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A q-Analogue of Derivations on the Tensor Algebra and the q-Schur–Weyl Duality

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Abstract. This paper presents a q-analogue of an extension of the tensor algebra given by the same author. This new algebra naturally contains the ordinary tensor algebra and the Iwahori–Hecke algebra type A of infinite degree. Namely, this algebra can be regarded as a natural mix of these two algebras. Moreover, we can consider natural "derivations" on this algebra. Using these derivations, we can easily prove the q-Schur–Weyl duality (the duality between the quantum enveloping algebra of the general linear Lie algebra and the Iwahori–Hecke algebra of type A).

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1. Introduction

This paper presents a q-analogue of an extension of the tensor algebra given in [4]. Using this algebra, we can easily prove the q-Schur–Weyl duality (the duality between the quantum enveloping algebra $U_q(\mathfrak{gl}_n)$ and the Iwahori–Hecke algebra of type A).

First, let us recall the algebra $\bar{T}(V)$ given in [4]. This algebra $\bar{T}(V)$ naturally contains the ordinary tensor algebra $T(V)$ and the infinite symmetric group S_∞. Moreover, we can consider natural "derivations" on this algebra, which satisfy an analogue of canonical commutation relations. This algebra and these derivations are useful to study representations on the tensor algebra. For example, we can prove the Schur–Weyl duality easily using this framework.

In this paper, we give a q-analogue of this algebra $\bar{T}(V)$. This new algebra $\hat{T}(V)$ naturally contains the ordinary tensor algebra $T(V)$ and the Iwahori–Hecke algebra $H_\infty(q)$ of type A_∞. Namely, we can regard this $\hat{T}(V)$ as a natural mix of $T(V)$ and $H_\infty(q)$. We can also consider natural "derivations" on the algebra $\hat{T}(V)$.

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These derivations are useful to describe the natural action of $U_q(\mathfrak{gl}_n)$ on $V^{\otimes p}$. Moreover, using these derivations, we can easily prove the q-Schur–Weyl duality.

Some applications of $\bar{T}(V)$ were given in [4]: (1) invariant theory in the tensor algebra (for example, a proof of the first fundamental theorem of invariant theory with respect to the natural action of the special linear group), and (2) application to immanants and the quantum immanants (a linear basis of the center of the universal enveloping algebra $U(\mathfrak{gl}_n)$; see [6,7]). The author hopes that the algebra $\hat{T}(V)$ will be useful to study representation theory and invariant theory related to $U_q(\mathfrak{gl}_n)$.

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2. Definition of $\hat{T}(V)$

Let us start with the definition of the algebra $\hat{T}(V)$ determined by a vector space $V = \mathbb{C}^n$. We recall that the ordinary tensor algebra is defined by

$$T(V) = \bigoplus_{p \geq 0} T_p(V)$$

with $T_p(V) = V^{\otimes p}$. Noting this, we define $\hat{T}(V)$ as a vector space by

$$\hat{T}(V) = \bigoplus_{p \geq 0} \hat{T}_p(V),$$

where $\hat{T}_p(V)$ is the following induced representation:

$$\hat{T}_p(V) = \operatorname{Ind}_{H_p(q)}^{H_\infty(q)} V^{\otimes p} = H_\infty(q) \otimes_{H_p(q)} V^{\otimes p}.$$

Here, the notation is as follows. First, $H_p(q)$ is the Iwahori–Hecke algebra of type A_{p-1}. Namely, this is the $\mathbb{C}$-algebra defined by the following generators and relations:

$$\begin{aligned} \text{generators: } & t_1, \ldots, t_{p-1}, \\ \text{relations: } & (t_r - q)(t_r + q^{-1}) = 0, \\ & t_r t_{r+1} t_r = t_{r+1} t_r t_{r+1}, \\ & t_r t_s = t_s t_r, \quad \text{for } |r-s| > 1. \end{aligned}$$

We define $H_\infty(q)$ as the inductive limit of the natural inclusions $H_0(q) \subset H_1(q) \subset \cdots$. Next, $H_p(q)$ naturally acts on $T_p(V) = V^{\otimes p}$ as follows [5]:

$$t_r = \overbrace{\mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V}^{r-1} \otimes t \otimes \overbrace{\mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V}^{n-r-1}.$$

Here, we define $t \in \operatorname{End}(V \otimes V)$ by

$$t e_i e_j = \begin{cases} q e_j e_i, & i = j, \\ e_j e_i, & i > j, \\ e_j e_i + (q - q^{-1}) e_i e_j, & i < j, \end{cases}$$

where $e_1, \dots, e_n$ mean the standard basis of V. Note that we omit the symbol "$\otimes$." Thus, we have explained the definition of $\hat{T}(V)$ as a vector space.

Moreover, we consider a natural algebra structure of $\hat{T}(V)$. Namely, for $\sigma v_1 \cdots v_k \in \hat{T}_k(V)$ and $\tau w_1 \cdots w_l \in \hat{T}_l(V)$, we define their product by

$$\sigma v_1 \cdots v_k \cdot \tau w_1 \cdots w_l = \sigma \alpha^k(\tau) v_1 \cdots v_k w_1 \cdots w_l.$$

Here, σ and τ are elements of $H_\infty(q)$, and $v_1, \dots, v_k, w_1, \dots, w_l$ are vectors in V. Moreover, α is the algebra endomorphism on $H_\infty(q)$ defined by

$$\alpha \colon H_\infty(q) \to H_\infty(q), \quad t_r \mapsto t_{r+1}.$$

This multiplication is well defined. With this multiplication, $\hat{T}(V)$ becomes an associative graded algebra.

Remark. In [4], the definition of $\bar{T}(V)$ was based on the *left* action of S_p on $V^{\otimes p}$. However, in this paper, we defined $\hat{T}(V)$ using the *right* action of $H_p(q)$ on $V^{\otimes p}$. Actually, we can also define a similar algebra using the left action, but we employ our definition because this is compatible with the action of $U_q(\mathfrak{gl}(V))$ (see Section 5).

3. The Multiplication by $v \in V$ and the Derivation by $v^* \in V^*$

In this section, we define two series of fundamental operators on $\hat{T}(V)$, namely the *multiplications* by vectors in V and the *derivations* by covectors in V^*.

First, let $R(\varphi)$ denote the right multiplication by $\varphi \in \hat{T}(V)$:

$$R(\varphi) \colon \hat{T}(V) \to \hat{T}(V), \quad \psi \mapsto \psi\varphi.$$

This operator is obviously fundamental, and the following two cases are particularly fundamental: (1) the case that φ is a vector in V, and (2) the case that φ is an element of $H_\infty(q)$. Indeed, the other cases can be generated by these two cases. Note that $R(v)$ for $v \in V \subset \hat{T}_1(V)$ raises the degree by one, and $R(\sigma)$ for $\sigma \in H_\infty(q) = \hat{T}_0(V)$ does not change the degree.

Next, we define an operator $R(v^*)$ associated to a covector $v^* \in V^*$. When v^* is a member of the dual basis $e_1^*, \dots, e_n^*$, we define $R(e_i^*) \in \operatorname{End}_{\mathbb{C}}(\hat{T}(V))$ by

$$\begin{aligned} R(e_i^*) \colon \hat{T}_p(V) &\to \hat{T}_{p-1}(V), \\ \sigma v_1 \cdots v_p &\mapsto \sum_{r=1}^{p} \sigma k_i^{-1}(v_1) \cdots k_i^{-1}(v_{r-1}) \langle e_i^*, v_r \rangle g_i(v_{r+1}) \cdots g_i(v_p). \end{aligned} \tag{3.1}$$

Here, k_i is the linear transformation on V, and g_i is the linear map from V to $H_2(q) \otimes V$ defined as follows:

$$k_i \colon V \to V, \quad e_j \mapsto q^{\delta_{ij}} e_j, \quad g_i \colon V \to H_2(q) \otimes V, \quad e_j \mapsto \begin{cases} t_1 e_j, & i \le j, \\ t_1^{-1} e_j, & i > j. \end{cases}$$

Based on this, we define $R(v^*)$ in such a way that $R\colon V^* \to \mathrm{End}_{\mathbb{C}}(\hat{T}(V))$ is linear. We call this $R(v^*)$ the *derivation* by $v^* \in V^*$.

For example, we have

$$\begin{aligned} R(e_1^*)e_1e_1e_2 &= \langle e_1^*, e_1\rangle t_1e_1t_1e_2 + q^{-1}e_1\langle e_1^*, e_1\rangle t_1e_2 + q^{-1}e_1q^{-1}e_1\langle e_1^*, e_2\rangle \\ &= t_1t_2e_1e_2 + q^{-1}t_1e_1e_2. \end{aligned}$$

Let us check the well-definedness of the definition (3.1) of $R(e_i^*)$. For this, we consider a linear map $f_r\colon T_p(V) \to \hat{T}_{p-1}(V)$ defined by

$$f_r(v_1 \cdots v_p) = k_i^{-1}(v_1)\cdots k_i^{-1}(v_{r-1})\langle e_i^*, v_r\rangle g_i(v_{r+1})\cdots g_i(v_p),$$

so that $R(e_i^*) = \sum_{r=1}^{p} f_r$. For the well-definedness of (3.1), it suffices to show that $t_1, \ldots, t_{p-1}$ commute with $\sum_{r=1}^{p} f_r$. Namely, we only have to show the following lemma:

LEMMA 3.1. *For $s = 1, \ldots, p-1$, the following hold*:

(1) *t_s commutes with f_r unless $r = s,\ s+1$.*
(2) *t_s commutes with $f_s + f_{s+1}$.*

Proof. We put $e_J = e_{j_1}\cdots e_{j_p}$ for $J = (j_1, \ldots, j_p)$. Let us fix $I = (i_1, \ldots, i_p)$ and $1 \le s \le p$, and put

$$\gamma = \begin{cases} 1, & i_s \ge i_{s+1}, \\ -1, & i_s < i_{s+1}, \end{cases}$$

so that $t_s^{\gamma} e_I = q^{\delta_{i_s i_{s+1}}} e_{I'}$ with $I' = (i_1, \ldots, i_{s-1}, i_{s+1}, i_s, i_{s+2}, \ldots, i_p)$. To show (1), it suffices to show $t_s^{\gamma} f_r(e_I) = q^{\delta_{i_s i_{s+1}}} f_r(e_I')$ for $r \ne s, s+1$. When $r > s+1$, this can be deduced from the relation $t_2^{\gamma} t_1^{\varepsilon} t_2^{\delta} t_1^{-\gamma} = t_1^{\delta} t_2^{\varepsilon}$ for $\gamma, \delta, \varepsilon \in \{1, -1\}$. We can show the the case $r < s$ by a direct calculation.

We can also show (2) by a direct calculation. □

Remark. This well-definedness means that $R(v^*)$ commutes with the action of $H_\infty(q)$.

4. Commutation Relations

For the multiplications and derivations introduced in the previous section, we have the following commutation relations.

THEOREM 4.1. *For $i < j$, we have*

$$\begin{aligned} &R(e_i)R(e_i) = q^{-1}R(e_i)R(e_i)R(t_1) = qR(e_i)R(e_i)R(t_1^{-1}), \\ &R(e_i)R(e_j) = R(e_j)R(e_i)R(t_1^{-1}), \\ &R(e_j)R(e_i) = R(e_i)R(e_j)R(t_1), \end{aligned}$$

$$
\begin{aligned}
&R(e_i^*)R(e_i^*)=q^{-1}R(t_1)R(e_i^*)R(e_i^*)=qR(t_1^{-1})R(e_i^*)R(e_i^*),\\
&R(e_i^*)R(e_j^*)=R(t_1^{-1})R(e_j^*)R(e_i^*),\\
&R(e_j^*)R(e_i^*)=R(t_1)R(e_i^*)R(e_j^*),
\end{aligned}
$$

$$
\begin{aligned}
&R(e_i^*)R(e_i)=R(e_i)R(t_1)R(e_i^*)+K_i^{-1}=R(e_i)R(t_1^{-1})R(e_i^*)+K_i,\\
&R(e_j^*)R(e_i)=R(e_i)R(t_1^{-1})R(e_j^*),\\
&R(e_i^*)R(e_j)=R(e_j)R(t_1)R(e_i^*),
\end{aligned}
$$

where K_i is the linear transformation on $\hat{T}(V)$ defined by

$$K_i : \sigma e_{i_1}\cdots e_{i_p} \mapsto q^{\delta_{ii_1}+\cdots+\delta_{ii_p}}\sigma e_{i_1}\cdots e_{i_p}.$$

Namely, we can exchange two multiplications by vectors putting t_1 or $t_1^{-1}\in H_2(q)$ on the right of these two operators. Similarly, we can exchange two derivations putting t_1 or t_1^{-1} on the left of two operators. The most interesting one is the commutation relation between a derivation and a multiplication. This time, t_1 or t_1^{-1} appears in the middle of these two operators. We can regard these relations as an analogue of the canonical commutation relations.

Proof of Theorem 4.1. These relations can be checked by direct calculations except for the commutation relations between two derivations. Thus, we here prove the sixth relation, from which the fifth relation is immediate. We can prove the fourth relation similarly (actually more easily).

To show the sixth relation, it suffices to prove

$$R(e_j^*)R(e_i^*)e_{k_1}\cdots e_{k_m}e_i^a e_j^b = R(t_1)R(e_i^*)R(e_j^*)e_{k_1}\cdots e_{k_m}e_i^a e_j^b$$

for $k_1,\ldots,k_m \neq i,j$, because the derivations commute with the action of $H_\infty(q)$. By the definition of derivations, we have

$$
\begin{aligned}
R(e_i^*)e_{k_1}\cdots e_{k_m}e_i^a e_j^b &= \sum_{r=1}^{a} e_{k_1}\cdots e_{k_m}e_i^{r-1}(t_1e_i)^{a-r}(t_1e_j)^b\\
&= \sum_{r=1}^{a} e_{k_1}\cdots e_{k_m}t_r^{(a+b-r)}e_i^{a-1}e_j^b,\\
R(e_j^*)e_{k_1}\cdots e_{k_m}e_i^a e_j^b &= \sum_{s=1}^{b} e_{k_1}\cdots e_{k_m}e_i^a e_j^{s-1}(t_1e_j)^{b-s} = \sum_{s=1}^{b} e_{k_1}\cdots e_{k_m}t_{a+s}^{(b-s)}e_i^a e_j^{b-1},
\end{aligned}
$$

where we put $t_k^{(c)}=t_kt_{k+1}\cdots t_{k+c-1}$. Using these, we have

$$
\begin{aligned}
R(e_j^*)R(e_i^*)e_{k_1}\cdots e_{k_m}e_i^a e_j^b &= \sum_{r=1}^{a}\sum_{s=1}^{b} e_{k_1}e_{k_1}\cdots e_{k_m}t_r^{(a+b-r)}t_{a-1+s}^{(b-s)}e_i^{a-1}e_j^{b-1},\\
R(t_1)R(e_i^*)R(e_j^*)e_{k_1}\cdots e_{k_m}e_i^a e_j^b &= \sum_{r=1}^{a}\sum_{s=1}^{b} e_{k_1}\cdots e_{k_m}t_{a+s}^{(b-s)}t_r^{(a+b-r)}e_i^{a-1}e_j^{b-1}.
\end{aligned}
$$

These are equal, because we have $t_r^{(a+b-r)} t_{a-1+s}^{(b-s)} = t_{a+s}^{(b-s)} t_r^{(a+b-r)}$ by a calculation. □

It is natural to consider the operator algebra generated by $R(v)$, $R(v^*)$ and $R(\sigma)$ with $v \in V$, $v^* \in V^*$ and $\sigma \in H_\infty(q)$. We can regard this operator algebra as an analogue of the Weyl algebras and the Clifford algebras.

The following commutation relations with K_i are also fundamental:

THEOREM 4.2. *We have*

$$K_j R(e_i) = q^{\delta_{ij}} R(e_i) K_j, \quad K_j R(e_i^*) = q^{-\delta_{ij}} R(e_i^*) K_j, \quad K_j R(t_r) = R(t_r) K_j.$$

5. The Natural Representation of $U_q(\mathfrak{gl}(V))$ on $V^{\otimes p}$

We can use the operators introduced in Section 3 to study the natural representation of the quantum enveloping algebra $U_q(\mathfrak{gl}(V))$ on $V^{\otimes p}$.

First, let us recall the definition of $U_q(\mathfrak{gl}(V))$. For $V = \mathbb{C}^n$, we define the $\mathbb{C}$-algebra $U_q(\mathfrak{gl}(V))$ by the following generators and relations [5]:

$$\begin{aligned}
\text{generators: } & q^{\pm\varepsilon_1/2}, \dots, q^{\pm\varepsilon_n/2}, \hat{e}_1, \dots, \hat{e}_{n-1}, \hat{f}_1, \dots, \hat{f}_{n-1}, \\
\text{relations: } & q^{\varepsilon_i/2} q^{\varepsilon_j/2} = q^{\varepsilon_j/2} q^{\varepsilon_i/2}, \qquad q^{\varepsilon_i/2} q^{-\varepsilon_i/2} = q^{-\varepsilon_i/2} q^{\varepsilon_i/2} = 1, \\
& q^{\varepsilon_i/2} \hat{e}_j q^{-\varepsilon_i/2} = q^{\delta_{ij} - \delta_{i,j+1}} \hat{e}_j, \qquad q^{\varepsilon_i/2} \hat{f}_j q^{-\varepsilon_i/2} = q^{-\delta_{ij} + \delta_{i,j+1}} \hat{f}_j, \\
& \hat{e}_i \hat{f}_j - \hat{f}_j \hat{e}_i = \delta_{ij} \frac{q^{(\varepsilon_i - \varepsilon_{i+1})/2} - q^{(-\varepsilon_i + \varepsilon_{i+1})/2}}{q - q^{-1}}, \\
& \hat{e}_i \hat{e}_j = \hat{e}_j \hat{e}_i, \qquad \hat{f}_i \hat{f}_j = \hat{f}_j \hat{f}_i \ \text{ for } |i-j| > 1, \\
& \hat{e}_i^2 \hat{e}_{i\pm1} - (q + q^{-1}) \hat{e}_i \hat{e}_{i\pm1} \hat{e}_i + \hat{e}_{i\pm1} \hat{e}_i^2 = 0, \\
& \hat{f}_i^2 \hat{f}_{i\pm1} - (q + q^{-1}) \hat{f}_i \hat{f}_{i\pm1} \hat{f}_i + \hat{f}_{i\pm1} \hat{f}_i^2 = 0.
\end{aligned}$$

Here, we denote $q^{a_1} \cdots q^{a_k}$ simply by $q^{a_1 + \cdots + a_k}$.

Next, we define $\hat{E}_{ij}$ and $\hat{E}_{ji} \in U_q(\mathfrak{gl}(V))$ for $1 \le i < j \le n$ by

$$\hat{E}_{i,i+1} = \hat{e}_i, \quad \hat{E}_{i+1,i} = \hat{f}_i$$

and recursive relations

$$\hat{E}_{ik} = \hat{E}_{ij} \hat{E}_{jk} - q \hat{E}_{jk} \hat{E}_{ij}, \quad \hat{E}_{ki} = \hat{E}_{kj} \hat{E}_{ji} - q^{-1} \hat{E}_{ji} \hat{E}_{kj}$$

for $i < j < k$. Moreover, for $i < j$ and $a \in \mathbb{C} \setminus \{0\}$, we put

$$\begin{aligned}
& \hat{E}_{ij}(a) = a^{-1} q^{-(\varepsilon_i + \varepsilon_j - 1)/2} \hat{E}_{ij}, \quad \hat{E}_{ji}(a) = a q^{(\varepsilon_j + \varepsilon_i - 1)/2} \hat{E}_{ji}, \\
& \hat{E}_{ii}(a) = \frac{a q^{\varepsilon_i} - a^{-1} q^{-\varepsilon_i}}{q - q^{-1}}.
\end{aligned}$$

We call this $\hat{E}_{ij}(a)$ for $1 \le i, j \le n$ the L-operator.

We denote by π the natural representation of the quantum enveloping algebra $U_q(\mathfrak{gl}(V))$ on $V^{\otimes p}$. This is determined by the following actions of generators [5]:

$$\pi(\hat{e}_i)=\sum_{r=1}^{p} k_i^{1/2}k_{i+1}^{-1/2}\otimes\cdots\otimes k_i^{1/2}k_{i+1}^{-1/2}\otimes \underbrace{E_{i,i+1}}_{r\text{th}}\otimes k_i^{-1/2}k_{i+1}^{1/2}\otimes\cdots\otimes k_i^{-1/2}k_{i+1}^{1/2},$$

$$\pi(\hat{f}_i)=\sum_{r=1}^{p} k_i^{1/2}k_{i+1}^{-1/2}\otimes\cdots\otimes k_i^{1/2}k_{i+1}^{-1/2}\otimes \underbrace{E_{i+1,i}}_{r\text{th}}\otimes k_i^{-1/2}k_{i+1}^{1/2}\otimes\cdots\otimes k_i^{-1/2}k_{i+1}^{1/2},$$

$$\pi(q^{\pm\varepsilon_i/2})=k_i^{\pm 1/2}\otimes\cdots\otimes k_i^{\pm 1/2}=K_i^{\pm 1/2}.$$

Here, $k_i^{1/2}$ and E_{ij} are the linear transformations on V defined by

$$k_i^{1/2}\colon e_h\mapsto q^{\delta_{ih}/2}e_h,\quad E_{ij}\colon e_h\mapsto \delta_{jh}e_i.$$

We can use our operators to express this representation π:

THEOREM 5.1. *We have*

$$\pi(\hat{E}_{ij}(1))=R(e_i)R(e_j^*).$$

Proof. We can check the assertion by a direct calculation when $i=j$.
Let us show the case $i\neq j$. We note that

$$\pi(\hat{E}_{ij}(1))=q^{-1/2}K_i^{-1/2}K_j^{-1/2}\pi(\hat{E}_{ij}),\quad \pi(\hat{E}_{ji}(1))=q^{1/2}K_j^{1/2}K_i^{1/2}\pi(\hat{E}_{ji})$$

for $i<j$. Thus, it suffices to show

$$\pi(\hat{E}_{ij})=K_j^{1/2}R(e_i)R(e_j^*)K_i^{1/2},\quad \pi(\hat{E}_{ji})=K_i^{-1/2}R(e_j)R(e_i^*)K_j^{-1/2} \tag{5.1}$$

for $i<j$. We can check these relations for $j=i+1$ by a direct calculation. To show the other cases, we put

$$F_{ij}=K_j^{1/2}R(e_i)R(e_j^*)K_i^{1/2},\quad F_{ji}=K_i^{-1/2}R(e_j)R(e_i^*)K_j^{-1/2}$$

for $i<j$. Then, we have

$$F_{ik}=F_{ij}F_{jk}-qF_{jk}F_{ij},\quad F_{ki}=F_{kj}F_{ji}-q^{-1}F_{ji}F_{kj}$$

for $i<j<k$. Indeed, using Theorems 4.1 and 4.2, we see the first relation as follows:

$$\begin{aligned}&F_{ij}F_{jk}-qF_{jk}F_{ij}\\&\quad=K_j^{1/2}R(e_i)R(e_j^*)K_i^{1/2}K_k^{1/2}R(e_j)R(e_k^*)K_j^{1/2}\\&\qquad-qK_k^{1/2}R(e_j)R(e_k^*)K_j^{1/2}K_j^{1/2}R(e_i)R(e_j^*)K_i^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= K_j^{1/2} K_k^{1/2} R(e_i) R(e_j^*) R(e_j) R(e_k^*) K_i^{1/2} K_j^{1/2} \\
&\quad - K_k^{1/2} K_j^{1/2} R(e_j) R(e_k^*) R(e_i) R(e_j^*) K_j^{1/2} K_i^{1/2} \\
&= K_j^{1/2} K_k^{1/2} R(e_i) (R(e_j) R(t_1) R(e_j^*) + K_j^{-1}) R(e_k^*) K_i^{1/2} K_j^{1/2} \\
&\quad - K_k^{1/2} K_j^{1/2} R(e_j) R(e_i) R(t_1^{-1}) R(e_k^*) R(e_j^*) K_j^{1/2} K_i^{1/2} \\
&= K_j^{1/2} K_k^{1/2} R(e_i) (R(e_j) R(t_1) R(e_j^*) + K_j^{-1}) R(e_k^*) K_i^{1/2} K_j^{1/2} \\
&\quad - K_k^{1/2} K_j^{1/2} R(e_i) R(e_j) R(t_1) R(e_j^*) R(e_k^*) K_j^{1/2} K_i^{1/2} \\
&= K_k^{1/2} R(e_i) R(e_k^*) K_i^{1/2} \\
&= F_{ik}.
\end{aligned}
$$

We can show the second relation similarly. Combining these, we have (5.1). □

Remark. Theorem 5.1 is quite similar to the natural action of the Lie algebra $\mathfrak{gl}(V)$ on $\mathcal{P}(V)$ the space of all polynomial functions on V. This action μ can be expressed as

$$\mu(E_{ij}) = x_i \frac{\partial}{\partial x_j}.$$

Here, x_i means the canonical coordinate of V, and E_{ij} means the standard basis of $\mathfrak{gl}(V)$.

Using Theorems 4.1 and 4.2, we have the following relations:

PROPOSITION 5.2 *We have*

$$
\begin{aligned}
R(e_i)R(e_j)R(e_k^*) &= R(e_j)R(e_k^*)R(e_i) && \text{when } i \leqslant j \text{ and } i \leqslant k,\\
R(e_i)R(e_j)R(e_k^*) &= R(e_j)R(e_k^*)R(e_i) \\
&\quad \pm (q - q^{-1}) R(e_i)R(e_k^*)R(e_j) && \text{when } j \leqslant i \leqslant k,\\
R(e_i)R(e_j)R(e_i^*) &= R(e_j)R(e_i^*)R(e_i) - K_i^{\pm 1} R(e_j) && \text{when } i \leqslant j,\\
R(e_i)R(e_i)R(e_j^*) &= q^{\pm 1} R(e_i)R(e_j^*)R(e_i) && \text{when } i \leqslant j,\\
R(e_i)R(e_i)R(e_i^*) &= q R(e_i)R(e_i^*)R(e_i) - K_i R(e_i).\\
&= q^{-1} R(e_i)R(e_i^*)R(e_i) - K_i^{-1} R(e_i).
\end{aligned}
$$

Moreover, we have the following proposition. Indeed, using Proposition 5.2, we can rewrite $R(v_k)\cdots R(v_1)R(v_1^*)\cdots R(v_k^*)$ as a sum of products of $R(v)R(v^*)$ and K_i.

PROPOSITION 5.3 *For any $v_1, \ldots, v_k \in V$ and $v_1^*, \ldots, v_k^* \in V^*$, we have*

$$R(v_k)\cdots R(v_1)R(v_1^*)\cdots R(v_k^*) \in \pi(U_q(\mathfrak{gl}(V))).$$

6. q-Schur–Weyl Duality

We can use our results to prove the following Jimbo duality, namely the q-analogue of the Schur–Weyl duality. This theorem was first given in [5], and several proofs have been given (see [3,8] for example).

THEOREM 6.1. *Assume that* $[p]! \neq 0$. *Let us denote by* ρ *the natural action of* $H_p(q)$ *on* $V^{\otimes p}$. *Then,* $\rho(H_p(q))$ *and* $\pi(U_q(\mathfrak{gl}(V)))$ *are mutual commutants of each other. Namely, we have*

$$\operatorname{End}(V^{\otimes p})^{\rho(H_p(q))} = \pi(U_q(\mathfrak{gl}(V))), \quad \operatorname{End}(V^{\otimes p})^{\pi(U_q(\mathfrak{gl}(V)))} = \rho(H_p(q)).$$

Here $[k]=[k]_q$ is a q-integer, and $[k]!=[k]_q!$ is a q-factorial:

$$[k]=\frac{q^k-q^{-k}}{q-q^{-1}}=q^{k-1}+q^{k-3}+\cdots+q^{-k+1}, \quad [k]!=[k][k-1]\cdots[1].$$

To prove this theorem, we consider the following analogue of the Euler operator:

$$\mathcal{E}=\sum_{J\in\mathcal{J}}\frac{1}{J!}R(e_{j_1})\cdots R(e_{j_p})R(e^*_{j_p})\cdots R(e^*_{j_1}).$$

Here, we put

$$\mathcal{J}=\{(j_1,\ldots,j_p)\in\mathbb{N}^p \mid 1\le j_1\le\cdots\le j_p\le n\}.$$

Moreover, we put $J!=[m_1]!\cdots[m_n]!$ for $J=(j_1,\ldots,j_p)\in\mathcal{J}$, where m_i is the multiplicity of $j_1,\ldots,j_p$ at i:

$$(j_1,\ldots,j_p)=(\underbrace{1,\ldots,1}_{m_1},\underbrace{2,\ldots,2}_{m_2},\ldots,\underbrace{n,\ldots,n}_{m_n}).$$

For this $\mathcal{E}$, the following relation holds:

LEMMA 6.2. *We have* $\mathcal{E}\varphi=\varphi$ *for any* $\varphi\in V^{\otimes p}$.

Proof. We put

$$\Phi_i(a)=\frac{q^aK_i-q^{-a}K_i^{-1}}{q-q^{-1}},$$

and moreover

$$\Phi_i^{(m)}=\Phi_i(0)\Phi_i(-1)\cdots\Phi_i(-m+2)\Phi_i(-m+1).$$

Then, we have $R(e_i)R(e_i^*)=\Phi_i(0)$ and $\Phi_i(a)R(e_i^*)=R(e_i^*)\Phi_i(a-1)$, so that

$$R(e_i)^mR(e_i^*)^m=\Phi_i^{(m)}.$$

Thus, for $1 \le j_1 \le \cdots \le j_p \le n$, we have

$$R(e_{j_1}) \cdots R(e_{j_p}) R(e^*_{j_p}) \cdots R(e^*_{j_1}) = \Phi^{(m_1)}_{j_1} \Phi^{(m_2)}_{j_2} \cdots \Phi^{(m_{p-1})}_{j_{p-1}} \Phi^{(m_p)}_{j_p},$$

where m_i is the multiplicity of $j_1, \ldots, j_p$ at i.

Moreover, we consider $1 \le i_1, \ldots, i_p \le n$, and let l_i be the multiplicity of $i_1, \ldots, i_p$ at i. Then, we have

$$\Phi_i(a) e_{i_1} \cdots e_{i_p} = [l_i + a] e_{i_1} \cdots e_{i_p}.$$

Hence, we have

$$\begin{aligned} &R(e_{j_1}) \cdots R(e_{j_p}) R(e^*_{j_p}) \cdots R(e^*_{j_1}) e_{i_1} \cdots e_{i_p} \\ &\quad = [l_1]^{(m_1)} \cdots [l_n]^{(m_n)} e_{i_1} \cdots e_{i_p} \\ &\quad = \begin{cases} [m_1]! \cdots [m_n]! e_{i_1} \cdots e_{i_p}, & (l_1, \ldots, l_n) = (m_1, \ldots, m_n), \\ 0, & (l_1, \ldots, l_n) \neq (m_1, \ldots, m_n). \end{cases} \end{aligned}$$

Here, we put $[l]^{(m)} = [l][l-1] \cdots [l-m+1]$. The assertion is immediate from this. □

Using this lemma, we can prove Theorem 6.1 as follows:

Proof of Theorem 6.1. We can check by a direct calculation that these two actions are commutative. When $[p]! \neq 0$, the algebra $H_p(q)$ is semisimple [2]. Thus, by the double commutant theorem [1], it suffices to show $\mathrm{End}(V^{\otimes p})^{\rho(H_p(q))} \subset \pi(U_q(\mathfrak{gl}(V)))$.

Assume that $f \in \mathrm{End}(V^{\otimes p})^{\rho(H_p(q))}$. Then, for any $\varphi \in V^{\otimes p}$, we have

$$\begin{aligned} f(\varphi) &= f(\mathcal{E}\varphi) \\ &= f\left(\sum_{J=(j_1,\ldots,j_p)\in\mathcal{J}} \frac{1}{[J]!} R(e_{j_p}) \cdots R(e_{j_1}) R(e^*_{j_1}) \cdots R(e^*_{j_p}) \varphi \right) \\ &= f\left(\sum_{J=(j_1,\ldots,j_p)\in\mathcal{J}} \frac{1}{[J]!} R(e_{j_p}) \cdots R(e_{j_1}) \sigma_J \right) \\ &= f\left(\sum_{J=(j_1,\ldots,j_p)\in\mathcal{J}} \frac{1}{[J]!} \sigma_J e_{j_1} \cdots e_{j_p} \right) \\ &= \sum_{J=(j_1,\ldots,j_p)\in\mathcal{J}} \frac{1}{[J]!} \sigma_J f(e_{j_1} \cdots e_{j_p}) \\ &= \sum_{J=(j_1,\ldots,j_p)\in\mathcal{J}} \frac{1}{[J]!} R(f(e_{j_1} \cdots e_{j_p})) R(e^*_{j_1}) \cdots R(e^*_{j_p}) \varphi. \end{aligned}$$

Here, we denote $R(e^*_{j_1}) \cdots R(e^*_{j_p})\varphi$ simply by σ_J. This σ_J is an element of $H_p(q)$, and f commutes with the action of $H_p(q)$, so that the fifth equality holds.

Thus, by Proposition 5.3, we see that $f \in \pi(U_q(\mathfrak{gl}(V)))$. □

Remark. For any group G, every map $f: G \to G$ commuting with all right translations is equal to a left translation. This fact is proved quickly as follows. Let e be the identity element of G. For any element x of G, we have $f(x) = f(ex) = f(e)x$, because f commutes with the right multiplication by x. Thus f is equal to the left multiplication by $f(e)$, as we claimed. It should be noted that our proof of Theorem 6.1 is based on the same principle (the operator $\mathcal{E}$ plays a role of the identity element e).

Theorem 6.1 holds, if and only if q satisfy $[p]! \neq 0$ (this condition is also equivalent with the condition that $H_p(q)$ is semisimple). Indeed, when $[p]! = 0$, this proof fails because there exists I such that $[I]! = 0$. It is interesting that the condition $[p]! = 0$ appears this way.

I hope that the algebra $\hat{T}(V)$ and the differential operators on $\hat{T}(V)$ will be useful to study invariant theory in quantum enveloping algebras.

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