ and

$$p_{01}' = p_0p_1 + p_tp_\infty - p_{01} - p_{0t}p_{1t}, \tag{2.22}$$

$$p_{1t}' = p_1p_t + p_0p_\infty - p_{1t} - p_{0t}p_{01}'. \tag{2.23}$$

Therefore, the change of the branch of $\tau(t)$ is encoded in the change of the structure constants. On the other hand, one can perform analytic continuation directly in the expansion (1.8). Up to an irrelevant overall factor, this amounts to multiplication of $C_n(\boldsymbol{\theta},\boldsymbol{\sigma})$ by $e^{4\pi i n\sigma_{0t}}$. Since both results should coincide, the structure constants have to satisfy the functional relation

$$C_n(\boldsymbol{\theta},\boldsymbol{\sigma}') = \kappa\cdot e^{4\pi i n\sigma_{0t}}C_n(\boldsymbol{\theta},\boldsymbol{\sigma}),$$

where κ is independent of n. The factor s^n in (1.11) is a minimal solution of this relation, as for $\boldsymbol{\sigma}'$ defined by (2.22)–(2.23) one has $s(\boldsymbol{\theta},\boldsymbol{\sigma}') = e^{4\pi i\sigma_{0t}}s(\boldsymbol{\theta},\boldsymbol{\sigma})$.

Remark 3. *Let us denote by $\dim\lambda$ the number of standard Young tableaux of shape $\lambda\in\mathbb{Y}$. It coincides with the dimension of the irreducible representation of symmetric group $S_{|\lambda|}$ associated to λ. Also write d_λ for the number of diagonal boxes in λ and introduce the Frobenius coordinates*

$$p_i^\lambda = \lambda_i - i, \qquad q_i^\lambda = \lambda_i' - i, \qquad i = 1,\ldots,d_\lambda,$$

which give the number of boxes to the right and above the ith diagonal box. It is well known/easy to show that

$$\begin{aligned}\frac{\dim\lambda}{|\lambda|!} &= \frac{1}{\prod_{(i,j)\in\lambda}h_\lambda(i,j)}\\ &= \frac{1}{\prod_{i=1}^{d_\lambda}\Gamma(p_i^\lambda+1)\,\Gamma(q_i^\lambda+1)}\det\left[\frac{1}{p_i^\lambda+q_j^\lambda+1}\right]_{i,j=1,\ldots,d_\lambda},\end{aligned}$$

$$\prod_{(i,j)\in\lambda}(i-j+z)(i-j+z') = (zz')^{d_\lambda}\prod_{i=1}^{d_\lambda}\Gamma\left[\begin{matrix}p_i^\lambda+1+z,\, q_i^\lambda+1-z,\, p_i^\lambda+1+z',\, q_i^\lambda+1-z'\\ 1+z,\, 1-z,\, 1+z',\, 1-z'\end{matrix}\right].$$

We now recognize in (1.10) typical pieces of z-measures on partitions [6]. It would be nice to understand this coincidence conceptually with the purpose to sum up the series for $\mathcal{B}(\boldsymbol{\theta},\sigma;t)$ and $\tau(t)$.

Remark 4. *Painlevé VI equation has a large group of hidden symmetries of affine Weyl type [33]. Almost all of them are manifest in the conformal expansion (1.8). For instance, the change of sign of any of parameters $\boldsymbol{\theta}$ has no effect on the tau function, since conformal blocks depend only on the dimensions $\Delta_\nu = \theta_\nu^2$ and the ratios of structure constants (1.11) also remain invariant.*

Conformal block symmetry $(\theta_0,\theta_t)\leftrightarrow(\theta_\infty,\theta_1)$ and its counterparts for expansions at $t=1,\infty$ yield further simple transformations of $\tau(t)$. Another, less trivial symmetry that can be found by inspection of (1.10) shifts the values of all $\boldsymbol{\theta}$ by $\delta = \dfrac{\theta_0+\theta_t+\theta_1+\theta_\infty}{2}$. Additional transformations come from crossing symmetry. In contrast to the previous ones, they also act on t by fractional linear transformations exchanging $0,1$ and ∞.

The action of generators of the above transformations on the parameters $\boldsymbol{\theta}$, conformal blocks and tau function expansions is recorded in table 1.

	θ_0	θ_t	θ_1	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\boldsymbol{s}_0$	$-\theta_0$	θ_t	θ_1	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\boldsymbol{s}_t$	θ_0	$-\theta_t$	θ_1	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\boldsymbol{s}_1$	θ_0	θ_t	$-\theta_1$	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\boldsymbol{s}_\infty$	θ_0	θ_t	θ_1	$-\theta_\infty$	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\boldsymbol{s}_\delta$	$\theta_0-\delta$	$\theta_t-\delta$	$\theta_1-\delta$	$\theta_\infty-\delta$	$(1-t)^{\delta_{1t}\delta}\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$t^{\delta_{0t}\delta}(1-t)^{\delta_{1t}\delta}\tau(t)$
$\boldsymbol{r}_{0t}$	θ_∞	θ_1	θ_t	θ_0	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$t^{\Delta_{0t}}\tau(t)$
$\boldsymbol{r}_{1t}$	θ_t	θ_0	θ_∞	θ_1	$(1-t)^{\Delta_{1t}}\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$(1-t)^{\Delta_{1t}}\tau(t)$
$\boldsymbol{r}_{01}$	θ_1	θ_∞	θ_0	θ_t	$(1-t)^{\Delta_{1t}}\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$t^{\Delta_{0t}}(1-t)^{\Delta_{1t}}\tau(t)$
$\boldsymbol{q}_{01}$	θ_1	θ_t	θ_0	θ_∞		$\tau(1-t)$
$\boldsymbol{q}_{0\infty}$	θ_∞	θ_t	θ_1	θ_0		$t^{-2\Delta_t}\tau\left(t^{-1}\right)$
$\boldsymbol{q}_{1\infty}$	θ_0	θ_t	θ_∞	θ_1	$(1-t)^{\Delta_0-\Delta_t-\Delta_\sigma}\mathcal{B}\big(\boldsymbol{\theta},\sigma;\frac{t}{t-1}\big)$	$(1-t)^{-2\Delta_t}\tau\big(\frac{t}{t-1}\big)$

Table 1. Backlund transformations of parameters, conformal blocks and tau functions.

Here we have introduced the notation

$$\delta_{0t}=\theta_0+\theta_t-\theta_1-\theta_\infty,\qquad \Delta_{0t}=\Delta_0+\Delta_t-\Delta_1-\Delta_\infty,$$
$$\delta_{1t}=\theta_1+\theta_t-\theta_0-\theta_\infty,\qquad \Delta_{1t}=\Delta_1+\Delta_t-\Delta_0-\Delta_\infty,$$

and $\Delta_\sigma=\sigma^2$*. Transformed tau functions in the last column are (in some cases) defined up to constant factors which depend on the choice of normalization of the structure constants.*

To complete the picture, it remains to understand the QFT meaning of an elementary Schlesinger transformation, e.g. the one shifting θ_0 *and* θ_1 *by* $\frac12$*. It may be expected that this symmetry arises from the fusion of auxiliary fields* $\bar\varphi_j$ *and* φ_k *with monodromy fields* $\mathcal{O}_{\mathcal{L}_0}$ *and* $\mathcal{O}_{\mathcal{L}_1}$ *in the correlator representation (2.14) of the fundamental solution* Φ*. Indeed,* $\bar\varphi$*'s and* φ*'s are degenerate at level 2, and therefore their OPEs with monodromy field of dimension* θ^2 *can only contain two conformal families generated by monodromy fields with dimensions* $\left(\theta\pm\frac12\right)^2$*.*

3 Painlevé VI recurrence

Painlevé VI tau function expansion (1.8)–(1.11) is, in the strict mathematical sense, a conjecture. On the other hand, $\tau(t)$ is completely fixed by its leading asymptotics as $t\to0$. Once the quantity σ_{0t} in the power-law exponent and the amplitude ratio $\frac{C_1(\boldsymbol{\theta},\boldsymbol{\sigma})}{C_0(\boldsymbol{\theta},\boldsymbol{\sigma})}$ are found from Jimbo's formula, the rest of the series can, at least in principle, be recursively reconstructed order by order from σPVI equation and checked against CFT predictions. Below we describe technical details of this procedure and derive several first terms of the conformal expansion.

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The equation (1.4) gives a quadrilinear 3rd order ODE for the tau function itself. Observe, however, that if we differentiate (1.4) one more time with respect to t, the resulting equation will be divisible by $\sigma''(t)$ and in the end turns out to be *bilinear* in $\tau(t)$. It is also convenient to work with a slightly modified function $\eta(t) = t^{\Delta_0+\Delta_t}(1-t)^{\Delta_t+\Delta_1}\tau(t)$, which partially takes into account the behaviour of $\tau(t)$ as $t \to 0, 1$. It satisfies the equation

$$\begin{aligned} t^2(t-1)^2 \Big\{ \Big[t(t-1)\eta^2 \Big]'''' - 8\Big[t(t-1)\left(\eta'\right)^2 \Big]'' - 4(2A-1)\eta\eta'' + 8(A-2)\left(\eta'\right)^2 \Big\} \\ + 16t^3(t-1)^3\left(\eta''\right)^2 + 4t(t-1)\eta\eta'' - 4(A-1)t(t-1)(2t-1)\eta\eta' + 4(Bt+C)\eta^2 = 0, \end{aligned} \tag{3.1}$$

where A, B, C are given by

$$\begin{aligned} A &= \Delta_0 + \Delta_t + \Delta_1 + \Delta_\infty, \\ B &= \left(\Delta_0 - \Delta_1\right)\left(\Delta_\infty - \Delta_t\right), \\ C &= \left(\Delta_0 - \Delta_t\right)\left(\Delta_1 - \Delta_\infty\right), \end{aligned}$$

and the fourth PVI parameter is killed by differentiation.

Now substitute into (3.1) the ansatz

$$\eta(t) = \sum_{n\in\mathbb{Z}} C_n t^{(\sigma+n)^2} \sum_{k=0}^{\infty} \eta_k^{(n)} t^k, \tag{3.2}$$

normalized by $\eta_0^{(n)} = 1$ for all $n \in \mathbb{Z}$. Picking up the coefficients of different powers of t in the result, one obtains an overdetermined system of equations for $\{C_n\}$, $\{\eta_k^{(n)}\}$. Namely, for any $\ell \in \mathbb{Z}$ and any $L \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \sum_{\substack{m,n\in\mathbb{Z}\\ m+n=\ell}} C_m C_n \Bigg[\sum_{\substack{k,k'\geq 0\\ k+k'+m^2+n^2=L+1}} \alpha_{mn}^{kk'} \eta_k^{(m)}\eta_{k'}^{(n)} + \sum_{\substack{k,k'\geq 0\\ k+k'+m^2+n^2=L}} \beta_{mn}^{kk'} \eta_k^{(m)}\eta_{k'}^{(n)} \\ + \sum_{\substack{k,k'\geq 0\\ k+k'+m^2+n^2=L-1}} \gamma_{mn}^{kk'} \eta_k^{(m)}\eta_{k'}^{(n)} + \sum_{\substack{k,k'\geq 0\\ k+k'+m^2+n^2=L-2}} \delta_{mn}^{kk'} \eta_k^{(m)}\eta_{k'}^{(n)} \Bigg] = 0, \end{aligned} \tag{3.3}$$

with somewhat cumbersome but completely explicit polynomial coefficients:

$$\begin{aligned} \alpha_{mn}^{kk'} &= \epsilon_{mn}^{kk'}\left(2P - \epsilon_{mn}^{kk'} + 1\right), \\ \beta_{mn}^{kk'} &= 3\left(\epsilon_{mn}^{kk'}\right)^2 - 2\left(P - 1 + 2A\right)\epsilon_{mn}^{kk'} - P\left(P - 2A\right) + 4C, \\ \gamma_{mn}^{kk'} &= -3\left(\epsilon_{mn}^{kk'}\right)^2 - 2\left(P - 4A\right)\epsilon_{mn}^{kk'} + \left(P-1\right)\left(P - 1 - 2A\right) + 4B, \\ \delta_{mn}^{kk'} &= \epsilon_{mn}^{kk'}\left(2P + \epsilon_{mn}^{kk'} - 3 - 4A\right), \\ \epsilon_{mn}^{kk'} &= \Big(k - k' + (m-n)(\ell + 2\sigma)\Big)^2, \\ P &= 2\sigma^2 + 2\sigma\ell + L. \end{aligned}$$

Let us write down the recurrence relations (3.3) and their consequences for some L and ℓ.

L = 0, ℓ = 0: from (3.3) it follows that

$$\left(\alpha_{0,0}^{1,0}+\alpha_{0,0}^{0,1}\right)\eta_1^{(0)}+\beta_{0,0}^{0,0}=0,$$

which in turn gives

$$\eta_1^{(0)}=-(\Delta_t+\Delta_1)+\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)}{2\Delta_\sigma},$$

with $\Delta_\sigma=\sigma^2$ as above. This reproduces $n=0$, level 1 descendant contribution to the expansion (1.8). The corresponding term was already found in [18].

L = 1, ℓ = ±1: in this case, (3.3) implies that

$$\left(\alpha_{\pm1,0}^{1,0}+\alpha_{0,\pm1}^{0,1}\right)\eta_1^{(\pm1)}+\left(\alpha_{\pm1,0}^{0,1}+\alpha_{0,\pm1}^{1,0}\right)\eta_1^{(0)}+\left(\beta_{\pm1,0}^{0,0}+\beta_{0,\pm1}^{0,0}\right)=0,$$

and we obtain

$$\eta_1^{(\pm1)}=-(\Delta_t+\Delta_1)+\frac{(\Delta_{\sigma\pm1}-\Delta_0+\Delta_t)\left(\Delta_{\sigma\pm1}-\Delta_\infty+\Delta_1\right)}{2\Delta_{\sigma\pm1}}.$$

The latter expression corresponds to level 1 descendants with $n=\pm1$. It coincides with $\eta_1^{(0)}$ with σ replaced by $\sigma\pm1$, but this is not surprising: the same should be correct for any $n\in\mathbb{Z}$ provided the conjectured periodicity of powers in (3.2) holds true.

L = 1, ℓ = 0: here we find

$$C_0^2\left[\left(\alpha_{0,0}^{2,0}+\alpha_{0,0}^{0,2}\right)\eta_2^{(0)}+\alpha_{0,0}^{1,1}\left(\eta_1^{(0)}\right)^2+\left(\beta_{0,0}^{1,0}+\beta_{0,0}^{0,1}\right)\eta_1^{(0)}+\gamma_{0,0}^{0,0}\right]$$
$$+C_1C_{-1}\left(\alpha_{1,-1}^{0,0}+\alpha_{-1,1}^{0,0}\right)=0, \tag{3.4}$$

which gives level 2 descendant contribution with $n=0$:

$$\begin{aligned}\eta_2^{(0)}=&-\frac{(\Delta_t+\Delta_1)(\Delta_t+\Delta_1+1)}{2}-(\Delta_t+\Delta_1)\eta_1^{(0)}\\&+\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)}{4\Delta_\sigma(1+2\Delta_\sigma)}\\&+\frac{(1+2\Delta_\sigma)\left(\Delta_0+\Delta_t+\frac{\Delta_\sigma(\Delta_\sigma-1)-3(\Delta_0-\Delta_t)^2}{1+2\Delta_\sigma}\right)\left(\Delta_\infty+\Delta_1+\frac{\Delta_\sigma(\Delta_\sigma-1)-3(\Delta_\infty-\Delta_1)^2}{1+2\Delta_\sigma}\right)}{2\left(1-4\Delta_\sigma\right)^2}.\end{aligned}$$

To obtain the last formula, one should use, in addition to (3.4) and the coefficients found above, the relation

$$\frac{C_1C_{-1}}{C_0^2}=\frac{\left((\Delta_0+\Delta_t-\Delta_\sigma)^2-4\Delta_0\Delta_t\right)\left((\Delta_\infty+\Delta_1-\Delta_\sigma)^2-4\Delta_\infty\Delta_1\right)}{16\Delta_\sigma^2\left(1-4\Delta_\sigma\right)^2}.$$

This piece of initial data disappears after the above differentiation but can be determined from the quadrilinear form of σPVI.

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ℓ \ L	0	1	2	3	4	5	6	7	8	9
0	$\eta_1^{(0)}$	$\eta_2^{(0)}$	$\eta_3^{(0)}$	$\eta_4^{(0)}$	$\eta_5^{(0)}$	$\eta_6^{(0)}$	$\eta_7^{(0)}$	$\eta_8^{(0)}$	$\eta_9^{(0)}$	$\eta_{10}^{(0)}$
±1		$\eta_1^{(\pm1)}$	$\eta_2^{(\pm1)}$	$\eta_3^{(\pm1)}$	$\eta_4^{(\pm1)}$	$\eta_5^{(\pm1)}$	$\eta_6^{(\pm1)}$	$\eta_7^{(\pm1)}$	$\eta_8^{(\pm1)}$	$\eta_9^{(\pm1)}$
±2				$C_{\pm2}$	$\eta_1^{(\pm2)}$	$\eta_2^{(\pm2)}$	$\eta_3^{(\pm2)}$	$\eta_4^{(\pm2)}$	$\eta_5^{(\pm2)}$	$\eta_6^{(\pm2)}$
±3									$C_{\pm3}$	$\eta_1^{(\pm3)}$

Table 2. Appearance of recurrence coefficients.

It is straightforward to compute more terms in (3.2) using computer algebra. The procedure works as follows. For a given L, one should start with maximal $|\ell|$ producing a nontrivial relation (3.3), and then repeatedly decrease $|\ell|$ by 1. When all possibilities are exhausted, increase L by 1. The coefficients determined at the first steps of this iterative procedure are listed in table 2. Empty entries correspond to the relations satisfied automatically or to no relations at all.

In this way, we have successfully checked the expansion (1.8)–(1.11) for $n = 0, \pm1, \pm2, \pm3$ going up to level 10 in descendants. To give the reader an idea of the computational complexity, we note that there are nearly 500 bipartitions of size 10, each of them producing a rational function of $\boldsymbol{\theta}$, σ_{0t} in the corresponding expansion coefficient.

4 Conformal blocks and special PVI solutions

4.1 Riccati solutions

These PVI solutions appear when the monodromy of (1.2) is equivalent to an upper triangular one. Parameters $\boldsymbol{\theta}$ can be Bäcklund transformed to satisfy $\theta_0+\theta_t+\theta_1+\theta_\infty = 0$. The initial conditions are also constrained and can be chosen as $\boldsymbol{\sigma} = (\theta_0+\theta_t, \theta_1+\theta_t, \theta_0+\theta_1)$.

This results into a one-parameter family of PVI transcendents $w(t)$ that may be written in terms of Gauss hypergeometric functions, see e.g. Proposition 49 in [25] for explicit formulas. The relevant tau function, however, is extremely simple:

$$\tau(t) = \text{const}\cdot t^{2\theta_0\theta_t}(1-t)^{2\theta_t\theta_1}.$$

We recognize in the r.h.s. of this relation the four-point correlator $\langle \mathcal{V}_{\theta_0}(0)\mathcal{V}_{\theta_t}(t)\mathcal{V}_{\theta_1}(1)\mathcal{V}_{\theta_\infty}(\infty)\rangle$ of the chiral vertex operators $\mathcal{V}_\theta(z) =: e^{i\sqrt{2}\theta\phi(z)}:$ made of free massless bosons. The contribution of conformal blocks $\mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n; t)$ with $n \neq 0$ to the expansion (1.8) is annihilated by the vanishing structure constants.

4.2 Chazy solutions

As is well-known [4, 10, 48] and already mentioned in section 2, the presence of degenerate states in the Virasoro module generated by a primary field ϕ leads to linear differential equations for the correlation functions involving this field. Possible conformal dimensions of such ϕ's are determined by the zeros of Kac determinant. They are labeled by two integers

$r, s \in \mathbb{Z}_{\geq 1}$ and, for $c = 1$, are explicitly given by $\Delta_{\phi_{r,s}} = \frac{(r-s)^2}{4}$. In the simplest nontrivial case $r = 2$, $s = 1$ the four-point correlator containing $\phi_{2,1}(z)$ and three arbitrary primaries solves a linear 2nd order ODE that can be reduced to the hypergeometric equation.

Thus if one of the monodromy matrices $\mathcal{M}_{0,t,1,\infty}$ is equal to $-\mathbf{1}$, there exists a PVI tau function given by a linear combination of two solutions of the latter equation. For example, for $\theta_\infty = \pm\frac{1}{2}$ (i.e. $\Delta_\infty = \frac{1}{4}$) the general solution of

$$\tau'' + 2\left[\frac{\theta_0^2 - \theta_1^2 + \theta_t^2 + \frac{1}{4}}{t} + \frac{\theta_1^2 - \theta_0^2 + \theta_t^2 + \frac{1}{4}}{t-1}\right]\tau'$$
$$+\left[\frac{(\theta_0+\theta_1)^2 + \theta_t^2 - \frac{1}{4}}{t(t-1)} + \prod_{\epsilon=\pm}\left(\frac{\theta_0^2 + \theta_t^2 - (\epsilon\theta_1 + \frac{1}{2})^2}{t} + \frac{\theta_t^2 + \theta_1^2 - (\epsilon\theta_0 + \frac{1}{2})^2}{t-1}\right)\right]\tau = 0$$

satisfies σPVI. Conformal expansion of this tau function at, say, $t = 0$ is determined by only two channels with dimensions $\left(\theta_1 \pm \frac{1}{2}\right)^2$. More precisely, one has

$$\tau(t) = \sum_{\epsilon=\pm} C_\epsilon t^{(\epsilon\theta_1+\frac{1}{2})^2 - \theta_0^2 - \theta_t^2} \mathcal{B}\left(\theta_0, \theta_t, \theta_1, \frac{1}{2}, \epsilon\theta_1 + \frac{1}{2}; t\right), \tag{4.1}$$

where $C_\pm$ are arbitrary constants. The relevant conformal blocks can be written as

$$\mathcal{B}\left(\theta_0, \theta_t, \theta_1, \frac{1}{2}, \theta_1 + \frac{1}{2}; t\right) = (1-t)^{(\theta_0+\frac{1}{2})^2 - \theta_t^2 - \theta_1^2}\, {}_2F_1\left[\begin{matrix}\theta_0+\theta_t+\theta_1+\frac{1}{2}, \theta_0-\theta_t+\theta_1+\frac{1}{2} \\ 1+2\theta_1\end{matrix}\,\middle|\, t\right].$$

The tau function (4.1) gives another known class of special function solutions of Painlevé VI, the so-called generalized Chazy solutions (cf Lemma 33 in [29]).

4.3 Algebraic solutions

Algebraic PVI solutions correspond to finite orbits of the braid/modular group action on monodromy of the associated linear system [11]. All such solutions have been classified in [25]. It turns out that there are 45 exceptional equivalence classes with fixed rational parameters $\boldsymbol{\theta}$, $\boldsymbol{\sigma}$ and three families continuously depending on some of them.

In the exceptional cases, the contributions of different conformal blocks overlap in the tau function expansion, which makes difficult their identification. On the other hand, this task is rather straightforward for continuous families, see examples given below. An infinite number of other explicit examples of $c = 1$ conformal blocks can be generated by Bäcklund transformations.

Example 5. *For $\boldsymbol{\theta} = \left(a, a, b, \frac{1}{2} - b\right)$, $\boldsymbol{\sigma} = \left(2a, \frac{1}{4}, \frac{1}{4}\right)$ there is a solution*

$$\tau(t) = \text{const} \cdot t^{2a^2} \underbrace{(1-t)^{-a^2-b^2+\frac{1}{16}} \left(\frac{1+\sqrt{1-t}}{2}\right)^{-4a^2+\left(2b-\frac{1}{2}\right)^2}}_{\mathcal{B}\left(a,a,b,\frac{1}{2}-b,2a;t\right)},$$

arising from the contribution of a single conformal block (Solution II in [25]).

Example 6. *For $\boldsymbol{\theta} = \left(2a, a, a, \frac{1}{3}\right)$, $\boldsymbol{\sigma} = \left(3a, \frac{1}{6}, \frac{1}{4}\right)$ there is a solution that can be parametrized as follows (Solution III in [25]):*

$$\tau(t(s)) = \text{const} \cdot (s-2)^{4a^2}(s-1)^{2a^2-\frac{7}{72}}(s+1)^{-10a^2+\frac{1}{8}}(s+2)^{10a^2-\frac{1}{9}},$$
$$t(s) = \frac{(s+1)^2(s-2)}{(s-1)^2(s+2)}.$$

Consider the interval $s \in (2, \infty)$ which maps to $t \in (0,1)$. Again only one block contributes to the tau function expansion on the corresponding branch at $t = 0$. It is explicitly given by

$$\mathcal{B}\left(2a, a, a, \frac{1}{3}, 3a; t\right) = (s_t - 1)^{10a^2-\frac{7}{72}} \left(\frac{s_t+1}{3}\right)^{-18a^2+\frac{1}{8}} \left(\frac{s_t+2}{4}\right)^{14a^2-\frac{1}{9}},$$

with $s_t = \frac{(1+\sqrt{t})^{\frac{2}{3}}+(1-\sqrt{t})^{\frac{2}{3}}}{(1-t)^{\frac{1}{3}}}$. Note that $t = 0$ is not really a branch point of s_t.

Example 7. *Taking $\boldsymbol{\theta} = \left(a, a, a, \frac{1}{4}\right)$, $\boldsymbol{\sigma} = \left(2a, \frac{1}{6}, \frac{1}{6}\right)$ one has a solution (corresponding to Solution IV in [25])*

$$\tau(t(s)) = \text{const} \cdot s^{-6a^2+\frac{1}{12}}(s-1)^{-6a^2+\frac{1}{12}}(s+1)^{2a^2}(2-s)^{2a^2}(2s-1)^{5a^2-\frac{1}{16}},$$
$$t(s) = \frac{s^3(2-s)}{2s-1}.$$

Under this parametrization, the interval $s \in (1,2)$ is mapped to $t \in (0,1)$. Tau function asymptotics near $t = 0$ on the relevant branch yields the conformal block

$$\mathcal{B}\left(a, a, a, \frac{1}{4}, 2a; t\right) = (s_t - 1)^{-6a^2+\frac{1}{12}} \left(\frac{s_t}{2}\right)^{-12a^2+\frac{1}{12}} \left(\frac{s_t+1}{3}\right)^{2a^2} \left(\frac{2s_t-1}{3}\right)^{7a^2-\frac{1}{16}},$$

where

$$s_t = \frac{1}{2}\left(1 + u_t + \sqrt{3 - u_t^2 + \frac{2-4t}{u_t}}\right), \qquad u_t = \sqrt{1 - (4t(1-t))^{\frac{1}{3}}}.$$

4.4 Picard solutions

The remainder of this section deals with Painlevé VI solutions of Picard type. Here the parameters are chosen to be $\boldsymbol{\theta}_{\text{Picard}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Up to Bäcklund transformations, this is the only case where the general two-parameter solution of PVI is available [14, 28]. Quite remarkably, for precisely these $\boldsymbol{\theta}$ there exists an explicit formula for $c = 1$ conformal block, found by Zamolodchikov [45]. We now briefly explain the relation between the two subjects.

First recall that complex torus can be seen as a two-sheeted covering of the four-punctured sphere, its period ratio being determined by the cross-ratio of the punctures. This suggests an elliptic parametrization of PVI variable t:

$$q = e^{i\pi\tau}, \qquad \tau = \frac{iK'(t)}{K(t)},$$

where $K(t)$, $K'(t)$ are complete elliptic integrals of the first kind:

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}, \qquad K'(t) = K(1-t).$$

Painlevé VI solutions	$c = 1$ conformal blocks
Riccati	vertex operators
Chazy	singular vectors
Picard	Ashkin-Teller
algebraic	?

Table 3. PVI-CFT correspondence.

In this parametrization, conformal block with external dimensions corresponding to $\boldsymbol{\theta}_{\text{Picard}}$ is given by (see eq. (2.28) in [45])

$$\mathcal{B}\left(\boldsymbol{\theta}_{\text{Picard}}, \sigma; t\right) = \frac{\left(16t^{-1}q\right)^{\sigma^2}}{(1-t)^{\frac{1}{8}}\vartheta_3(0|\tau)}.$$

Here and below $\vartheta_3(z|\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau+2inz}$ denotes Jacobi theta function. On the other hand, Barnes function duplication identity reduces the formula (1.11) for the structure constant to

$$(C_n)_{\text{Picard}} = \frac{\pi^2 G^4\left(\frac{1}{2}\right)}{\cos\pi\sigma_{0t}}(-s)^n 2^{-4(\sigma_{0t}+n)^2},$$

The expression for s also simplifies drastically. Jimbo-Fricke relation (1.7) implies that for fixed σ_{0t}, σ_{1t} the two possible values of $\cos 2\pi\sigma_{01}$ are given by $-\cos 2\pi(\sigma_{0t}\mp\sigma_{1t})$ and therefore $s = -e^{\pm 2\pi i\sigma_{1t}}$.

Summation over conformal families in (1.8) now gives theta function series so that we get an explicit expression

$$\tau_{\text{Picard}}(t) = \text{const}\cdot\frac{q^{\sigma_{0t}^2}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\frac{\vartheta_3\left(\sigma_{0t}\pi\tau\pm\sigma_{1t}\pi|\tau\right)}{\vartheta_3(0|\tau)}.$$

Straightforward check (of the literature [22]) shows that this function indeed satisfies PVI with Picard parameters, i.e. the expansion (1.8) is complete.

The above observations on the correspondence between special solutions of Painlevé VI equation and $c = 1$ conformal blocks are collected in table 3.

5 Further questions

To summarize, we have obtained a complete series expansion of Painlevé VI tau function near the singular points $t = 0, 1, \infty$. This series involves summation over conformal families arising in the OPEs of monodromy fields and labeled by $n \in \mathbb{Z}$, as well as a double sum over Young diagrams which encode the contribution of Virasoro descendants. The field theory meaning of $\tau(t)$ now becomes clear: *it is a generating function of $c = 1$ conformal blocks.*

This opens a way to intriguing applications of AGT correspondence in the theory of monodromy preserving deformations. To increase the number of singular points of the linear system (1.2), one should deal with AGT representation for the n-point conformal block with $n > 4$. Increasing rank N leads to other integer values of central charge, and it

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can be expected that intermediate conformal families will be labeled by $(N-1)$-tuples of integers. Another interesting option is the study of isomonodromic tau functions on higher genus Riemann surfaces.

Conversely, isomonodromy problems may provide useful insight into CFT. For instance, it would be interesting to understand if conformal blocks associated to exceptional algebraic Painlevé VI solutions can be computed in explicit form and identify the corresponding theories, orbifold CFTs [15] being the most natural candidates. One may also try to generate new explicit examples of conformal blocks for higher values of c from tau functions associated to branched covers of $\mathbb{P}^1$ [24].

It is in principle straightforward to obtain from (1.8)–(1.11) similar expansions for Painlevé V and Painlevé III tau functions. In particular, this gives full short-distance (conformal perturbation theory) expansion of two-point functions in the Ising and free-fermion sine-Gordon field theory, and of the PV tau function describing universal part of the process of formation of Fisher-Hartwig singularities in the asymptotics of Toeplitz determinants [8]. This also seems to shed some light on recent results of [1, 17]. We hope to return to these issues elsewhere.

Acknowledgments

The authors are grateful to B. Doyon, P. Gavrylenko, K. Kozlowski, A. Marshakov, A. Mironov, V. Roubtsov and V. Shadura for useful discussions and comments. This work was supported by the ERC grant no. 279738 – NEDFOQ (O. Gamayun), the Program of Fundamental Research of the Physics and Astronomy Division of NASU, Joint Ukrainian-Russian SFFR-RFBR project F40.2/108 (N. Iorgov), IRSES project "Random and Integrable Models in Mathematical Physics" (O. Lisovyy), and the joint program of bilateral seminars of CNRS and NASU. A part of this work was done when N. Iorgov was a visiting researcher at Tours University.

References

[1] A.G. Abanov, D.A. Ivanov and Y. Qian, *Quantum fluctuations of one-dimensional free fermions and Fisher-Hartwig formula for Toeplitz determinants*, *J. Phys.* **A 44** (2011) 485001 [arXiv:1108.1355] [INSPIRE].

[2] V.A. Alba, V.A. Fateev, A.V. Litvinov and G.M. Tarnopolskiy, *On combinatorial expansion of the conformal blocks arising from AGT conjecture*, *Lett. Math. Phys.* **98** (2011) 33 [arXiv:1012.1312] [INSPIRE].

[3] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[4] A. Belavin, A.M. Polyakov and A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B 241** (1984) 333 [INSPIRE].

[5] P. Boalch, *The fifty-two icosahedral solutions to Painleve VI*, *J. Reine Angew. Math.* **596** (2006) 183 [math/0406281]. .

[6] A. Borodin and G. Olshanski, *Z-measures on partitions, Robinson-Schensted-Knuth correspondence and $\beta = 2$ random matrix ensembles*, in *Random matrix models and their applications*, P.M. Bleher and A.R. Its eds., Cambridge University Press, Cambridge U.K. (2001), math/9905189.

[7] P.A. Clarkson, *Painlevé transcendents*, Digital library of special functions, chapter 32, http://dlmf.nist.gov/32.

[8] T. Claeys, A. Its and I. Krasovsky, *Emergence of a singularity for Toeplitz determinants and Painlevé V*, Duke Math. J. **160** (2011) 207 [arXiv:1004.3696] [INSPIRE].

[9] R. Conte, *The Painlevé property: one century later*, Springer, U.S.A. (1999).

[10] Ph. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Springer U.S.A. (1997).

[11] B. Dubrovin and M. Mazzocco, *Monodromy of certain Painleve' VI transcendents and reflection groups*, *Inv. Math.* **141** (2000) 55 [math/9806056].

[12] P. Fendley and H. Saleur, *$N = 2$ supersymmetry, Painlevé III and exact scaling functions in 2D polymers*, Nucl. Phys. **B 388** (1992) 609 [hep-th/9204094] [INSPIRE].

[13] A.S. Fokas, A.R. Its, A.A. Kapaev and V.Yu. Novokshenov, *Painlevé transcendents: the Riemann-Hilbert approach*, Mathematical Surveys and Monographs volume 128, American Mathematical Society, U.S.A. (2006).

[14] R. Fuchs, *Über lineare homogene Differentialgleichungen zweiter Ordnung mit im drei im Endrichen gelegene wesentlich singulären Stellen*, *Math. Ann.* **63** (1907) 301.

[15] P.H. Ginsparg, *Curiosities at $c = 1$*, Nucl. Phys. **B 295** (1988) 153 [INSPIRE].

[16] D. Harlow, J. Maltz and E. Witten, *Analytic continuation of Liouville theory*, *JHEP* **12** (2011) 071 [arXiv:1108.4417] [INSPIRE].

[17] D.A. Ivanov, A.G. Abanov and V.V. Cheianov, *Counting free fermions on a line: a Fisher-Hartwig asymptotic expansion for the Toeplitz determinant in the double-scaling limit*, arXiv:1112.2530 [INSPIRE].

[18] M. Jimbo, *Monodromy problem and the boundary condition for some Painlevé equations*, *Publ. RIMS Kyoto Univ.* **18** (1982) 1137.

[19] M. Jimbo, T. Miwa, Y. Môri and M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*, *Physica* **1D** (1980) 80.

[20] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I*, *Physica* **2D** (1981) 306.

[21] B. Hanlon and G.C. Joshi, *A noncommutative geometric approach to left-right symmetric weak interactions*, Lett. Math. Phys. **27** (1993) 105 [hep-ph/9212209] [INSPIRE].

[22] A.V. Kitaev and D.A. Korotkin, *On solutions of the Schlesinger equations in terms of Θ-functions*, *Int. Math. Res. Notices* **17** (1998) 877.

[23] V. Knizhnik and A. Zamolodchikov, *Current algebra and Wess-Zumino model in two-dimensions*, Nucl. Phys. **B 247** (1984) 83 [INSPIRE].

[24] D. Korotkin, *Solution of matrix Riemann-Hilbert problems with quasi-permutation monodromy matrices*, *Math. Ann.* **329** (2004) 335 [math-ph/0306061].

[25] O. Lisovyy and Y. Tykhyy, *Algebraic solutions of the sixth Painleve equation*, arXiv:0809.4873.

[26] O. Lisovyy, *Dyson's constant for the hypergeometric kernel*, arXiv:0910.1914 [INSPIRE].

[27] S.L. Lukyanov, *Critical values of the Yang-Yang functional in the quantum sine-Gordon model*, Nucl. Phys. **B 853** (2011) 475 [arXiv:1105.2836] [INSPIRE].

[28] M. Mazzocco, *Picard and Chazy solutions to the Painleve' VI equation*, *Math. Ann.* **321** (2001) 157 [math/9901054].

[29] M. Mazzocco, *The geometry of the classical solutions of the Garnier systems*, *Int. Math. Res. Notices* **12** (2002) 613 [math/0106208].

[30] A. Mironov, A. Morozov and S. Shakirov, *A direct proof of AGT conjecture at $\beta = 1$*, *JHEP* **02** (2011) 067 [arXiv:1012.3137] [INSPIRE].

[31] G. Moore, *Geometry of the string equations*, Commun. Math. Phys. **133** (1990) 261 [INSPIRE].

[32] D. P. Novikov, *The 2×2 matrix Schlesinger system and the Belavin-Polyakov-Zamolodchikov system*, Theor. Math. Phys. **161** (2009) 1485.

[33] K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation PVI*, *Ann. Mat. Pura Appl.* **146** (1986) 337.

[34] J. Palmer, *Determinants of Cauchy-Riemann operators as τ-functions*, *Acta Appl. Math.* **18** (1990) 199.

[35] M. Sato, T. Miwa and M. Jimbo, *Holonomic quantum fields I*, *Publ. RIMS Kyoto Univ.* **14** (1978) 223.

[36] M. Sato, T. Miwa and M. Jimbo, *Holonomic quantum fields II*, *Publ. RIMS Kyoto Univ.* **14** (1978) 201.

[37] M. Sato, T. Miwa and M. Jimbo, *Holonomic quantum fields III*, *Publ. RIMS Kyoto Univ.* **14** (1978) 577.

[38] M. Sato, T. Miwa and M. Jimbo, *Holonomic quantum fields IV*, *Publ. RIMS Kyoto Univ.* **14** (1978) 871.

[39] M. Sato, T. Miwa and M. Jimbo, *Holonomic quantum fields V*, *Publ. RIMS Kyoto Univ.* **14** (1978) 531.

[40] R. Schiappa and N. Wyllard, *An $A(r)$ threesome: matrix models, 2d CFTs and 4d $N = 2$ gauge theories*, *J. Math. Phys.* **51** (2010) 082304 [arXiv:0911.5337] [INSPIRE].

[41] C.A. Tracy and H. Widom, *Level spacing distributions and the Airy kernel*, *Commun. Math. Phys.* **159** (1994) 151 [hep-th/9211141] [INSPIRE].

[42] C. Tracy and H. Widom, *Fredholm determinants, differential equations and matrix models*, *Commun. Math. Phys.* **163** (1994) 33 [hep-th/9306042] [INSPIRE].

[43] C.A. Tracy and H. Widom, *Painlevé functions in statistical physics*, *Publ. RIMS Kyoto Univ.* **47** (2011) 361 [arXiv:0912.2362]. .

[44] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch, *Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region*, Phys. Rev. **B 13** (1976) 316 [INSPIRE].

[45] Al.B. Zamolodchikov, *Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model*, *Zh. Eksp. Teor. Fiz.* **90** (1986) 1808.

[46] A.B. Zamolodchikov, *Painlevé III and 2D polymers*, Nucl. Phys. **B 432** (1994) 427 [hep-th/9409108] [INSPIRE].

[47] A.B. Zamolodchikov, *On the three-point function in minimal Liouville gravity*, Theor. Math. Phys. **142** (2005) 183 [hep-th/0505063] [INSPIRE].

[48] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Conformal field theory and critical phenomena in two-dimensional systems*, MCNMO, Moscow Russia (2009).