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Research Article

Best Proximity Points of Cyclic φ -Contractions on Reflexive Banach Spaces

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We provide a positive answer to a question raised by Al-Thagafi and Shahzad (Nonlinear Analysis, 70 (2009), 3665-3671) about best proximity points of cyclic φ -contractions on reflexive Banach spaces.

1. Introduction

As a generalization of Banach contraction principle, Kirk et al. proved, in 2003, the following fixed point result; see [1].

Theorem 1.1. *Let A and B be nonempty closed subsets of a complete metric space (X, d) . Suppose that $T : A \cup B \rightarrow A \cup B$ is a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

Let A and B be nonempty closed subsets of a metric space (X, d) and $\varphi : [0, \infty) \rightarrow [0, \infty)$ a strictly increasing map. We say that $T : A \cup B \rightarrow A \cup B$ is a cyclic φ -contraction map [2] whenever $T(A) \subseteq B$, $T(B) \subseteq A$ and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) \quad (1.1)$$

for all $x \in A$ and $y \in B$, where $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. Also, $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$. As a special case, when $\varphi(t) = (1 - \alpha)t$, in which $\alpha \in (0, 1)$ is a constant, T is called cyclic contraction.

In 2005, Petrusel proved some periodic point results for cyclic contraction maps [3]. Then, Eldered and Veeramani proved some results on best proximity points of cyclic

contraction maps in 2006 [4]. They raised a question about the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space. In 2009, Al-Thagafi and Shahzad gave a positive answer to the question [2]. More precisely, they proved some results on the existence and convergence of best proximity points of cyclic contraction maps defined on reflexive (and strictly convex) Banach spaces [2, Theorems 9, 10, 11, and 12]. They also introduced cyclic φ -contraction maps and raised the following question in [2].

Question 1. It is interesting to ask whether Theorems 9 and 10 (resp., Theorems 11 and 12) hold for cyclic φ -contraction maps where the space is only reflexive (resp., reflexive and strictly convex) Banach space.

In this paper, we provide a positive answer to the above question. For obtaining the answer, we use some results of [2].

2. Main Results

First, we give the following extension of [4, Proposition 3.3] for cyclic φ -contraction maps, where φ is unbounded.

Theorem 2.1. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing unbounded map. Also, let A and B be nonempty subsets of a metric space (X, d) , $T : A \cup B \rightarrow A \cup B$ a cyclic φ -contraction map, $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose that $x_0 \in A$ (the proof when $x_0 \in B$ is similar). By [2, Theorem 3], $d(x_{2n}, x_{2n+1}) \rightarrow d(A, B)$. Hence, it is sufficient to prove that $\{x_{2n+1}\}$ is bounded. Since φ is unbounded, there exists $M > 0$ such that

$$\varphi(M) > d(x_0, Tx_0) + \varphi(d(A, B)). \quad (2.1)$$

If $\{x_{2n+1}\}$ is not bounded, then there exists a natural number n_0 such that

$$d(T^2x_0, T^{2n_0+1}x_0) > M, \quad d(T^2x_0, T^{2n_0-1}x_0) \leq M. \quad (2.2)$$

Then, we have

$$\begin{aligned} M &< d(T^2x_0, T^{2n_0+1}x_0) \leq d(Tx_0, T^{2n_0}x_0) - \varphi(d(Tx_0, T^{2n_0}x_0)) + \varphi(d(A, B)) \\ &\leq d(x_0, T^{2n_0-1}x_0) - \left[\varphi(d(Tx_0, T^{2n_0}x_0)) + \varphi(d(x_0, T^{2n_0-1}x_0)) \right] + 2\varphi(d(A, B)) \\ &\leq d(x_0, T^2x_0) + d(T^2x_0, T^{2n_0-1}x_0) - \left[\varphi(d(Tx_0, T^{2n_0}x_0)) + \varphi(d(x_0, T^{2n_0-1}x_0)) \right] \\ &\quad + 2\varphi(d(A, B)) \\ &\leq d(x_0, Tx_0) + d(Tx_0, T^2x_0) + M - \left[\varphi(d(Tx_0, T^{2n_0}x_0)) + \varphi(d(x_0, T^{2n_0-1}x_0)) \right] \\ &\quad + 2\varphi(d(A, B)). \end{aligned} \quad (2.3)$$

Since $d(Tx, Ty) \leq d(x, y)$ for all $x \in A$ and $y \in B$, we obtain

$$M < 2d(x_0, Tx_0) + M - \left[\varphi\left(d\left(Tx_0, T^{2n_0}x_0\right)\right) + \varphi\left(d\left(x_0, T^{2n_0-1}x_0\right)\right) \right] + 2\varphi(d(A, B)). \quad (2.4)$$

Since $d(Tx_0, T^{2n_0}x_0) \leq d(x_0, T^{2n_0-1}x_0)$, we have

$$\varphi\left(d\left(Tx_0, T^{2n_0}x_0\right)\right) \leq \varphi\left(d\left(x_0, T^{2n_0-1}x_0\right)\right). \quad (2.5)$$

Thus, we obtain $\varphi(d(Tx_0, T^{2n_0}x_0)) < d(x_0, Tx_0) + \varphi(d(A, B))$. Since

$$M < d\left(T^2x_0, T^{2n_0+1}x_0\right) \leq d\left(Tx_0, T^{2n_0}x_0\right), \quad (2.6)$$

$\varphi(M) < \varphi(d(Tx_0, T^{2n_0}x_0))$. Hence, $\varphi(M) < d(x_0, Tx_0) + \varphi(d(A, B))$. This contradiction completes the proof. \square

Since the proof of last result was classic, we presented it separately. Here, we provide our key result via a special proof which is a general case of Theorem 2.1.

Theorem 2.2. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty subsets of a metric space (X, d) , $T : A \cup B \rightarrow A \cup B$ a cyclic φ -contraction map, $x_0 \in A \cup B$, and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose that $x_0 \in A$ (the proof when $x_0 \in B$ is similar). By [2, Theorem 3], $d(x_{2n}, x_{2n+1}) \rightarrow d(A, B)$. Hence, either $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are bounded or both are unbounded. Suppose that both sequences are unbounded. Fix $n_1 \in \mathbb{N}$ and define

$$e_{n,k} = d\left(T^{2n}x_0, T^{2(n_1+k)+1}x_0\right) \quad (2.7)$$

for all $n, k \geq 1$. Since $\{x_{2n+1}\}$ is unbounded, $\limsup_{k \rightarrow \infty} e_{n,k} = \infty$ for all $n \geq 1$. Thus, we can choose a strictly increasing subsequence $\{e_{1,k_i^1}\}_{i \geq 1}$ of the sequence $\{e_{1,k}\}_{k \geq 1}$. Since $d(T^2x_0, T^{2(n_1+k_i^1)+1}x_0) \leq d(T^2x_0, T^4x_0) + d(T^4x_0, T^{2(n_1+k_i^1)+1}x_0)$, we have

$$\limsup_{i \rightarrow \infty} e_{2,k_i^1} = \infty. \quad (2.8)$$

Again, we can choose a strictly increasing subsequence $\{e_{2,k_i^2}\}_{i \geq 1}$ of the sequence $\{e_{2,k_i^1}\}_{i \geq 1}$ such that $\limsup_{i \rightarrow \infty} e_{2,k_i^2} = \infty$. By continuing this process, for each natural number n , we can choose a strictly increasing subsequence $\{e_{n,k_i^n}\}_{i \geq 1}$ of the sequence $\{e_{n,k_i^{n-1}}\}_{i \geq 1}$ such that $\limsup_{i \rightarrow \infty} e_{n,k_i^n} = \infty$. By the construction, if we consider the sequence $\{k_i^i\}_{i \geq 1}$, then $\lim_{i \rightarrow \infty} k_i^i = \infty$, $\{e_{n,k_i^i}\}_{i \geq 1}$ is a strictly increasing subsequence of $\{e_{n,k_i^n}\}_{i \geq 1}$ and $\limsup_{i \rightarrow \infty} e_{n,k_i^i} = \infty$ for all $n \geq 1$. Now, define $n_2 = n_1 + k_2^2 - k_1^1$. Also, by induction define the sequence $\{n_m\}_{m \geq 1}$ by $n_m = n_1 + k_m^m - k_1^1$. Note that, the sequence $\{n_m\}_{m \geq 1}$ is strictly increasing and $\limsup_{m \rightarrow \infty} n_m = \infty$. Since T is a cyclic φ -contraction map, $\{d(T^{2n_m}x_0, T^{2(n_m+k_1^1)+1}x_0)\}_{m \geq 1}$ is a decreasing

sequence. Hence by the construction of the sequence $\{n_m\}_{m \geq 1}$, $\{d(T^{2n_m}x_0, T^{2(n_1+k_m^m)+1}x_0)\}_{m \geq 1}$ is a decreasing sequence. Let $m \geq 1$ be given. Since $e_{n_m, k_1^1} \leq e_{n_m, k_m^m}$, we have

$$d(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0) \leq d(T^{2n_m}x_0, T^{2(n_1+k_m^m)+1}x_0). \quad (2.9)$$

Thus,

$$d(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0) \leq d(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0) \quad (2.10)$$

for all $m \geq 1$. Hence, we have

$$\begin{aligned} d(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)+1}x_0) &\leq d(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(T^{2n_m}x_0, T^{2(n_1+k_m^m)+1}x_0) \\ &= d(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(T^{2n_m}x_0, T^{2(n_m+k_1^1)+1}x_0) \\ &\leq d(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(T^{2n_m}x_0, T^{2(n_m+k_1^1)+1}x_0) \end{aligned} \quad (2.11)$$

for all $m \geq 1$. Since $d(Tx, Ty) \leq d(x, y)$ for all $x \in A$ and $y \in B$, we obtain

$$\begin{aligned} d(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)+1}x_0) &\leq d(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(T^{2n_m-1}x_0, T^{2(n_m+k_1^1)}x_0) \\ &\leq d(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(x_0, T^{2k_1^1+1}x_0) \end{aligned} \quad (2.12)$$

for all $m \geq 1$. Consequently

$$\begin{aligned} d(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)}x_0) &\leq d(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(T^{2(n_1+k_1^1)+1}x_0, T^{2(n_1+k_1^1)}x_0) \\ &\leq d(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(x_0, T^{2k_1^1+1}x_0) \\ &\quad + d(T^{2(n_1+k_1^1)+1}x_0, T^{2(n_1+k_1^1)}x_0) \end{aligned} \quad (2.13)$$

for all $m \geq 1$. This implies that

$$e_{(n_1+k_1^1), k_m^m} \leq \mu \quad (2.14)$$

for all $m \geq 1$, where

$$\mu = d(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0) + d(x_0, T^{2k_1^1+1}x_0) + d(T^{2(n_1+k_1^1)+1}x_0, T^{2(n_1+k_1^1)}x_0) \quad (2.15)$$

is a constant. But, $\limsup_{i \rightarrow \infty} e_{n, k_i^1} = \infty$ for all $n \geq 1$. This contradiction completes the proof. \square

Now by using this key result, we provide our main results which give positive answer to the question. Their proofs are basically due to Al-Thagafi and Shahzad [2]. However, the crucial role is played by our key result. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \xrightarrow{w} x$.

Theorem 2.3. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty weakly closed subsets of a reflexive Banach space and $T : A \cup B \rightarrow A \cup B$ a cyclic φ -contraction map. Then there exists $(x, y) \in A \times B$ such that*

$$\|x - y\| = d(A, B). \quad (2.16)$$

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for all $n \geq 1$. By Theorem 2.2, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ such that $x_{2n_k} \xrightarrow{w} x \in A$. As $\{x_{2n_{k+1}}\}$ is bounded and B is weakly closed, we can say, without loss of generality, that $x_{2n_{k+1}} \xrightarrow{w} y \in B$ as $k \rightarrow \infty$. Since $x_{2n_k} - x_{2n_{k+1}} \xrightarrow{w} x - y \neq 0$ as $k \rightarrow \infty$, there exists a bounded linear functional $f : X \rightarrow [0, \infty)$ such that $\|f\| = 1$ and $f(x - y) = \|x - y\|$. For each $k \geq 1$, we have

$$|f(x_{2n_k} - x_{2n_{k+1}})| \leq \|f\| \|x_{2n_k} - x_{2n_{k+1}}\| = \|x_{2n_k} - x_{2n_{k+1}}\|. \quad (2.17)$$

Since $\lim_{k \rightarrow \infty} f(x_{2n_k} - x_{2n_{k+1}}) = f(x - y) = \|x - y\|$, by using [2, Theorem 3] we obtain

$$\|x - y\| = \lim_{k \rightarrow \infty} |f(x_{2n_k} - x_{2n_{k+1}})| \leq \lim_{k \rightarrow \infty} \|x_{2n_k} - x_{2n_{k+1}}\| = \|x_{2n_k} - x_{2n_{k+1}}\| = d(A, B). \quad (2.18)$$

Hence, $\|x - y\| = d(A, B)$. □

Definition 2.4. (see [2]) Let A and B be nonempty subsets of a normed space X , $T : A \cup B \rightarrow A \cup B$, $T(A) \subseteq B$, and $T(B) \subseteq A$. We say that T satisfies the proximal property if

$$x_n \xrightarrow{w} x \in A \cup B, \quad \|x_n - Tx_n\| \rightarrow d(A, B) \implies \|x - Tx\| = d(A, B). \quad (2.19)$$

Theorem 2.5. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty subsets of a reflexive Banach space X such that A is weakly closed and $T : A \cup B \rightarrow A \cup B$ a cyclic φ -contraction map. Then, there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*

- (a) T is weakly continuous on A .
- (b) T satisfies the proximal property.

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for all $n \geq 1$. By Theorem 2.2, the sequence $\{x_{2n}\}$ is bounded. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ such that $x_{2n_k} \xrightarrow{w} x \in A$ as $k \rightarrow \infty$.

- (a) Since T is weakly continuous on A and $T(A) \subseteq B$, we have $x_{2n_{k+1}} \xrightarrow{w} Tx \in B$ as $k \rightarrow \infty$. So $x_{2n_k} - x_{2n_{k+1}} \xrightarrow{w} x - Tx \neq 0$ as $k \rightarrow \infty$. The rest of the proof is similar to that of Theorem 2.3.

(b) By [2, Theorem 3], we have

$$\|x_{2n_k} - Tx_{2n_k}\| = \|x_{2n_k} - x_{2n_k+1}\| \longrightarrow d(A, B) \quad (2.20)$$

as $k \rightarrow \infty$. Since T satisfies the proximal property, we have $\|x - Tx\| = d(A, B)$. \square

Theorem 2.6. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty closed and convex subsets of a reflexive and strictly convex Banach space and $T : A \cup B \rightarrow A \cup B$ a cyclic φ -contraction map. If $(A - A) \cap (B - B) = \{0\}$, then there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$.*

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.3, there exists $(x, y) \in A \times B$ with $\|x - y\| = d(A, B)$. To show the uniqueness of (x, y) , suppose that there exists another $(x', y') \in A \times B$ with $\|x' - y'\| = d(A, B)$. Since $(A - A) \cap (B - B) = \{0\}$, we conclude that $x - y \neq x' - y'$. As both A and B are convex, by the strict convexity of X , we have

$$\left\| \frac{x + x'}{2} - \frac{y + y'}{2} \right\| = \left\| \frac{x - y}{2} + \frac{x' - y'}{2} \right\| < d(A, B), \quad (2.21)$$

which is a contradiction. Since $\|Ty - Tx\| = \|Tx - Ty\| = \|x - y\| = d(A, B)$, we obtain, from the uniqueness of (x, y) , that $(Ty, Tx) = (x, y)$. Hence $Tx = y$, $Ty = x$ and $T^2x = x$. \square

Theorem 2.7. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty subsets of a reflexive and strictly convex Banach space X such that A is closed and convex and $T : A \cup B \rightarrow A \cup B$ a cyclic φ -contraction map. Then, there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*

- (a) T is weakly continuous on A .
- (b) T satisfies the proximal property.

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.5 that there exists $x \in A$ with $\|x - Tx\| = d(A, B)$. Thus, $T^2x = x$. Indeed, if we assume that $T^2x - Tx \neq x - Tx$. Then from the convexity of A and the strict convexity of X , we have

$$\left\| \frac{T^2x + x}{2} - Tx \right\| = \left\| \frac{T^2x - Tx}{2} + \frac{x - Tx}{2} \right\| < d(A, B), \quad (2.22)$$

which is a contradiction. The uniqueness of x follows as in the proof of [2, Theorem 8]. \square

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References

- [1] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, "Fixed points for mappings satisfying cyclical contractive conditions," *Fixed Point Theory*, vol. 4, no. 1, pp. 79–89, 2003.
- [2] M. A. Al-Thagafi and N. Shahzad, "Convergence and existence results for best proximity points," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 10, pp. 3665–3671, 2009.
- [3] G. Petrușel, "Cyclic representations and periodic points," *Universitatis Babeș-Bolyai. Studia Mathematica*, vol. 50, no. 3, pp. 107–112, 2005.
- [4] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.