Ramanujan J (2009) 20: 179–188
DOI 10.1007/s11139-009-9169-x

On an asymptotic series of Ramanujan

Yaming Yu

Received: 22 May 2008 / Accepted: 1 April 2009 / Published online: 12 August 2009

Abstract An asymptotic series in Ramanujan's second notebook (Entry 10, Chap. 3) is concerned with the behavior of the expected value of $\phi(X)$ for large λ where X is a Poisson random variable with mean λ and ϕ is a function satisfying certain growth conditions. We generalize this by studying the asymptotics of the expected value of $\phi(X)$ when the distribution of X belongs to a suitable family indexed by a convolution parameter. Examples include the binomial, negative binomial, and gamma families. Some formulas associated with the negative binomial appear new.

Keywords Asymptotic expansion · Binomial distribution · Central moments · Cumulants · Gamma distribution · Negative binomial distribution · Poisson distribution · Ramanujan's notebooks

Mathematics Subject Classification (2000) Primary 34E05 · Secondary 60E05

1 An asymptotic series of Ramanujan

A version (modified from [2]) of Entry 10 in Chap. 3 of Ramanujan's second notebook reads

Theorem 1 *Let $\phi(x)$, $x \in [0, \infty)$, denote a function of at most polynomial growth as x tends to ∞. Suppose there exist constants $x_0 > 0$ and $A \geq 1$, and a function $G(x)$ of at most polynomial growth as $x \to \infty$ such that for each nonnegative integer m and all $x > x_0$, the derivatives $\phi^{(m)}(x)$ exist and satisfy*

$$\left|\frac{\phi^{(m)}(x)}{m!}\right| \leq G(x)\left(\frac{A}{x}\right)^m. \tag{1}$$

Y. Yu (✉)
Department of Statistics, University of California, Irvine 92697, USA
e-mail: yamingy@uci.edu

Assume that there exists a positive constant c such that

$$G(x) \gg e^{-c\sqrt{x}} \tag{2}$$

as $x \to \infty$. *Put*

$$\phi_\infty(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k \phi(k)}{k!}.$$

Then for any fixed positive integer M,

$$\phi_\infty(x) = \phi(x) + \sum_{n=2}^{2M-2} \sum_{k=\lfloor (n+1)/2 \rfloor + 1}^{n} b_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} + O\left(G(x) x^{-M}\right), \tag{3}$$

as x *tends to* ∞, *where* $\lfloor (n+1)/2 \rfloor$ *denotes the integer part of* $(n+1)/2$ *and the numbers* b_{kn} *are defined recursively by*

$$b_{kk} = 1, \quad k \geq 2;$$
$$b_{kn} = 0, \quad n < k \text{ or } n > 2k-2;$$
$$b_{k+1,n+1} = n b_{k,n-1} + (n-k+1) b_{kn}, \quad k \leq n \leq 2k-1.$$

This result may seem hard to penetrate at first glance. Its relevance, however, is easily appreciated through interesting examples such as [2, 3]

$$e^{-x} \sum_{k=0}^{\infty} \frac{\sqrt{k} x^k}{k!} = \sqrt{x} \left(1 - \frac{1}{8x} - \frac{7}{128x^2} + O\left(\frac{1}{x^3}\right)\right) \tag{4}$$

and

$$e^{-x} \sum_{k=0}^{\infty} \frac{x^k \log(k+1)}{k!} = \log(x) + \frac{1}{2x} + \frac{1}{12x^2} + O\left(\frac{1}{x^3}\right),$$

both valid as $x \to \infty$; by choosing $\phi(x) = \log \Gamma(x+1)$ in (3), an asymptotic formula for the Shannon entropy of the Poisson distribution can also be obtained (see [6]).

The first goal of this note is a formal, probabilistic derivation of Theorem 1. The starting point is the observation that $\phi_\infty(x) = E\phi(U)$, where U is a Poisson random variable with mean x and E denotes expectation. Based on this we present in Sect. 2 a more general version (Theorem 2) of Theorem 1 by considering U distributed as some distribution other than the Poisson, e.g., a gamma distribution or a binomial distribution. As illustrations, we derive asymptotic expansions for digamma functions and for inverse moments of certain positive random variables. We prove Theorem 2 in Sect 3.

Noting that $\phi_\infty(x) = E\phi(U)$ where U has the $\mathrm{Po}(x)$ distribution, we expand $\phi(U)$ as a Taylor series

$$\phi(U) = \phi(x) + \sum_{n=1}^{2M-2} \frac{(U-x)^n \phi^{(n)}(x)}{n!} + \cdots,$$

and formally take the expectation term by term:

$$E\phi(U)=\phi(x)+\sum_{n=1}^{2M-2}\frac{E(U-x)^n\phi^{(n)}(x)}{n!}+\cdots \tag{5}$$

The quantity $\mu_n=E(U-x)^n$ is the nth central moment of the Po(x) distribution. The first few μ_n's are

$$(\mu_1,\mu_2,\mu_3,\mu_4)=(0,x,x,3x^2+x),$$

and they obey the well-known recursion (see [19], Lemma 3, for example)

$$\mu_{n+1}=x\left(\frac{\mathrm{d}\mu_n}{\mathrm{d}x}+n\mu_{n-1}\right),\quad n\geq 2,$$

from which we obtain, by comparing the coefficients of x^{n-k+1} and using the definition of b_{kn},

$$\mu_n=\sum_{k=0}^{n}b_{kn}x^{n-k+1},\quad n\geq 2. \tag{6}$$

The double sum in (3) is the result of substituting (6) in (5) and noting that $b_{kn}=0$ if $k\leq\lfloor(n+1)/2\rfloor$. Based on this it is also clear that μ_n is a polynomial in x of degree $\lfloor n/2\rfloor$, which implies that, given the condition (1), the "leading term of the remainder" in (5),

$$\frac{E(U-x)^{2M-1}\phi^{(2M-1)}(x)}{(2M-1)!}=\frac{\mu_{2M-1}\phi^{(2M-1)}(x)}{(2M-1)!},$$

is $O(G(x)x^{-M})$.

The above derivation is, of course, strictly formal. However, it can be made rigorous under the stated conditions; see Berndt [2] and Evans [5]. Berndt actually proved a modification of (3) where the order of summation over k and n on the right hand side is inverted and certain higher order terms of the resulting sum are absorbed in the $O(G(x)x^{-M})$ term.

2 A general version

The formal derivation in Sect. 1 suggests that it is possible to generalize Theorem 1 if we let U have a suitable distribution other than the Poisson. Noting the key role played by the central moments of U, we give a version of Theorem 1 by imposing conditions on the moment generating function (mgf) of U. An introduction to moment generating functions can be found in probability texts such as Gut [9]. A useful property is that, if an mgf exists in a neighborhood of zero, then for all $m\geq 1$, the mth moment exists and can be obtained by differentiating the mgf m times.

Theorem 2 *Let $\phi(x), x \in [0, \infty)$, denote a Borel measurable function that can be bounded in absolute value by a polynomial in x. Let M be a fixed positive integer. Suppose there exist a constant $A \geq 1$ and a function $G(x)$ of at most polynomial growth such that for $1 \leq m \leq 2M$ and all sufficiently large x, the derivatives $\phi^{(m)}(x)$ exist and satisfy*

$$\left|\frac{\phi^{(m)}(x)}{m!}\right| \leq G(x)\left(\frac{A}{x}\right)^m. \tag{7}$$

Assume there exist $\eta \in (0, 1)$ and a constant B such that for all sufficiently large x,

$$G(y) \leq BG(x) \quad \text{whenever } |y - x| \leq \eta x. \tag{8}$$

Let Ω be an unbounded subset of $[0, \infty)$ and let $U_x, x \in \Omega$, be a family of nonnegative random variables. Assume there exist a constant $\delta > 0$ and a function $g(s)$ such that for all $x \in \Omega$, the mgf of U_x exists in the interval $(-\delta, \delta)$ and satisfies

$$Ee^{sU_x} = e^{xg(s)}, \quad s \in (-\delta, \delta). \tag{9}$$

Assume $g'(0) = 1$ in addition. Then

$$E\phi(U_x) = \phi(x) + \sum_{n=2}^{2M-2} \sum_{k=\lfloor (n+1)/2 \rfloor + 1}^{n} c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} + O\left(G(x)x^{-M}\right), \tag{10}$$

as x tends to ∞, where c_{kn} are constants that depend only on the function $g(s)$, and are determined by

$$E(U_x - x)^n = \sum_{k=0}^{n} c_{kn} x^{n-k+1}, \quad n \geq 2.$$

Evidently, Theorem 1 is the special case $g(s) = e^s - 1$, except for the assumption (8) on $G(x)$ which replaces (2). This new assumption does not appear very restrictive as we shall see from the examples later in this section; however it does make the proof of Theorem 2 more straightforward. We also relax the assumption on $\phi(x)$ slightly by requiring only $2M$ derivatives.

It should be emphasized that the function $g(s)$ in (9) does not depend on x. Also note that $g(s)$ is analytic in $s \in (-\delta, \delta)$ given the existence of the mgf. Aside from the Poisson, examples of distribution families that satisfy (9) include the binomial, negative binomial, and gamma families. In general, suppose $Y_1, Y_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.), nonnegative, nondegenerate random variables whose mgf exists in a neighborhood of zero. Then the family of random variables $\{\sum_{k=1}^{n} Y_k,\ n = 1, 2, \ldots\}$ (n is known as a convolution parameter) has mgf

$$E\exp\left(s\sum_{k=1}^{n} Y_k\right) = \exp\left(n\log\left(Ee^{sY_1}\right)\right), \quad n = 1, 2, \ldots,$$

which is of the form (9) with $g(s) = (EY_1)^{-1}\log(Ee^{sY_1})$, if we index the family by its mean $x = nEY_1$. This shows that Theorem 2 is potentially applicable to a wide range of problems.

Example 1 For a fixed $p \in (0, 1)$, consider the family $U_x, x = p, 2p, 3p, \ldots$, where U_x has the binomial distribution $\mathrm{Bi}(n, p)$, $n = x/p$. The first few central moments of U_x are given by ($q = 1 - p$)

$$(\mu_2, \mu_3, \mu_4) = (qx, (q-p)qx, 3q^2x^2 + q(1-6pq)x).$$

Since $\mathrm{Bi}(n, p)$ is a sum of n i.i.d. Bernoulli(p) random variables, (9) is satisfied.

- Given r (real) and $a > 0$, let $\phi(x) = G(x) = (x+a)^{-r}$. It is easy to verify (7) and (8); thus we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}(k+a)^{-r} = (np+a)^{-r} &+ \frac{qr(r+1)}{2}(np)(np+a)^{-r-2}\\ &- \frac{(q-p)qr(r+1)(r+2)}{6}(np)(np+a)^{-r-3}\\ &+ \frac{q^2r(r+1)(r+2)(r+3)}{8}(np)^2(np+a)^{-r-4}\\ &+ O(n^{-r-3}),\end{aligned} \tag{11}$$

  as $n \to \infty$.
- If we let $\phi(x) = x^{-r}$, $x \ge 1$ and $\phi(x) = 0$, $x < 1$, then we obtain an asymptotic expansion for

$$\sum_{k=1}^{n}\binom{n}{k}p^kq^{n-k}k^{-r} \tag{12}$$

  by simply substituting $a = 0$ in the right hand side of (11). When r is a positive integer, (12) is sometimes known as the rth inverse moment of the binomial. The problem of inverse moments has a long history in statistics (see, for example, Stephan [16], Grab and Savage [8], and David and Johnson [4]). More recently, expansions for (12) have been considered by Marciniak and Wesolowski [14] (see also Rempala [15]) for $r = 1$, and by Žnidarič [19] for general r. Žnidarič [19] also gives a brief historical account with many references.
  A special case corresponding to $r = -1/2$ is

$$\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}\sqrt{k} = \sqrt{np}\left[1 - \frac{q/8}{np} + \frac{(q-p)q/16 - 15q^2/128}{(np)^2} + O\left(\frac{1}{n^3}\right)\right],$$

  which is the binomial analog of (4) considered by Ramanujan [2].
- Let $\phi(x) = \log(x+\beta)$ for a fixed $\beta > 0$ and let $G(x) \equiv 1$. We have (as $n \to \infty$)

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}\log(k+\beta)\\ &\quad = \log(np+\beta) - \frac{npq}{2}(np+\beta)^{-2}\end{aligned}$$

$$+\frac{(q-p)q}{3}(np)(np+\beta)^{-3}-\frac{3q^2}{4}(np)^2(np+\beta)^{-4}$$
$$+O(n^{-3}). \tag{13}$$

The problem of approximating the left hand side of (13) appears in Krichevskiy [13] in an information theoretic context; see also Jacquet and Szpankowski [10, 11], who give an alternative derivation of (13) using the method of analytic poissonization and depoissonization. Flajolet [7] also considers similar problems using singularity analysis.

Example 2 For a fixed $p \in (0,1)$, consider the negative binomial family NB(n,p) whose probability mass function is $f(k;n,p)=\binom{n+k-1}{k}p^n q^k$, $k=0,1,\ldots,$ where $q=1-p$. The mean is nq/p and the first few central moments are

$$(\mu_2,\mu_3,\mu_4)=\left(\frac{nq}{p^2},\frac{nq(1+q)}{p^3},\left[3+\frac{6}{n}+\frac{p^2}{nq}\right]\frac{(nq)^2}{p^4}\right).$$

Similar to the binomial case, as NB(n,p) is a sum of n i.i.d. geometric(p) random variables, (9) is satisfied and Theorem 2 is applicable for an appropriate $\phi(x)$.

- Take $\phi(x)=(x+a)^{-r}$, $a>0$. We have, as $n\to\infty$,

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}p^n q^k (k+a)^{-r}$$
$$=\left(\frac{nq}{p}+a\right)^{-r}+\frac{r(r+1)}{2p}\left(\frac{nq}{p}\right)\left(\frac{nq}{p}+a\right)^{-r-2}$$
$$-\frac{(1+q)r(r+1)(r+2)}{6p^2}\left(\frac{nq}{p}\right)\left(\frac{nq}{p}+a\right)^{-r-3}$$
$$+\frac{r(r+1)(r+2)(r+3)}{8p^2}\left(\frac{nq}{p}\right)^2\left(\frac{nq}{p}+a\right)^{-r-4}$$
$$+O(n^{-r-3}). \tag{14}$$

- As in the binomial case, we obtain an asymptotic expansion for

$$\sum_{k=1}^{\infty}\binom{n+k-1}{k}p^n q^k k^{-r} \tag{15}$$

for real r by substituting $a=0$ in the right hand side of (14). Expansions for (15) have been considered by Marciniak and Wesolowski [14] and Rempala [15] for the special case $r=1$, and by Wuyungaowa and Wang [18] for integer $r\geq 0$.
- Let $\phi(x)=\log(x+\beta)$ for a fixed $\beta>0$. We have

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}p^n q^k \log(k+\beta)=\log\left(\frac{nq}{p}+\beta\right)-\frac{1}{2p}\left(\frac{nq}{p}\right)\left(\frac{nq}{p}+\beta\right)^{-2}$$

$$+\frac{(1+q)}{3p^2}\left(\frac{nq}{p}\right)\left(\frac{nq}{p}+\beta\right)^{-3}$$
$$-\frac{3}{4p^2}\left(\frac{nq}{p}\right)^2\left(\frac{nq}{p}+\beta\right)^{-4}+O(n^{-3}).$$

Example 3 Consider the gamma family Gam$(x, 1)$, $x > 0$, whose density function is $f(u; x) = u^{x-1}e^{-u}/\Gamma(x)$, $u > 0$. The mean is x and the first few central moments are

$$(\mu_2, \mu_3, \mu_4) = (x, 2x, 3x^2 + 6x).$$

The moment generating function is $(1 - s)^{-x}$, $s < 1$, which is of the form (9) with $g(s) = -\log(1 - s)$.

Take $\phi(x) = G(x) = x\log(x)$. We have

$$\frac{1}{\Gamma(x)}\int_0^{\infty} u\log(u)u^{x-1}e^{-u}\mathrm{d}u = x\log(x) + \frac{1}{2} - \frac{1}{12x} + O\left(\frac{\log(x)}{x^2}\right),$$

as $x \to \infty$. Noting $\Gamma(x + 1) = x\Gamma(x)$, we may write

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \log(x) + \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{\log(x)}{x^3}\right), \tag{16}$$

which is a familiar asymptotic formula for the digamma function ([1], p. 259). By expanding for one more term we can replace $O(x^{-3}\log(x))$ by $O(x^{-3})$ in (16). A full asymptotic expansion can be recovered by applying (10) and using the following recursion between the central moments of Gam$(x, 1)$ (see [17]):

$$\mu_k = (k-1)(\mu_{k-1} + x\mu_{k-2}), \quad k \geq 2.$$

3 Proof of Theorem 2

Our proof follows Berndt [2]. In the setting of Theorem 2 we have

Lemma 1 *Let $h(u)$, $u \in [0, \infty)$, be a Borel measurable function that can be bounded in absolute value by a polynomial. Then for a fixed $t \in (0, 1)$, both $EI(U_x < tx)h(U_x)$ and $EI(U_x > x/t)h(U_x)$ tend to 0 exponentially fast as x tends to ∞, where $I(\cdot)$ is the indicator function.*

Proof Observe that $xg(s)$, the cumulant generating function of U_x, is an analytic function of s (real) in a neighborhood of zero. Because $g(0) = 0$, $g'(0) = 1$ and $t \in (0, 1)$, we may choose $r, \epsilon > 0$ small enough such that both $g(r) < r/t$ and $g(r + \epsilon) < r/t$. Since $|h(u)|$ is bounded by a polynomial, there exists a constant D such that $|h(u)| < e^{\epsilon u} + D$ for all $u \in [0, \infty)$. We have

$$|EI(U_x > x/t)h(U_x)| \leq E(e^{\epsilon U_x} + D)e^{r(U_x - x/t)}$$
$$= e^{x[g(r+\epsilon)-r/t]} + De^{x[g(r)-r/t]},$$

which tends to zero exponentially as $x \to \infty$. The proof for $EI(U_x < tx)h(U_x)$ is similar and hence omitted. □

Proof of Theorem 2 Throughout we assume that x is sufficiently large. Define intervals $I_1 = [0, (1-\eta)x)$, $I_2 = [(1-\eta)x, (1+\eta)x)$ and $I_3 = [(1+\eta)x, \infty)$, where η is as specified in (8).

By Lemma 1, both

$$EI(U_x \in I_1)\phi(U_x) \tag{17}$$

and

$$EI(U_x \in I_3)\phi(U_x) \tag{18}$$

tend to zero exponentially as $x \to \infty$.

Consider the Taylor polynomial

$$\psi(y) = \sum_{k=0}^{2M-1} \frac{\phi^{(k)}(x)}{k!}(y-x)^k.$$

Since for any $y \in I_1$,

$$|\psi(y)| \le \sum_{k=0}^{2M-1} \left|\frac{\phi^{(k)}(x)}{k!}\right| x^k \equiv q(x),$$

we have

$$|EI(U_x \in I_1)\psi(U_x)| \le q(x)EI(U_x \in I_1).$$

From (7) it follows that $q(x)$ has at most polynomial growth as $x \to \infty$; by Lemma 1 we know that

$$EI(U_x \in I_1)\psi(U_x) \tag{19}$$

tends to zero exponentially as $x \to \infty$.

Similarly, for any $y \in I_3$, we have

$$|\psi(y)| \le \sum_{k=0}^{2M-1} \left|\frac{\phi^{(k)}(x)}{k!}\right| y^k,$$

and hence

$$|EI(U_x \in I_3)\psi(U_x)| \le \sum_{k=0}^{2M-1} \left|\frac{\phi^{(k)}(x)}{k!}\right| EI(U_x \in I_3)U_x^k.$$

By Lemma 1, each of $EI(U_x \in I_3)U_x^k$, $k \le 2M-1$, tends to zero exponentially as $x \to \infty$. By (7), each of $\phi^{(k)}(x)$ has at most polynomial growth as $x \to \infty$. Overall

$$EI(U_x \in I_3)\psi(U_x) \tag{20}$$

tends to zero exponentially as $x \to \infty$.

For any $y \in I_2$, there exists some point ζ between x and y such that

$$\begin{aligned}|\phi(y) - \psi(y)| &= \left|\frac{\phi^{(2M)}(\zeta)}{(2M)!}\right| |y - x|^{2M} \\ &\le G(\zeta)\left(\frac{A}{(1-\eta)x}\right)^{2M} |y - x|^{2M} \\ &\le BG(x)\left(\frac{A}{(1-\eta)x}\right)^{2M} |y - x|^{2M},\end{aligned}$$

where (7) and (8) are used in the inequalities. Letting $C = B[A/(1-\eta)]^{2M}$, we have

$$|EI(U_x \in I_2)[\phi(U_x) - \psi(U_x)]| \le C\frac{G(x)}{x^{2M}} E|U_x - x|^{2M}.$$

We now consider the nth central moment of U_x, $\mu_n = E(U_x - x)^n$, as a function of x. (Note that the mean of U_x is x as $EU_x = xg'(0) = x$.) Expand $xg(s)$ around $s = 0$ to get

$$xg(s) = \sum_{j=1}^{\infty} \frac{xg^{(j)}(0)s^j}{j!}.$$

Note that the coefficient $xg^{(j)}(0)$ is the jth cumulant of U_x, and, according to the well-known relation between central moments and cumulants (see [12] or [17], for example)

$$\mu_n = \sum_{j=0}^{n-2} \binom{n-1}{j} \mu_j x g^{(n-j)}(0), \quad n \ge 2, \tag{21}$$

with $\mu_0 = 1$ and $\mu_1 = 0$. Based on (21), it is easy to show by induction that μ_n is a polynomial in x of degree at most $\lfloor n/2 \rfloor$, its coefficients depending only on the function $g(s)$. Hence, for large x we have $E(U_x - x)^{2M} = O(x^M)$ and

$$EI(U_x \in I_2)[\phi(U_x) - \psi(U_x)] = O(G(x)x^{-M}).$$

Combined with the exponentially small items (17), (18), (19) and (20), this gives

$$E[\phi(U_x) - \psi(U_x)] = O(G(x)x^{-M}).$$

It remains to calculate $E\psi(U_x)$. We have, by the definition of c_{kn},

$$\begin{aligned}E\psi(U_x) &= \sum_{n=0}^{2M-1} E(U_x - x)^n \frac{\phi^{(n)}(x)}{n!} \\ &= \phi(x) + \sum_{n=2}^{2M-1} \sum_{k=0}^{n} c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!}\end{aligned}$$

$$= \phi(x) + \sum_{n=2}^{2M-1} \sum_{k=\lfloor (n+1)/2 \rfloor+1}^{n} c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!}$$

$$= \phi(x) + \sum_{n=2}^{2M-2} \sum_{k=\lfloor (n+1)/2 \rfloor+1}^{n} c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} + O(G(x)x^{-M}).$$

Note that the inner sum over k is curtailed because the degree of μ_n is at most $\lfloor n/2 \rfloor$, i.e., $c_{kn} = 0$ if $k \le \lfloor (n+1)/2 \rfloor$. As a consequence of (7), the term corresponding to $n = 2M - 1$ in the outer sum is written as $O(G(x)x^{-M})$ in the last equality. The proof of (10) is now complete. □

Acknowledgements The author would like to thank the reviewer for his/her valuable comments.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1964)
2. Berndt, B.C.: Ramanujan's Notebooks, Part I. Springer, New York (1985)
3. Berndt, B.C.: Ramanujan's Notebooks, Part II. Springer, New York (1989)
4. David, F.N., Johnson, N.L.: Reciprocal Bernoulli and Poisson variables. Metron **18**, 77–81 (1956)
5. Evans, R.J.: Ramanujan's second notebook: asymptotic expansions for hypergeometric series and related functions. In: Proc. Ramanujan Centennial Conference. Academic Press, New York (1988)
6. Evans, R.J., Boersma, J., Blachman, N.M., Jagers, A.A.: The entropy of a Poisson distribution: problem 87-6. SIAM Rev. **30**, 314–317 (1988)
7. Flajolet, P.: Singularity analysis and asymptotics of Bernoulli sums. Theor. Comput. Sci. **215**, 371–381 (1999)
8. Grab, E.L., Savage, I.R.: Tables for the expected value of $1/x$ for positive Bernoulli and Poisson variables. J. Am. Stat. Assoc. **49**, 169–177 (1954)
9. Gut, A.: Probability: A Graduate Course. Springer, New York (2005)
10. Jacquet, P., Szpankowski, W.: Analytical depoissonization and its applications. Theor. Comput. Sci. **201**, 1–62 (1998)
11. Jacquet, P., Szpankowski, W.: Entropy computations via analytic depoissonization. IEEE Trans. Inf. Theory **45**, 1072–1081 (1999)
12. Johnson, N.L., Kemp, A.W., Kotz, S.: Univariate Discrete Distributions, 3rd edn. Wiley, Hoboken (2005)
13. Krichevskiy, R.E.: Laplace's law of succession and universal encoding. IEEE Trans. Inf. Theory **44**, 296–303 (1998)
14. Marciniak, E., Wesolowski, J.: Asymptotic Eulerian expansions for binomial and negative binomial reciprocals. Proc. Am. Math. Soc. **127**, 3329–3338 (1999)
15. Rempala, G.A.: Asymptotic factorial powers expansions for binomial and negative binomial reciprocals. Proc. Am. Math. Soc. **132**, 261–272 (2004)
16. Stephan, F.F.: The expected value and variance of the reciprocal and other negative powers of a positive Bernoullian variate. Ann. Math. Stat. **16**, 50–61 (1945)
17. Willink, R.: Relationships between central moments and cumulants, with formulae for the central moments of gamma distributions. Commun. Stat. Theory Methods **32**, 701–704 (2003)
18. Wuyungaowa and Wang, T.: Asymptotic expansions for inverse moments of binomial and negative binomial. Stat. Probab. Lett. **78**, 3018–3022 (2008)
19. Žnidarič, M.: Asymptotic expansion for inverse moments of binomial and Poisson distributions. Open Stat. Probab. J. **1**, 7–10 (2009)