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Upper estimates of transition densities for stable-dominated semigroups

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Abstract. We derive upper estimates of transition densities for Feller semigroups with jump intensities lighter than that of the rotation invariant stable Lévy process.

1. Introduction and preliminaries

Let $\alpha \in (0, 2)$ and $d = 1, 2, \ldots$. For the rotation invariant α-stable Lévy process on $\mathbb{R}^d$ with the Lévy measure

$$\nu(\mathrm{d}y) = \frac{c}{|y|^{\alpha+d}}\,\mathrm{d}y, \quad y \in \mathbb{R}^d \setminus \{0\}, \tag{1}$$

the asymptotic behavior of its transition densities $p(t, x, y)$ is well known (see, e.g., [2]), i.e.,

$$p(t, x, y) \approx \min\left(t^{-d/\alpha}, \frac{t}{|y-x|^{\alpha+d}}\right), \quad t > 0,\ x, y \in \mathbb{R}^d.$$

Estimates of densities for more general classes of stable and other jump Lévy processes gradually extended. We would like to mention some of the recent results. Estimates for general stable processes were obtained in [4,32] and for tempered and layered stable processes in [28] and [30]. Other estimates of Lévy and Lévy-type transition densities are discussed by Knopova and Kulik in [21], by Knopova and Schilling in [22] and by Jacob et al. in [20]. Estimates of heat kernels on metric measure spaces having the volume doubling property were obtained by Barlow et al. [1], Chen and Kumagai [7,8], and Grigor'yan et al. [10]. Upper estimates for heat kernels of symmetric jump processes with small jumps of high intensity were obtained by Mimica in [25]. In [24,27], the derivatives of stable densities have been considered, while bounds of heat kernels of the fractional Laplacian perturbed by gradient operators were studied by Bogdan and Jakubowski in [3].

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In [29], estimates of semigroups of stable-dominated Feller operators are given. The corresponding Markov process is a Feller process and not necessarily a Lévy process. The name *stable-dominated* refers to the fact that the intensity of jumps for the investigated semigroup is dominated by (1). In the present paper, we extend the results obtained in [29] and give estimates from above for a wider class of semigroups with the intensity of jumps lighter than stable processes. We will now describe our results.

Let $f : \mathbb{R}^d \times \mathbb{R}^d \mapsto [0, \infty]$ be a Borel function. We consider the following assumptions on f.

(A.1) There exists a constant $M > 0$ such that

$$f(x, y) \le M \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}, \quad x, y \in \mathbb{R}^d,\ y \ne x,$$

where $\phi : [0, \infty) \to (0, 1]$ is a Borel measurable function such that

(a) $\phi(a) = 1$ for $a \in [0, 1]$ and there is a constant $c_1 = c_1(\phi)$ such that

$$\phi(a) \le c_1 \phi(b), \quad |a-b| \le 1,$$

(b) $\phi \in C^2(1, \infty)$ and there is a constant $c_2 = c_2(\phi, \alpha, d)$ such that

$$\max\left(\left|\phi'(a)\right|, \left|\phi''(a)\right|\right) \le c_2 \phi(a)$$

for every $a > 1$.

(c) there is $c_3 = c_3(\phi, \alpha, d)$ such that

$$\int_{|x-z|\ge 1, |y-z|\ge 1} \frac{\phi(|y-z|)}{|y-z|^{\alpha+d}} \frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} \, dz \le c_3 \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}},$$

for every $|x-y| > 2$.

(A.2) $f(x, x+h) = f(x, x-h)$ for all $x, h \in \mathbb{R}^d$ if $\alpha \ge 1$.

(A.3) $f(x, y) = f(y, x)$ for all $x, y \in \mathbb{R}^d$.

(A.4) There exists a constant $c_4 = c_4(\phi, \alpha, d)$ such that

$$\inf_{x \in \mathbb{R}^d} \int_{|y-x|>\varepsilon} \frac{f(x, y)}{\phi(|y-x|)} \, dy \ge c_4 \varepsilon^{-\alpha}, \quad \varepsilon > 0.$$

Denote

$$b_\varepsilon(x) = \int_{|y-x|>\varepsilon} f(x, y) \, dy, \quad \varepsilon > 0,\ x \in \mathbb{R}^d.$$

It follows from (A.1) that there is also the constant $c_5 = c_5(\phi, M, \alpha, d)$ such that

$$\bar{b}_\varepsilon := \sup_{x \in \mathbb{R}^d} b_\varepsilon(x) \le c_5 \varepsilon^{-\alpha}, \quad 0 < \varepsilon \le 1.$$

Thus, (A.4) is a partial converse of (A.1) and we have

$$\underline{b}_\varepsilon := \inf_{x \in \mathbb{R}^d} b_\varepsilon(x) \ge c_6 \varepsilon^{-\alpha}, \quad 0 < \varepsilon \le \varepsilon_0,$$

for constants $\varepsilon_0 = \varepsilon_0(\phi, M, \alpha, d)$, $c_6 = c_6(\phi, M, \alpha, d)$, since

$$\begin{aligned} b_\varepsilon(x) &= \int_{|y-x|>\varepsilon} \frac{f(x,y)}{\phi(|y-x|)}\,\mathrm{d}y - \int_{|y-x|>1} \frac{f(x,y)}{\phi(|y-x|)}(1-\phi(|y-x|))\,\mathrm{d}y \\ &\geq c_4\varepsilon^{-\alpha} - M\int_{|y-x|>1} |y-x|^{-\alpha-d}\,\mathrm{d}y \\ &= c_4\varepsilon^{-\alpha} - Mc(\alpha,d) \geq c_6\varepsilon^{-\alpha}, \end{aligned}$$

provided $\varepsilon^\alpha \leq \frac{c_4 - c_6}{Mc(\alpha,d)}$.

We note that the assumption (A.1)(c) is satisfied for every nonincreasing function $\phi : (0,\infty) \to (0,1]$ such that

$$\phi(a)\phi(b) \leq c\,\phi(a+b), \quad a, b > 1,$$

for some positive constant c. Therefore, it is easy to verify that all the assumptions on ϕ are satisfied, e.g., for functions $\phi(s) = \mathrm{e}^{(1-s^\beta)} \wedge 1$, where $\beta \in (0,1]$, $\phi(s) = (1 \vee s)^{-\gamma}$, where $\gamma > 0$, $\phi(s) = 1/\log(e(s \vee 1))$, $\phi(s) = 1/\log\log(e^e(s \vee 1))$, and all their products and positive powers.

It is also reasonable to ask whether the conditions in the assumption (A.1) are satisfied by more general functions of the form

$$\phi(s) = \begin{cases} 1 & \text{if } s \in [0,1], \\ \mathrm{e}^{-ms^\beta} s^\gamma & \text{if } s > 1, \text{ with } m, \beta > 0, \gamma \in \mathbb{R}. \end{cases} \tag{2}$$

In this case, both conditions (a) and (b) on ϕ hold for $\beta \in (0,1]$ with no further restrictions on parameters m and γ, while, as proven in Sect. 3, the condition (c) is satisfied when $\beta \in (0,1]$ and $\gamma < d/2 + \alpha - 1/2$. Furthermore, this restriction on parameters is essential (see Remark 1 in Section 3). Note also that this range of β and γ in (2) covers, e.g., jump intensities dominated by those of isotropic relativistic stable processes (see, e.g., [23, Lemma 2.3]).

For $x \in \mathbb{R}^d$ and $r > 0$, we let $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$. $B_b(\mathbb{R}^d)$ denotes the set of bounded Borel measurable functions, $C_c^k(\mathbb{R}^d)$ denotes the set of k times continuously differentiable functions with compact support, and $C_\infty(\mathbb{R}^d)$ is the set of continuous functions vanishing at infinity. We use c, C (with subscripts) to denote finite positive constants which depend only on ϕ, M, α, and the dimension d. Any *additional* dependence is explicitly indicated by writing, e.g., $c = c(n)$. The value of c, C, when used without subscripts, may change from place to place. We write $f(x) \approx g(x)$ to indicate that there is a constant c such that $c^{-1}f(x) \leq g(x) \leq cf(x)$.

Under the assumptions (A.1) and (A.2), we may consider the operator

$$\begin{aligned} \mathcal{A}\varphi(x) &= \lim_{\varepsilon\downarrow 0} \int_{|y-x|>\varepsilon} (\varphi(y) - \varphi(x))\,f(x,y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left(\varphi(x+h) - \varphi(x) - h\cdot\nabla\varphi(x)\mathbf{1}_{|h|<1}\right) f(x,x+h)\,\mathrm{d}h \\ &\quad + \frac{1}{2}\int_{|h|<1} h\cdot\nabla\varphi(x)\,(f(x,x+h) - f(x,x-h))\,\mathrm{d}h, \quad \varphi \in C_c^2(\mathbb{R}^d). \end{aligned}$$

Recall the following basic fact (see [29, Lemma 1]).

LEMMA 1. *If (A.1) and (A.2) hold and the function* $x \to f(x, y)$ *is continuous on* $\mathbb{R}^d \setminus \{y\}$ *for every* $y \in \mathbb{R}^d$*, then* $\mathcal{A}$ *maps* $C_c^2(\mathbb{R}^d)$ *into* $C_\infty(\mathbb{R}^d)$.

In the following, we always assume that the condition (A.1) is satisfied. For every $\varepsilon > 0$, we denote

$$f_\varepsilon(x, y) = \mathbf{1}_{B(0,\varepsilon)^c}(y - x) f(x, y), \quad x, y \in \mathbb{R}^d,$$

and

$$\mathcal{A}_\varepsilon \varphi(x) = \int (\varphi(y) - \varphi(x))\, f_\varepsilon(x, y)\, \mathrm{d}y, \quad \varphi \in B_b(\mathbb{R}^d).$$

Note that the operators $\mathcal{A}_\varepsilon$ are bounded since $|\mathcal{A}_\varepsilon \varphi(x)| \leq 2\|\varphi\|_\infty b_\varepsilon(x) \leq 2\bar{b}_\varepsilon \|\varphi\|_\infty$. Therefore, the operator

$$\mathrm{e}^{t\mathcal{A}_\varepsilon} = \sum_{n=0}^{\infty} \frac{t^n \mathcal{A}_\varepsilon^n}{n!}, \quad t \geq 0,\ \varepsilon > 0,$$

is well defined and bounded from $B_b(\mathbb{R}^d)$ to $B_b(\mathbb{R}^d)$. In fact, for every $\varepsilon > 0$, the family of operators $\{\mathrm{e}^{t\mathcal{A}_\varepsilon},\ t \leq 0\}$ is a semigroup on $B_b(\mathbb{R}^d)$, i.e., $\mathrm{e}^{(t+s)\mathcal{A}_\varepsilon} = \mathrm{e}^{t\mathcal{A}_\varepsilon}\mathrm{e}^{s\mathcal{A}_\varepsilon}$ for all $t, s \geq 0$, $\varphi \in B_b(\mathbb{R}^d)$. We note that $\mathrm{e}^{t\mathcal{A}_\varepsilon}$ is positive for all $t \geq 0$, $\varepsilon > 0$ (see (5)).

Our first result is the following theorem.

THEOREM 1. *If (A.1)–(A.4) are satisfied, then there exists the constants* C_1 *and* C_2 *such that for every nonnegative* $\varphi \in B_b(\mathbb{R}^d)$ *and* $\varepsilon \in (0, \varepsilon_0 \wedge 1)$*, we have*

$$\mathrm{e}^{t\mathcal{A}_\varepsilon}\varphi(x) \leq C_1 \mathrm{e}^{C_2 t} \int \varphi(y) \min\left(t^{-d/\alpha}, \frac{t\phi(|y - x|)}{|y - x|^{\alpha+d}}\right) \mathrm{d}y + \mathrm{e}^{-tb_\varepsilon(x)}\varphi(x),$$

for every $x \in \mathbb{R}^d$.

The proof of Theorem 1 is given in Sect. 2. To study a limiting semigroup, we will need some additional assumptions.

(A.5) The function $x \to f(x, y)$ is continuous on $\mathbb{R}^d \setminus \{y\}$ for every $y \in \mathbb{R}^d$.

(A.6) $\mathcal{A}$ regarded as an operator on $C_\infty(\mathbb{R}^d)$ is closable and its closure $\bar{\mathcal{A}}$ is a generator of a strongly continuous contraction semigroup of operators $\{P_t,\ t \geq 0\}$ on $C_\infty(\mathbb{R}^d)$.

Clearly, for every $\varphi \in C_c^2(\mathbb{R}^d)$ with $\sup_{x \in \mathbb{R}^d} \varphi(x) = \varphi(x_0) \geq 0$, we have $\mathcal{A}\varphi(x_0) \leq 0$, i.e., $\mathcal{A}$ satisfies *the positive maximum principle*. This implies that all P_t $(t \geq 0)$ are positive operators (see [9, Theorems 1.2.12 and 4.2.2]). Thus, by our assumptions, $\{P_t,\ t \geq 0\}$ is *a Feller semigroup*.

The following theorem is our main result.

THEOREM 2. *If (A.1)–(A.6) hold, then there is* $p : (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ *such that*

$$P_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)p(t,x,y)\,dy, \quad x\in\mathbb{R}^d,\ t>0,\ \varphi\in C_\infty(\mathbb{R}^d),$$

and

$$p(t,x,y) \le C_1 e^{C_2 t}\min\left(t^{-d/\alpha}, \frac{t\phi(|y-x|)}{|y-x|^{\alpha+d}}\right), \quad x,y\in\mathbb{R}^d,\ t>0. \tag{3}$$

We note that $\mathcal{A}$ is conservative, i.e., for $\varphi\in C_c^\infty(\mathbb{R}^d)$ such that $0\le\varphi\le 1$, $\varphi(0)=1$, and $\varphi_k(x)=\varphi(x/k)$, we have $\sup_{k\in\mathbb{N}}\|\mathcal{A}\varphi_k\|_\infty<\infty$, and $\lim_{k\to\infty}(\mathcal{A}\varphi_k)(x)=0$, for every $x\in\mathbb{R}^d$. It follows from Theorem 4.2.7 in [9] that there exists a Markov process $\{X_t,\ t\ge 0\}$ such that $\mathbb{E}[\varphi(X_t)|X_0=x]=P_t\varphi(x)$.

It is known that every generator G of a Feller semigroup with $C_c^\infty(\mathbb{R}^d)\subset\mathcal{D}(G)$ is necessarily of the form

$$\begin{aligned} G\varphi(x) = {} & \sum_{i,j=1}^{d} q_{ij}(x)D_{x_i}D_{x_j}\varphi(x) + l(x)\nabla\varphi(x) - c(x)\varphi(x) \\ & + \int_{\mathbb{R}^d}\left(\varphi(x+h)-\varphi(x)-h\cdot\nabla\varphi(x)\,\mathbf{1}_{|h|<1}\right)\,\nu(x,\mathrm{d}h), \end{aligned} \tag{4}$$

where $\varphi\in C_c^\infty(\mathbb{R}^d)$, $q(x)=(q_{ij}(x))_{i,j=1}^n$ is a nonnegative definite real symmetric matrix, the vector $l(x)=(l_i(x))_{i=1}^d$ has real coordinates, $c(x)\ge 0$, and $\nu(x,\cdot)$ is a Lévy measure (see [17, Chapter 4.5]).

The converse problem whether a given operator G generates a Feller semigroup is not completely resolved yet. For the interested reader, we remark that criteria are given, e.g., in [13–16,18]. Generally, smoothness of the coefficients q, l, c, ν in (4) is sufficient for the existence (see Theorem 5.24 in [12], Theorem 4.6.7 in [19] and Lemma 2 in [29]). Other conditions are given also in [26].

Chen et al. [6] and Chen and Kumagai [7,8] investigate the case of symmetric jump-type Markov processes on metric measure spaces by using Dirichlet forms. Under the assumption that the corresponding jump kernels are *comparable* with certain rotation invariant functions, they prove the existence and obtain estimates of the densities (see Theorem 1.2 in [6]) analogous to (3). In the present paper, we propose completely different approach which is based on general approximation scheme recently devised in [29]. In Theorem 2, we assume the estimate (A.1) from above but we use (A.4) as the only estimate for the size of f from below. We also emphasize that we obtain exactly $\phi(|x-y|)$ in (3) and from [8,6] follow estimates with $\phi(c|x-y|)$ for some constant $c\in(0,1)$. This seems to be essential especially in the case of exponentially localized Lévy measures. Our general framework, including a layout of lemmas, is similar to that in [29]. However, in the present case, the decay of the jump intensity may be significantly lighter than stable, and therefore, much more subtle argument is needed. Note that the new condition (A.1)(c), which is pivotal for our further investigations, is necessary for the two-sided sharp bounds similar to the right-hand side of (3).

2. Approximation

In this section, we apply an approximation scheme recently devised in [29] (we note that an alternative approximation scheme is given in [5]). We recall that

$$f_\varepsilon(x, y) = \mathbf{1}_{B(0,\varepsilon)^c}(y - x) f(x, y), \quad \varepsilon > 0, x, y \in \mathbb{R}^d,$$

and

$$b_\varepsilon(x) = \int_{|y-x|>\varepsilon} f(x, y)\, \mathrm{d}y = \int f_\varepsilon(x, y)\, \mathrm{d}y, \quad \varepsilon > 0,\ x \in \mathbb{R}^d.$$

We have

$$\begin{aligned} \mathcal{A}_\varepsilon\varphi(x) &= \int (\varphi(y) - \varphi(x))\, f_\varepsilon(x, y)\, \mathrm{d}y + (\bar{b}_\varepsilon - b_\varepsilon(x)) \int (\varphi(y) - \varphi(x))\delta_x(\mathrm{d}y) \\ &= \int (\varphi(y) - \varphi(x))\tilde{\nu}_\varepsilon(x, \mathrm{d}y) \\ &= \Gamma_\varepsilon\varphi(x) - \bar{b}_\varepsilon\varphi(x), \quad \varphi \in B_b(\mathbb{R}^d),\ x \in \mathbb{R}^d, \end{aligned}$$

where

$$\tilde{\nu}_\varepsilon(x, \mathrm{d}y) = f_\varepsilon(x, y)\, \mathrm{d}y + (\bar{b}_\varepsilon - b_\varepsilon(x))\delta_x(\mathrm{d}y),$$

and

$$\Gamma_\varepsilon\varphi(x) = \int \varphi(y)\tilde{\nu}_\varepsilon(x, \mathrm{d}y), \quad \varphi \in B_b(\mathbb{R}^d), x \in \mathbb{R}^d.$$

This yields that

$$\mathrm{e}^{t\mathcal{A}_\varepsilon}\varphi(x) = \mathrm{e}^{t(\Gamma_\varepsilon - \bar{b}_\varepsilon I)}\varphi(x) = \mathrm{e}^{-t\bar{b}_\varepsilon}\mathrm{e}^{t\Gamma_\varepsilon}\varphi(x). \tag{5}$$

A consequence of (5) is that we may consider the operator Γ_ε and its powers instead of $\mathcal{A}_\varepsilon$. The fact that Γ_ε is positive enables for more precise estimates.

For $n \in \mathbb{N}$, we define

$$\begin{aligned} f_{n+1,\varepsilon}(x, y) = &\int f_{n,\varepsilon}(x, z) f_\varepsilon(z, y)\, \mathrm{d}z \\ &+ \left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) f_{n,\varepsilon}(x, y) + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n f_\varepsilon(x, y), \end{aligned}$$

where we let $f_{1,\varepsilon} = f_\varepsilon$. By induction and Fubini–Tonelli theorem, we get

$$\int f_{n,\varepsilon}(x, y)\, \mathrm{d}y = \bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n, \quad x \in \mathbb{R}^d,\ n \in \mathbb{N}. \tag{6}$$

Also, it was proved in [29, Lemma 3] that for all $\varepsilon > 0$, $x \in \mathbb{R}^d$, and $n \in \mathbb{N}$

$$\Gamma_\varepsilon^n\varphi(x) = \int \varphi(z) f_{n,\varepsilon}(x, z)\, \mathrm{d}z + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n \varphi(x), \tag{7}$$

whenever $\varphi \in B_b(\mathbb{R}^d)$.

The next lemma is crucial for our further investigation. It is essential that we obtain precisely the constants equal to one before $b_\varepsilon(y)$ below.

LEMMA 2. *We have the following.*

(1) *If (A.1), (A.2) and (A.4) hold then there is a constant* $c_7 = c_7(\phi, M, \alpha, d)$ *and the number* $\kappa \in (0, 1)$ *such that*

$$\int_{B(y,\kappa|y-x|)} |z-x|^{-\alpha-d} f_\varepsilon(y,z)\mathrm{d}z \le (b_\varepsilon(y) + c_7)\,|y-x|^{-\alpha-d},$$

for every $\varepsilon \in (0, 1)$ *and for every* $x, y \in \mathbb{R}^d$.

(2) *If (A.1) and (A.2) hold then there is a constant* $c_8 = c_8(\phi, M, \alpha, d)$ *such that*

$$\int_{B(y,1)} \frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} f_\varepsilon(y,z)\mathrm{d}z \le (b_\varepsilon(y) + c_8)\,\frac{\phi(|y-x|)}{|y-x|^{\alpha+d}},$$

for every $\varepsilon \in (0, 1)$ *and for every* $|x-y| > 2$.

Proof. First, we prove the statement (1). We have

$$\begin{aligned}
&\int_{B(y,\kappa|y-x|)} |z-x|^{-\alpha-d} f_\varepsilon(y,z)\mathrm{d}z \\
&\quad = \int_{B(y,\kappa|y-x|)} \left[|z-x|^{-\alpha-d} - |y-x|^{-\alpha-d}\right] f_\varepsilon(y,z)\mathrm{d}z \\
&\qquad + |y-x|^{-\alpha-d} \int_{B(y,\kappa|y-x|)} f_\varepsilon(y,z)\mathrm{d}z.
\end{aligned}$$

We only need to estimate the first integral on the right-hand side of the above equality. Denote $\theta(z) := |z-x|^{-\alpha-d}$, $|z-x| > 0$.

$$\partial_j \theta(z) = (\alpha+d)|z-x|^{-\alpha-d-2}(x_j - z_j),$$

and

$$\partial_{j,k}\theta(z) = (\alpha+d)|z-x|^{-\alpha-d-2}\left[(\alpha+d+2)\frac{(x_j-z_j)(x_k-z_k)}{|x-z|^2} - \delta_{jk}\right].$$

This yields

$$\sup_{\substack{z\in B(y,\kappa|y-x|),\\ j\in\{1,\dots,d\}}} |\partial_j\theta(z)| \le (\alpha+d)(1-\kappa)^{-\alpha-d-1}|y-x|^{-\alpha-d-1}, \tag{8}$$

and

$$\sup_{\substack{z\in B(y,\kappa|y-x|),\\ j,k\in\{1,\dots,d\}}} |\partial_{j,k}\theta(z)| \le (\alpha+d)(\alpha+d+3)(1-\kappa)^{-\alpha-d-2}|y-x|^{-\alpha-d-2}, \tag{9}$$

for every $\kappa \in (0,1)$. We consider now two cases. Let first $\alpha < 1$. Using the Taylor expansion for θ, (8) and (A.1), we get

$$\begin{aligned} &\left| \int_{B(y,\kappa|y-x|)} \left[|z-x|^{-\alpha-d} - |y-x|^{-\alpha-d} \right] f_\varepsilon(y,z)\,\mathrm{d}z \right| \\ &\quad = \left| \int_{B(0,\kappa|y-x|)} (\theta(y+h)-\theta(y))\, f_\varepsilon(y,y+h)\,\mathrm{d}h \right| \\ &\quad \le C(1-\kappa)^{-\alpha-d-1}|y-x|^{-\alpha-d-1} \int_{B(0,\kappa|y-x|)} |h| f_\varepsilon(y,y+h)\,\mathrm{d}h \\ &\quad \le C\kappa^{1-\alpha}(1-\kappa)^{-\alpha-d-1}|y-x|^{-\alpha-d}|y-x|^{-\alpha}. \end{aligned}$$

Let now $\alpha \ge 1$. Again, using the Taylor expansion for θ, (9), (A.1) and (A.2), we obtain

$$\begin{aligned} &\left| \int_{B(y,\kappa|y-x|)} \left[|z-x|^{-\alpha-d} - |y-x|^{-\alpha-d} \right] f_\varepsilon(y,z)\,\mathrm{d}z \right| \\ &\quad = \left| \int_{B(0,\kappa|y-x|)} (\theta(y+h)-\theta(y)-\nabla\theta(y)\cdot h)\, f_\varepsilon(y,y+h)\,\mathrm{d}h \right| \\ &\quad \le C(1-\kappa)^{-\alpha-d-2}|y-x|^{-\alpha-d-2} \int_{B(0,\kappa|y-x|)} |h|^2 f_\varepsilon(y,y+h)\,\mathrm{d}h \\ &\quad \le C\kappa^{2-\alpha}(1-\kappa)^{-\alpha-d-2}|y-x|^{-\alpha-d}|y-x|^{-\alpha}. \end{aligned}$$

We thus see that by the two above estimates and by (A.4), we finally have

$$\begin{aligned} &\left| \int_{B(y,\kappa|y-x|)} \left[|z-x|^{-\alpha-d} - |y-x|^{-\alpha-d} \right] f_\varepsilon(y,z)\,\mathrm{d}z \right| \\ &\quad \le |y-x|^{-\alpha-d} \left(\mathbf{1}_{\{|y-x|<\varepsilon_0\}} \int_{|z-y|>\kappa|y-x|} f(y,z)\,\mathrm{d}z + c_7 \mathbf{1}_{\{|y-x|\ge\varepsilon_0\}} \right), \end{aligned}$$

for sufficiently small $\kappa \in (0,1)$. This ends the proof of (1).

We now show the statement (2). Let $|x-y| > 2$. Similarly as before, we have

$$\begin{aligned} \int_{B(y,1)} \frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} f_\varepsilon(y,z)\,\mathrm{d}z = &\int_{B(y,1)} \left[\frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} - \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}} \right] f_\varepsilon(y,z)\mathrm{d}z \\ &+ \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}} \int_{B(y,1)} f_\varepsilon(y,z)\mathrm{d}z. \end{aligned}$$

Observe that it is enough to estimate the first integral on the right-hand side of the above-displayed equality. Denote $\eta(z) := \phi(|z-x|)|z-x|^{-\alpha-d}$. Clearly, by (A.1) (a)–(b), we have

$$\max\left(\sup_{\substack{z\in B(y,1),\\ j\in\{1,\ldots,d\}}} |\partial_j \eta(z)|, \sup_{\substack{z\in B(y,1),\\ j,k\in\{1,\ldots,d\}}} |\partial_{j,k}\eta(z)| \right) \le C\eta(y). \qquad (10)$$

Using the Taylor expansion for η, (10), (A.1) and (A.2), we obtain

$$\begin{aligned}
&\left|\int_{B(y,1)} \left[\frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} - \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}\right] f_\varepsilon(y,z)\,\mathrm{d}z\right| \\
&= \left|\int_{B(0,1)} (\eta(y+h)-\eta(y))\, f_\varepsilon(y,y+h)\,\mathrm{d}h\right| \\
&\leq \left|\int_{B(0,1)} (\eta(y+h)-\eta(y)-\nabla\eta(y)\cdot h)\, f_\varepsilon(y,y+h)\,\mathrm{d}h\right| \\
&\quad + \left|\int_{B(0,1)} \nabla\eta(y)\cdot h\, \frac{f_\varepsilon(y,y+h)-f_\varepsilon(y,y-h)}{2}\,\mathrm{d}h\right| \\
&\leq c_8\eta(y).
\end{aligned}$$

which ends the proof.

□

We now obtain the estimates of $f_{n,\varepsilon}(x,y)$. Our argument in the proof of the following lemma shows the significance of assumptions on the dominating function ϕ.

LEMMA 3. *If (A.1)–(A.4) hold then:*

(1) there exists a constant $c_9 = c_9(\phi, M, \alpha, d)$ *such that*

$$f_{n,\varepsilon}(x,y) \leq c_9 n \left(\bar{b}_\varepsilon + c_7\right)^{n-1} |y-x|^{-\alpha-d},$$

for every $x, y \in \mathbb{R}^d$, $\varepsilon \in (0,1)$, $n \in \mathbb{N}$,

(2) there exists the constants $c_{10} = c_{10}(\phi, M, \alpha, d)$ *and* $c_{11} = c_{11}(\phi, M, \alpha, d)$ *such that*

$$f_{n,\varepsilon}(x,y) \leq c_{10} n \left(\bar{b}_\varepsilon + c_{11}\right)^{n-1} \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}},$$

for every $x, y \in \mathbb{R}^d$, $\varepsilon \in (0,1)$, $n \in \mathbb{N}$.

Proof. We use induction. Clearly, for $n = 1$, both inequalities hold with constants $c_9 = M$, $c_{10} = M$ (and an arbitrary positive c_{11}), respectively. Consider first the inequality in (1). We will prove that it holds with constant $c_9 = M\kappa^{-\alpha-d}$, where $\kappa \in (0,1)$ is the number from previous lemma.

Let

$$\int f_{n,\varepsilon}(x,z) f_\varepsilon(z,y)\mathrm{d}z = \int_{B(y,\kappa|y-x|)^c} + \int_{B(y,\kappa|y-x|)} = I + II.$$

By (A.1) (a) and (6), we have

$$\begin{aligned}
I &\leq \kappa^{-\alpha-d} M\, |y-x|^{-\alpha-d} \int f_{n,\varepsilon}(x,z)\mathrm{d}z \\
&= \kappa^{-\alpha-d} M\, |y-x|^{-\alpha-d} \left[\bar{b}_\varepsilon^n - (\bar{b}_\varepsilon - b_\varepsilon(x))^n\right].
\end{aligned}$$

By symmetry of f (see (A.3)), induction, and Lemma 2 (1), we also have

$$II \le c_9 n(\bar{b}_\varepsilon + c_7)^{n-1} \int_{B(y,\kappa|y-x|)} |x-z|^{-\alpha-d} f_\varepsilon(y,z)\mathrm{d}z$$
$$\le c_9 n(\bar{b}_\varepsilon + c_7)^{n-1}(b_\varepsilon(y)+c_7)|x-y|^{-\alpha-d}.$$

We get

$$\begin{aligned} f_{n+1,\varepsilon}(x,y) &= I + II + \left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) f_{n,\varepsilon}(x,y) + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n f_\varepsilon(x,y) \\ &\le M\kappa^{-\alpha-d}\left[\bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n\right]|y-x|^{-\alpha-d} \\ &\quad + c_9 n(\bar{b}_\varepsilon + c_7)^{n-1}(b_\varepsilon(y)+c_7)|x-y|^{-\alpha-d} \\ &\quad + \left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) c_9 n(\bar{b}_\varepsilon + c_7)^{n-1}|x-y|^{-\alpha-d} \\ &\quad + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n M|x-y|^{-\alpha-d} \\ &\le c_9(n+1)(\bar{b}_\varepsilon + c_7)^n|x-y|^{-\alpha-d}, \end{aligned}$$

which ends the proof of part (1).

We now complete the proof of the inequality in (2). We will prove that it holds with constants $c_{10} = c_1 \max(c_9, 2^{\alpha+d}M)$ and $c_{11} = \max(c_7, c_8 + Mc_3)$. When $|x-y| \le 2$, then it directly follows from the part (1) and (A.1)(a). Assume now that $|x-y| > 2$. We have

$$\int f_{n,\varepsilon}(x,z) f_\varepsilon(z,y)\mathrm{d}z = \int_{B(x,1)} + \int_{B(x,1)^c} = I + II.$$

By (A.1) (a) and (6), we get

$$I \le 2^{\alpha+d} M c_1 \frac{\phi(|x-y|)}{|y-x|^{\alpha+d}} \int f_{n,\varepsilon}(x,z)\mathrm{d}z$$
$$= 2^{\alpha+d} M c_1 \frac{\phi(|x-y|)}{|y-x|^{\alpha+d}} \left[\bar{b}_\varepsilon^n - (\bar{b}_\varepsilon - b_\varepsilon(x))^n\right].$$

By symmetry of f (see (A.3)), induction, and Lemma 2 (2) and (A.1) (c), we also have

$$II \le c_{10} n(\bar{b}_\varepsilon + c_{11})^{n-1} \int_{B(x,1)^c} \frac{\phi(|x-z|)}{|x-z|^{\alpha+d}} f_\varepsilon(y,z)\mathrm{d}z$$
$$\le c_{10} n(\bar{b}_\varepsilon + c_{11})^{n-1}(b_\varepsilon(y) + c_8 + Mc_3)\frac{\phi(|x-y|)}{|x-y|^{\alpha+d}}.$$

We get

$$\begin{aligned} f_{n+1,\varepsilon}(x,y) &= I + II + \left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) f_{n,\varepsilon}(x,y) + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n f_\varepsilon(x,y) \\ &\le 2^{\alpha+d} M c_1 \left[\bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n\right] \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}} \\ &\quad + c_{10} n(\bar{b}_\varepsilon + c_{11})^{n-1}(b_\varepsilon(y) + c_8 + Mc_3)\frac{\phi(|x-y|)}{|x-y|^{\alpha+d}} \end{aligned}$$

$$
\begin{aligned}
&+\left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) c_{10} n (\bar{b}_\varepsilon + c_{11})^{n-1} \frac{\phi(|x-y|)}{|x-y|^{\alpha+d}} \\
&+\left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n M \frac{\phi(|x-y|)}{|x-y|^{\alpha+d}} \\
&\le c_{10}(n+1)(\bar{b}_\varepsilon + c_{11})^n \frac{\phi(|x-y|)}{|x-y|^{\alpha+d}}.
\end{aligned}
$$

□

LEMMA 4. *Assume (A.1), (A.3), and (A.4). Then, there exists* $c_{12} = c_{12}(\phi, M, \alpha, d)$ *such that*

$$
f_{n,\varepsilon}(x,y) \le c_{12} \bar{b}_\varepsilon^{d/\alpha} \left(\bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n\right), \quad x, y \in \mathbb{R}^d,\ \varepsilon \in (0, \varepsilon_0),\ n \in \mathbb{N}. \tag{11}
$$

Proof. For $n = 1$ by (A.1) and (A.4), we have

$$
f_\varepsilon(x,y) \le \frac{M}{\varepsilon^{\alpha+d}} \le M \left(\frac{b_\varepsilon(x)}{c_6}\right)^{(\alpha+d)/\alpha} \le M \left(\frac{b_\varepsilon(x)}{c_6}\right)\left(\frac{\bar{b}_\varepsilon}{c_6}\right)^{d/\alpha},
$$

and so (11) holds with $c_{12} = M\, c_6{}^{-d/\alpha-1}$. Let (11) holds for some $n \in \mathbb{N}$ with $c_{12} = M\, c_6{}^{-d/\alpha-1}$. By induction and the symmetry of f_ε, we get

$$
\begin{aligned}
f_{n+1,\varepsilon}(x,y) &\le c_{12} \bar{b}_\varepsilon^{d/\alpha} \left(\bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n\right) \left(\int f_\varepsilon(y,z)\,\mathrm{d}z + \bar{b}_\varepsilon - b_\varepsilon(y)\right) \\
&\quad + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n c_{12} \bar{b}_\varepsilon^{d/\alpha} b_\varepsilon(x) \\
&= c_{12} (\bar{b}_\varepsilon)^{d/\alpha} \left(\bar{b}_\varepsilon^{n+1} - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^{n+1}\right).
\end{aligned}
$$

□

In the following lemma, we will need some additional notations. For a function g, we denote $b_\varepsilon^g(x) := \int_{|y-x|>\varepsilon} g(|y-x|) f_\varepsilon(x,y)\mathrm{d}y$ and $\bar{b}_\varepsilon^g = \sup_{x\in\mathbb{R}^d} b_\varepsilon^g(x)$. We note that it follows from (A.1) that

$$
\bar{b}_\varepsilon^{\frac{1}{\phi}} \le c_{13} \varepsilon^{-\alpha}.
$$

LEMMA 5. *If (A.1), (A.3), and (A.4) are satisfied, then there exists* $c_{14} = c_{14}(\phi, M, \alpha, d)$ *and* $c_{15} = c_{15}(\phi, \alpha, d)$ *such that*

$$
f_{n,\varepsilon}(x,y) \le c_{14} \left(\bar{b}_\varepsilon + c_{15}\right)^{n+d/\alpha} n^{-d/\alpha}, \quad x, y \in \mathbb{R}^d,\ \varepsilon \in (0, \varepsilon_0 \wedge 1),\ n \in \mathbb{N}. \tag{12}
$$

Proof. We may choose $n_0 \in \mathbb{N}$ such that

$$
(1 - c_6/c_5)^n (n+1)^{d/\alpha} < \frac{1}{n+1} \tag{13}
$$

for every $n \geq n_0$. For $n \leq n_0$ by Lemma 4, we have

$$f_{n,\varepsilon}(x, y) \leq c_{12}\bar{b}_\varepsilon^{d/\alpha}\bar{b}_\varepsilon^{n} \leq c_{12}\bar{b}_\varepsilon^{n+d/\alpha}n^{-d/\alpha}n_0^{d/\alpha},$$

which yields the inequality (12) with $c_{14} = c_{12}n_0^{d/\alpha}$ in this case. For $n \geq n_0$, we use induction. We assume that (12) holds for some $n \geq n_0$ with $c_{14} = \max(c_{12}n_0^{d/\alpha}, M\,\eta^{-\alpha-d}c_6{}^{-1-d/\alpha})$ and $c_{15} = \bar{b}_1^{\frac{1}{\phi}-1}$, where

$$p = \frac{d\,2^{\max(d/\alpha,1)-1}}{\alpha}, \quad \text{and} \quad \eta = \left(\frac{c_4^2/(c_{13}(c_5+c_{15}))}{2+2p}\right)^{\frac{1}{\alpha}}.$$

We have

$$\int f_{n,\varepsilon}(x,z)f_\varepsilon(z,y)\,\mathrm{d}z = \int_{B(y,\eta\varepsilon(n+1)^{1/\alpha})^c} + \int_{B(y,\eta\varepsilon(n+1)^{1/\alpha})} = I + II.$$

By (A.1), (A.4), and (6), we get

$$\begin{aligned}
I &= \int_{B(y,\eta\varepsilon(n+1)^{1/\alpha})^c} f_{n,\varepsilon}(x,z)f_\varepsilon(z,y)\,\mathrm{d}z\\
&\leq M\int_{B(y,\eta\varepsilon(n+1)^{1/\alpha})^c} f_{n,\varepsilon}(x,z)|y-z|^{-\alpha-d}\,\mathrm{d}z\\
&\leq M\,\eta^{-\alpha-d}\varepsilon^{-\alpha-d}(n+1)^{-1-d/\alpha}\int f_{n,\varepsilon}(x,z)\,\mathrm{d}z\\
&\leq M\,\eta^{-\alpha-d}c_6{}^{-1-d/\alpha}\bar{b}_\varepsilon^{1+d/\alpha}(n+1)^{-1-d/\alpha}\left[\bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n\right].
\end{aligned}$$

By induction, the symmetry of f_ε and (A.4), we obtain

$$\begin{aligned}
II &= \int_{B(y,\eta\varepsilon(n+1)^{1/\alpha})} f_{n,\varepsilon}(x,z)f_\varepsilon(z,y)\,\mathrm{d}z\\
&\leq c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+d/\alpha} n^{-d/\alpha}\int_{B(y,\eta\varepsilon(n+1)^{1/\alpha})}\frac{f_\varepsilon(y,z)}{\phi(|z-y|)}\,\mathrm{d}z\\
&= c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+d/\alpha} n^{-d/\alpha}\left(b_\varepsilon^{\frac{1}{\phi}}(y) - b_{\eta\varepsilon(n+1)^{1/\alpha}}^{\frac{1}{\phi}}(y)\right)\\
&\leq c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+d/\alpha} n^{-d/\alpha}b_\varepsilon^{\frac{1}{\phi}}(y)\left(1 - \frac{c_4\eta^{-\alpha}}{c_{13}(n+1)}\right).
\end{aligned}$$

By (13), we also have

$$\left(1 - \frac{b_\varepsilon(x)}{\bar{b}_\varepsilon}\right)^n (n+1)^{d/\alpha} \leq (1 - c_6/c_5)^n(n+1)^{d/\alpha} \leq \frac{1}{n+1}. \tag{14}$$

Using the fact that $\phi(a) = 1$ for $a \in [0, 1]$ and $b_\varepsilon^{\frac{1}{\phi}}(y) - b_\varepsilon(y) = b_1^{\frac{1}{\phi}-1}(y) \le c_{15}$, we get

$$\begin{aligned} f_{n+1,\varepsilon}(x, y) &= I + II + \left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) f_{n,\varepsilon}(x, y) + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n f_\varepsilon(x, y) \\ &\le c_{14}\bar{b}_\varepsilon^{1+d/\alpha}(n+1)^{-1-d/\alpha}\left[\bar{b}_\varepsilon^n - \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n\right] \\ &\quad + c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+d/\alpha} n^{-d/\alpha} b_\varepsilon^{\frac{1}{\phi}}(y)\left(1 - \frac{c_4\eta^{-\alpha}}{c_{13}(n+1)}\right) \\ &\quad + c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+d/\alpha} n^{-d/\alpha}\left(\bar{b}_\varepsilon - b_\varepsilon(y)\right) \\ &\quad + c_{14}\bar{b}_\varepsilon^{1+d/\alpha}\left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n \\ &\le c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+1+d/\alpha}(n+1)^{-d/\alpha}\left[\frac{1}{n+1}\left(1 - \left(1 - \frac{b_\varepsilon(x)}{\bar{b}_\varepsilon}\right)^n\right)\right. \\ &\quad - \frac{b_\varepsilon^{\frac{1}{\phi}}(y)}{\bar{b}_\varepsilon + c_{15}}\left(1 + \frac{1}{n}\right)^{d/\alpha}\frac{c_4\eta^{-\alpha}}{c_{13}(n+1)} + \left(1 + \frac{1}{n}\right)^{d/\alpha} \\ &\quad \left. + \left(1 - \frac{b_\varepsilon(x)}{\bar{b}_\varepsilon}\right)^n (n+1)^{d/\alpha}\right]. \end{aligned}$$

By (A.1), (A.4), (14), and the following inequality

$$\frac{b_\varepsilon^{\frac{1}{\phi}}(y)}{\bar{b}_\varepsilon + c_{15}} \ge \frac{c_4\varepsilon^{-\alpha}}{c_5\varepsilon^{-\alpha} + c_{15}} \ge \frac{c_4}{c_5 + c_{15}},$$

the last expression is bounded above by

$$\begin{aligned} &c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+1+d/\alpha}(n+1)^{-d/\alpha} \\ &\quad \times\left[\frac{2}{n+1} + \left(1 + \frac{1}{n}\right)^{d/\alpha}\left(1 - \frac{\eta^{-\alpha}c_4^2/(c_{13}(c_5 + c_{15}))}{n+1}\right)\right] \end{aligned}$$

and, finally, by the inequality

$$\left(1 + \frac{1}{n}\right)^{d/\alpha} \le \left(1 + \frac{p}{n}\right),$$

this is smaller or equal to

$$\begin{aligned} &c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+1+d/\alpha}(n+1)^{-d/\alpha} \\ &\quad \times\left[\frac{2}{n+1} + \left(1 + \frac{p}{n}\right)\left(1 - \frac{\eta^{-\alpha}c_4^2/(c_{13}(c_5 + c_{15}))}{n+1}\right)\right] \\ &\le c_{14}\left(\bar{b}_\varepsilon + c_{15}\right)^{n+1+d/\alpha}(n+1)^{-d/\alpha} \\ &\quad \times\left[1 - \frac{1}{n+1}\left(\eta^{-\alpha}c_4^2/(c_{13}(c_5 + c_{15})) - 2 - 2p\right)\right], \end{aligned}$$

which gives

$$f_{n+1,\varepsilon}(x, y) \le c_{14} \left(\bar{b}_\varepsilon + c_{15}\right)^{n+1+d/\alpha} (n+1)^{-d/\alpha}.$$

□

Using the above lemmas, we may estimate Γ_ε^n and in consequence also the exponent operator $\mathrm{e}^{t\mathcal{A}_\varepsilon} = \mathrm{e}^{-t\bar{b}_\varepsilon}\mathrm{e}^{t\Gamma_\varepsilon}$.

LEMMA 6. *Assume (A.1)–(A.4). Then, for all $x \in \mathbb{R}^d$ and all nonnegative $\varphi \in B_b(\mathbb{R}^d)$ such that $x \notin \operatorname{supp}(\varphi)$, we have*

$$\mathrm{e}^{t\mathcal{A}_\varepsilon}\varphi(x) \le c_{10} t \exp(c_{11}t) \int \varphi(y) \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}\,\mathrm{d}y, \quad \varepsilon \in (0,1).$$

Proof. By (7) and Lemma 3 for every φ such that $x \notin \operatorname{supp}(\varphi)$, we get

$$\Gamma_\varepsilon^n \varphi(x) \le \int \varphi(y) c_{10} n \left(\bar{b}_\varepsilon + c_{11}\right)^{n-1} \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}\,\mathrm{d}y,$$

and

$$\begin{aligned}
\mathrm{e}^{t\mathcal{A}_\varepsilon}\varphi(x) &\le c_{10}\mathrm{e}^{-t\bar{b}_\varepsilon} \sum_{n=1}^{\infty} \frac{t^n n \left(\bar{b}_\varepsilon + c_{11}\right)^{n-1}}{n!} \int \varphi(y) \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}\,\mathrm{d}y \\
&= c_{10}\mathrm{e}^{-t\bar{b}_\varepsilon} t \sum_{n=0}^{\infty} \frac{t^n \left(\bar{b}_\varepsilon + c_{11}\right)^{n}}{n!} \int \varphi(y) \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}\,\mathrm{d}y \\
&= c_{10} t \exp\left(c_{11}t\right) \int \varphi(y) \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}\,\mathrm{d}y.
\end{aligned}$$

□

LEMMA 7. *Assume (A.1), (A.3), and (A.4). Then, there is a constant $c_{16} = c_{16}(\phi, M, \alpha, d)$ such that for every nonnegative $\varphi \in B_b(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$, we have*

$$\mathrm{e}^{t\mathcal{A}_\varepsilon}\varphi(x) \le c_{16} \exp(c_{15}t) t^{-d/\alpha} \int \varphi(y)\,\mathrm{d}y + \mathrm{e}^{-tb_\varepsilon(x)}\varphi(x),$$

for $x \in \mathbb{R}^d$, $\varepsilon \in (0, \varepsilon_0 \wedge 1)$, $t > 0$.

Proof. We directly deduce from Lemma 5 that for every $\varphi \in B_b(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$

$$\Gamma_\varepsilon^n \varphi(x) \le c_{14}(\bar{b}_\varepsilon + c_{15})^{n+d/\alpha} n^{-d/\alpha} \int \varphi(y)\,\mathrm{d}y + \left(\bar{b}_\varepsilon - b_\varepsilon(x)\right)^n \varphi(x),$$

and, consequently, by [29, Lemma 9], we obtain

$$\begin{aligned}
\mathrm{e}^{t\mathcal{A}_\varepsilon}\varphi(x) &\le \mathrm{e}^{-t\bar{b}_\varepsilon} \left[c_{14} \int \varphi(y)\,\mathrm{d}y \sum_{n=1}^{\infty} \frac{t^n (\bar{b}_\varepsilon + c_{15})^{n+d/\alpha}}{n! n^{d/\alpha}} + \mathrm{e}^{t(\bar{b}_\varepsilon - b_\varepsilon(x))}\varphi(x) \right] \\
&\le c_{16} \exp(c_{15}t) t^{-d/\alpha} \int \varphi(y)\,\mathrm{d}y + \mathrm{e}^{-tb_\varepsilon(x)}\varphi(x).
\end{aligned}$$

□

Proof of Theorem 1. Let $t > 0, \varphi \in B_b(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$. Denote $D = \{y \in \mathbb{R}^d : \phi(|y-x|)|y-x|^{-\alpha-d} < t^{-1-d/\alpha}\}$. Using Lemma 6 for $\mathbf{1}_D\varphi$ and Lemma 7 for $\mathbf{1}_{D^c}\varphi$, we obtain

$$\begin{aligned} e^{t\mathcal{A}_\varepsilon}\varphi(x) &= e^{t\mathcal{A}_\varepsilon}[\mathbf{1}_D\varphi](x) + e^{t\mathcal{A}_\varepsilon}[\mathbf{1}_{D^c}\varphi](x) \\ &\le C_1 e^{C_2 t}\left[\int_D \varphi(y)\frac{t\phi(|y-x|)}{|y-x|^{\alpha+d}}\,dy + \int_{D^c}\varphi(y)t^{-d/\alpha}\,dy\right] \\ &\quad + e^{-tb_\varepsilon(x)}\varphi(x) \\ &\le C_1 e^{C_2 t}\int \varphi(y)\min\left(t^{-d/\alpha}, \frac{t\phi(|y-x|)}{|y-x|^{\alpha+d}}\right)dy + e^{-tb_\varepsilon(x)}\varphi(x). \end{aligned}$$

□

Proof of Theorem 2. By Lemma 12 in [29], we have

$$\lim_{\varepsilon\to 0}\|\mathcal{A}\varphi - \mathcal{A}_\varepsilon\varphi\|_\infty = 0$$

for every $\varphi \in C^2_\infty(\mathbb{R}^d)$. A closure of $\mathcal{A}$ is a generator of a semigroup and from the Hille-Yosida theorem, it follows that the range of $\lambda - \mathcal{A}$ is dense in $C_\infty(\mathbb{R}^d)$ and therefore by Theorem 5.2 in [31] (see also [11]), we get

$$\lim_{\varepsilon\downarrow 0}\|e^{t\mathcal{A}_\varepsilon}\varphi - P_t\varphi\|_\infty = 0,$$

for every $\varphi \in C_\infty(\mathbb{R}^d)$. By Theorem 1, this yields

$$P_t\varphi(x) \le C_1 e^{C_2 t}\int \varphi(z)\min\left(t^{-d/\alpha}, \frac{t\phi(|z-x|)}{|z-x|^{\alpha+d}}\right)dz,$$

for every nonnegative $\varphi \in C_\infty(\mathbb{R}^d)$. □

3. Discussion of examples

We now prove the condition (A.1) (c) for functions ϕ of the form (2) for restricted set of parameters β and γ. First, we recall some well-known geometric fact, see, e.g., [23, Lemma 5.3].

LEMMA 8. *The volume of intersection of two balls $B(x, p+k)$ and $B(y, n-p)$ such that $|y-x| = n \in \mathbb{N}, 1 \le p \le n-1, 0 < k \le n-p$ is less than $ck^{\frac{d+1}{2}}(\min\{p+k, n-p\})^{\frac{d-1}{2}}$.*

PROPOSITION 1. *Let the function ϕ be of the form (2). Then, the assumption (A.1) (c) is satisfied if $\beta \in (0,1]$ and $\gamma < d/2 + \alpha - 1/2$.*

Proof. Let $\beta \in (0,1]$ and $\gamma < d/2 + \alpha - 1/2$. First, note that there is an absolute constant $C = C(m, \beta, \gamma, \alpha, d)$ such that $\phi(s)s^{-d-\alpha} \le C\phi(u)u^{-d-\alpha}$ for $|s-u| \le 1$

whenever $s, u \geq 1$. By this fact, with no loss of generality, we may and do consider only the case when $|x-y| = n$ for some even natural number $n \geq 4$. Let

$$\begin{aligned}
&\int_{B(x,1)^c \cap B(y,1)^c} \frac{\phi(|y-z|)}{|y-z|^{\alpha+d}} \frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} \,\mathrm{d}z \\
&\quad \leq 2 \int_{B(x,1)^c \cap B(y,n-1)^c} + \int_{(B(x,1)^c \cap B(x,n-1)) \cup (B(y,1)^c \cap B(y,n-1))} \\
&\quad = 2I + II.
\end{aligned}$$

We have

$$I \leq \frac{\phi(n-1)}{(n-1)^{\alpha+d}} \int_{B(0,1)^c} \frac{\phi(|z|)}{|z|^{\alpha+d}} \,\mathrm{d}z \leq C \frac{\phi(|y-x|)}{|y-x|^{\alpha+d}}$$

with some constant $C = C(m, \beta, \gamma, \alpha, d)$.

To estimate the term II, we will need the additional notations. For $1 \leq p < n/2$ and $0 \leq k < n-p$, we denote

- $D_p := \left\{z \in \mathbb{R}^d : n-p-1 \leq |z-y| < n-p, |x-z| < |y-z|\right\}$,
- $D_{p,k} = D_p \cap \left\{z \in \mathbb{R}^d : p+k \leq |z-x| < p+k+1\right\}$,
- $n_p := \max\left\{k \in \mathbb{N} : D_{p,k} \neq \emptyset\right\}$.

Clearly, $D_p \subset \bigcup_{k=0}^{n_p} D_{p,k}$ and $D_{p,k} \subset B(x, p+k+1) \cap B(y, n-p)$. We have

$$\begin{aligned}
II &\leq 2^{\alpha+d-\gamma+1} |y-x|^{-\alpha-d+\gamma} \int_{1 \leq |y-z| < n-1, |x-z| < |y-z|} \frac{\mathrm{e}^{-m|x-z|^\beta} \mathrm{e}^{-m|y-z|^\beta}}{|x-z|^{\alpha+d-\gamma}} \mathrm{d}z \\
&= 2^{\alpha+d-\gamma+1} |y-x|^{-\alpha-d+\gamma} \sum_{p=1}^{n/2-1} \int_{D_p} \frac{\mathrm{e}^{-m|x-z|^\beta} \mathrm{e}^{-m|y-z|^\beta}}{|x-z|^{\alpha+d-\gamma}} \mathrm{d}z \\
&\leq 2^{\alpha+d-\gamma+1} |y-x|^{-\alpha-d+\gamma} \sum_{p=1}^{n/2-1} \sum_{k=0}^{n_p} \int_{D_{p,k}} \frac{\mathrm{e}^{-m|x-z|^\beta} \mathrm{e}^{-m|y-z|^\beta}}{|x-z|^{\alpha+d-\gamma}} \mathrm{d}z \\
&\leq 2^{\alpha+d-\gamma+1} |y-x|^{-\alpha-d+\gamma} \sum_{p=1}^{n/2-1} \sum_{k=0}^{n_p} \frac{\mathrm{e}^{-m(p+k)^\beta} \mathrm{e}^{-m(n-p-1)^\beta}}{(p+k)^{\alpha+d-\gamma}} |D_{p,k}|.
\end{aligned}$$

Notice that $(n-p)^\beta - (n-p-1)^\beta \leq \beta$ when $\beta \in (0, 1]$. Furthermore, since $p+k \leq p+n_p < n-p$, we also have $k^\beta + n^\beta \leq (p+k)^\beta + (n-p)^\beta$. These inequalities and Lemma 8 thus yield

$$\begin{aligned}
II &\leq C \frac{\mathrm{e}^{-mn^\beta}}{|y-x|^{\alpha+d-\gamma}} \sum_{p=1}^{n/2-1} \sum_{k=0}^{n_p} \mathrm{e}^{-mk^\beta} k^{\frac{d+1}{2}} (p+k)^{-\alpha-d+\gamma} \, (p+k)^{\frac{d-1}{2}} \\
&\leq C \frac{\mathrm{e}^{-m|y-x|^\beta}}{|y-x|^{\alpha+d-\gamma}} \sum_{p=1}^{\infty} p^{-\frac{d+1}{2}-\alpha+\gamma} \sum_{k=0}^{\infty} \mathrm{e}^{-mk^\beta} k^{\frac{d+1}{2}},
\end{aligned}$$

for some $C = C(m, \beta, \gamma, \alpha, d)$. We conclude by observing that for $\beta > 0$ and $\gamma < d/2 + \alpha - 1/2$, the last two sums are bounded by constant. □

REMARK 1. *(1) When $\beta > 1$, then the condition (c) in assumption (A.1) fails. This can be shown by estimating from below the integral*

$$\int_{B((x+y)/2,1)} \frac{\phi(|y-z|)}{|y-z|^{\alpha+d}} \frac{\phi(|z-x|)}{|z-x|^{\alpha+d}} \, dz$$

for $|y-x|$ big enough.

(2) Also, if $\beta = 1$ and $\gamma = d/2 + \alpha - 1/2$, then at least for $d = 1$ the condition (c) in assumption (A.1) does not hold. In this case we have

$$\int_1^{x-1} \mathrm{e}^{-(x-z)}(x-z)^{-1}\mathrm{e}^{-z}z^{-1} \, dz = 2\log(x-1)\mathrm{e}^{-x}x^{-1}, \quad x > 2.$$

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