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A Wald test with enhanced selectivity properties in homogeneous environments

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Abstract

A Wald test with enhanced selectivity capabilities is proposed in homogeneous environments. At the design stage, we assume that the cell under test contains a noise-like interferer in addition to colored noise and possible signal of interest. We show that the Wald test is equivalent to a recently proposed Rao test. We also observe that this Rao/Wald test possesses constant false alarm rate property in homogeneous environments.

Keywords: Adaptive detection, Rao test, Wald test, Mismatched signals, Constant false alarm rate (CFAR)

1. Introduction

Detection of a deterministic signal known up to an unknown scaling factor in the presence of colored noise is a fundamental problem in many applications including wireless communications, seismic analysis, hyperspectral imaging, sonar, radar, and others. However, there is no uniformly most powerful test for the quoted problem since the covariance matrix of the noise and the amplitude of the signal are both unknown. Consequently, a variety of different solutions have been explored in open literature under slightly different settings. The most prominent and pioneering detection approaches are Kelly's generalized likelihood ratio test (GLRT) [1], adaptive matched filter (AMF) [2], and adaptive coherence estimator (ACE) [3].

However, the above-cited detectors have been designed without taking into account the possible presence of signal mismatch, and they behave quite differently in this situation. A mismatched signal may arise due to several reasons, for example, imperfect array calibration, spatial multipath, pointing errors, etc. Since it is difficult to find a decision scheme capable of successfully detecting slightly mismatched mainlobe targets and effectively rejecting sidelobe targets simultaneously, it becomes important to achieve a good tradeoff between a high sensitivity of mainlobe targets and perfect rejection of sidelobe targets. In order to meet this goal, several strategies have been exploited. One solution is to design a two-stage detector, which is formed by cascading two detectors: the first-stage detector, usually with perfect sensitivity properties, judges if there is enough received energy entering into the receivers; the second-stage detector, usually with perfect selectivity properties, makes the decision as to whether or not the received signal is to be considered as the signal of interest (SOI). One declares the presence of a target only when the received signal survives both detection thresholds. This is the principle underlying the adaptive sidelobe blanker (ASB) and its improved versions [4-7]. Another solution is to modify the hypothesis test problem by adding a fictitious signal under the null hypothesis; this fictitious signal is assumed to be orthogonal to the presumed signal steering vector. When there is no target in the presumed direction but one in another direction, e.g., a sidelobe target, the detector will incline towards the null hypothesis, which is the desired result. This is the rationale of the adaptive beamformer orthogonal rejection test (ABORT) [8] and whitened ABORT (W-ABORT) [9]. A third solution is to design a tunable detector. For example, in [10], a tunable detector is proposed, which consists of a blend of Kelly's GLRT and AMF through a so-called sensitivity parameter. This parameter controls the degree to which sidelobe targets are rejected. This approach is also used in [11,12], where different tunable detectors are devised through similar sensitivity parameters. A fourth solution is to assume that a noise-like interferer exists in the cell under test (CUT) but not present in the training data. More precisely, the GLRT in this setting is proposed in [13], which is found to be the ACE, while the Rao test is proposed in [14], with the name—double-normalized AMF (DN-AMF). It is shown that the ACE has excellent

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sidelobe signals rejection capabilities, at the price of a certain loss in terms of detection of matched signals. Compared to its natural competitor, the DN-AMF provides both enhanced sidelobe targets rejection capabilities and high-detection performance of mainlobe targets.

The solutions mentioned above are either based on GLRT criterion or based on Rao criterion. As is well known, there are usually three design criteria; besides the GLRT and Rao criteria, another one is the Wald test criterion. Thus, we resort to the Wald test criterion to devise a detector with enhanced selectivity properties in homogeneous environments. At the design stage, we assume that the CUT contains a noise-like interferer in addition to colored noise and possible SOI. In particular, we show that the Wald test is equivalent to the Rao test, i.e., DN-AMF. This shades a new light on the fact that the Rao/Wald in this situation has excellent sidelobe targets rejection capabilities.

The remainder of this article is organized as follows. Section 2 deals with the problem formulation. The Wald test is presented in Section 3, while the equivalence of the Wald and Rao tests is exploited in Section 4. Finally, Section 5 concludes the article.

2. Problem formulation

We assume that data are collected from sensors and denote the complex vector of the primary data by x, with dimension N. As customary, we assume that a secondary dataset, x_l, $l = 1,...,L$, is available, that each of them does not contain any useful signal, and shares the same covariance matrix with the primary data. The detection problem can be formulated as the following binary hypothesis test:

$$H_0 : \begin{cases} x = n; \\ x_l = n_l; \quad l = 1, \ldots, L \end{cases}$$
$$H_1 : \begin{cases} x = as + n; \\ x_l = n_l; \quad l = 1, \ldots, L \end{cases} \tag{1}$$

where $E\{x_l x_l^H\} = R$, $E\{nn^H\} = R + qq^H$. The signal amplitude a, the covariance matrix R, and the noise-like interferer q are all unknown. For notational convenience, let $S = XX^H$, which is L times the sample covariance matrix, with $X = [x_1, x_2,...,x_L]$.

3. The Wald test

Denote by $\theta \in \mathbb{C}^{(1+N+N^2)\times 1}$ the parameter vector, namely,

$$\theta = \left[\theta_r, \theta_s^T\right] = \left[a, q^T, \mathrm{vec}^T(R)\right]^T \tag{2}$$

where $\theta_r = a \in \mathbb{C}^{1\times 1}$ and $\theta_s = \left[q^T, \mathrm{vec}^T(R)\right]^T \in \mathbb{C}^{[N(N+1)]\times 1}$, the notation $\mathrm{vec}(\cdot)$ stands for vectorization operator. Note that θ_s is the so-called nuisance parameter.

The Fisher information matrix (FIM) for real-valued signal is well known, see, for example, [15]. In fact, we can analogically obtain the complex FIM described as follows[a]

$$I(\theta) = E\left[\left(\partial \ln f\left(\frac{x, X|\theta)}{\partial\theta}\right)\left(\partial \ln f\left(\frac{x, X|\theta)}{\partial\theta^*}\right)^H\right] \tag{3}$$

where $f(x, X|\theta)$ is the joint probability density function (PDF) of x and X, under hypothesis H_1, with θ fixed.

Then we partition the Fisher information matrix (FIM) $I(\theta)$ as follows

$$I(\theta) = \begin{bmatrix} I_{\theta_r\theta_r^*} & I_{\theta_r\theta_s^*} \\ I_{\theta_s\theta_r^*} & I_{\theta_s\theta_s^*} \end{bmatrix} \tag{4}$$

The Wald test for real-valued signal is well known [15]. For complex-valued signal, the Wald test is analogously given by

$$t_{\mathrm{Wald}} = \left(\hat{\theta}_{r1} - \theta_{r0}\right)^* \left\{\left[I^{-1}\left(\hat{\theta}_1\right)\right]_{\theta_r\theta_r^*}\right\}^{-1} \left(\hat{\theta}_{r1} - \theta_{r0}\right) \tag{5}$$

where $\hat{\theta}_{r1}$ is the maximum likelihood estimate (MLE) of θ_r under hypothesis H_1,θ_{r0} is the value of θ_r under hypothesis H_0, and $\left\{\left[I^{-1}\left(\hat{\theta}_1\right)\right]_{\theta_r\theta_r^*}\right\}^{-1}$ is the Schur complement of $I_{\theta_s\theta_s^*}$ evaluated at $\hat{\theta}_1$ namely,

$$\left\{\left[I^{-1}\left(\hat{\theta}_1\right)\right]_{\theta_r\theta_r^*}\right\}^{-1} = \left(I_{\theta_r\theta_r^*} - -I_{\theta_r\theta_s^*} I^{-1}_{\theta_s\theta_s^*} I_{\theta_s\theta_r^*}\right)\Big|_{\theta=\hat{\theta}_1} \tag{6}$$

In order to calculate (6), we need the joint probability density function (PDF) of x and X under H_1, which is found to be

$$f(x, X|\theta) = \frac{\exp\left\{-tr\left[R^{-1}\left(S + (x - as)(x - as)^H\right)\right]\right\}}{\pi^{N(L+1)}|R|^{L+1}(1 + q^H R^{-1} q)} \exp\left[\frac{|(x - as)^H R^{-1} q|^2}{(1 + q^H R^{-1} q)}\right] \tag{7}$$

Take the gradient with respect to a, and equate to zero, we obtain the MLE of a described as

$$\hat{a} = \frac{s^H R^{-1} q q^H R^{-1} x - (1 + q^H R^{-1} q) s^H R^{-1} x}{|s^H R^{-1} q|^2 - (1 + q^H R^{-1} q) s^H R^{-1} s} \tag{8}$$

According to (7), we have

$$\frac{\partial^2 \ln f}{\partial a \partial a^*} = \frac{|s^H R^{-1} q|2}{1 + q^H R^{-1} q} - s^H R^{-1} s \tag{9}$$

Note that $I_{\theta_r\theta_r^*} = -\partial^2 \ln f/\partial a \partial a$. Consequently, we arrive at

$$I_{\theta_r\theta_r^*} = s^H R^{-1} s - \left| s^H R^{-1} q \right|^2 \frac{1}{1 + q^H R^{-1} q} \tag{10}$$

$I_{\theta_r\theta_s^*}$ is found to be a null row vector, thus, $\left\{ \left[I^{-1}\left(\hat{\theta}_1\right) \right]_{\theta_r\theta_r^*} \right\}^{-1} = I_{\theta_r\theta_r^*}$. Notice that $\hat{\theta}_{r1} = \hat{a}$ and $\theta_{r0} = 0$, consequently, the intermediate Wald test is given by

$$t_{\text{Wald}} = \frac{|s^H R^{-1} q q^H R^{-1} x - (1 + q^H R^{--1} q) s^H R^{--1} x|^2}{(1 + q^H R^{-1} q)[(1 + q^H R^{-1} q) s^H R^{-1} s - |s^H R^{-1} q|^2]} \tag{11}$$

In order to obtain the explicit Wald test, we need the MLE's of R and q, which are given by [13]

$$\hat{R} = \frac{1}{L+1}\left(S + \frac{x x^H}{L x^H S^{--1} x} \right), \hat{q} = \gamma_0 x, \tag{12}$$

respectively, with γ_0 satisfying the following equation

$$\left| \gamma_0 \right|^2 = \left(x^H \hat{R}^{-1} x - 1 \right) / x^H \hat{R}^{-1} x \tag{13}$$

Plugging (12) and (13) into (11), after some algebraic manipulations, yields the final Wald test as

$$t_{\text{Wald}} = \frac{t_{\text{ACE}}}{x^H S^{-1} x \cdot (1 - t_{\text{ACE}}) + t_{\text{ACE}}} \tag{14}$$

in which t_{ACE} is the ACE statistic with expression as

$$t_{\text{ACE}} = \frac{|s^H S^{-1} x|^2}{s^H S^{-1} s \cdot x^H S^{-1} x} \tag{15}$$

One can easily verify that this Wald test possesses constant false alarm rate (CFAR) property in homogeneous environments.

4. The equivalence of the Wald and Rao tests

Dividing the numerator and denominator of (14) by the quantity $(1 - t_{\text{ACE}})$, we have

$$t_{\text{Wald}} = \frac{\widetilde{t}_{\text{ACE}}}{x^H S^{-1} x + \widetilde{t}_{\text{ACE}}} \tag{16}$$

where $\widetilde{t}_{\text{ACE}} = t_{\text{ACE}}/(1 - t_{\text{ACE}})$

Let $\widetilde{x} = S^{-1/2} x$, $\widetilde{s} = S^{-1/2} s$, then $\widetilde{t}_{\text{ACE}}$ can be rewritten as

$$\widetilde{t}_{\text{ACE}} = \frac{\widetilde{x}^H P_{\widetilde{s}} \widetilde{x}}{\widetilde{x}^H P_{\widetilde{s}}^{\perp} \widetilde{x}} = \frac{P_{\widetilde{s}} \widetilde{x}^2}{P_{\widetilde{s}}^{\perp} \widetilde{x}^2} \tag{17}$$

where $P_{\widetilde{s}}$ is the projection matrix onto $\widetilde{s}$, and $P_{\widetilde{s}}^{\perp}$ is the orthogonal complement of $P_{\widetilde{s}}$.

Using (17), Equation (16) can be rewritten as

$$t_{\text{Wald}} = \frac{1}{1 + \widetilde{x}^2 P_{\widetilde{s}}^{\perp} \widetilde{x}^2 / P_{\widetilde{s}} \widetilde{x}^2} \tag{18}$$

Therefore, t_{Wald} is statistically equivalent to

$$\widetilde{t}_{\text{Wald}} = \frac{P_{\widetilde{s}} \widetilde{x}^2}{P_{\widetilde{s}}^{\perp} \widetilde{x}^2 \widetilde{x}^2} \tag{19}$$

which is exactly the Rao test of [14].

5. Conclusions

We have considered the design of a detector with improved mismatched signals rejection capabilities. To this end, at the design stage it is assumed that the CUT contains a random interferer with its steering vector unknown. Under the above assumption, a Wald test has been designed, which is found equivalent to the Rao test proposed by Orlando and Ricci, which, in turn, is found to yield better performance in terms of mismatched signals rejection.

Endnotes

[a]A different but equivalent complex FIM is given in [16].

Competing interests

The authors declare that they have on competing interests.

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