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RESEARCH ARTICLE

Completely inverse AG^{**}-groupoids

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Abstract A completely inverse AG^{**}-groupoid is a groupoid satisfying the identities $(xy)z = (zy)x$, $x(yz) = y(xz)$ and $xx^{-1} = x^{-1}x$, where x^{-1} is a unique inverse of x, that is, $x = (xx^{-1})x$ and $x^{-1} = (x^{-1}x)x^{-1}$. First we study some fundamental properties of such groupoids. Then we determine certain fundamental congruences on a completely inverse AG^{**}-groupoid; namely: the maximum idempotent-separating congruence, the least AG-group congruence and the least E-unitary congruence. Finally, we investigate the complete lattice of congruences of a completely inverse AG^{**}-groupoids. In particular, we describe congruences on completely inverse AG^{**}-groupoids by their kernel and trace.

Keywords Completely inverse AG^{**}-groupoid · AG-group · Semilattice of AG-groups · Trace of congruence · Kernel of congruence · AG-group congruence · E-unitary congruence · Idempotent-separating congruence · Idempotent pure congruence · Fundamental congruence

1 Introduction

By an *Abel-Grassmann's groupoid* (briefly an *AG-groupoid*) we shall mean any groupoid which satisfies the identity $(xy)z = (zy)x$. Such a groupoid is also called a *left almost semigroup* (briefly an *LA-semigroup*) or a *left invertive groupoid* or a *right modular groupoid* (cf. [4, 5, 7]). This structure is closely related to a commuta-

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tive semigroup, because if an AG-groupoid contains a right identity, then it becomes a commutative monoid. Also, if an AG-groupoid A with a left zero z is finite, then (under certain conditions) $A \setminus \{z\}$ is a commutative group [8].

The name *Abel-Grassmann's groupoids* was suggested by Stojan Bogdanović at a seminar in Niš. The first time that this name appeared was in the paper [15] and in the book [2].

An AG-groupoid A satisfying the identity $x(yz) = y(xz)$ is called an *AG^{**}-groupoid*. Such groupoids were studied by many authors. For example, in [6] it has been proved that an AG^{**}-groupoid containing a left cancellative AG^{**}-subgroupoid can be embedded in a commutative monoid whose cancellative elements form a commutative group whose identity coincides with the identity of the commutative monoid. Also, each AG^{**}-groupoid satisfying the identity $(xx)x = x(xx)$ can be uniquely expressed as a semilattice of certain Archimedean AG^{**}-groupoids [10]. Some other decompositions of certain AG^{**}-groupoids are given in [12, 16]. Further, certain fundamental congruences on AG^{**}-groupoids are described in [9, 14]. Finally, the kernel normal system of an inversive AG^{**}-groupoid has been studied in [1].

In this paper we investigate *completely inverse AG^{**}-groupoids*, i.e., AG^{**}-groupoids in which every element a has a unique inverse a^{-1} such that $aa^{-1} = a^{-1}a$. In Sect. 2 we establish some necessary definitions and facts concerning AG^{**}-groupoids. In Sect. 3 we give a few interesting results about completely inverse AG^{**}-groupoids. Recall from [3] that any completely inverse AG^{**}-groupoid satisfies Lallement's lemma for regular semigroups. Using this fact, we describe the maximum idempotent-separating congruence μ (which is equal to the least semilattice congruence) on a completely inverse AG^{**}-groupoid A. In particular, A is a semilattice E_A of AG-groups $e\mu$ $(e \in E_A)$. Also, we show that the interval $[1_A, \mu]$ is a modular lattice. The main result of this section says that any AG-groupoid A is a completely inverse AG^{**}-groupoid if and only if A is a strong semilattice of AG-groups. On the one hand, in the light of this fact, we are able to construct completely inverse AG^{**}-groupoids. On the other hand, completely inverse AG^{**}-groupoids are very similar to Clifford semigroups (i.e., (strong) semilattices of groups).

At the beginning of Sect. 4 we prove that any congruence ρ on a completely inverse AG^{**}-groupoid is uniquely determined by (i) its kernel and trace; (ii) the set of ρ-classes containing idempotents. Furthermore, we determine the least AG-group congruence σ and describe all AG-group congruences in terms of their kernels. Also, we give some equivalent conditions for a completely inverse AG^{**}-groupoid A to be E-unitary and we describe all E-unitary congruences on A.

In Sect. 5 we characterize abstractly congruences on an arbitrary completely inverse AG^{**}-groupoid A via the so-called congruences pairs for A. Furthermore, we study the trace classes of the complete lattice $\mathcal{C}(A)$ of all congruences on A. The main result of this section says that the map $\rho \to \mathrm{tr}(\rho)$ $(\rho \in \mathcal{C}(A))$ is a complete lattice homomorphism of $\mathcal{C}(A)$ onto the lattice of all congruences on the semilattice E_A. Also, if θ denotes the congruence on $\mathcal{C}(A)$ induced by this map, then for every $\rho \in \mathcal{C}(A)$, $\rho\theta$ is a modular lattice (with commutating elements). Moreover, $\rho\theta = [\rho_\theta, \mu(\rho)]$. If in addition, A is E-unitary, then $\rho_\theta = \rho \cap \sigma$, and the mapping $\rho \to \rho \cap \sigma$ $(\rho \in \mathcal{C}(A))$ is a complete lattice homomorphism of $\mathcal{C}(A)$ onto the lattice of idempotent pure congruences. Finally, we investigate the lattice $\mathcal{FC}(A)$ of all fundamental congruences on A. We prove that $\mathcal{FC}(A) = \{\mu(\rho) : \rho \in \mathcal{C}(A)\} \cong \mathcal{C}(E_A)$.

In Sect. 6 we show first that each completely inverse AG^{**}-groupoid A possesses a largest idempotent pure congruence τ. Also, we study the kernel classes of $\mathcal{C}(A)$. We prove a result analogous to a result from the previous section. In particular, we show that the interval $[\rho \cap \mu, \tau(\rho)]$ consist of all congruences on A such that their kernels are equal to $\ker(\rho)$. Further, we go back to study E-unitary congruences. We determine all E-unitary congruences on A; that is, we show that a congruence is E-unitary if and only if its kernel is equal to the kernel of some AG-group congruence on A. Finally, we give once again necessary and sufficient conditions for a completely inverse AG^{**}-groupoid to be E-unitary.

The terminology used in this paper coincides with semigroup terminology (see the book [11]).

2 Preliminaries

One can easily check that in an arbitrary AG-groupoid A, the *medial law* is valid, that is, the equality

$$(ab)(cd) = (ac)(bd) \tag{1}$$

holds for all $a, b, c, d \in A$.

Recall from [16] that an *AG-band* A is an AG-groupoid satisfying the identity $x^2 = x$. If in addition, $ab = ba$ for all $a, b \in A$, then we say that A is an *AG-semilattice*.

Let A be an AG-groupoid and $B \subseteq A$. Denote the set of all idempotents of B by E_B, that is, $E_B = \{b \in B : b^2 = b\}$. From (1) it follows that if $E_A \neq \emptyset$, then $E_A E_A \subseteq E_A$, therefore, E_A is an AG-band.

An AG-groupoid satisfying the identity $x(yz) = y(xz)$ is said to be an *AG^{**}-groupoid*. Any AG^{**}-groupoid is *paramedial*, i.e., it satisfies the identity

$$(wx)(yz) = (zx)(yw). \tag{2}$$

Notice that each AG-groupoid with a left identity is an AG^{**}-groupoid. Further, observe that if A is an AG^{**}-groupoid, then (2) implies that if $E_A \neq \emptyset$, then E_A is an AG-semilattice. Indeed, in this case E_A is an AG-band and $ef = (ee)(ff) = (fe)(fe) = fe$ for all $e, f \in E_A$. Moreover for $a, b \in A$ and $e \in E_A$, using (1) and (2) we have

$$e(ab) = (ee)(ab) = (ea)(eb) = (ba)e = (ea)b.$$

We have just proved the following result (its second part was proved earlier in [14]).

Proposition 2.1 *Let A be an AG^{**}-groupoid. Then*

$$e \cdot ab = ea \cdot b \tag{3}$$

for all $a, b \in A$ and $e \in E_A$.

In particular, the set of all idempotents of an arbitrary AG^{**}-groupoid is either empty or a semilattice.

We say that an AG^{**}-groupoid A is *completely regular* if for every $a \in A$ there exists $x \in A$ such that $a = (ax)a$ and $ax = xa$. Observe that in such a case,

$$(ax)(ax) = (aa)(xx) = x(aa \cdot x) = x(xa \cdot a) = x(ax \cdot a) = xa = ax \in E_A,$$

therefore, E_A forms a semilattice.

Let A be an AG-groupoid with a left identity e and $a \in A$. An element a^* of A is said to be a *left* (*right*) *inverse* of a if $a^*a = e$ (resp. $aa^* = e$), and an element of A which is both a left and right inverse of a is called an *inverse* of a. Let a^* be a left inverse of a. Then $aa^* = (ea)a^* = (a^*a)e = e$. It follows that any left inverse a^* of a is also its right inverse, therefore, it is its inverse. In particular, if a^{**} is another left inverse of a, then $a^* = (a^*a)a^* = (a^{**}a)a^* = (a^*a)a^{**} = (a^{**}a)a^{**} = a^{**}$. The conclusion is that each left inverse of a is its unique inverse. Further, if f is a left identity of A, then $fe = e = ee$, so $e = f$, i.e., e is a unique left identity of A. Dually, any right inverse of a is its unique inverse. Denote as usual the inverse of a by a^{-1}. Finally, it is clear that $a = (a^{-1})^{-1}$, $(ab)^{-1} = a^{-1}b^{-1}$.

An AG-groupoid with a left identity in which every element has a left inverse is called an *AG-group*.

Proposition 2.2 *Let A be an AG-groupoid with a left identity e. Then the following conditions are equivalent*:

(a) *A is an AG-group*;
(b) *every element of A has a right inverse*;
(c) *every element a of A has a unique inverse a^{-1}*;
(d) *the equation $xa = b$ has a unique solution for all $a, b \in A$.*

Proof By above (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

(c) $\Longrightarrow$ (d). Let $a, b \in A$. Then $b = eb = (aa^{-1})b = (ba^{-1})a$, i.e., ba^{-1} is a solution of the equation $xa = b$. Also, if c and d are solutions of this equation, then

$$c = ec = (a^{-1}a)c = (ca)a^{-1} = (da)a^{-1} = d.$$

(d) $\Longrightarrow$ (a). This is obvious. □

Notice that if g is an arbitrary idempotent of an AG-group A with a left identity e, then $gg = g = eg$. Hence $e = g$, therefore, $E_A = \{e\}$.

Denote by $V(a)$ the set of all *inverses* of a, that is,

$$V(a) = \{a^* \in A : a = (aa^*)a,\ a^* = (a^*a)a^*\}.$$

An AG-groupoid A is called *regular* (in [1] it is called *inverse*) if $V(a) \neq \emptyset$ for all $a \in A$. Note that AG-groups are of course regular AG-groupoids, but the class of all regular AG-groupoids is vastly more extensive than the class of all AG-groups. For example, every AG-band A is evidently regular, since $a = (aa)a$ for every $a \in A$. In [1] it has been proved that in any regular AG^{**}-groupoid, $|V(a)| = 1$ ($a \in A$), therefore, we call it an *inverse AG^{**}-groupoid*. In that case, we denote a unique inverse of $a \in A$ by a^{-1}. Furthermore, recall from [1] that in any regular AG-groupoid A, $V(a)V(b) \subseteq V(ab)$ for all $a, b \in A$. Indeed, let $a^* \in V(a)$ and $b^* \in V(b)$. Then

$$\begin{aligned}(ab)(a^*b^*) \cdot ab &= (ab)a \cdot (a^*b^*)b = (ab)a \cdot (bb^*)a^* \\ &= (ab)(bb^*) \cdot aa^* = (bb^* \cdot b)a \cdot aa^*,\end{aligned}$$

so

$$(ab)(a^*b^*) \cdot ab = (ba)(aa^*) = (aa^* \cdot a)b = ab.$$

By symmetry, $a^*b^* = (a^*b^*)(ab) \cdot (a^*b^*)$, as exactly required. Finally, there are regular AG-groupoids without idempotents. On the other hand, if $a^* \in V(a)$ and $aa^* = a^*a$ in the AG-groupoid A, then $aa^* \in E_A$ (cf. [1]).

3 Completely inverse AG^{**}-groupoids

One can prove (cf. [1]) that in an inverse AG^{**}-groupoid A, $aa^{-1} = a^{-1}a$ if and only if $aa^{-1}, a^{-1}a \in E_A$. Also, in [1] the authors studied congruences on inverse AG^{**}-groupoids satisfying the identity $xx^{-1} = x^{-1}x$. We will call such groupoids *completely inverse AG^{**}-groupoids*. Each AG-group is a completely inverse AG^{**}-groupoid.

Example 3.1 Let A be a commutative inverse semigroup. Put $a \cdot b = a^{-1}b$ for all $a, b \in A$, where a^{-1} is a unique inverse of a in the inverse semigroup A. Then it is easy to check that $(A, \cdot)$ is an AG^{**}-groupoid and $E_{(A,\cdot)} = E_A$. Furthermore, $(a \cdot a) \cdot a = a$, so a is its own unique inverse in $(A, \cdot)$ for every $a \in A$, so $a \cdot a \in E_{(A,\cdot)}$ for all $a \in A$ and $(A, \cdot)$ is a completely inverse AG^{**}-groupoid. Also, we have that $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot a^{-1}b = aa^{-1}b$. Hence

$$a^{-1} \cdot (a^{-1} \cdot (a \cdot b)) = a^{-1} \cdot aa^{-1}b = aaa^{-1}b = aa^{-1}ab = ab,$$

that is,

$$ab = a^{-1} \cdot (a^{-1} \cdot (a \cdot b)) = a \cdot (a^{-1} \cdot (a^{-1} \cdot b))$$

for all $a, b \in A$.

Let ρ be a congruence on $(A, \cdot)$. From the above equalities follows easily that ρ is a congruence on the commutative inverse semigroup A. Also, if $(a, a \cdot a) \in \rho$ in $(A, \cdot)$, then $(a, a^{-1}a) \in \rho$ in A, since $a \cdot a = a^{-1}a$. Thus $(a^2, aa^{-1}a) \in \rho$ in A, so $(a^2, a) \in \rho$ in A. Lallement's Lemma implies that there exists $e \in E_A \cap a\rho$ and so $e \in E_{(A,\cdot)} \cap a\rho$. On the other hand, trivially $a \cdot a \in E_{a\rho}$ in $(A, \cdot)$.

Conversely, one can easily see that if ρ is a congruence on A, then ρ is also a congruence on $(A, \cdot)$. Further, if $(a, a^2) \in \rho$ in A, then $(a, e) \in \rho$ in A for some $e \in E_A$. Since $e \cdot e = e$, then $(a, a \cdot a) \in \rho$ in $(A, \cdot)$.

A groupoid A is said to be *idempotent-surjective* if for each congruence ρ on A, every idempotent ρ-class contains an idempotent of A.

The following theorem was proved in [3]. Now we give another proof.

Theorem 3.2 *Completely inverse AG^{**}-groupoids are idempotent-surjective.*

Proof Let ρ be a congruence on a completely inversive AG^{**}-groupoid A, $a \in A$ and $a\rho a^2$. We know that there exists an element $x \in A$ such that $a^2 = (a^2x)a^2$, $x = (xa^2)x$ and $a^2x = xa^2 \in E_A$. Note that

$$(a^2x)(aa) = a(a^2x \cdot a) = a(xa^2 \cdot a) = a(aa^2 \cdot x) = (aa^2)(ax) = a^2(a^2x) = a^2(xa^2),$$

that is, $a^2 = a^2(xa^2)$. Put $e = a(xa)$. Then $e\rho a^2(xa^2) = a^2\rho a$. Also,

$$e^2 = (a \cdot xa)(a \cdot xa) = a \cdot (a \cdot xa)(xa) = a \cdot (ax)(xa \cdot a) = a \cdot (ax)(a^2x).$$

Furthermore, using (2)

$$(ax)(a^2x) = (ax)(xa^2) = (a^2x)(xa) = (xa^2)(xa) = (xa^2 \cdot x)a$$

by (3), since $xa^2 \in E_A$. Hence $(ax)(a^2x) = xa$. Consequently,

$$e^2 = a(xa) = e \in E_A,$$

as required. □

Let ρ be a congruence on a completely inverse AG^{**}-groupoid A and $a, b \in A$. It is evident that $(a\rho)^{-1} = a^{-1}\rho$. Hence if $(a, b) \in \rho$, then $(a^{-1}, b^{-1}) \in \rho$. Moreover, A/ρ is a completely inverse AG^{**}-groupoid.

Further, let A be an arbitrary groupoid and $\mathcal{V}$ be a fixed class of groupoids. We say that a congruence ρ on A is a $\mathcal{V}$*-congruence* if $A/\rho \in \mathcal{V}$. For example, if $\mathcal{V}$ is the class of all semilattices, then ρ is a *semilattice* congruence on A if A/ρ is a semilattice. Moreover, A is called a *semilattice A/ρ of AG-groups* if there is a semilattice congruence ρ on A such that every ρ-class is an AG-group. In that case, A is a *semilattice $Y = A/\rho$ of AG-groups A_α*, $\alpha \in Y$, where A_α are the ρ-classes of A, or briefly a *semilattice $Y = A/\rho$ of AG-groups A_α*. Notice that in such a case, $A_\alpha A_\beta \subseteq A_{\alpha\beta}$, where $\alpha\beta$ is the product of α and β in Y. Also, $A_{\alpha\beta} = A_{\beta\alpha}$ and $A_{(\alpha\beta)\gamma} = A_{\alpha(\beta\gamma)}$.

Finally, we say that a congruence ρ on a groupoid A is *idempotent-separating* if every ρ-class contains at most one idempotent of A.

The following simple result will at times be useful.

Lemma 3.3 *A completely inverse AG^{**}-groupoid containing only one idempotent is an AG-group.*

Proof Let $E_A = \{e\}$, $a \in A$. Then $e = aa^{-1} = a^{-1}a$. Hence $ea = (aa^{-1})a = a$. Thus A is an AG-group. □

For elementary facts about (inverse) semigroups the reader is referred to the book of Petrich [11]. It is well known that each completely regular inverse semigroup is a semilattice of groups. We prove now an analogous result.

Theorem 3.4 *Let A be a completely inverse AG^{**}-groupoid. Define on A the relation μ by*

$$(a,b)\in\mu \quad\Longleftrightarrow\quad aa^{-1}=bb^{-1}$$

for all $a,b\in A$. Then:

(a) *μ is the least semilattice congruence on A*;
(b) *every μ-class is an AG-group*;
(c) *μ is the maximum idempotent-separating congruence on A*;
(d) *A is a semilattice A/μ of AG-groups*;
(e) *$E_A\cong A/\mu$.*

Hence A is a semilattice E_A of AG-groups G_e, where $G_e=\{a\in A: aa^{-1}=e\}$ for $e\in E_A$.

Proof (a) Clearly, μ is an equivalence relation on A. Let $(a,b)\in\mu$ and $c\in A$. Then

$$(ca)(ca)^{-1}=(ca)(c^{-1}a^{-1})=(cc^{-1})(aa^{-1})=(cc^{-1})(bb^{-1})=(cb)(cb)^{-1}$$

and similarly $(ac)(ac)^{-1}=(bc)(bc)^{-1}$. Hence μ is a congruence on A. Also,

$$(aa^{-1})(aa^{-1})^{-1}=(aa^{-1})(a^{-1}(a^{-1})^{-1})=(aa^{-1})(a^{-1}a)=(aa^{-1})(aa^{-1})=aa^{-1},$$

so $(a,aa^{-1})\in\mu$, where $aa^{-1}\in E_A$. Since E_A is a semilattice, then S/μ is a semilattice, too. Consequently, μ is a semilattice congruence on A. Moreover, since $e^{-1}=e$ for every $e\in E_A$, then μ is idempotent-separating.

Finally, suppose that there is a semilattice congruence ρ on A such that $\mu\nsubseteq\rho$. Then the relation $\mu\cap\rho$ is a semilattice congruence on A which is properly contained in μ, so not every $(\mu\cap\rho)$-class contains an idempotent of A, since each μ-class contains exactly one idempotent, a contradiction with Theorem 3.2. Consequently, μ must be the least semilattice congruence on A.

(b) We have noticed above that μ is idempotent-separating. It is evident that every μ-class is itself a completely inverse AG^{**}-groupoid, since $a^{-1}\in a\mu$ for all $a\in A$. In view of Lemma 3.3, every μ-class is an AG-group.

(c) Let ρ be an idempotent-separating congruence on $A, (a,b)\in\rho$. Then $a^{-1}\rho b^{-1}$. It follows that $(aa^{-1},bb^{-1})\in\rho$. Thus $aa^{-1}=bb^{-1}$, so $(a,b)\in\mu$. Consequently, $\rho\subseteq\mu$.

The rest is obvious. □

Let $\mathcal{C}(A)$ denote the complete lattice of all congruences on a groupoid A. It is well known that if a sublattice $\mathcal{L}$ of $\mathcal{C}(A)$ has the property that $\alpha\beta=\beta\alpha$ for all $\alpha,\beta\in\mathcal{L}$, then $\mathcal{L}$ is a modular lattice.

Let A be a completely inverse AG^{**}-groupoid. Consider the complete lattice $[1_A,\mu]$ of all idempotent-separating congruences on A (see Theorem 3.4(c)). Let $\rho_1,\rho_2\in[1_A,\mu]$ and $(a,b)\in\rho_1\rho_2$. Then there is $c\in A$ such that $a\rho_1c\rho_2b$. In particular, $(a,c),(c,b)\in\mu$. Hence

$$a=aa^{-1}\cdot a=cc^{-1}\cdot a\rho_2bc^{-1}\cdot a=ac^{-1}\cdot b\rho_1cc^{-1}\cdot b=bb^{-1}\cdot b=b,$$

so $(a,b) \in \rho_2\rho_1$. Thus $\rho_1\rho_2 \subseteq \rho_2\rho_1$. By symmetry, $\rho_2\rho_1 \subseteq \rho_1\rho_2$. We have just shown the following theorem.

Theorem 3.5 *Let A be a completely inverse AG^{**}-groupoid. Then the interval $[1_A, \mu]$, consisting of all idempotent-separating congruences on A, is a modular lattice.*

Corollary 3.6 *The lattice of congruences on an AG-group is modular.*

A completely inverse AG^{**}-groupoid A is a semilattice E_A of AG-groups G_e ($e \in E_A$), where $G_e = \{a \in A : aa^{-1} = e\}$ (Theorem 3.4). The relation $\leq$ defined on the semilattice E_A by $e \leq f \Leftrightarrow e = ef$ is the so-called *natural partial order* on E_A.

Let $e \geq f$ and $a_e \in G_e$. Then $fa_e \in G_f G_e \subseteq G_{fe} = G_f$. Hence we may define a map $\phi_{e,f} : G_e \to G_f$ by

$$a_e\phi_{e,f} = fa_e \quad (a_e \in G_e).$$

Also, for all $a_e, b_e \in G_e$, $(fa_e)(fb_e) = (ff)(a_eb_e) = f(a_eb_e)$, so

$$(a_e\phi_{e,f})(b_e\phi_{e,f}) = (a_eb_e)\phi_{e,f}, \tag{4}$$

i.e., $\phi_{e,f}$ is a homomorphism between the AG-groups G_e and G_f. In particular, $e\phi_{e,f} = f$ (this follows also from $e \geq f$). Observe that $\phi_{e,e}$ is the identical automorphism of the AG-group G_e.

Suppose now that $e \geq f \geq g$. Then for every $a_e \in G_e$,

$$(a_e\phi_{e,f})\phi_{f,g} = g(fa_e) = (gg)(fa_e) = (gf)(ga_e) = g(ga_e) = ga_e = a_e\phi_{e,g},$$

since $ga_e \in G_gG_e \subseteq G_{ge} = G_g$, that is,

$$\phi_{e,f}\phi_{f,g} = \phi_{e,g} \tag{5}$$

for every $e, f, g \in E_A$ such that $e \geq f \geq g$.

Finally, let $a_e \in G_e$ and $a_f \in G_f$ (and so $a_ea_f \in G_{ef}$; also $e, f \geq ef$). Then we get $a_ea_f = (ef)(a_ea_f) = (ef \cdot ef)(a_ea_f) = ((ef)a_e)((ef)a_f)$, i.e.,

$$a_ea_f = (a_e\phi_{e,ef})(a_f\phi_{f,ef}). \tag{6}$$

Remark that we have used only the medial law in the proof of the equalities (4), (5) and (6), therefore, if an AG-groupoid A is a semilattice E_A of the AG-groups G_e ($e \in E_A$), then these equalities hold true.

Let now Y be a semilattice, $\mathcal{F} = \{A_\alpha : \alpha \in Y\}$ be a family of disjoint AG-groupoids of type $\mathcal{T}$, indexed by the set Y ($\mathcal{F}$ may be a family of disjoint AG-groups). Suppose also that for each pair $(\alpha, \beta) \in Y \times Y$ such that $\alpha \geq \beta$ there is an associated homomorphism $\phi_{\alpha,\beta} : A_\alpha \to A_\beta$ such that

(a) $\phi_{\alpha,\alpha}$ is the identical automorphism of A_α for every $\alpha \in Y$, and
(b) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$.

Put $A = \bigcup\{A_\alpha : \alpha \in Y\}$, and define a binary operation $\cdot$ on A by the rule that if $a_\alpha \in A_\alpha$ and $a_\beta \in A_\beta$, then

$$a_\alpha \cdot a_\beta = (a_\alpha \phi_{\alpha,\alpha\beta})(a_\beta \phi_{\beta,\alpha\beta}),$$

where the multiplication on the right side takes place in the AG-groupoid $A_{\alpha\beta}$.

It is a matter of routine to check that $(A, \cdot)$ is an AG-groupoid. If in addition, each AG-groupoid A_α is an AG^{**}-groupoid (in particular, an AG-group), then $(A, \cdot)$ is itself an AG^{**}-groupoid. Finally, in the light of the condition (a), the new multiplication coincides with the given of each A_α, so A is certainly a semilattice Y of AG-groupoids A_α. We usually denote the product in A also by juxtaposition, and write $A = [Y; A_\alpha; \phi_{\alpha,\beta}]$.

We call the AG-groupoid $[Y; A_\alpha; \phi_{\alpha,\beta}]$ a *strong semilattice of AG-groupoids* A_α.

In fact, we have proved the following theorem (see (4), (5) and (6)).

Theorem 3.7 *Let an AG-groupoid A be a semilattice A/ρ of AG-groups. Then A is a strong semilattice of AG-groups. In fact,*

$$A = [E_A; G_e; \phi_{e,f}],$$

where for all $e, f \in E_A$, $G_e = e\rho$; $\phi_{e,f} : G_e \to G_f$ is given by

$$a_e \phi_{e,f} = f a_e \quad (a_e \in G_e),$$

and

$$a_e a_f = (a_e \phi_{e,ef})(a_f \phi_{f,ef}) \quad (a_e \in G_e, a_f \in G_f).$$

In particular, A is an AG^{**}-groupoid.

Proof Let A be a semilattice A/ρ of AG-groups, then ρ is idempotent-separating. Hence $E_A \cong A/\rho$, so E_A is necessarily a semilattice. Thus A is a semilattice E_A of AG-groups $G_e = e\rho$ $(e \in E_A)$. This implies the thesis of the theorem. □

It is well known that if a semigroup S is a semilattice of groups, then its idempotents are *central*, that is, $se = es$ for all $s \in S$ and $e \in E_S$. The following proposition says particularly that there is no non-associative AG-groupoids which are a semilattice of AG-groups and their idempotents are central.

Proposition 3.8 *Let A be an AG-groupoid which is a semilattice of AG-groups. If the idempotents of A are central, then A is a strong semilattice of Abelian groups. In particular, A is a commutative semigroup.*

Proof Let $A = [E_A; G_e; \phi_{e,f}]$. If the idempotents of A are central, then particularly for all $e \in E_A$, $ae = ea$ for every $a \in G_e$. This implies that every G_e is a commutative group, so A is a strong semilattice of Abelian groups. From the definition of the multiplication in $[E_A; G_e; \phi_{e,f}]$ and from the fact that Abelian groups are commutative semigroups follows that A is a commutative semigroup. □

Remark 1 Let A be a completely inverse AG^{**}-groupoid. Then $ae = ea$ for all $a \in A$, $e \in E_A$ if and only if $a = a(a^{-1}a)$ for every $a \in A$. Indeed,

$$ea = e(aa^{-1} \cdot a) = (e \cdot aa^{-1})a = (a \cdot aa^{-1})e = (a(a^{-1}a))e.$$

This implies that if $a = a(a^{-1}a)$, then the idempotents of A are central. The converse implication is obvious.

In the proof of Theorem 3.2 we have shown that $a^2 = a^2(xa^2)$ for every $a \in A$, where $x \in V(a^2)$. Furthermore, $A^{(2)} = \{a^2 : a \in A\}$ is an AG^{**}-groupoid, since $a^2b^2 = (ab)^2$ for all $a, b \in A$. Also, $(a^{-1})^2 \in V(a^2)$ for every $a \in A$. Evidently, $E_A \subseteq A^{(2)}$. Consequently, $A^{(2)}$ is a completely inverse AG^{**}-groupoid in which the idempotents are central. From Proposition 3.8 we obtain the following theorem.

Theorem 3.9 *If A is a completely inverse AG^{**}-groupoid, then $A^{(2)}$ is a strong semilattice of Abelian groups with semilattice E_A of idempotents.*

The next theorem gives necessary and sufficient conditions for an AG-groupoid to be a completely inverse AG^{**}-groupoid.

Theorem 3.10 *The following conditions concerning an AG-groupoid A are equivalent*:

(a) *A is a completely inverse AG^{**}-groupoid*;
(b) *A is a semilattice of AG-groups*;
(c) *A is a strong semilattice of AG-groups.*

Proof (a) $\Longrightarrow$ (b) by Theorem 3.4 and (b) $\Longrightarrow$ (c) by Theorem 3.7.

(c) $\Longrightarrow$ (a). In that case, A is an AG^{**}-groupoid (see again Theorem 3.7). Also, let $a \in A$. Then a belongs to some AG-group G_e, where e is a left identity of G_e. Consider now a unique inverse a^{-1} of a in G_e. Then evidently $a = (aa^{-1})a$, $a^{-1} = (a^{-1}a)a^{-1}$ and $aa^{-1} = a^{-1}a = e$. Consequently, A is a completely inverse AG^{**}-groupoid. □

Remark 2 In view of the above theorem, we are able to construct completely inverse AG^{**}-groupoids.

Let A be a completely inverse AG^{**}-groupoid. The relation $\leq_A$ defined on A by $a \leq_A b$ if $a \in E_A b$ is the *natural partial order* on A. Notice the restriction of $\leq_A$ to E_A is equal to the natural partial order $\leq$ on E_A, therefore, we will be write briefly $\leq$ instead of $\leq_A$.

The following result can be deduced from [12].

Lemma 3.11 *In any completely inverse AG^{**}-groupoid A, the relation $\leq$ is a compatible partial order on A. Also, $a \leq b$ implies $a^{-1} \leq b^{-1}$ for all $a, b \in A$.*

Proof We include a simple proof. It is evident that $\leq$ is reflexive and preserves inverses. Let $a \leq b$ and $b \leq a$, i.e., $a = eb$ and $b = fa$ for some $e, f \in E_A$. Then by Proposition 2.1, $ea = a$. Using again Proposition 2.1, $a = eb = e(fa) = (ef)a = (fe)a = f(ea) = fa = b$. Hence $\leq$ is antisymmetric. From Proposition 2.1 it follows also that $\leq$ is transitive. Finally, if $a \leq b$ and $c \leq d$, that is, $a = eb$ and $c = fd$ for some $e, f \in E_A$, then we obtain that $ac = (eb)(fd) = (ef)(bd)$. Thus $ac \leq bd$. □

For some equivalent definitions of the relation $\leq$, consult [12]. Moreover, we have the following proposition.

Proposition 3.12 *In any completely inverse AG^{**}-groupoid A, $\leq \cap\, \mu = 1_A$, that is, if $A = [E_A; G_e; \phi_{e,f}]$, then $\leq_{|G_e} = 1_{G_e}$ for every $e \in E_A$.*

Proof Let $a(\leq \cap \mu)b$. Then $aa^{-1} = bb^{-1}$ and $a = eb$ for some $e \in E_A$, therefore we get $aa^{-1} = (eb)(eb^{-1}) = (ee)(bb^{-1}) = e(bb^{-1}) = (eb)b^{-1} = ab^{-1}$. Consequently,

$$a = (aa^{-1})a = (bb^{-1})a = (ab^{-1})b = (aa^{-1})b = (bb^{-1})b = b,$$

as required. □

Finally, for any nonempty subset B of a completely inverse AG^{**}-groupoid A, we call

$$B\omega = \{a \in A : \exists(b \in B)\ b \leq a\}$$

the *closure of B in A*; if $B = B\omega$, then B is *closed in A*. Note that $B\omega$ is closed in A.

It is clear that a subgroupoid B of a completely inverse AG^{**}-groupoid A is itself a completely inverse AG^{**}-groupoid if and only if $b \in B$ implies $b^{-1} \in B$ for every $b \in B$. In such a case, B is a *completely inverse AG^{**}-subgroupoid* of A. Using Lemma 3.11, one can prove the following proposition.

Proposition 3.13 *If B is a completely inverse AG^{**}-subgroupoid of a completely inverse AG^{**}-groupoid A, then $B\omega$ is a closed completely inverse AG^{**}-subgroupoid of A.*

In particular, $E_A\omega$ is a closed completely inverse AG^{**}-subgroupoid of A. It is easy to see that

$$E_A\omega = \{a \in A : (\exists e \in E_A)\ ea \in E_A\}.$$

4 Certain E-unitary congruences

Let ρ be a congruence on a completely inverse AG^{**}-groupoid A. By the *kernel* $\ker(\rho)$ (respectively the *trace* $\operatorname{tr}(\rho)$) of ρ we shall mean the set $\{a \in A : (a, a^2) \in \rho\}$ (respectively the restriction of ρ to the set E_A). Note that $\operatorname{tr}(\rho)$ is a congruence on the semilattice E_A. Also, in the light of Theorem 3.2,

$$\ker(\rho) = \{a \in A : \exists(e \in E_A)\ (a, e) \in \rho\} = \bigcup\{e\rho : e \in E_A\}.$$

The following proposition may be sometimes useful.

Proposition 4.1 *Let $A = [E_A; G_e; \phi_{e,f}]$ be a completely inverse AG^{**}-groupoid and let $a, b \in A$ be such that $ab \in E_A$. Then $ab = ba$.*

Proof Let $ab = e \in E_A$. Then

$$ba = b(aa^{-1} \cdot a) = (aa^{-1})(ba) = (ab)(a^{-1}a) \in E_A.$$

Since $ab, ba \in G_e$, then $ab = ba$. □

The following theorem says particularly that each congruence on a completely inverse AG^{**}-groupoid is uniquely determined by its kernel and trace.

Theorem 4.2 *If ρ is a congruence on a completely inverse AG^{**}-groupoid A, then*

$$(a, b) \in \rho \quad \Longleftrightarrow \quad (aa^{-1}, bb^{-1}) \in \mathrm{tr}(\rho) \ \&\ ab^{-1} \in \ker(\rho).$$

Thus for all $\rho_1, \rho_2 \in \mathcal{C}(A)$,

$$\rho_1 \subseteq \rho_2 \quad \Longleftrightarrow \quad \mathrm{tr}(\rho_1) \subseteq \mathrm{tr}(\rho_2) \ \&\ \ker(\rho_1) \subseteq \ker(\rho_2).$$

In particular, each congruence on a completely inverse AG^{**}-groupoid is uniquely determined by its kernel and trace.

Proof Let $(a, b) \in \rho$. Then evidently (a^{-1}, b^{-1}), $(ab^{-1}, bb^{-1}) \in \rho$, so $(aa^{-1}, bb^{-1}) \in \mathrm{tr}(\rho)$ and $ab^{-1} \in \ker(\rho)$.

Conversely, let now $(aa^{-1}, bb^{-1}) \in \mathrm{tr}(\rho), ab^{-1} \in \ker(\rho)$. In view of Theorem 3.4, $(a\rho, b\rho) \in \mu_{S/\rho}$, so $((ab^{-1})\rho, (bb^{-1})\rho) \in \mu_{S/\rho}$. Since $ab^{-1} \in \ker(\rho)$, then $(ab^{-1})\rho \in E_{A/\rho}$. Evidently, $(bb^{-1})\rho \in E_{A/\rho}$. Hence $(ab^{-1})\rho = (bb^{-1})\rho$ (by Theorem 3.4(c)). Thus

$$a\rho = (aa^{-1} \cdot a)\rho = (bb^{-1} \cdot a)\rho = (ab^{-1} \cdot b)\rho = (bb^{-1} \cdot b)\rho = b\rho,$$

as required. The rest of the theorem follows from the first equivalence. □

Remark 3 Note that the first part of the above theorem is true for an arbitrary Clifford semigroup, the proof is very similar. In fact, if $ab^{-1} \in \ker(\rho)$, then

$$(ab^{-1})\rho = (b^{-1}a)\rho = (b^{-1}b)\rho,$$

so $a\rho = (aa^{-1} \cdot a)\rho = (bb^{-1} \cdot a)\rho = (b \cdot b^{-1}a)\rho = (b \cdot b^{-1}b)\rho = b\rho$.

Clearly, the condition $ab^{-1} \in \ker(\rho)$ from Theorem 4.2 is equivalent to the condition $a^{-1}b \in \ker(\rho)$. In the light of Proposition 4.1, it is also equivalent to $b^{-1}a \in \ker(\rho)$.

Theorem 4.3 *Let ρ_1, ρ_2 be congruences on a completely inverse AG^{**}-groupoid A. Then the following statements are equivalent*:

(a) $e\rho_1 \subseteq e\rho_2$ *for every* $e \in E_A$;
(b) $\rho_1 \subseteq \rho_2$.

In particular, every congruence ρ *on a completely inverse* AG^{**}*-groupoid is uniquely determined by the set of* ρ*-classes containing idempotents.*

Proof (a) $\Longrightarrow$ (b). Let $a \in b\rho_1$. Then

$$aa^{-1} \in (bb^{-1})\rho_1 \subseteq (bb^{-1})\rho_2 \quad \& \quad ab^{-1} \in (bb^{-1})\rho_1 \subseteq (bb^{-1})\rho_2.$$

In the light of Theorem 4.2, $a \in b\rho_2$, that is, $\rho_1 \subseteq \rho_2$.

(b) $\Longrightarrow$ (a). This is trivial. □

In Sect. 5 we shall characterize abstractly the congruences on a completely inverse AG^{**}-groupoid A via the congruence pairs for A.

A nonempty subset B of a groupoid A is called *left* (*right*) *unitary* if $ba \in B$ (resp. $ab \in B$) implies $a \in B$ for every $b \in B, a \in A$. Also, we say that B is *unitary* if it is both left and right unitary. Finally, a groupoid A is said to be E*-unitary* if E_A is unitary.

Proposition 4.4 *Let* E_A *be a left unitary subset of an AG-groupoid. Then* E_A *is also right unitary. If in addition*, A *is an* AG^{**}*-groupoid*, *then the following conditions are equivalent*:

(a) A *is* E*-unitary*;
(b) E_A *is left unitary*;
(c) E_A *is right unitary.*

Proof (a) $\Longrightarrow$ (b), (c). Obvious.

(b) $\Longrightarrow$ (a). Let $a \in A, e \in E_A$ and let $ae = f \in E_A$. Then $(ae)f \in E_A$, therefore, $(fe)a \in E_A$. Thus $a \in E_A$, since $fe \in E_A$ and E_A is left unitary.

(c) $\Longrightarrow$ (a). Let $a \in A, e \in E_A$ and $ea = f \in E_A$. Then, using (3),

$$f = f(ea) = (fe)a = (ae)f.$$

Hence $ae \in E_A$. Thus $a \in E_A$. □

AG-groups are examples of E-unitary completely inverse AG^{**}-groupoids.

A congruence ρ on a completely inverse AG^{**}-groupoid is a *AG-group* congruence if A/ρ is an AG-group. By Lemma 3.3, ρ is an AG-group congruence if and only if $\mathrm{tr}(\rho) = E_A \times E_A$. Since $A \times A$ is an AG-group congruence on A, then the intersection of all the AG-group congruences on A is the least AG-group congruence on A.

A more useful characterization of the least AG-group congruence on A is given in the following theorem.

Theorem 4.5 *In any completely inverse* AG^{**}*-groupoid* A,

$$\sigma = \{(a, b) \in A \times A : (\exists e \in E_A)\ ea = eb\}$$

is the least AG-group congruence with the kernel $E_A\omega$.

Proof It is evident that σ is reflexive and symmetric. Let $(a,b),(b,c)\in\sigma$, so that $ea=eb$ and $fb=fc$ for some $e,f\in E_A$. Using Proposition 2.1, we have that

$$(fe)a=f(ea)=f(eb)=(fe)b=(ef)b=e(fb)=e(fc)=(ef)c=(fe)c,$$

where $fe\in E_A$. Thus $(a,c)\in\sigma$. Consequently, σ is an equivalence relation on A. Further, let $(a,b)\in\sigma$, that is, $ea=eb$, where $e\in E_A$, and let $c\in A$. Then again in the light of Proposition 2.1, $e(ac)=(ea)c=(eb)c=e(bc)$. Also,

$$(cc^{-1})e\cdot ca=(cc^{-1})c\cdot ea=(cc^{-1})c\cdot eb=(cc^{-1})e\cdot cb,$$

where $(cc^{-1})e\in E_A$. Hence σ is a congruence on A. Since $(ef)e=(ef)f$ and $ef\in E_A$ for all $e,f\in E_A$, then σ is an *AG*-group congruence on A. Also, let ρ be an *AG*-group congruence on A and $(a,b)\in\sigma$. Then $ea=eb$, where $e\in E_A$, so $(e\rho)(a\rho)=(e\rho)(b\rho)$. Hence $a\rho=b\rho$, since $e\rho$ is a left identity of the *AG*-group A/ρ. Thus $\sigma\subseteq\rho$. Consequently, σ is the least *AG*-group congruence on A. Finally,

$$\begin{aligned} a\in\ker(\sigma) &\iff (\exists f\in E_A)\ (a,f)\in\sigma \iff (\exists e,f\in E_A)\ ea=ef\\ &\iff a\in E_A\omega, \end{aligned}$$

as required. □

From Theorem 4.2 follows that $(a,b)\in\sigma\Leftrightarrow ab^{-1}\in E_A\omega$. Also, in the light of the end of Sect. 5, $E_A\omega$ is a closed completely inverse AG^{**}-subgroupoid of A. Evidently, $E_A\subseteq E_A\omega$ and if $ab\in E_A\omega$, then $ba\in E_A\omega$.

A nonempty subset B of a completely inverse AG^{**}-groupoid A is called:

(F) *full* if $E_A\subseteq B$;
(S) *symmetric* if $xy\in B$ implies $yx\in B$ for all $x,y\in A$.

A completely inverse AG^{**}-subgroupoid N of A is said to be *normal* if it full, closed and symmetric. In that case, we shall write $N\lhd A$.

Denote the set of all *AG*-group congruences on a completely inverse AG^{**}-groupoid A by $\mathcal{GC}(A)$. It is clear that $\mathcal{GC}(A)=[\sigma,A\times A]$ is a complete sublattice of $\mathcal{C}(A)$. Note that $\mathcal{GC}(A)\cong\mathcal{C}(A/\sigma)$ and so the lattice $\mathcal{GC}(A)$ is modular (by Corollary 3.6). Further, let $\mathcal{N}(A)$ be the set of all normal completely inverse AG^{**}-subgroupoids of A. It is obvious that $E_A\omega\subseteq N$ for every $N\lhd A$, and if $\emptyset\neq\mathcal{F}\subseteq\mathcal{N}(A)$, then $\bigcap\{B:B\in\mathcal{F}\}\in\mathcal{N}(A)$. Consequently, $\mathcal{N}(A)$ is a complete lattice.

The following theorem (proved in [13]) describes the *AG*-group congruences on a completely inverse AG^{**}-groupoid in the terms of its normal completely inverse AG^{**}-subgroupoids.

Theorem 4.6 *Let A be a completely inverse AG^{**}-groupoid, $N\lhd A$. Then the relation*

$$\rho_N=\{(a,b)\in A\times A: ab^{-1}\in N\}$$

is the unique AG-group congruence ρ on A for which $\ker(\rho)=N$.

Conversely, if $\rho\in\mathcal{GC}(A)$, then $\ker(\rho)\in\mathcal{N}(A)$ *and $\rho=\rho_N$ for $N=\ker(\rho)$.*

Consequently, the map $\phi : \mathcal{N}(A) \to \mathcal{GC}(A)$ *given by* $N\phi = \rho_N$ $(N \in \mathcal{N}(A))$ *is a complete lattice isomorphism of* $\mathcal{N}(A)$ *onto* $\mathcal{GC}(A)$. *In particular, the lattice* $\mathcal{N}(A)$ *is modular.*

We say that a congruence ρ on a groupoid A is *idempotent pure* if $e\rho \subseteq E_A$ for all $e \in E_A$. Notice that any idempotent pure congruence ρ on an arbitrary completely inverse AG^{**}-groupoid A is contained in σ. Indeed, if $(a, b) \in \rho$, then $(ab^{-1}, bb^{-1}) \in \rho$, so $ab^{-1} \in E_A \subseteq E_A\omega$. Thus $(a, b) \in \sigma$, as required.

The following theorem gives necessary and sufficient conditions for a completely inverse AG^{**}-groupoid to be E-unitary.

Theorem 4.7 *Let* $A = [E_A; G_e; \phi_{e,f}]$ *be a completely inverse* AG^{**}*-groupoid. Then the following conditions are equivalent*:

(a) *A is E-unitary*;
(b) $\ker(\sigma) = E_A$;
(c) *σ is the maximum idempotent pure congruence on A*;
(d) $\sigma \cap \mu = 1_A$;
(e) *$\phi_{e,f}$ is a monomorphism for all $e, f \in E_A$ such that $e \geq f$.*

Proof In view of Proposition 4.4, (a) and (b) are equivalent, since $\ker(\sigma) = E_A\omega$.

(b) $\Longrightarrow$ (c). This follows from the preceding remark.

(c) $\Longrightarrow$ (d). Indeed, $\mathrm{tr}(\sigma \cap \mu) \subseteq \mathrm{tr}(\mu) = 1_{E_A}$ (by Theorem 3.4(c)). Furthermore, $\ker(\sigma \cap \mu) \subseteq \ker(\sigma) = E_A$. In the light of Theorem 4.2, $\sigma \cap \mu = 1_A$.

(d) $\Longrightarrow$ (e). Let $a_e, b_e \in G_e$ be such that $a_e\phi_{e,f} = b_e\phi_{e,f}$. Then $fa_e = fb_e$, therefore, $(a_e, b_e) \in \sigma$. Since clearly $(a_e, b_e) \in \mu$, then $a_e = b_e$.

(e) $\Longrightarrow$ (a). Let $a_f \in G_f$ be such that $ea_f = g$ $(e, f, g \in E_A)$. Then $ef = g$. Hence $eg = g$, that is, $e \geq g$, so $a_f\phi_{e,g} = g\phi_{e,g}$, therefore, $a_f = g \in E_A$. Thus A is E-unitary (by Proposition 4.4). □

Let ρ, υ be congruences on A such that $\rho \subseteq \upsilon$. Then the map $\Phi : A/\rho \to A/\upsilon$, where $(a\rho)\Phi = a\upsilon$ for every $a \in A$, is a well-defined epimorphism between these groupoids. Denote its kernel by

$$\upsilon/\rho = \{(a\rho, b\rho) \in A/\rho \times A/\rho : (a, b) \in \upsilon\}.$$

Then $(A/\rho)/(\upsilon/\rho) \cong A/\upsilon$. Moreover, every congruence α on A/ρ is of the form υ/ρ, where $\upsilon \supseteq \rho$ is a congruence on A. Indeed, the relation υ, defined on A by: $(a, b) \in \upsilon$ if and only if $(a\rho, b\rho) \in \alpha$, is a congruence on A such that $\rho \subseteq \upsilon$ and $\alpha = \upsilon/\rho$.

We are able now to determine all E-unitary congruences on any completely inverse AG^{**}-groupoid.

Theorem 4.8 *The intersection of an AG-group congruence and a semilattice congruence on a completely inverse* AG^{**}*-groupoid A is an E-unitary congruence on A. Moreover, any E-unitary congruence on a completely inverse* AG^{**}*-groupoid A can be expressed uniquely in this way.*

Proof Let ρ_N be an *AG*-group congruence ($N \lhd A$) and υ be a semilattice congruence on A. Put for simplicity $\rho = \rho_N \cap \upsilon$, and observe that ρ_N/ρ is an *AG*-group congruence on A/ρ and υ/ρ is a semilattice congruence on A/ρ. Since $\rho_N/\rho \cap \upsilon/\rho = 1_{A/\rho}$, then $\sigma_{A/\rho} \cap \mu_{A/\rho} = 1_{A/\rho}$ (see Theorem 3.4(a)). In the light of Theorem 4.7, ρ is an E-unitary congruence on A.

Conversely, let ρ be an E-unitary congruence on A, $\rho_N/\rho = \sigma_{A/\rho}$ and let $\upsilon/\rho = \mu_{A/\rho}$, where $\rho \subseteq \rho_N, \upsilon$. Then ρ_N is an *AG*-group congruence and υ is a semilattice congruence on A. Also, $(\rho_N \cap \upsilon)/\rho = \sigma_{A/\rho} \cap \mu_{A/\rho} = 1_{A/\rho}$ (again by Theorem 4.7). Thus $\rho = \rho_N \cap \upsilon$, as required.

Finally, let $\rho = \rho_{N_1} \cap \upsilon_1 = \rho_{N_2} \cap \upsilon_2$, where $N_i \lhd A$ and υ_i is a semilattice congruence on A ($i = 1, 2$). Let $(a, b) \in \upsilon_1$. Since $\upsilon_1 \cap \upsilon_2$ is a semilattice congruence on A, then there exists $e, f \in E_A$ such that $(a, e) \in \upsilon_1 \cap \upsilon_2$, $(e, f) \in \rho_{N_1}$, $(f, b) \in \upsilon_1 \cap \upsilon_2$ (Theorem 3.2), so $(e, f) \in \upsilon_1 \cap \rho_{N_1} = \upsilon_2 \cap \rho_{N_2} \subseteq \upsilon_2$. Hence $(a, b) \in \upsilon_2$, i.e., $\upsilon_1 \subseteq \upsilon_2$. By symmetry, we deduce that $\upsilon_1 = \upsilon_2$. Put $\upsilon_1 = \upsilon_2 = \upsilon$, so that $\rho = \rho_{N_1} \cap \upsilon = \rho_{N_2} \cap \upsilon$. If $(a, b) \in \rho_{N_1}$, then $(aab, abb) \in \upsilon \cap \rho_{N_1} \subseteq \rho_{N_2}$, therefore, $(a, b) \in \rho_{N_2}$ (by cancellation). Hence $\rho_{N_1} \subseteq \rho_{N_2}$. By symmetry, $\rho_{N_2} \subseteq \rho_{N_1}$. Thus $\rho_{N_1} = \rho_{N_2}$, as required. □

Corollary 4.9 *In any completely inverse AG^{**}-groupoid A, the relation*

$$\pi = \sigma \cap \mu$$

is the least E-unitary congruence on A.

Observe that if ρ is an E-unitary congruence on A, then $\ker(\rho) = \ker(\rho_N)$ for some $N \lhd A$. In the last section we will show that the converse implication is also true, that is, for any *AG*-group congruence ρ_N on A ($N \lhd A$), the family

$$\mathcal{U}_N = \{\rho_N \cap \upsilon : \mu \subseteq \upsilon\}$$

coincides with the set of all (E-unitary) congruence ρ on A such that

$$\ker(\rho) = \ker(\rho_N).$$

Finally, denote by $\mathcal{U}(A)$ the set of all E-unitary congruences on a completely inverse AG^{**}-groupoid A. Since the intersection of an arbitrary nonempty family of E-unitary congruences on A is again an E-unitary congruence on A, and $\mathcal{U}(A)$ has a least element, then the following corollary is valid.

Corollary 4.10 *Let A be a completely inverse AG^{**}-groupoid. Then the set $\mathcal{U}(A)$ is a complete $\cap$-sublattice of $\mathcal{C}(A)$ with the least element π and the greatest element $A \times A$.*

Moreover, $\mathcal{U}_N = \{\rho_N \cap \upsilon : \mu \subseteq \upsilon\}$ ($N \lhd A$) is a complete sublattice of $\mathcal{U}(A)$ with the least element $\rho_N \cap \mu$ and the greatest element ρ_N.

In view of the corollary, for each $\rho \in \mathcal{C}(A)$, there is the least E-unitary congruence π_ρ containing ρ. We will show in Sect. 6 that $\pi_\rho = \sigma\rho\sigma \cap \mu\rho\mu$.

5 The trace classes of $\mathcal{C}(A)$

Let ρ be a congruence on A, where A denotes (unless otherwise stated) an arbitrary completely inverse AG^{**}-groupoid. Put $K = \ker(\rho)$. It is immediate that K is a full completely inverse AG^{**}-subgroupoid of A. In the light of Proposition 4.1, K is also symmetric. Finally, put $\rho_{(K,\tau)} = \rho$, where $\tau = \mathrm{tr}(\rho)$. Theorem 4.2 states that

$$(a,b) \in \rho_{(K,\tau)} \iff (aa^{-1}, bb^{-1}) \in \tau \;\&\; ab^{-1} \in K. \tag{7}$$

Notice that if $a \in \ker(\rho_{(K,\tau)})$, that is, $(a,e) \in \rho_{(K,\tau)}$, where $e \in E_A$, then

$$ea \in K \;\&\; (e, aa^{-1}) \in \mathrm{tr}(\rho_{(K,\tau)}).$$

Observe further that if $ea \in K$ and $(e, aa^{-1}) \in \mathrm{tr}(\rho)$, then $a = (aa^{-1})a\rho ea$, therefore, $a \in K$.

Also, the following special case is of particular interest.

Proposition 5.1 *Let A be a completely inverse AG^{**}-groupoid. Then $\rho \in \mathcal{U}(A)$ if and only if $\ker(\rho)$ is closed in A.*

Proof Let $\rho \in \mathcal{U}(A)$ and $a \in (\ker(\rho))\omega$. Then $b = ea$ for some $b \in \ker(\rho)$ and $e \in E_A$. Hence $b\rho = (e\rho)(a\rho)$, where $e\rho, b\rho \in E_{A/\rho}$ and so $a\rho \in E_{A/\rho}$, since A/ρ is E-unitary. Thus $a \in \ker(\rho)$. Consequently, $(\ker(\rho))\omega = \ker(\rho)$.

Conversely, let $(e\rho)(a\rho) = f\rho$, where $a \in A$ and $e, f \in E_A$, then $ea \in \ker(\rho)$. Hence $a \in (\ker(\rho))\omega = \ker(\rho)$, that is, $a\rho \in E_{A/\rho}$. Thus ρ is E-unitary. $\square$

In Sect. 3 we have called a completely inverse AG^{**}-subgroupoid of A normal if it is full, symmetric and closed in A. Also, we say that a completely inverse AG^{**}-subgroupoid K is *seminormal* if K is full and symmetric.

Finally, for any ordered pair (K,τ), where K is a seminormal completely inverse AG^{**}-subgroupoid of A and τ is a congruence on E_A such that

(CP) if $ea \in K$ and $(e, aa^{-1}) \in \tau$, then $a \in K$ $(a \in A, e \in E_A)$,

define a relation $\rho_{(K,\tau)}$ like the above. In that case, (K,τ) is a *congruence pair* for A and we can define a relation $\rho_{(K,\tau)}$ as in (7) above.

The following theorem together with the above consideration and Theorem 4.2 says that any congruence on A is of the form $\rho_{(K,\tau)}$, where (K,τ) is a congruence pair for A, and this expression is unique.

Theorem 5.2 *If (K,τ) is a congruence pair for a completely inverse AG^{**}-groupoid A, then $\rho_{(K,\tau)}$ is the unique congruence on A with $\ker(\rho_{(K,\tau)}) = K$ and $\mathrm{tr}(\rho_{(K,\tau)}) = \tau$.*

Conversely, if ρ is a congruence on A, then $(\ker(\rho), \mathrm{tr}(\rho))$ is a congruence pair for A and $\rho_{(\ker(\rho),\mathrm{tr}(\rho))} = \rho$.

Proof It is sufficient to show the direct part of the theorem. Put $\rho = \rho_{(K,\tau)}$. It is clear that ρ is reflexive and symmetric. Let now $(a,b), (b,c) \in \rho$. Then $(aa^{-1}, cc^{-1}) \in \tau$

and $(b^{-1}a)(bc^{-1}) = (b^{-1}b)(ac^{-1}) = (bb^{-1})(ac^{-1}) \in K$. Also,

$$bb^{-1}\tau(aa^{-1})(c^{-1}c) = (ac^{-1})(a^{-1}c) = (ac^{-1})(ac^{-1})^{-1}.$$

In the light of the condition (CP), $ac^{-1} \in K$. Thus ρ is transitive. Let $(a,b) \in \rho$ and $c \in A$. Then

$$(ca)(ca)^{-1} = (ca)(c^{-1}a^{-1}) = (cc^{-1})(aa^{-1})\tau(cc^{-1})(bb^{-1}) = (cb)(cb)^{-1},$$
$$(ac)(ac)^{-1} = (ac)(a^{-1}c^{-1}) = (aa^{-1})(cc^{-1})\tau(bb^{-1})(cc^{-1}) = (bc)(bc)^{-1}.$$

Also,

$$(ca)(cb)^{-1} = (ca)(c^{-1}b^{-1}) = (cc^{-1})(ab^{-1}) \in E_A K \subseteq KK \subseteq K,$$
$$(ac)(bc)^{-1} = (ac)(b^{-1}c^{-1}) = (ab^{-1})(cc^{-1}) \in K E_A \subseteq KK \subseteq K.$$

Consequently, ρ is a congruence on A.

Finally, let $a \in \ker(\rho)$, that is, $(a,e) \in \rho$ for some $e \in E_A$. Then clearly $ea \in K$ and $(e, aa^{-1}) \in \tau$. Hence $a \in K$ (by (CP)). Thus $\ker(\rho) \subseteq K$. Conversely, let $a \in K$. Then $a^{-1} \in K$. Hence $(a^{-1}a, a) \in \rho$ and so $a \in \ker(\rho)$. Thus $\ker(\rho) = K$. Evidently, $\mathrm{tr}(\rho) = \tau$. In view of Theorem 4.2, $\rho_{(K,\tau)}$ is uniquely determined by the congruence pair (K,τ). □

It is easy to see that if K is closed in A, then the condition (CP) is not necessary in the proof of the direct part of Theorem 5.2. Combining this fact with Proposition 5.1 and Theorem 4.2 we obtain the following corollary.

Corollary 5.3 *Each E-unitary congruence ρ on a completely inverse AG^{**}-groupoid A is of the form $\rho_{(K,\tau)}$, where $K \lhd A$ and $\tau \in \mathcal{C}(E_A)$, and this expression is unique.*

Remark 4 One can modify Proposition III.2.3 [11] for completely inverse AG^{**}-groupoids.

Further, let ρ be a congruence on A. Put

$$\mu(\rho) = \{(a,b) \in A \times A : (a\rho, b\rho) \in \mu_{A/\rho}\}.$$

Clearly, $\mu(\rho) \in \mathcal{C}(A)$ and $\rho \subseteq \mu(\rho)$. From Theorem 3.4 follows that

$$(a,b) \in \mu(\rho) \quad \Longleftrightarrow \quad (aa^{-1}, bb^{-1}) \in \rho.$$

Put $\mu(\rho) = \rho^\theta$. It is clear that $\mathrm{tr}(\rho) = \mathrm{tr}(\rho^\theta)$. Also, if $\mathrm{tr}(\rho_1) = \mathrm{tr}(\rho_2)$ $(\rho_1, \rho_2 \in \mathcal{C}(A))$, then from the above equality follows that $\rho_1^\theta = \rho_2^\theta$. Consequently, ρ^θ is the maximum congruence with respect to $\mathrm{tr}(\rho)$.

Also, put (see Theorem 4.5)

$$\rho_\theta = \{(a,b) \in A \times A : (aa^{-1}, bb^{-1}) \in \rho \ \& \ (\exists e \in E_{(aa^{-1})\rho})\ ea = eb\}.$$

Since $a=(aa^{-1})a$, then ρ_θ is reflexive. Obviously, ρ_θ is symmetric. The proof that ρ_θ is transitive and left compatible is closely similar to the corresponding proof for the relation σ (see Theorem 4.5). Let $(a,b)\in\rho_\theta$ and $c\in A$. Then

$$(ac)(ac)^{-1}=(ac)(a^{-1}c^{-1})=(aa^{-1})(cc^{-1})\rho(bb^{-1})(cc^{-1})=(bc)(bc)^{-1}.$$

Also, $e(cc^{-1})\rho(aa^{-1})(cc^{-1})=(ac)(ac)^{-1}$ and

$$e(cc^{-1})\cdot ac=ea\cdot(cc^{-1})c=eb\cdot(cc^{-1})c=e(cc^{-1})\cdot bc.$$

Consequently, ρ_θ is a congruence on A. Finally, from the definition of ρ_θ follows that $\operatorname{tr}(\rho)=\operatorname{tr}(\rho_\theta)$, and since the definition of ρ_θ depends only on idempotents, then ρ_θ is the minimum congruence with respect to $\operatorname{tr}(\rho)$.

We have just proved part of the following theorem.

Theorem 5.4 *Let A be an arbitrary completely inverse AG^{**}-groupoid. Define a map $\Theta:\mathcal{C}(A)\to\mathcal{C}(E_A)$ by*

$$\rho\Theta=\operatorname{tr}(\rho)\quad(\rho\in\mathcal{C}(A)).$$

Then Θ is a complete lattice homomorphism of $\mathcal{C}(A)$ onto $\mathcal{C}(E_A)$. Also, if θ denotes the congruence on $\mathcal{C}(A)$ induced by Θ, that is,

$$\theta=\{(\rho_1,\rho_2)\in\mathcal{C}(A)\times\mathcal{C}(A):\operatorname{tr}(\rho_1)=\operatorname{tr}(\rho_2)\},$$

then for every $\rho\in\mathcal{C}(A)$,

$$\rho\theta=[\rho_\theta,\rho^\theta]$$

is a complete modular sublattice (with commuting elements) of $\mathcal{C}(A)$.

Proof The proof that Θ is a complete homomorphism is closely similar to the corresponding proof of Theorem III.2.5 [11], since the join of any nonempty family $\mathcal{F}$ of congruences in an arbitrary universal algebra is given by $\bigcup_{n\in\mathbb{N}}(\bigcup\mathcal{F})^n$. Further, let τ be a congruence on E_A. Define an equivalence relation ρ on A by

$$\rho=\{(a,b)\in A\times A:(aa^{-1},bb^{-1})\in\tau\}.$$

It is easy to check that ρ is compatible with the operation on A. Consequently, $\rho\in\mathcal{C}(A)$. Obviously, $\operatorname{tr}(\rho)=\tau$. Thus Θ maps $\mathcal{C}(A)$ onto $\mathcal{C}(E_A)$.

Finally, $\rho\theta$ is an interval of a complete lattice, so it is itself a complete lattice. Let $\rho_1,\rho_2\in\rho\theta$ and $a(\rho_1\rho_2)b$. Then $a\rho_1c\rho_2b$, where $c\in A$, so $(aa^{-1})\rho_1(cc^{-1})\rho_2(bb^{-1})$. Hence $(aa^{-1})\rho_2(cc^{-1})\rho_1(bb^{-1})$, since $\operatorname{tr}(\rho_1)=\operatorname{tr}(\rho_2)$. Moreover, $(cc^{-1})\rho_2(bc^{-1})$. It follows that $(aa^{-1})\rho_2(bc^{-1})$. Consequently,

$$a=(aa^{-1}\cdot a)\rho_2(bc^{-1}\cdot a)=(ac^{-1})b.$$

Further, $(ac^{-1})\rho_1(cc^{-1})$ and so $(ac^{-1})\rho_1(bb^{-1})$. Hence $(ac^{-1}\cdot b)\rho_1(bb^{-1}\cdot b)=b$. We have just shown that $a\rho_2(ac^{-1}\cdot b)\rho_1b$, that is, $\rho_1\rho_2\subseteq\rho_2\rho_1$. By symmetry, we deduce that $\rho_1\rho_2=\rho_2\rho_1$, therefore, the lattice $\rho\theta$ is modular. □

We call the classes of θ in the above theorem, the *trace classes* of A.

Lemma 5.5 *Let A be a completely inverse AG^{**}-groupoid. Then*

$$\rho_\theta \subseteq \gamma_\theta \quad \Longleftrightarrow \quad \operatorname{tr}(\rho) \subseteq \operatorname{tr}(\gamma) \quad \Longleftrightarrow \quad \rho^\theta \subseteq \gamma^\theta$$

for all $\rho, \gamma \in \mathcal{C}(A)$. Also, if $\rho \subseteq \gamma$, then $\rho_\theta \subseteq \gamma_\theta$ and $\rho^\theta \subseteq \gamma^\theta$.

Proof This follows directly from the definitions of ρ_θ and ρ^θ. □

Lemma 5.6 *Let $\mathcal{F}$ be an arbitrary nonempty family of congruences on a completely inverse AG^{**}-groupoid. Put*

$$\mathcal{F}_\theta = \{\rho_\theta : \rho \in \mathcal{F}\}, \qquad \mathcal{F}^\theta = \{\rho^\theta : \rho \in \mathcal{F}\}.$$

Then

$$\bigvee \mathcal{F}_\theta = \Big(\bigvee \mathcal{F}\Big)_\theta \quad \& \quad \bigcap \mathcal{F}^\theta = \Big(\bigcap \mathcal{F}\Big)^\theta.$$

Proof The proof is similar to the proof of Lemma III.2.9 [11]. □

Lemma 5.7 *Let A a completely inverse AG^{**}-groupoid. Then $\sigma = (A \times A)_\theta$.*

Proof This is obvious. □

The following corollary gives another equivalent conditions for a completely inverse AG^{**}-groupoid to be E-unitary.

Corollary 5.8 *Let A be a completely inverse AG^{**}-groupoid. The following conditions are equivalent*:

(a) *A is E-unitary*;
(b) *$\rho_\theta = \rho \cap \sigma$ for every $\rho \in \mathcal{C}(A)$*;
(c) *ρ_θ is an idempotent pure congruence on A for every $\rho \in \mathcal{C}(A)$.*

Proof Recall that A is E-unitary if and only if σ is the maximum idempotent pure congruence on A (Theorem 4.7).

(a) $\Longrightarrow$ (b). If $\rho \in \mathcal{C}(A)$, then $\rho_\theta \subseteq \rho \cap (A \times A)_\theta = \rho \cap \sigma$ (Lemmas 5.5, 5.7). On the other hand,

$$\operatorname{tr}(\rho \cap \sigma) = \operatorname{tr}(\rho) \cap \operatorname{tr}(\sigma) = \operatorname{tr}(\rho) \cap (E_A \times E_A) = \operatorname{tr}(\rho) = \operatorname{tr}(\rho_\theta)$$

and

$$\ker(\rho \cap \sigma) = \ker(\rho) \cap \ker(\sigma) = \ker(\rho) \cap E_A = E_A \subseteq \ker(\rho_\theta).$$

Thus $\rho \cap \sigma \subseteq \rho_\theta$ (Theorem 4.2). Consequently, $\rho_\theta = \rho \cap \sigma$.

(b) $\Longrightarrow$ (a). Clearly, $\mu_\theta = 1_A$. Moreover, $\mu_\theta = \mu \cap \sigma = \pi$ (Corollary 4.9), therefore, $\pi = 1_A$, so A is E-unitary.

It is now clear that (a) implies (c). We show the opposite implication. Indeed, if (c) holds, then $(A \times A)_\theta = \sigma$ is idempotent pure. Since $\rho_\theta \subseteq \sigma$ for every $\rho \in \mathcal{C}(A)$, then each ρ_θ is idempotent pure, too, as required. □

We have mentioned in the above proof that if A is E-unitary, then σ is the maximum idempotent pure congruence on A, therefore, the set of all idempotent pure congruences $[1_A, \sigma]$ on an E-unitary completely inverse AG^{**}-groupoid A forms a complete sublattice of the lattice $\mathcal{C}(A)$.

From the above corollary we obtain the following proposition.

Proposition 5.9 *Let A be an E-unitary completely inverse AG^{**}-groupoid. Then the mapping $\chi : \mathcal{C}(A) \to \mathcal{C}(A)$ defined by*

$$\rho\chi = \rho \cap \sigma \quad (\rho \in \mathcal{C}(A))$$

is a complete lattice homomorphism of $\mathcal{C}(A)$ onto the lattice of all idempotent pure congruences on A.

Proof In view of Corollary 5.8, $\rho_\theta = \rho \cap \sigma$ for every $\rho \in \mathcal{C}(A)$. Hence χ is a complete $\vee$-homomorphism (by Lemma 5.6). It is evident that χ is a complete $\cap$-homomorphism. Finally, if ρ is idempotent pure, then $\rho \subseteq \sigma$ and so $\rho\chi = \rho$. Thus χ maps $\mathcal{C}(A)$ onto the lattice of all idempotent pure congruences on A, as exactly required. □

We now investigate the θ-classes of A.

Lemma 5.10 *In any completely inverse AG^{**}-groupoid A, $\mu_{A/\rho} = \mu(\rho)/\rho$ for every $\rho \in \mathcal{C}(A)$. In particular, $[\rho/\rho, \rho^\theta/\rho]$ is the modular lattice of all idempotent-separating congruences on A/ρ $(\rho \in \mathcal{C}(A))$.*

Proof It is easy to see that $\mu(\rho)/\rho$ is idempotent-separating, so $\mu(\rho)/\rho \subseteq \mu_{A/\rho}$. On the other hand, if γ/ρ, where $\rho \subseteq \gamma$, is an idempotent-separating congruence on A/ρ, then $\mathrm{tr}(\gamma) \subseteq \mathrm{tr}(\rho)$ and so $\mathrm{tr}(\gamma) = \mathrm{tr}(\rho)$. Hence $\rho \subseteq \gamma \subseteq \mu(\rho)$, therefore, $\gamma/\rho \subseteq \mu(\rho)/\rho$. Thus $\mu_{A/\rho} = \mu(\rho)/\rho$. The second part of the lemma follows from Theorem 3.5. □

The following theorem follows easily from the above lemma.

Theorem 5.11 *Let A be a completely inverse AG^{**}-groupoid, $\rho \in \mathcal{C}(A)$. Define a map $\phi : [\rho_\theta, \rho^\theta] \to A/\rho_\theta$ by $\rho\phi = \rho/\rho_\theta$ for all $\rho \in [\rho_\theta, \rho^\theta]$. Then ϕ is a complete isomorphism of the trace class $[\rho_\theta, \rho^\theta]$ onto the modular lattice of all idempotent-separating congruences on A/ρ_θ.*

Remark 5 Note that $\phi_{|[\gamma,\mu(\rho)]}$, where $\gamma \in \rho\theta$, is a complete isomorphism of the interval $[\gamma, \mu(\rho)]$ onto the lattice of all idempotent-separating congruences on A/γ.

Recall that A is *fundamental* if and only if $\mu = 1_A$. By the above remark we have the following corollary.

Corollary 5.12 *Let ρ be a congruence on a completely inverse AG^{**}-groupoid A. Then A/ρ is fundamental if and only if $\rho = \mu(\rho)$.*

Denote by $\mathcal{FC}(A)$ the set of all fundamental congruences on A, that is,

$$\mathcal{FC}(A) = \{\mu(\rho) : \rho \in \mathcal{C}(A)\}.$$

Since $1_A \subseteq \rho$, then $\mu = \mu(1_A) \subseteq \mu(\rho)$ for all $\rho \in \mathcal{C}(A)$, what means that μ is the least fundamental congruence on A. Also, from Lemma 5.6 follows that $\mathcal{FC}(A)$ is a complete $\cap$-sublattice of $\mathcal{C}(A)$.

We have just proved a part of the following theorem.

Theorem 5.13 *Let A be a completely inverse AG^{**}-groupoid. Then $\mathcal{FC}(A)$ is a complete $\cap$-sublattice of $\mathcal{C}(A)$ with the least element μ and the greatest element $A \times A$. For any nonempty family $\{\rho_i : i \in I\}$ of fundamental congruences on A, the join of $\{\rho_i : i \in I\}$ in $\mathcal{FC}(A)$ is given by $\mu(\bigvee\{\rho_i : i \in I\})$. Also, $\mathcal{FC}(A) \cong \mathcal{C}(E_A)$.*

Proof Let $\emptyset \neq \{\rho_i : i \in I\} \subseteq \mathcal{FC}(A)$. Then

$$\Big(\bigvee\{\rho_i : i \in I\}, \mu\Big(\bigvee\{\rho_i : i \in I\}\Big)\Big) \in \theta.$$

On the other hand, if $\rho \in [\bigvee\{\rho_i : i \in I\}, \mu(\bigvee\{\rho_i : i \in I\})]$, then $\mu(\rho) = \rho$ if and only if $\rho = \mu(\bigvee\{\rho_i : i \in I\})$. Consequently, $\mu(\bigvee\{\rho_i : i \in I\})$ is the join of $\{\rho_i : i \in I\}$ in $\mathcal{F}(S)$.

Finally, if $\mu(\rho_1) \neq \mu(\rho_2)$, where $\rho_1, \rho_2 \in \mathcal{C}(A)$, then $\mathrm{tr}(\rho_1) \neq \mathrm{tr}(\rho_2)$, therefore, the restriction of the map Θ from Theorem 5.4 to the set $\mathcal{FC}(A)$ is the required complete lattice isomorphism. □

6 The kernel classes of $\mathcal{C}(A)$

Let A be an AG^{**}-groupoid. For every nonempty subset Q of A there exists an associated equivalence relation $\mathcal{Q}$ on A which is induced by the partition: $\{Q, A \setminus Q\}$. Define on A an equivalence relation τ^Q by

$$\tau^Q = \{(a,b) \in A \times A : (\forall x, y \in A^1) x(ay) \in Q \iff x(by) \in Q\},$$

where $A^1 = A \cup \{1\}$, $1 \notin A$ and $1a = a1 = a$ for all $a \in A$.

Observe that if $(a,b) \in \tau^Q$, then putting $x = y = 1$ in the definition of τ^Q, we obtain that either $a, b \in Q$ or $a, b \notin Q$. Thus $\tau^Q \subseteq \mathcal{Q}$.

Proposition 6.1 *Let Q be a nonempty subset of an AG^{**}-groupoid A. Then τ^Q is the largest congruence ρ on A for which Q is the union of some ρ-classes.*

Proof Let $(a,b) \in \tau^Q$, $x, y \in A^1$ and $c \in A$. Observe that

$$x(ac \cdot y) = (ac)(xy) = (xy \cdot c)a.$$

Hence if $x(ac \cdot y) \in Q$, then $(xy \cdot c)b \in Q$, since $(a,b) \in \tau^Q$. Thus we get $x(bc \cdot y) \in Q$. By symmetry, we conclude that τ^Q is right compatible. Further, the equality

$$x(ca \cdot y) = (ca)(xy) = (cx)(ay)$$

implies that τ^Q is also left compatible. Consequently, τ^Q is a congruence on A and Q is the union of some τ^Q-classes, since $\tau^Q \subseteq \mathcal{Q}$. Finally, if ρ is any congruence on A for which Q is the union of some ρ-classes, then $\rho \subseteq \mathcal{Q}$. Hence if $(a,b) \in \rho$, then either $a, b \in Q$ or $a, b \notin Q$. Thus for all $x, y \in A^1$, $x(ay) \in Q \Leftrightarrow x(by) \in Q$, so $a\tau^Q b$. Consequently, $\rho \subseteq \tau^Q$. □

Corollary 6.2 *In any completely inverse AG^{**}-groupoid A, the relation τ^{E_A} is the largest idempotent pure congruence on A.*

We shall write τ instead of τ^{E_A}, or τ_A if necessary.

Let ρ be a congruence on A, where A denotes (unless otherwise stated) an arbitrary completely inverse AG^{**}-groupoid. Put

$$\tau(\rho) = \{(a,b) \in A \times A : (a\rho, b\rho) \in \tau_{A/\rho}\}.$$

Clearly, $\tau(\rho) \in \mathcal{C}(A)$ and $\rho \subseteq \tau(\rho)$. Using Theorem 3.2, one can prove without difficulty that $\tau(\rho) = \tau^{\ker(\rho)}$. Thus $\tau(\rho)$ is the maximum congruence with respect to $\ker(\rho)$. Denote it by ρ^κ.

Further, put $\rho_\kappa = \rho \cap \mu$. Then $\ker(\rho_\kappa) = \ker(\rho)$, since μ is a semilattice congruence. On the other hand, μ is idempotent-separating, so ρ_κ is the minimum congruence with respect to $\ker(\rho)$.

Finally if K is a seminormal completely inverse AG^{**}-subgroupoid of A. Then the pair $(K, 1_{E_A})$ is a congruence pair for A, since then the condition (CP) is trivially met for this pair, and $\ker(\rho_{(K,1_{E_A})}) = K$. Consequently, K is seminormal if and only if K is a kernel of some congruence on A. Denote by $\mathcal{SN}(A)$ the set of seminormal completely inverse AG^{**}-subgroupoids of A. It is easy to see that $\mathcal{SN}(A)$ is a lattice under inclusion.

It is clear that if $\emptyset \neq \{\rho_i : i \in I\} \subseteq \mathcal{C}(A)$, then

$$\ker\Big(\bigcap\{\rho_i : i \in I\}\Big) = \bigcap\{\ker(\rho_i) : i \in I\},$$

therefore, we have just proved the following theorem.

Theorem 6.3 *Let A be an arbitrary completely inverse AG^{**}-groupoid. Define a map $K : \mathcal{C}(A) \to \mathcal{P}(A)$ by*

$$\rho K = \ker(\rho) \quad (\rho \in \mathcal{C}(A)).$$

Then K is a complete lattice $\cap$-homomorphism of $\mathcal{C}(A)$ onto $\mathcal{SN}(A)$. Also, if κ denotes the $\cap$-congruence on $\mathcal{C}(A)$ induced by K, that is,

$$\kappa = \{(\rho_1, \rho_2) \in \mathcal{C}(A) \times \mathcal{C}(A) : \ker(\rho_1) = \ker(\rho_2)\},$$

then for ever $\rho \in \mathcal{C}(A)$,

$$\rho\kappa = [\rho_\kappa, \rho^\kappa]$$

is a complete sublattice of $\mathcal{C}(A)$.

We call the classes of κ in the above theorem, the *kernel classes* of A.

Example 6.4 The following example shows that $\ker(\rho) \subseteq \ker(\gamma)$ (or even $\rho \subseteq \gamma$) does not imply (in general) that $\rho^\kappa \subseteq \gamma^\kappa$. Indeed, let $A = \{a, b, e, f\}$ be a commutative inverse semigroup with the multiplication table given below:

$\cdot$	a	b	e	f
a	e	e	a	a
b	e	f	a	b
e	a	a	e	e
f	a	b	e	f

Then clearly $1_A \subseteq \rho = 1_A \cup \{(a, e), (e, a)\}$. On the other hand, $\rho^\kappa = \rho \cup \{(b, f), (f, b)\}$ and $1_A^\kappa = 1_A \cup \{(e, f), (f, e), (a, b), (b, a)\}$ and so $1_A^\kappa = \tau \nsubseteq \rho^\kappa = \tau(\rho)$. Notice also that $\tau \cap \tau(\rho) = 1_A$.

Using Theorem 4.2 one can easily prove the following proposition.

Proposition 6.5 *If ρ is a congruence on a completely inverse AG^{**}-groupoid, then*

$$\rho = \rho_\theta \vee \rho_\kappa = \rho^\theta \cap \rho^\kappa.$$

We now investigate the κ-classes of A.

Lemma 6.6 *In any completely inverse AG^{**}-groupoid A, $\tau_{A/\rho} = \tau(\rho)/\rho$ for every $\rho \in \mathcal{C}(A)$. In particular, $[\rho/\rho, \rho^\kappa/\rho]$ is the lattice of all idempotent pure congruences on A/ρ $(\rho \in \mathcal{C}(A))$.*

Proof One can easily see that $\tau(\rho)/\rho$ is idempotent pure and so $\tau(\rho)/\rho \subseteq \tau_{A/\rho}$. On the other hand, if γ/ρ, where $\rho \subseteq \gamma$, is idempotent pure, then $\ker(\gamma) \subseteq \ker(\rho)$, therefore, $\ker(\gamma) = \ker(\rho)$. Hence $\rho \subseteq \gamma \subseteq \tau(\rho)$, so $\gamma/\rho \subseteq \tau(\rho)/\rho$. Consequently, $\tau_{A/\rho} = \tau(\rho)/\rho$. □

From the above lemma follows the following theorem.

Theorem 6.7 *Let A be a completely inverse AG^{**}-groupoid, $\rho \in \mathcal{C}(A)$. Define a map $\phi : [\rho_\kappa, \rho^\kappa] \to A/\rho_\kappa$ by $\rho\phi = \rho/\rho_\kappa$ for all $\rho \in [\rho_\kappa, \rho^\kappa]$. Then ϕ is a complete isomorphism of the kernel class $[\rho_\kappa, \rho^\kappa]$ onto the lattice of all idempotent pure congruences on A/ρ_κ.*

Note that $\phi_{|[\gamma,\tau(\rho)]}$, where $\gamma \in \rho\kappa$, is a complete isomorphism of the interval $[\gamma, \tau(\rho)]$ onto the lattice of all idempotent pure congruences on A/γ.

Recall that A is *E-disjunctive* if and only if $\tau = 1_A$. By the above remark we have the following corollary.

Corollary 6.8 *Let ρ be a congruence on a completely inverse AG^{**}-groupoid A. Then A/ρ is E-disjunctive if and only if $\rho = \tau(\rho)$.*

Remark 6 Note that in view of the end of Example 6.4, the set of all E-disjunctive congruences on a commutative inverse semigroup (in particular, on a completely inverse AG^{**}-groupoid) A does not form (in general) a sublattice of $\mathcal{C}(A)$.

Also, a completely inverse AG^{**}-groupoid A is an AG-group if and only if A is both E-unitary and E-disjunctive.

Finally, notice that a congruence ρ on A is idempotent pure if and only if $\rho \cap \mu = 1_A$. In particular, $\tau \cap \mu = 1_A$, therefore, A is a subdirect product of A/τ and E_A, where A/τ is an E-disjunctive completely inverse AG^{**}-groupoid.

Finally, we go back to study the lattice $\mathcal{U}(A)$ of all E-unitary congruences on a completely inverse AG^{**}-groupoid A. First, we prove the following useful result.

Lemma 6.9 *The following conditions are valid for a congruence ρ on a completely inverse AG^{**}-groupoid A:*

(a) $\rho \vee \sigma = \sigma\rho\sigma$;
(b) $a(\rho \vee \sigma)b \Leftrightarrow (ea)\rho(eb)$ *for some* $e \in E_A$;
(c) $\ker(\rho \vee \sigma) = (\ker(\rho))\omega$.

Proof Using Proposition 2.1, we may show, in a very similar way like in the proof of Lemma III.5.4(i) [11], the condition (a). Furthermore, the condition (b) follows directly from Proposition 2.1 and (a). Finally, the proof of (c) is closely similar to the corresponding proof of Corollary III.5.5 [11]. □

Using Proposition 2.1 and Lemma 6.9(b), we are able to show the following theorem.

Theorem 6.10 *Let A be an arbitrary completely inverse AG^{**}-groupoid. Then the map $\phi : \mathcal{C}(A) \to \mathcal{C}(A)$ defined by*

$$\rho\phi = \rho \vee \sigma$$

is a homomorphism of $\mathcal{C}(A)$ onto the lattice $[\sigma, A \times A]$ of all AG-group congruences on A.

Define the relation $\bar{\sigma}$ on $\mathcal{C}(A)$ by putting

$$(\rho_1, \rho_2) \in \bar{\sigma} \quad \Longleftrightarrow \quad \rho_1 \vee \sigma = \rho_2 \vee \sigma.$$

In the light of the above theorem, $\bar{\sigma}$ is a congruence on $\mathcal{C}(A)$, since $\phi\phi^{-1} = \bar{\sigma}$.

Proposition 6.11 *Let A be a completely inverse AG^{**}-groupoid and $\rho \in \mathcal{C}(A)$. Then the elements ρ, π_ρ and $\rho \vee \sigma$ are $\bar{\sigma}$-equivalent and $\rho \subseteq \pi_\rho \subseteq \rho \vee \sigma$. Moreover, the element $\rho \vee \sigma$ is the largest in the $\bar{\sigma}$-class $\rho\bar{\sigma}$.*

Proof Since π_ρ is the least E-unitary congruence containing ρ and $\rho \vee \sigma$ is E-unitary, then $\rho \subseteq \pi_\rho \subseteq \rho \vee \sigma$. Hence we get $\rho \vee \sigma \subseteq \pi_\rho \vee \sigma \subseteq \rho \vee \sigma$, so

$\rho \vee \sigma = \pi_\rho \vee \sigma$, therefore, $(\rho, \pi_\rho) \in \bar{\sigma}$. Evidently, $(\rho, \rho \vee \sigma) \in \bar{\sigma}$. This implies the first part of the proposition. The second part is clear. □

Further, let $a, b \in A$ and $\rho \in \mathcal{C}(A)$. If $(a, b) \in \sigma$, then evidently $(a\rho)\sigma(b\rho)$ in S/ρ. If in addition, $\rho \subseteq \sigma$, then $(a\rho)\sigma(b\rho)$ in S/ρ implies that $(a, b) \in \sigma$ in S. It follows that $A/\sigma \cong (A/\rho)/\sigma$, i.e., A and A/ρ have isomorphic maximal *AG*-group homomorphic images. In that case, we may say that ρ *preserves the maximal AG-group homomorphic images*. Since for every $\rho \in \mathcal{C}(A)$ we have $\rho_\theta \subseteq \rho$, then we obtain the following factorization:

$$A \to A/\rho_\theta \to A/\rho \cong (A/\rho_\theta)/(\rho/\rho_\theta).$$

Using the obvious terminology, we have the following proposition.

Proposition 6.12 *Every homomorphism of completely inverse AG**-groupoids can be factored into a homomorphism preserving the maximal AG-group homomorphic images and an idempotent-separating homomorphism.*

Proof The proof is similar to the proof of Proposition III.5.10 [11]. □

The following theorem gives another equivalent conditions for a congruence to be E-unitary (cf. the end of Sect. 4).

Theorem 6.13 *Let ρ be a congruence on a completely inverse AG**-groupoid A. Then the following conditions are equivalent*:

(a) ρ *is* E-*unitary*;
(b) $\ker(\rho)$ *is closed*;
(c) $\ker(\rho) = \ker(\rho \vee \sigma)$;
(d) $\rho \vee \sigma = \tau(\rho)$;
(e) $\tau(\rho) \in \mathcal{GC}(A)$.

Proof In the light of Proposition 5.1, (a) and (b) are equivalent.

(b) $\Longrightarrow$ (c). This follows from Lemma 6.9(c).

(c) $\Longrightarrow$ (d). Indeed, $\ker(\tau(\rho)) = \ker(\rho) = \ker(\rho \vee \sigma)$ and so $\rho \vee \sigma \subseteq \tau(\rho)$, therefore, $\tau(\rho) \in \mathcal{GC}(A)$.

(d) $\Longrightarrow$ (a). Let $\tau(\rho) \in \mathcal{GC}(A)$. Then $\tau(\rho)$ is E-unitary. Since the conditions (a) and (b) are equivalent, we get $\ker(\rho) = \ker(\tau(\rho))$ is closed. Thus ρ is E-unitary.

(c) $\Longrightarrow$ (d). By the above $\rho \vee \sigma \subseteq \tau(\rho)$ and so $\rho \vee \sigma, \tau(\rho) \in \mathcal{GC}(S)$. Furthermore, $\ker(\tau(\rho)) = \ker(\rho) = \ker(\rho \vee \sigma)$. Hence $\rho \vee \sigma = \tau(\rho)$ (by Theorem 4.6).

(d) $\Longrightarrow$ (e). This is trivial. □

In view of the above theorem, Theorem 6.3 and Corollary 4.10,

$$\rho_N \kappa = \{\rho_N \cap \nu : \mu \subseteq \nu\} = [\rho_N \cap \mu, \rho_N]$$

for every $N \lhd A$. Consequently,

$$\mathcal{U}(A) = \bigcup_{N \lhd A} \{\rho_N \cap \nu : \mu \subseteq \nu\}.$$

Thus we have the following statement (see the end of Sect. 4).

Proposition 6.14 *Let ρ be a congruence on a completely inverse AG^{**}-groupoid A. Then*:

(a) $\rho \vee \mu = \mu\rho\mu$;
(b) $a(\rho \vee \mu)b \Leftrightarrow (aa^{-1})\rho(bb^{-1})$;
(c) $\pi_\rho = \sigma\rho\sigma \cap \mu\rho\mu$.

Proof (a) It is clear that $\mu\rho\mu \subseteq \rho \vee \mu$ is a reflexive, symmetric and compatible relation on A. We show that it is also transitive. Let $a(\mu\rho\mu)b(\mu\rho\mu)c$. Then there exist elements $r, s, t, w \in A$ such that

$$\begin{aligned} &aa^{-1} = rr^{-1}, &&(r, s) \in \rho, &&ss^{-1} = bb^{-1}, \\ &bb^{-1} = tt^{-1}, &&(t, w) \in \rho, &&ww^{-1} = cc^{-1}. \end{aligned}$$

Also, $(rr^{-1})\rho(ss^{-1}) = (tt^{-1})\rho(ww^{-1})$. Consequently,

$$a\mu(aa^{-1}) = (rr^{-1})\rho(ww^{-1}) = (cc^{-1})\mu c.$$

Hence $(a, c) \in \mu\rho\mu$, as required, and so $\mu\rho\mu$ is a congruence on A contained in $\rho \vee \mu$. Since evidently $\rho, \mu \subseteq \mu\rho\mu$, then (a) holds.

(b) ($\Longrightarrow$) Let $a(\rho \vee \mu)b$. Then by (a), $aa^{-1} = cc^{-1}$, $(c, d) \in \rho$ and $dd^{-1} = bb^{-1}$ for some $c, d \in A$. Hence $(cc^{-1})\rho(dd^{-1})$. Thus $(aa^{-1})\rho(bb^{-1})$.

($\Longleftarrow$) If $(aa^{-1})\rho(bb^{-1})$, then $a\mu(aa^{-1})\rho(bb^{-1})\mu b$. Thus $a(\rho \vee \mu)b$.

(c) In the light of Lemma 6.9 (a) and the condition (b), $\alpha = \sigma\rho\sigma \cap \mu\rho\mu$ is a congruence on A. It is evident that $\rho \subseteq \alpha$ and $\ker(\alpha) = \ker(\sigma\rho\sigma)$, therefore, α is an E-unitary congruence on A which contains ρ. Finally, let $\rho \subseteq \beta = \rho_N \cap \nu \in \mathcal{U}(A)$, where $N \lhd A$ and $\mu \subseteq \nu$. Then $\rho \vee \sigma \subseteq \rho_N \vee \sigma = \rho_N$ and $\rho \vee \mu \subseteq \nu \vee \mu = \nu$. It follows that $\alpha \subseteq \rho_N \cap \nu = \beta$, as required. □

Using the condition (b) one can prove the following theorem.

Theorem 6.15 *Let A be an arbitrary completely inverse AG^{**}-groupoid. Then the map $\phi : \mathcal{C}(A) \to \mathcal{C}(A)$ defined by*

$$\rho\phi = \rho \vee \mu$$

is a homomorphism of $\mathcal{C}(A)$ onto the lattice $[\mu, A \times A]$ of semilattice congruences on A.

Let $\rho \in \mathcal{C}(A)$. Since $\rho \subseteq A \times A$, then there is the least semilattice congruence μ_ρ containing ρ (note that $\mu_\rho = \pi_\rho$, see the proof of Proposition 6.14(c)).

Define the relation $\overline{\mu}$ on $\mathcal{C}(A)$ by putting

$$(\rho_1, \rho_2) \in \overline{\mu} \quad \Longleftrightarrow \quad \rho_1 \vee \mu = \rho_2 \vee \mu.$$

In view of the above theorem, $\overline{\mu}$ is a congruence on $\mathcal{C}(A)$.

Proposition 6.16 *Let A be a completely inverse AG^{**}-groupoid, $\rho \in \mathcal{C}(A)$. Then the elements ρ, μ_ρ and $\rho \vee \mu$ are $\overline{\mu}$-equivalent and $\rho \subseteq \mu_\rho \subseteq \rho \vee \mu$. Moreover, the element $\rho \vee \mu$ is the largest in the $\overline{\mu}$-class $\rho\overline{\mu}$.*

Also, let $a, b \in A$ and $\rho \in \mathcal{C}(A)$. If $(a, b) \in \mu$, then clearly $(a\rho)\mu(b\rho)$ in S/ρ. If in addition, $\rho \subseteq \mu$, then $(a\rho)\mu(b\rho)$ in S/ρ implies that $(a, b) \in \mu$, since μ is idempotent-separating. It follows that $A/\mu \cong (A/\rho)/\mu$, that is, A and A/ρ have isomorphic minimal idempotent-separating homomorphic images. We may say that ρ *preserves the minimal idempotent-separating homomorphic images*. Since for all $\rho \in \mathcal{C}(A)$, $\rho_\kappa \subseteq \rho$, then we have the following factorization:

$$A \to A/\rho_\kappa \to A/\rho \cong (A/\rho_\kappa)/(\rho/\rho_\kappa).$$

We get the following proposition.

Proposition 6.17 *Every homomorphism of completely inverse AG^{**}-groupoids can be factored into a homomorphism preserving the minimal idempotent-separating homomorphic images and an idempotent pure homomorphism.*

Proof Let ρ be a congruence on a completely inverse AG^{**}-groupoid A. Then obviously $\rho_\kappa \subseteq \mu$, and hence the canonical homomorphism of A onto A/ρ_κ preserves the minimal idempotent-separating homomorphic images. Also, the mapping $a\rho_\kappa \to a\rho$ $(a \in A)$ is an idempotent pure homomorphism of A/ρ_κ onto A/ρ, since $\ker(\rho) = \ker(\rho_\kappa)$. The thesis of the proposition follows now from the above factorization. □

Since μ is also the least semilattice congruence on A (Theorem 3.4), then we may replace in the above proposition the words "minimal idempotent-separating" by the words "maximal semilattice".

Once again we prove some equivalent conditions for A to be E-unitary.

Theorem 6.18 *Let A be a completely inverse AG^{**}-groupoid. The following conditions are equivalent*:

(a) *A is E-unitary*;
(b) $\sigma = \tau$;
(c) *every idempotent pure congruence on A is E-unitary*;
(d) *there exists an idempotent pure E-unitary congruence on A*;
(e) *τ is E-unitary.*

Proof (a) $\Longrightarrow$ (b). Let $\pi = 1_A$. Then $\sigma = \tau(\pi) = \tau(1_A) = \tau$.

(b) $\Longrightarrow$ (a). Let $\sigma = \tau$. Then $\pi = \sigma_\kappa = \tau_\kappa = 1_A$.

(a) $\Longrightarrow$ (c). Firstly, A/τ is E-unitary. Indeed, if $(e\tau)(a\tau) = f\tau \in E_{A/\tau}$, where $a \in A$ and $e, f \in E_A$, then $(ea, f) \in \tau$. Hence $ea \in E_A$. Thus $a \in E_A$. Secondly, if $\rho \in \mathcal{C}(A)$ is idempotent pure, then $\rho \subseteq \tau$. Consequently, ρ is E-unitary.

(c) $\Longrightarrow$ (d). Obvious.

(d) $\Longrightarrow$ (e). If ρ is an idempotent pure E-unitary congruence on A, then we get $\pi \subseteq \rho \subseteq \tau \subseteq \sigma$, so τ is E-unitary (Theorem 6.13).

(e) $\Longrightarrow$ (a). Let $ea = f$, where $a \in A$ and $e, f \in E_A$. Then $(e\tau)(a\tau) = f\tau$ and so $a \in \ker(\tau) = E_A$. Thus S is E-unitary. □

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