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On the interval differential equation: novel solution methodology

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Abstract

In this study, by a parametric representation of interval numbers, two parametric representations for interval-valued functions are presented. Then, using these representations, the calculus of interval-valued functions and interval differential equations are investigated with two different approaches. In the first approach, the interval differential is transformed into a crisp problem. In the second approach, two solutions are obtained with the characterization of the solutions of two ordinary differential equation systems.

Keywords: interval arithmetic; interval-valued function; interval differential equation

1 Introduction

The interval-valued analysis and interval differential equations are specific cases of set-valued analysis and set differential equations, respectively [1–3]. Interval analysis is introduced as an attempt to handle interval uncertainty, while interval differential equations are natural models for describing dynamic systems under uncertainty, and this approach is useful in many applications areas, such as physics and engineering [4, 5]. As in classical real analysis, the importance of the derivative of an interval-valued function in the study of interval differential equations is well known. On the other hand, the inversions of addition and multiplication are fundamental in interval arithmetic, interval analysis, and the concept of interval differentiability.

In interval arithmetic, standard Minkowski addition and multiplication are not invertible operations, and the idea of finding some inverse operations has been a field that has long been of interest [6–8]. The Hukuhara difference, which was introduced by Hukuhara in [8], has been a starting point for this purpose, which exists under restrictive conditions; for more details, see [6]. To overcome this shortcoming, Stefanini and Bede [6] proposed the generalized Hukuhara difference of two interval numbers, which has a large advantage over the peer concept, namely, that it always exists. In a similar discussion, the concept of Hukuhara derivative is extended to the generalized Hukuhara derivative. The same remark holds if the concept of differentiability for differential equations in the interval setting is regarded. The diameter of the solution of an interval differential equation with a Hukuhara derivative is a nondecreasing function with respect to (w.r.t.) time [9]. The generalized Hukuhara differentiability for an interval-valued function allows us to obtain the solutions to interval differential equations with a decreasing diameter [6, 7].

Recently, several studies [6, 10] have been performed on generalized Hukuhara differentiability.

A key point in our investigation is the parametric representations for interval numbers. By this notation, interval arithmetic operations are defined, and two parametric representations for interval-valued functions are expressed. The results are applied to introduce the concepts of the derivative and integral of interval-valued functions. Additionally, several properties of the new concepts are investigated and compared with other recently presented results. Moreover, the relationship between the new interval derivative and the interval integral is studied, and Newton-Leibniz type formulas are nontrivially extended to the interval case. Finally, an application to solve interval differential equations is shown in two different aspects: in the first approach, the interval problem has a unique solution, which is obtained by solving an ordinary differential equation, and the results in the second approach are equivalent to the obtained results using the gH-differentiability concept. Furthermore, the existence and uniqueness of the solutions are discussed.

This paper is organized in five sections. In Section 2, the parametric representations of interval numbers are defined, and interval arithmetic operations are presented. The interval-valued functions in the parametric form and their properties, derivatives, integrals and the relations between them are studied in Section 3. Section 4 describes an application of the new results to interval differential equations, and the paper ends with conclusions in Section 5.

2 Arithmetic of intervals

In this section, some basic definitions and existing arithmetic operations between intervals in the parametric form are presented. Let K_c be the space of nonempty compact and convex sets of $\mathbb{R}$, *i.e.*,

$$K_c = \{[a,b] \mid a,b \in \mathbb{R} \text{ and } a \leq b\}.$$

Suppose $A = [\underline{a},\overline{a}] \in K_c$, where $\underline{a}$ and $\overline{a}$ mean the lower and upper bounds of A, respectively. Obviously, two parametric representations for any $A \in K_c$ can be considered:

i. $A = \{a(t) \mid a(t) = \underline{a} + t(\overline{a} - \underline{a}), t \in [0,1]\}$ (increasing representation or *IR*),
ii. $A = \{a(t) \mid a(t) = \overline{a} + t(\underline{a} - \overline{a}), t \in [0,1]\}$ (decreasing representation or *DR*).

Definition 2.1 Let $\{a(t) \mid t \in [0,1]\}$ and $\{b(t) \mid t \in [0,1]\}$ be the *IR*s (*DR*s) of $A = [\underline{a},\overline{a}]$ and $B = [\underline{b},\overline{b}]$, respectively, and λ be a real number. The parametric arithmetics in K_c can be defined as

1. $A \oplus B = \{a(t_1) + b(t_2) \mid t_1, t_2 \in [0,1]\}$,
2. $A \otimes B = \{a(t_1)b(t_2) \mid t_1, t_2 \in [0,1]\}$,
3. $A \oslash B = \{a(t_1)/b(t_2) \mid b(t_2) \neq 0, t_1, t_2 \in [0,1]\}$,
4. $\lambda \odot A = \{\lambda a(t) \mid t \in [0,1]\}$,
5. $A \ominus_p B = \{a(t) - b(t) \mid t \in [0,1]\}$,
6. $A = B \Leftrightarrow \{a(t) \mid t \in [0,1]\} = \{b(t) \mid t \in [0,1]\}$.

Remark 2.2 Note that if A has *IR* (*DR*) and $\lambda \geq 0$, then $\lambda \odot A$ has *IR* (*DR*), and if A has *IR* (*DR*) and $\lambda \leq 0$, then $\lambda \odot A$ has *DR* (*IR*). Moreover, if $a(t) - b(t)$ is an increasing (a decreasing) function, then $A \ominus_p B$ has *IR* (*DR*).

Remark 2.3 By Definition 2.1, two intervals A and B are equal if and only if $a(t) = b(t)$, for all $t \in [0,1]$.

Example 2.1 Assume that $A = [-1,2]$ and $B = [1,5]$. According to *IR*

$$A = \{-1 + 3t \mid t \in [0,1]\}, \qquad B = \{1 + 4t \mid t \in [0,1]\},$$

and thus

1. $A \oplus B = \{(-1 + 3t_1) + (1 + 4t_2) \mid t_1, t_2 \in [0,1]\} = \{3t_1 + 4t_2 \mid t_1, t_2 \in [0,1]\}$, because $3t_1 + 4t_2$ is a continuous function w.r.t. t_1, t_2, $A \oplus B = [\min_{t_1,t_2} 3t_1 + 4t_2, \max_{t_1,t_2} 3t_1 + 4t_2] = [0,7]$,
2. $A \otimes B = \{(-1 + 3t_1)(1 + 4t_2) \mid t_1, t_2 \in [0,1]\} = [-5,10]$, because of continuity of the function $-1 - 4t_2 + 3t_1 + 12t_1t_2$ w.r.t. t_1, t_2,
3. $A \oslash B = \{(-1 + 3t_1)/(1 + 4t_2) \mid t_1, t_2 \in [0,1]\} = [-1,2]$,
4. $-2 \odot A = \{-2(-1 + 3t) \mid t \in [0,1]\} = \{2 - 6t \mid t \in [0,]\}$, due to the decreasing function $2 - 6t$ and *DR*, $-2 \odot A = [-4,2]$,
5. $A \ominus_p B = \{(-1 + 3t) - (1 + 4t) \mid t \in [0,1]\} = \{-2 - t \mid t \in [0,1]\} = [-3,-2]$, from *DR*,
6. $B \ominus_p A = \{(1 + 4t) - (-1 + 3t) \mid t \in [0,1]\} = \{2 + t \mid t \in [0,1]\} = [2,3]$, from *IR*.

If $\lambda = -1$, then the interval $(-1)A$ gives the additive inverse of A, which is denoted by $-A$, where $-A = \{-a(t) \mid t \in [0,1]\}$. Note that, generally, $A \oplus (-1)A \neq \{0\}$. Therefore, another difference, $A - B = A \oplus (-1)B$, which is constructed with the following parametric form, can be defined:

$$A - B = \{a(t_1) - b(t_2) \mid a(t_1) = \underline{a} + t_1(\overline{a} - \underline{a}), b(t_2) = \underline{b} + t_2(\overline{b} - \underline{b}), t_1, t_2 \in [0,1]\}.$$

Because $a(t_1) - b(t_2)$ is a continuous function w.r.t. t_1 and t_2, we have

$$A - B = \Bigl[\min_{t_1,t_2 \in [0,1]} \bigl(a(t_1) - b(t_2)\bigr), \max_{t_1,t_2 \in [0,1]} \bigl(a(t_1) - b(t_2)\bigr)\Bigr].$$

If $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, then $a(t_1) - b(t_2) = \underline{a} - \underline{b} + t_1(\overline{a} - \underline{a}) - t_2(\overline{b} - \underline{b})$, thus

$$\min_{t_1,t_2 \in [0,1]} \bigl(a(t_1) - b(t_2)\bigr) = \underline{a} - \underline{b} - (\overline{b} - \underline{b}) = \underline{a} - \overline{b}$$

and

$$\max_{t_1,t_2 \in [0,1]} \bigl(a(t_1) - b(t_2)\bigr) = \underline{a} - \underline{b} + (\overline{a} - \underline{a}) = \overline{a} - \underline{b}.$$

Hence, $A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$, which is the same as the Minkowski difference [11]. Note that, in general, $(A \oplus B) - B \neq A$, while this equality is valid for the parametric difference.

Remark 2.4 Obviously, $A \ominus_p B = A - B$ and $B \ominus_p A = B - A$, whenever $A \in \mathbb{R}$ and $B \in K_c$.

Proposition 2.5 *The parametric difference* $\ominus_p$ *has the following properties*:

(i) $A \ominus_p A = \{0\}$,
(ii) $A \ominus_p B = -(B \ominus_p A)$,

(iii) $(A \oplus B) \ominus_p B = A$, $A \ominus_p (A \oplus B) = -B$, $A \ominus_p (A - B) = B$,
(iv) $A \ominus_p B = B \ominus_p A$ *if and only if* $C = -C$ *and* $A \ominus_p B = C$, *and particulary* $C = \{0\}$ *if and only if* $A = B$,
(v) $A \oplus (B \ominus_p A) = B$ *or* $B - (B \ominus_p A) = A$,
(vi) *if* $A \ominus_p B$ *has* IR (DR) *and* $A \ominus_p B = C$, *then* $A \ominus_p C = B$ $(A - C = B)$,
(vii) *if* $A - B = C$, *then* $A \ominus_p C = B$.

Proof Suppose $A = [\underline{a}, \overline{a}] = \{a(t) \mid a(t) = \underline{a} + t(\overline{a} - \underline{a})\}$ and $B = [\underline{b}, \overline{b}] = \{b(t) \mid b(t) = \underline{b} + t(\overline{b} - \underline{b})\}$. Property (i) is an immediate consequence of Definition 2.1. To prove (ii), one has to show that

$$\{a(t) - b(t) \mid t \in [0,1]\} = \{-(b(t) - a(t)) \mid t \in [0,1]\},$$

which can be obtained as

$$\begin{aligned} \{a(t) - b(t) \mid t \in [0,1]\} &= \{\underline{a} + t(\overline{a} - \underline{a}) - \underline{b} - t(\overline{b} - \underline{b}) \mid t \in [0,1]\} \\ &= \{-(\underline{b} - \underline{a} - t(\overline{a} - \underline{a}) + t(\overline{b} - \underline{b})) \mid t \in [0,1]\} \\ &= \{-(\underline{b} + t(\overline{b} - \underline{b}) - (\underline{a} + t(\overline{a} - \underline{a}))) \mid t \in [0,1]\} \\ &= \{-(b(t) - a(t)) \mid t \in [0,1]\}. \end{aligned}$$

The first part of (iii) is evidently proved by

$$(A \oplus B) \ominus_p B = \{\underline{a} + \underline{b} + t((\overline{a} + \overline{b}) - (\underline{a} + \underline{b})) - \underline{b} - t(\overline{b} - \underline{b}) \mid t \in [0,1]\} = \{a(t) \mid t \in [0,1]\} = A.$$

The second part of (iii) follows from a similar argument. To prove the third part, it is sufficient to note that $A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$. To denote the first part of (vi), suppose $A \ominus_p B = B \ominus_p A$; then by Definition 2.1

$$\{a(t) - b(t) \mid t \in [0,1]\} = \{b(t) - a(t) \mid t \in [0,1]\}.$$

On the other hand, $A \ominus_p B = C$, and thus it follows that $\{c(t) \mid t \in [0,1]\} = \{-c(t) \mid t \in [0,1]\}$. This gives $C = -C$, which proves our assertion, and because of property (ii) the reverse holds true. For the second part of (vi), from $C = \{0\}$ and $A \ominus_p B = C$, it can be concluded that $\{a(t) - b(t) \mid t \in [0,1]\} = \{0\}$, and thus, by Remark 2.3, it can be deduced that $a(t) - b(t) = 0$ and $a(t) = b(t)$, for all $t \in [0,1]$. Hence,

$$\{a(t) \mid t \in [0,1]\} = \{b(t) \mid t \in [0,1]\},$$

which shows $A = B$. Conversely, let $A = B$; thus by property 1, $C = A \ominus_p B = \{0\}$. To deduce (v), if $B \ominus_p A$ has *IR* then $A \oplus (B \ominus_p A) = B$, and if it has *DR*, then $B - (B \ominus_p A) = A$. Next, let us to prove (vi). First, consider that $A \ominus_p B = C$ has *IR* thus $C = A \ominus_p B = \{c(t) \mid c(t) = a(t) - b(t) = \underline{a} - \underline{b} + t(\overline{a} - \underline{a} - \overline{b} - \underline{b})\}$, where $\overline{a} - \underline{a} - \overline{b} - \underline{b}$ is positive. Therefore,

$$\begin{aligned} A \ominus_p C &= \{a(t) - c(t) \mid t \in [0,1]\} \\ &= \{\underline{a} + t(\overline{a} - \underline{a}) - (\underline{a} - \underline{b} + t(\overline{a} - \underline{a} - \overline{b} - \underline{b})) \mid t \in [0,1]\} \\ &= \{\underline{b} + t(\overline{b} - \underline{b}) \mid t \in [0,1]\} = B. \end{aligned}$$

With a similar argument as above, one can prove the second case. To see (vii), assume that $A - B = C$. Thus, $C = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$, and by Definition 2.1

$$\begin{aligned} A \ominus_p C &= \{a(t) - c(t) \mid t \in [0,1]\} \\ &= \{\underline{a} + t(\overline{a} - \underline{a}) - (\underline{a} - \overline{b} + t(\overline{a} - \underline{b} - \underline{a} + \overline{b})) \mid t \in [0,1]\} \\ &= \{\overline{b} + t(\underline{b} - \overline{b}) \mid t \in [0,1]\} = B. \end{aligned}$$

The last equality follows from *DR* of *B*. □

Stefanini [11] proposed a generalization of the Hukuhara difference for interval numbers, namely gH-difference. In the next preposition, it will be shown that the concepts of gH-difference and parametric difference are equivalent. So the parametric difference is equivalent with the concept of difference defined by Markov [12] and with π-difference [13].

Proposition 2.6 *The parametric difference is equivalent to the gH-difference.*

Proof Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, and $C = [\underline{c}, \overline{c}] = A \ominus_p B$. It is sufficient to show that $\underline{c} = \min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}$ and $\overline{c} = \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}$. There are two cases, which are dependent on whether the function $a(t) - b(t) = \underline{a} - \underline{b} + t(\overline{a} - \underline{a} - \overline{b} - \underline{b})$ is increasing or decreasing w.r.t. t.

1. In the increasing case, we have $\overline{a} - \overline{b} \geq \underline{a} - \underline{b}$ and by *IR*, and thus

$$\begin{aligned} &\underline{c} = \underline{a} - \underline{b} = \min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \\ &\overline{c} - \underline{c} = \overline{a} - \underline{a} - \overline{b} + \underline{b} \quad \Rightarrow \quad \overline{c} = \overline{a} - \overline{b} = \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}. \end{aligned}$$

2. In the decreasing case, we have $\overline{a} - \overline{b} \leq \underline{a} - \underline{b}$ and by *DR*, it can be concluded that

$$\begin{aligned} &\overline{c} = \underline{a} - \underline{b} = \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \\ &\underline{c} - \overline{c} = \overline{a} - \underline{a} - \overline{b} + \underline{b} \quad \Rightarrow \quad \underline{c} = \overline{a} - \overline{b} = \min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \end{aligned}$$

which is the desired conclusion. □

Proposition 2.7 *For any two intervals $A, B \in K_c$, the parametric difference $A \ominus_p B = C$ always exists and is unique.*

Proof Clearly, from the definition of a parametric difference

$$C = \{a(t) - b(t) \mid a(t) - b(t) = \underline{a} - \underline{b} + t(\overline{a} - \overline{b} - (\underline{a} - \underline{b})), t \in [0,1]\}.$$

First, suppose that $a(t) - b(t)$ be an increasing function. Then $\overline{a} - \overline{b} \geq \underline{a} - \underline{b}$ and by Remark 2.2 the interval number C has *IR* and $C = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$, which is a unique interval in K_c. Secondly, suppose $a(t) - b(t)$ be a decreasing function then the assertion can be proved in a similar way. □

For arbitrary intervals $A, B \in K_c$, the metric $D : K_c \times K_c \to \mathbb{R}_+ \cup \{0\}$ is defined as

$$D(A,B) = \max\Bigl\{\min_t |a(t) - b(t)|, \max_t |a(t) - b(t)|\Bigr\}, \tag{1}$$

where $A = \{a(t) \mid a(t) = \underline{a} + t(\overline{a} - \underline{a}), t \in [0,1]\}$ and $B = \{b(t) \mid b(t) = \underline{b} + t(\overline{b} - \underline{b}), t \in [0,1]\}$.

Proposition 2.8 (K_c, D) *is a complete metric space that have following properties*:
1. $D(A \oplus C, B \oplus C) = D(A, B)$,
2. $D(\lambda A, \lambda B) = |\lambda| D(A, B)$,
3. $D(A \oplus B, C \oplus E) \leq D(A, C) + D(B, E)$,
4. $D(A, B) = D(A \ominus_p B, \{0\})$.

Proof We prove only the final part; the other parts are trivial. Suppose

$$D\big(A \ominus_p B, \{0\}\big) = \max\Big\{\min_t \big|c(t)\big|, \max_t \big|c(t)\big|\Big\},$$

where $C = A \ominus_p B$ and $c(t)$ is *IR* of C. If $a(t) - b(t)$ is an increasing function, then $c(t) = a(t) - b(t)$ and $D(A \ominus_p B, \{0\}) = D(A, B)$. If $a(t) - b(t)$ is a decreasing function, then

$$\max_t \big|c(t)\big| = \min_t \big|a(t) - b(t)\big|, \qquad \min_t \big|c(t)\big| = \max_t \big|a(t) - b(t)\big|,$$

and $D(A \ominus_p B, \{0\}) = D(A, B)$. □

The Hausdorff distance between two intervals $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ is defined as follows (see [6]):

$$D_H(A, B) = \big\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\big\}$$

Proposition 2.9 *The Hausdorff distance is equivalent to the metric D in* (1).

Proof The proof is clearly depending on whether $a(t) - b(t)$ is an increasing or a decreasing function. □

3 Parametric representation of interval-valued function

Let $F : T \subseteq \mathbb{R} \to K_c$ be an interval-valued function with lower bound $\underline{f}$ and upper bound $\overline{f}$, *i.e.*, $F(x) = [\underline{f}(x), \overline{f}(x)]$. Then it is trivial to see that a parametric representation of $F(x)$ is

$$\big\{\underline{f}(x) + t\big(\overline{f}(x) - \underline{f}(x)\big) \mid t \in [0,1]\big\}. \tag{2}$$

An interval-valued function with real independent variable and interval coefficients is special case of $F : T \subseteq \mathbb{R} \to K_c$. Using *IR*s of coefficients, it is possible to write this function as a set of classical functions. As an example, consider $F(x) = [-2, 3] \odot x \oplus [-3, -1] \odot x^2 \otimes e^{[1,4] \odot x} \oplus \sin([3, 6] \odot x)$. If $\mathbf{C}_\nu^4 = ([-2, 3], [-3, -1], [1, 4], [3, 6])^T$ is a column vector in $(K_c)^4$, whose elements are intervals which, respectively, appear in the function F, then by *IR*s

$$\mathbf{C}_\nu^4 = \big\{(-2 + 5t_1, -3 + 2t_2, 1 + 3t_3, 3 + 3t_4)^T \mid \mathbf{t} = (t_1, t_2, t_3, t_4) \in [0,1]^4\big\}.$$

Hence, $F(x)$ can be redefined as

$$F_{\mathbf{C}_\nu^4}(x) = \big\{f_{\mathbf{c(t)}}(x) \mid f_{\mathbf{c(t)}}(x) = (-2 + 5t_1)x + (-3 + 2t_2)x^2 e^{(1+3t_3)x} + \sin\big((3 + 3t_4)x\big); \mathbf{c(t)} \in \mathbf{C}_\nu^4\big\}.$$

In the light of the concept of this example, if $\mathbf{C}_\nu^k \in (K_c)^k$ denotes the intervals that are presented in an interval-valued function, then another parametric representation for this class of function, *i.e.*, $F_{\mathbf{C}_\nu^k} : T \subseteq \mathbb{R} \to K_c$, can be written as

$$F_{\mathbf{C}_\nu^k}(x) = \{f_{\mathbf{c}(\mathbf{t})}(x) \mid f_{\mathbf{c}(\mathbf{t})} : T \subseteq \mathbb{R} \to \mathbb{R}; \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\}, \tag{3}$$

where $\mathbf{C}_\nu^k = \{\mathbf{c}(\mathbf{t}) \mid \mathbf{c}(\mathbf{t}) = (c_1(t_1), c_2(t_2), \dots, c_k(t_k))^T; c_j(t_j) = \underline{c}_j + t_j(\overline{c}_j - \underline{c}_j), \mathbf{t} = (t_1, t_2, \dots, t_k) \in [0,1]^k\}$. Clearly, $f_{\mathbf{c}(\mathbf{t})}(x)$ is a continuous function in $\mathbf{t}$ for every fixed x.

Note that the parametric variables t_j, $j = 1, 2, \dots, k$ in (3) are real numbers in the interval $[0,1]$; however, the parameter t in (2) is a function of x, *i.e.*, $t : \mathbb{R} \to [0,1]$. For more precise look, consider the interval-valued function $F_{\mathbf{C}_\nu^2}(x) = [1,2] \odot x \oplus [2,4] \odot x^2$, $0 < x < 2$. By (2) and (3), two parametric representations can be considered:

$$F_{\mathbf{C}_\nu^2}(x) = \{x + 2x^2 + t(x + 2x^2) \mid t \in [0,1]\}, \tag{4}$$

$$F_{\mathbf{C}_\nu^2}(x) = \{(1 + t_1)x + (2 + 2t_2)x^2 \mid \mathbf{t} = (t_1, t_2) \in [0,1]^2\}, \tag{5}$$

respectively. By putting $t_1 = 0$ and $t_2 = \frac{1}{2}$ in (5), the function $x + 3x^2$ is obtained, while this function can be acquired by setting $t = \frac{x^2}{x+2x^2}$ in the representation (4).

Next, by the parametric representations (2) and (3), some concepts such as limit, continuity, derivative, and integral for interval-valued functions will be defined in a parametric form. Using these concepts, interval differential equations will be studied by two approaches. Considering the parametric representation (2), the results in [6, 10] are obtained, as we expected.

Definition 3.1 Let $F : T \subseteq \mathbb{R} \to K_c$ be an interval-valued function and $x_0 \in T$. $L \in K_c$ is the limit of F at x_0, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $D(F(x), L) < \varepsilon$, whenever $|x - x_0| < \delta$ and it is denoted by $\lim_{x \to x_0} F(x) = L$.

Proposition 3.2 *For an interval-valued function* $F : T \subseteq \mathbb{R} \to K_c$,

$$\lim_{x \to x_0} F(x) = L \quad \Leftrightarrow \quad \lim_{x \to x_0} (F(x) \ominus_p L) = \{0\}.$$

Proof It can be proved from Definition 3.1 and Proposition 2.8. □

From the definition of limit for interval-valued functions, it is clear that $F : T \subseteq \mathbb{R} \to K_c$ is continuous at point $x_0 \in T$, if $\lim_{x \to x_0} F(x) = F(x_0)$. Furthermore, F is continuous on T if it is continuous at any point in T.

Proposition 3.3 *Suppose* $F_{\mathbf{C}_\nu^k} : T \subseteq \mathbb{R} \to K_c$ *be an interval-valued function and*

$$F_{\mathbf{C}_\nu^k}(x) = \{f_{\mathbf{c}(\mathbf{t})}(x) \mid f_{\mathbf{c}(\mathbf{t})} : T \subseteq \mathbb{R} \to \mathbb{R}, \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\}.$$

If $\lim_{x \to x_0} f_{\mathbf{c}(\mathbf{t})}(x)$ *exists for every* $\mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k$, *then* $\lim_{x \to x_0} F_{\mathbf{C}_\nu^k}(x)$ *exists and*

$$\lim_{x \to x_0} F_{\mathbf{C}_\nu^k}(x) = \Big\{\lim_{x \to x_0} f_{\mathbf{c}(\mathbf{t})}(x) \,\Big|\, f_{\mathbf{c}(\mathbf{t})} : T \subseteq \mathbb{R} \to \mathbb{R}, \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\Big\}.$$

Moreover, $F_{\mathbf{C}_\nu^k}(x)$ is continuous at x_0, if for every $\mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k$ the function $f_{\mathbf{c}(\mathbf{t})}$ is continuous at x_0.

Proof Assume that $\lim_{x\to x_0} f_{\mathbf{c}(\mathbf{t})}(x) = a(\mathbf{t})$ for every $\mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k$. Thus, for every $\mathbf{t} \in [0,1]^k$, it can be concluded that

$$\forall \varepsilon > 0, \exists \delta > 0, \quad 0 < |x - x_0| < \delta \quad \Rightarrow \quad \left|f_{\mathbf{c}(\mathbf{t})}(x) - a(\mathbf{t})\right| < \varepsilon. \tag{6}$$

Because of continuity of $f_{\mathbf{c}(\mathbf{t})}$ and $a(\mathbf{t})$ in $\mathbf{t}$, there exist $\mathbf{t}', \mathbf{t}'' \in [0,1]^k$ such that

$$\min_{\mathbf{t}} \left|f_{\mathbf{c}(\mathbf{t})}(x) - a(\mathbf{t})\right| = \left|f_{\mathbf{c}(\mathbf{t}')}(x) - a(\mathbf{t}')\right|, \qquad \max_{\mathbf{t}} \left|f_{\mathbf{c}(\mathbf{t})}(x) - a(\mathbf{t})\right| = \left|f_{\mathbf{c}(\mathbf{t}'')}(x) - a(\mathbf{t}'')\right|.$$

By (6), for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$, such that

$$0 < |x - x_0| < \delta_1 \quad \Rightarrow \quad \left|f_{\mathbf{c}(\mathbf{t}')}(x) - a(\mathbf{t}')\right| < \varepsilon,$$
$$0 < |x - x_0| < \delta_1 \quad \Rightarrow \quad \left|f_{\mathbf{c}(\mathbf{t}'')}(x) - a(\mathbf{t}'')\right| < \varepsilon.$$

Next, by choosing $\overline{\delta} = \min\{\delta_1, \delta_2\}$,

$$D\left(F_{\mathbf{C}_\nu^k}(x), A\right) = \max\left\{\min_{\mathbf{t}} \left|f_{\mathbf{c}(\mathbf{t})}(x) - a(\mathbf{t})\right|, \max_{\mathbf{t}} \left|f_{\mathbf{c}(\mathbf{t})}(x) - a(\mathbf{t})\right|\right\} < \varepsilon,$$

whenever $0 < |x - x_0| < \overline{\delta}$ and $A = [\min_{\mathbf{t}} a(\mathbf{t}), \max_{\mathbf{t}} a(\mathbf{t})]$. Because ε is arbitrary, this means that $\lim_{x\to x_0} F_{\mathbf{C}_\nu^k}(x) = A$, which is due to Definition 3.1. In the same manner, it can be proved that the interval-valued function $F_{\mathbf{C}_\nu^k}$ is continuous at x_0, when for every $\mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k$ the function $f_{\mathbf{c}(\mathbf{t})}$ is continuous at x_0. □

Proposition 3.4 *If $F_{\mathbf{C}_\nu^k}(x)$ is continuous at $x_0 \in T$ and $F_{\mathbf{C}_\nu^k}(x) = \{f_{\mathbf{c}(\mathbf{t})}(x) \mid f_{\mathbf{c}(\mathbf{t})} : T \subseteq \mathbb{R} \to \mathbb{R}, \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\}$, then for all $\mathbf{t} \in [0,1]^k$, $f_{\mathbf{c}(\mathbf{t})}(x)$ is continuous at x_0.*

Proof Let $F_{\mathbf{C}_\nu^k}(x)$ be continuous at x_0. By the definition of continuity and Proposition 3.2

$$\lim_{x\to x_0} F_{\mathbf{C}_\nu^k}(x) = F_{\mathbf{C}_\nu^k}(x_0)$$
$$\Leftrightarrow \quad \lim_{x\to x_0} \left(F_{\mathbf{C}_\nu^k}(x) \ominus_p F_{\mathbf{C}_\nu^k}(x_0)\right) = \{0\}$$
$$\Leftrightarrow \quad \left\{\lim_{x\to x_0} \left(f_{\mathbf{c}(\mathbf{t})}(x) - f_{\mathbf{c}(\mathbf{t})}(x_0)\right) \,\middle|\, f_{\mathbf{c}(\mathbf{t})} : T \subseteq \mathbb{R} \to \mathbb{R}, \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\right\} = \{0\}.$$

Hence, for every $\mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k$,

$$\lim_{x\to x_0} \left(f_{\mathbf{c}(\mathbf{t})}(x) - f_{\mathbf{c}(\mathbf{t})}(x_0)\right) = 0,$$

and finally

$$\lim_{x\to x_0} f_{\mathbf{c}(\mathbf{t})}(x) = f_{\mathbf{c}(\mathbf{t})}(x_0),$$

which is the desired conclusion. □

Remark 3.5 Let $F(x) = \{\underline{f}(x) + t(\overline{f}(x) - \underline{f}(x)) \mid t \in [0,1]\}$ be a continuous function at x_0. Then, similar to Proposition 3.4, one can prove that $\underline{f}(x)$ and $\overline{f}(x)$ are continuous functions at x_0.

Markov defined the differential calculus for interval-valued function $F(x)$ including differentiability [12, 14]. In these forms an interval-valued function is expressed as its boundary function $\underline{f}(x)$ and $\overline{f}(x)$. So the existence of its derivative depends upon the existence of the derivative of the boundary functions. In the following definition, the concept of p-derivative for an interval-valued function will be defined, which depends upon the existence of the derivative of $f_{\mathbf{c}(\mathbf{t})}(x)$ for every value $\mathbf{t}$.

Definition 3.6 The p-derivative of an interval-valued function $F : (a,b) \to K_c$ at $x_0 \in (a,b)$ is denoted by $F'(x_0)$ and is defined as

$$F'(x_0) = \lim_{h\to 0} \frac{1}{h}\big[F(x_0 + h) \ominus_p F_{\mathbf{C}_\nu^k}(x_0)\big], \tag{7}$$

where h is such that $x_0 + h \in (a,b)$. In this case, F is called p-differentiable at x_0.

Proposition 3.7 *The interval-valued function $F_{\mathbf{C}_\nu^k} : (a,b) \to K_c$ with the parametric representation $F_{\mathbf{C}_\nu^k}(x) = \{f_{\mathbf{c}(\mathbf{t})}(x) \mid f_{\mathbf{c}(\mathbf{t})} : (a,b) \to \mathbb{R}, \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\}$ is p-differentiable at $x_0 \in (a,b)$ if for every $\mathbf{t} \in [0,1]^k$, $f_{\mathbf{c}(\mathbf{t})}$ is differentiable at x_0 and additionally*

$$F'_{\mathbf{C}_\nu^k}(x_0) = \big\{f'_{\mathbf{c}(\mathbf{t})}(x_0) \mid f_{\mathbf{c}(\mathbf{t})} : (a,b) \to \mathbb{R}; \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\big\}.$$

Proof By Definition 2.1

$$F_{\mathbf{C}_\nu^k}(x_0 + h) \ominus_p F_{\mathbf{C}_\nu^k}(x_0) = \big\{f_{\mathbf{c}(\mathbf{t})}(x_0 + h) - f_{\mathbf{c}(\mathbf{t})}(x_0) \mid f_{\mathbf{c}(\mathbf{t})} : (a,b) \to \mathbb{R}; \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\big\}.$$

Because $f_{\mathbf{c}(\mathbf{t})}(x)$ is differentiable at x_0, $\lim_{h\to 0} \frac{f_{\mathbf{c}(\mathbf{t})}(x_0+h) - f_{\mathbf{c}(\mathbf{t})}(x_0)}{h}$ exists. Hence, by Proposition 3.3, $\lim_{h\to 0} \frac{F_{\mathbf{C}_\nu^k}(x_0+h)\ominus_p F_{\mathbf{C}_\nu^k}(x_0)}{h}$ exists and using Definition 3.6

$$\begin{aligned}
F'_{\mathbf{C}_\nu^k}(x_0) &= \lim_{h\to 0} \frac{F_{\mathbf{C}_\nu^k}(x_0 + h) \ominus_p F_{\mathbf{C}_\nu^k}(x_0)}{h} \\
&= \left\{\lim_{h\to 0} \frac{f_{\mathbf{c}(\mathbf{t})}(x_0 + h) - f_{\mathbf{c}(\mathbf{t})}(x_0)}{h} \,\middle|\, f_{\mathbf{c}(\mathbf{t})} : (a,b) \to \mathbb{R}; \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\right\} \\
&= \big\{f'_{\mathbf{c}(\mathbf{t})}(x_0) \mid f_{\mathbf{c}(\mathbf{t})} : (a,b) \to \mathbb{R}; \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k\big\},
\end{aligned}$$

which proves the proposition. □

The converse of Proposition 3.7 is not true, *i.e.*, it is possible that $F_{\mathbf{C}_\nu^k}(x)$ is p-differentiable at given point x_0 whenever for every $\mathbf{t} \in [0,1]^k$, the function $f_{\mathbf{c}(\mathbf{t})}$ is not differentiable at x_0. For example, assume that $F_{\mathbf{C}_\nu^1}(x) = [-2,2] \odot g(x)$, where $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function such that

$$g(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Obviously, $F_{\mathbf{C}_v^1}(x)$ is p-differentiable at $x_0 = 0$ and due to Definition 3.6

$$\begin{aligned} F'_{\mathbf{C}_v^1}(0) &= \lim_{h\to 0}\frac{F_{\mathbf{C}_v^1}(h)\ominus_p F_{\mathbf{C}_v^1}(0)}{h} \\ &= \left\{\lim_{h\to 0}\frac{f_{\mathbf{c}(t)}(h)-f_{\mathbf{c}(t)}(0)}{h}\,\middle|\, f_{\mathbf{c}(t)}(x) = \begin{cases}(-2+4t)(x), & x\geq 0,\\ (-2+4t)(-x), & x<0;\end{cases} t\in[0,1]\right\} \\ &= \left\{f_{\mathbf{c}(t)}(x)\,\middle|\, f_{\mathbf{c}(t)}(x) = \begin{cases}-2+4t, & x\geq 0,\\ 2-4t, & x<0;\end{cases} t\in[0,1]\right\} = [-2,2]. \end{aligned}$$

But

$$f_{\mathbf{c(t)}} = \begin{cases}(-2+4t)(x), & x\geq 0,\\ (-2+4t)(-x), & x<0,\end{cases}$$

is not differentiable at 0 for every $t\in[0,1]$.

In the next proposition, the relationship between gH-differentiability concept [6] and p-differentiability is expressed.

Proposition 3.8 *If functions $\underline{f}$ and $\overline{f}$ are differentiable at x_0, then the interval-valued function $F:(a,b)\to K_c$ with $F(x) = \{\underline{f}(x) + t(\overline{f}(x) - \underline{f}(x)) \mid t\in[0,1]\}$ is p-differentiable at x_0, and $F'(x) = \{\underline{f}'(x) + t(\overline{f}'(x) - \underline{f}'(x)) \mid t\in[0,1]\}$. Moreover, in this case, the concepts of p-differentiability and gH-differentiability are equivalent.*

Proof It can be deduced that

$$\begin{aligned} F'(x_0) &= \left\{\lim_{h\to 0}\frac{\underline{f}(x_0+h)+t(\overline{f}(x_0+h)-\underline{f}(x_0+h))-\underline{f}(x_0)-t(\overline{f}(x_0)-\underline{f}(x_0))}{h}\,\middle|\, t\in[0,1]\right\} \\ &= \left\{\lim_{h\to 0}\frac{\underline{f}(x_0+h)-\underline{f}(x_0)}{h} + t\lim_{h\to 0}\frac{\overline{f}(x_0+h)-\overline{f}(x_0)}{h}\right. \\ &\quad \left. - t\lim_{h\to 0}\frac{\underline{f}(x_0+h)-\underline{f}(x_0)}{h}\,\middle|\, t\in[0,1]\right\} \\ &= \left\{\underline{f}'(x_0)+t\big(\overline{f}'(x_0)-\underline{f}'(x_0)\big)\mid t\in[0,1]\right\}. \end{aligned}$$

Next, if $\overline{f}'(x_0) - \underline{f}'(x_0)\geq 0$, then by *IR*

$$F'(x_0) = \big[\underline{f}'(x_0), \overline{f}'(x_0)\big],$$

that is consistent with (i)-gH-differentiability. If $\overline{f}'(x_0) - \underline{f}'(x_0)\leq 0$, then by *DR*

$$F'(x_0) = \big[\overline{f}'(x_0), \underline{f}'(x_0)\big],$$

which is in accordance with (ii)-gH-differentiability. □

Remark 3.9 Due to the definition of the interval derivative (7) and two parametric representations (2) and (3) for interval-valued functions, in general, it cannot be expected that the derivative of an interval-valued function be equal based on these representations. This outcome occurs because the derivative for representation (3) is calculated regardless of the sign of the independent variable x, while the derivative for representation (2) depends on the sign of x. For example, consider the interval-valued function $F_{\mathbf{C}_v^2}(x) = [-1,2] \odot x \oplus [2,4]e^x$, $x \in [-2,5]$. By the parametric representation (3), $F_{\mathbf{C}_v^2}(x)$ can be written as $F_{\mathbf{C}_v^2}(x) = \{(-1+3t_1)x + (2+2t_2)e^x \mid t_1, t_2 \in [0,1]\}$, and it is possible to calculate its p-derivative from Proposition 3.7 as

$$\begin{aligned} F'_{\mathbf{C}_v^2}(x) &= \bigl\{(-1+3t_1) + (2+2t_2)e^x \mid t_1, t_2 \in [0,1]\bigr\} \\ &= [-1,2] \oplus [2,4] \odot e^x = \bigl[-1+2e^x, 2+4e^x\bigr]. \end{aligned} \tag{8}$$

Next, by the parametric representation (2), it follows that

$$F_{\mathbf{C}_v^2}(x) = \begin{cases} \{2x + 2e^x + t(2e^x - 3x) \mid t \in [0,1]\}, & -2 \le x \le 0, \\ \{-x + 2e^x + t(2e^x + 3x) \mid t \in [0,1]\}, & 0 < x \le 5, \end{cases}$$

and its derivative is given by

$$\begin{aligned} F'_{\mathbf{C}_v^2}(x) &= \begin{cases} \{2 + 2e^x + t(2e^x - 3) \mid t \in [0,1]\}, & -2 \le x \le 0, \\ \{-1 + 2e^x + t(2e^x + 3) \mid t \in [0,1]\}, & 0 < x \le 5 \end{cases} \\ &= \begin{cases} [-1+4e^x, 2+2e^x], & -2 \le x \le 0, \\ [-1+2e^x, 2+4e^x], & 0 < x \le 5, \end{cases} \end{aligned}$$

which is different from (8).

According to Proposition 3.8, two cases corresponding to increasing and decreasing representations are distinguished for the definition of p-differentiability.

Definition 3.10 Suppose that $F : (a,b) \to K_c$ is an interval-valued function with parametric representation $F = \{\underline{f}(x) + t(\overline{f}(x) - \underline{f}(x)) \mid t \in [0,1]\}$. Then F is (i)-p-differentiable at x_0 if

$$F'(x_0) = \bigl\{\underline{f}'(x_0) + t\bigl(\overline{f}'(x_0) - \underline{f}'(x_0)\bigr) \mid t \in [0,1]\bigr\}, \tag{9}$$

and it is (d)-p-differentiable at x_0 if

$$F'(x_0) = \bigl\{\overline{f}'(x_0) + t\bigl(\underline{f}'(x_0) - \overline{f}'(x_0)\bigr) \mid t \in [0,1]\bigr\}. \tag{10}$$

The concept of a switching point can be extended by the above definition.

Definition 3.11 A point $x_0 \in (a,b)$ is said to be a switching point for the differentiability of interval-valued function F, if in any neighborhood N of x_0 there exist points $x_1 < x_0 < x_2$ such that

(type I) at x_1 (9) holds, while (10) does not hold and at x_2 (10) holds and (9) does not hold, or (type II) at x_1 (10) holds, while (9) does not hold and at x_2 (9) holds and (10) does not hold.

Remark 3.12 Using Definition 3.11 and Proposition 3.8, it is easy to find switching points. For this purpose, it is sufficient to determine the sign of $\overline{f}' - \underline{f}'$ in the parametric representation of F'. More precisely, if $\overline{f}'(x) - \underline{f}'(x)$ is positive for $x < x_0$ and negative for $x > x_0$, then x_0 is a switching point of type I; and x_0 is a switching point of type II, if $\overline{f}'(x) - \underline{f}'(x)$ is negative for $x < x_0$ and positive for $x > x_0$.

Example 3.1 Let us consider the interval-valued function $F_{\mathbf{C}^1_\nu} : (-10, 10) \to K_c$ defined by (see Figure 1(a))

$$F_{\mathbf{C}^1_\nu}(x) = [2, 4] \odot \left(\cos(x) - \frac{x^2}{32}\right).$$

Clearly,

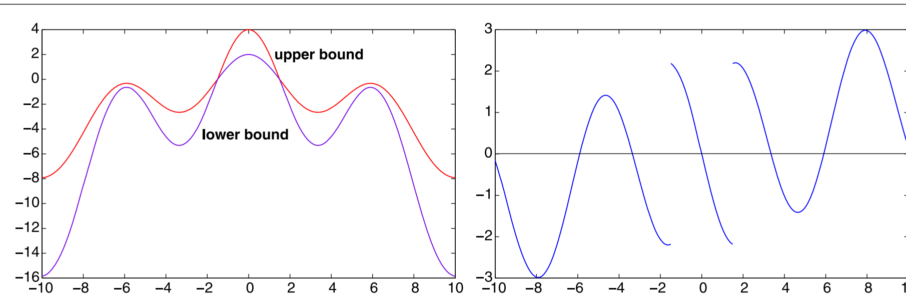
$$\underline{f}(x) = \begin{cases} 4\cos(x) - \frac{x^2}{8}, & -10 \le x < -1.5004, \\ 2\cos(x) - \frac{x^2}{16}, & -1.5004 \le x < 1.5004, \\ 4\cos(x) - \frac{x^2}{8}, & 1.5004 \le x < -10, \end{cases}$$

$$\overline{f}(x) = \begin{cases} 2\cos(x) - \frac{x^2}{16}, & -10 \le x < -1.5004, \\ 4\cos(x) - \frac{x^2}{8}, & -1.5004 \le x < 1.5004, \\ 2\cos(x) - \frac{x^2}{16}, & 1.5004 \le x < -10. \end{cases}$$

By Remark 3.12 and determining the sign of

$$\overline{f}'(x) - \underline{f}'(x) = \begin{cases} 2\sin(x) + \frac{x}{8}, & -10 \le x < -1.5004, \\ -2\sin(x) - \frac{x}{8}, & -1.5004 \le x < 1.5004, \\ 2\sin(x) + \frac{x}{8}, & 1.5004 \le x < -10, \end{cases}$$

which is represented in Figure 1(b), it can be deduced that $x_1 = -1.5004$, $x_2 = 1.5004$, $x_3 = -5.9052$, $x_4 = 5.9052$ are switching points of type II; and $x_5 = 0$, $x_6 = -3.3527$, $x_7 = 3.3527$ are switching points of type I.



(a) (b)

Figure 1 The function (a) $F_{\mathbf{C}^1_\nu}(x)$, (b) $\overline{f}'(x) - \underline{f}'(x)$ in Example 3.1.

Finally, the concept of integral for an interval-valued function in terms of its parameters is introduced as follows.

Definition 3.13 For a given interval-valued function $F_{\mathbf{C}_\nu^k} : T \subseteq \mathbb{R} \to K_c$ with $F_{\mathbf{C}_\nu^k}(x) = \{f_{\mathbf{c(t)}}(x) \mid f_{\mathbf{c(t)}} : T \subseteq \mathbb{R} \to \mathbb{R}; \mathbf{c(t)} \in \mathbf{C}_\nu^k\}$ and an interval $[\alpha, \beta] \subseteq T$, the definite integral $\int_\alpha^\beta F_{\mathbf{C}_\nu^k}(x)\,dx$ is defined as

$$\int_\alpha^\beta F_{\mathbf{C}_\nu^k}(x)\,dx = \left\{\int_\alpha^\beta f_{\mathbf{c(t)}}(x)\,dx \,\middle|\, f_{\mathbf{c(t)}}(x) \text{ is integrable w.r.t. } x \text{ for every } \mathbf{c(t)} \in \mathbf{C}_\nu^k\right\}.$$

The mentioned issue in Remark 3.9 holds for the concept of a definite integral for interval-valued functions. See the following example.

Example 3.2 Consider $F_{\mathbf{C}_\nu^2}(x) = [-1,2] \odot (1-x) \oplus [-2,4] \odot (3-x)$ and the interval of integration $[0,4]$. Then, by Definition 3.13,

$$\begin{aligned}
&\int_0^4 [-1,2] \odot (1-x) \oplus [-2,4] \odot (3-x)\,dx \\
&\quad = \left\{\int_0^4 (-1+3t_1)(1-x) + (-2+6t_2)(3-x)\,dx \,\middle|\, t_1, t_2 \in [0,1]\right\} \\
&\quad = \left\{-4 - 12t_1 + 24t_2 \mid t_1, t_2 \in [0,1]\right\} = [-16, 20],
\end{aligned}$$

while by considering the parametric representation (2) for $F_{\mathbf{C}_\nu^2}(x)$, it follows that

$$\begin{aligned}
&\int_0^4 [-1,2] \odot (1-x) \oplus [-2,4] \odot (3-x)\,dx \\
&\quad = \int_0^1 \big((3x-7) + t(21-9x)\big)\,dx + \int_1^3 \big(-4 + t(15-3x)\big)\,dx \\
&\qquad + \int_3^4 \big(14 - 6x + t(-21+9x)\big)\,dx = \left\{\frac{-41}{2} + \frac{90}{2}t \,\middle|\, t \in [0,1]\right\} \\
&\quad = \left[\frac{-41}{2}, \frac{49}{2}\right].
\end{aligned}$$

Proposition 3.14 *A continuous interval-valued function $F(x)$ is integrable.*

Proof The proof is an immediate consequence of Proposition 3.4 and Remark 3.5. □

The concept of definite integral satisfies following properties:

Proposition 3.15 *Let $F(x)$ and $G(x)$ be two integrable interval-valued functions. Then*:
1. $\int_\alpha^\beta F(x)\,dx = \int_\alpha^\gamma F(x)\,dx + \int_\gamma^\beta F(x)\,dx;\ \alpha \le \gamma \le \beta$,
2. $\int_\alpha^\beta (aF(x) + bG(x))\,dx = a\int_\alpha^\beta F(x)\,dx + b\int_\alpha^\beta G(x)\,dx;\ a, b \in \mathbb{R}$.

Proposition 3.16 *Let $F_{\mathbf{C}_\nu^k} : T \subseteq \mathbb{R} \to K_c$ be continuous. Then*:
1. *the function $G_{\mathbf{D}_\nu^n}(x) = \int_\alpha^x F_{\mathbf{C}_\nu^k}(z)\,dz$ is p-differentiable, and $G'_{\mathbf{D}_\nu^n}(x) = F_{\mathbf{C}_\nu^k}(x)$,*
2. *the function $H_{\mathbf{E}_\nu^m}(x) = \int_x^\beta F_{\mathbf{C}_\nu^k}(z)\,dz$ is p-differentiable, and $H'_{\mathbf{E}_\nu^m}(x) = -F_{\mathbf{C}_\nu^k}(x)$.*

Proof First, assume that

$$G_{\mathbf{D}_\nu^n}(x) = \int_\alpha^x F_{\mathbf{C}_\nu^k}(z)\,dz = \left\{ \int_a^x f_{\mathbf{c}(\mathbf{t})}(z)\,dz \,\middle|\, f_{\mathbf{c}(\mathbf{t})} : T \to \mathbb{R}; \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k \right\},$$

because of the continuity of $F_{\mathbf{C}_\nu^k}$ and Proposition 3.4, the function $\int_\alpha^x f_{\mathbf{c}(\mathbf{t})}(z)\,dz$ is differentiable for every $\mathbf{t} \in [0,1]^k$. Hence, according to Proposition 3.7, $G_{\mathbf{D}_\nu^n}(x)$ is p-differentiable, and

$$G'_{\mathbf{D}_\nu^n}(x) = \{ f_{\mathbf{c}(\mathbf{t})}(x) \mid \mathbf{c}(\mathbf{t}) \in \mathbf{C}_\nu^k \} = F_{\mathbf{C}_\nu^k}(x).$$

The second part is proved in the same way as the first one. □

In addition, Proposition 3.16 holds for the interval-valued function $F(x) = \{\underline{f}(x) + t(\overline{f}(x) - \underline{f}(x)) \mid t \in [0,1]\}$, by using Remark 3.5 and Proposition 3.8.

Proposition 3.17 *If* $F(x) = \{\underline{f}(x) + t(\overline{f}(x) - \underline{f}(x)) \mid t \in [0,1]\}$ *is p-differentiable with no switching point in the interval* $[\alpha, \beta]$, *then*

$$\int_\alpha^\beta F'(x)\,dx = F(\beta) \ominus_p F(\alpha).$$

Proof By Definition 3.10, because there is no switching point, F is (i)-p-differentiable or (d)-p-differentiable in the interval $[\alpha, \beta]$. Let F be (i)-p-differentiable (the proof for the (d)-p-differentiable case being similar), then

$$\begin{aligned}\int_\alpha^\beta F'(x)\,dx &= \left\{ \int_\alpha^\beta \left(\underline{f}'(x) + t\left(\overline{f}'(x) - \underline{f}'(x)\right)\right) dx \,\middle|\, t \in [0,1] \right\} \\ &= \left\{ \underline{f}(\beta) - \underline{f}(\alpha) + t\left(\overline{f}(\beta) - \overline{f}(\alpha) - \underline{f}(\beta) + \underline{f}(\alpha)\right) \mid t \in [0,1] \right\} \\ &= F(\beta) \ominus_p F(\alpha),\end{aligned}$$

which completes the proof. □

Theorem 3.18 *Let* $F(x) = \{\underline{f}(x) + t(\overline{f}(x) - \underline{f}(x)) | t \in [0,1]\}$ *be p-differentiable with n switching point at* γ_i, $i = 1, 2, \ldots, n$, $\alpha = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n < \gamma_{n+1} = \beta$ *and exactly at these points. Then*

$$F(\beta) \ominus_p F(\alpha) = \sum_{i=1}^n \left[\int_{\gamma_{i-1}}^{\gamma_i} F'(x)\,dx \ominus_p (-1) \int_{\gamma_i}^{\gamma_{i+1}} F'(x)\,dx \right].$$

Moreover,

$$\int_\alpha^\beta F'(x)\,dx = \sum_{i=1}^{n+1} \left(F(\gamma_i) \ominus_p F(\gamma_{i-1})\right),$$

and if $F(\gamma_i)$ *is a real number for* $i = 1, 2, \ldots, n$ *then* $\int_\alpha^\beta F'(x)\,dx = F(\beta) - F(\alpha)$.

Proof To prove this, consider only one switching point, the case of a finite number of switching points follows similarly. Let F be (i)-p-differentiable on $[\alpha,\gamma]$ and (d)-p-differentiable on $[\gamma,\beta]$. Then, by Proposition 3.17,

$$\begin{aligned}
&\int_\alpha^\gamma F'(x)\,dx \ominus_p (-1)\int_\gamma^\beta F'(x)\,dx \\
&\quad = \bigl(F(\gamma)\ominus_p F(\alpha)\bigr)\ominus_p (-1)\bigl(F(\beta)\ominus_p F(\gamma)\bigr) \\
&\quad = \bigl(F(\gamma)\ominus_p F(\alpha)\bigr)\ominus_p \bigl(F(\gamma)\ominus_p F(\beta)\bigr) \\
&\quad = \bigl\{\bigl(\underline{f}(\gamma)-\underline{f}(\alpha)+t\bigl(\overline{f}(\gamma)-\overline{f}(\alpha)-\underline{f}(\gamma)+\underline{f}(\alpha)\bigr)\bigr) \\
&\qquad - \bigl(\underline{f}(\gamma)-\underline{f}(\beta)+t\bigl((\gamma)-\overline{f}(\beta)-\underline{f}(\gamma)+\underline{f}(\beta)\bigr)\bigr) \mid t\in[0,1]\bigr\} \\
&\quad = \bigl\{\bigl(\underline{f}(\beta)+t\bigl(\overline{f}(\beta)-\underline{f}(\beta)\bigr)\bigr)-\bigl(\underline{f}(\alpha)+t\bigl(\overline{f}(\alpha)-\underline{f}(\alpha)\bigr)\bigr) \mid t\in[0,1]\bigr\} \\
&\quad = F(\beta)\ominus_p F(\alpha).
\end{aligned}$$

Additionally, by Proposition 3.15 and Proposition 3.17,

$$\begin{aligned}
\int_\alpha^\beta F'(x)\,dx &= \int_\alpha^\gamma F'(x)\,dx + \int_\gamma^\beta F'(x)\,dx \\
&= \bigl(F(\gamma)\ominus_p F(\alpha)\bigr) + \bigl(F(\beta)\ominus_p F(\gamma)\bigr).
\end{aligned}$$

If the values $F(\gamma_i)$ at all the switching points γ_i, $i = 1,2,\ldots,n$ are real numbers, then by Remark 2.4

$$\begin{aligned}
\int_\alpha^\beta F'(x)\,dx &= \sum_{i=1}^{n+1}\bigl(F(\gamma_i)\ominus_p F(\gamma_{i-1})\bigr) \\
&= \bigl(F(\beta)-F(\gamma_n)\bigr)+\bigl(F(\gamma_n)-F(\gamma_{n-1})\bigr)+\cdots \\
&\quad + \bigl(F(\gamma_2)-F(\gamma_1)\bigr)+\bigl(F(\gamma_1)-F(\alpha)\bigr) \\
&= F(\beta)-F(\alpha).
\end{aligned}$$

□

4 Interval differential equation

In this section, two approaches are proposed to find the solution of the interval differential equation. In the first approach, by using the parametric representation (3) and its corresponding definitions of the p-derivative and integral, the interval differential equation is converted to a crisp problem. The second follows from the notation of the p-derivative based on the parametric representation (2). It is noteworthy that the solutions obtained by the two approaches are distinctive.

4.1 The first approach

In this approach, the following interval differential equation is considered:

$$\begin{aligned}
&Y'_{\mathbf{D}_\nu^n}(x) = F_{\mathbf{C}_\nu^k}\bigl(x, Y_{\mathbf{D}_\nu^n}(x)\bigr), \\
&Y_{\mathbf{D}_\nu^n}(x_0) = Y_0,
\end{aligned} \tag{11}$$

where $Y_0 \in K_c$, $F_{\mathbf{C}_\nu^k} : [a,b]\times K_c \to K_c$ and $Y_{\mathbf{D}_\nu^n} : [a,b] \to K_c$.

Let $\{y_{\mathbf{d(t)}}(x) \mid y_{\mathbf{d(t)}} : [a,b] \to \mathbb{R}; \mathbf{d(t)} \in \mathbf{D}_\nu^n\}$, $\{f_{\mathbf{c(t')}}(x, y_{\mathbf{d(t)}}(x)) \mid f_{\mathbf{c(t')}} : [a,b] \times \mathbb{R} \to \mathbb{R}; \mathbf{c(t')} \in \mathbf{C}_\nu^k\}$ and $\{\underline{y}_0 + t''(\overline{y}_0 - \underline{y}_0) \mid t'' \in [0,1]\}$ be parametric representations of $Y_{\mathbf{D}_\nu^n} : [a,b] \to K_c$, $F_{\mathbf{C}_\nu^k}(x, Y_{\mathbf{D}_\nu^n}(x))$, and Y_0, respectively. From definition of equality given in Definition 2.1, the differential equation (11) can be considered as

$$\begin{aligned}
&\{y'_{\mathbf{d(t)}}(x) \mid y_{\mathbf{d(t)}} : [a,b] \to \mathbb{R}; \mathbf{d(t)} \in \mathbf{D}_\nu^n\} \\
&\quad = \{f_{\mathbf{c(t')}}(x, y_{\mathbf{d(t)}}(x)) \mid f_{\mathbf{c(t')}} : [a,b] \times \mathbb{R} \to \mathbb{R}; \mathbf{c}(\mathbf{t}') \in \mathbf{C}_\nu^k\}, \\
&\{y_{\mathbf{d(t)}}(x_0) \mid y_{\mathbf{d(t)}} : [a,b] \to \mathbb{R}; \mathbf{d(t)} \in \mathbf{D}_\nu^n\} = \{\underline{y}_0 + t''(\overline{y}_0 - \underline{y}_0) \mid t'' \in [0,1]\}.
\end{aligned}$$

By Remark 2.3, it follows that

$$\begin{aligned}
&y'_{\mathbf{d(t)}}(x) = f_{\mathbf{c(t')}}(x, y_{\mathbf{d(t)}}(x)), \\
&y_{\mathbf{d(t)}}(x_0) = \underline{y}_0 + t''(\overline{y}_0 - \underline{y}_0).
\end{aligned} \tag{12}$$

Theorem 4.1 *Let* $f : [x_0, x_0 + p] \times \overline{B}([\underline{y}_0, \overline{y}_0], r) \times \overline{B}([0,1]^k, q) \to \mathbb{R}$ *be Lipschitz in its second and third variables*; *i.e.*,

$$\begin{aligned}
&\exists L_1 \quad \text{s.t.}\ \|f(x,y,\mathbf{t}') - f(x,w,\mathbf{t}')\| \le L_1 \|y - w\|, \\
&\exists L_2 \quad \text{s.t.}\ \|f(x,y,\mathbf{t}') - f(x,y,\mathbf{s})\| \le L_2 \|\mathbf{t}' - \mathbf{s}\|,
\end{aligned}$$

respectively. Then the initial valued problem

$$\begin{aligned}
&y'(x) = f(x, y(x), \mathbf{t}'), \\
&y(x_0) = \underline{y}_0 + t''(\overline{y}_0 - \underline{y}_0)
\end{aligned}$$

has a unique solution. Moreover, if f *is continuous in* $\mathbf{t}'$ *then the solution* $y(x, \mathbf{t}', t'')$ *is continuous in* $\mathbf{t}'$ *and* t''.

Proof See [15], Theorem 9.6, for the first part. The second part follows immediately from the continuity of f and $y(x_0)$ w.r.t. $\mathbf{t}'$ and t'', respectively. □

Let $Y_{\mathbf{D}_\nu^n}(x)$ and $F_{\mathbf{C}_\nu^k}(x, Y_{\mathbf{D}_\nu^n}(x))$ be interval-valued functions with parametric representations $\{y_{\mathbf{d(t)}}(x) \mid y_{\mathbf{d(t)}} : [a,b] \to \mathbb{R}; \mathbf{d(t)} \in \mathbf{D}_\nu^n\}$ and $\{f_{\mathbf{c(t')}}(x, y_{\mathbf{d(t)}}(x)) \mid f_{\mathbf{c(t')}} : [a,b] \times \mathbb{R} \to \mathbb{R}; \mathbf{c(t')} \in \mathbf{C}_\nu^k\}$, respectively. Suppose that $f_{\mathbf{c(t')}}(x, y_{\mathbf{d(t)}}(x))$ is Lipschitz in y and $\mathbf{t}'$, because $f_{\mathbf{c(t')}}(x, y_{\mathbf{d(t)}}(x))$ and $y_{d(t'')}(x_0) = \underline{y}_0 + t''(\overline{y}_0 - \underline{y}_0)$ are continuous functions in $\mathbf{t}'$ and t'', respectively, by Theorem 4.1, problem (12) has a unique solution $y_{\mathbf{d(t)}}(x) = y(x, \mathbf{t}', t'')$, which is a continuous function in $\mathbf{t}'$ and t''. Therefore, $\min_{\mathbf{t}',t''} y(x, \mathbf{t}', t'')$ and $\max_{\mathbf{t}',t''} y(x, \mathbf{t}', t'')$ exist, and

$$Y_{\mathbf{D}_\nu^n}(x) = \Big[\min_{\mathbf{t}',t''} y(x, \mathbf{t}', t''), \max_{\mathbf{t}',t''} y(x, \mathbf{t}', t'')\Big]. \tag{13}$$

Example 4.1 Consider the following interval differential equation:

$$\begin{cases} Y'_{\mathbf{D}_\nu^n}(x) = -Y_{\mathbf{D}_\nu^n}(x) \oplus [1,2] \odot \sin(x), \\ Y_{\mathbf{D}_\nu^n}(0) = [1,3], \quad x \in [0,6]. \end{cases} \tag{14}$$

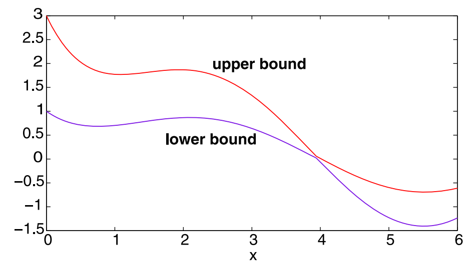


Figure 2 Upper and lower bounds of solution of the interval equation (14).

Its corresponding ordinary differential equation is

$$y'_{\mathbf{d(t)}}(x) = -y_{\mathbf{d(t)}}(x) + (1 + t')\sin(x), \quad (15)$$
$$y_{\mathbf{d(t)}}(0) = 1 + 2t'', \quad t', t'' \in [0,1], x \in [0,6].$$

The general solution (15) is provided by

$$y_{\mathbf{d(t)}}(x) = (1 + t')\left(\frac{-\cos(x) + \sin(x)}{2}\right) + ce^{-x},$$

and so the particular solution would be

$$y_{\mathbf{d(t)}}(x) = (1 + t')\left(\frac{-\cos(x) + \sin(x) + e^{-x}}{2}\right) + (1 + 2t'')e^{-x}.$$

Hence, the unique solution of interval differential equation (14) is obtained as

$$Y_{\mathbf{D}_v^2}(x) = [1,2] \odot \left(\frac{-\cos(x) + \sin(x) + e^{-x}}{2}\right) \oplus [1,3] \odot e^{-x}.$$

Moreover, by (13),

$$Y_{\mathbf{D}_v^n}(x) = \begin{cases} [\frac{-\cos(x)+\sin(x)+e^{-x}}{2} + e^{-x}, -\cos(x) + \sin(x) + 4e^{-x}], & x \in [0, 3.941], \\ [-\cos(x) + \sin(x) + 4e^{-x}, \frac{-\cos(x)+\sin(x)+e^{-x}}{2} + 3e^{-x}], & x \in [3.941, 6], \end{cases}$$

which is illustrated in Figure 2.

4.2 The second approach

Now, consider the interval differential equation

$$Y'(x) = F(x, Y(x)), \quad (16)$$
$$Y(x_0) = Y_0,$$

where $Y_0 \in K_c$, $F : [a,b] \times K_c \to K_c$, and $Y : [a,b] \to K_c$. By the parametric representation (2) and rewriting (16), two following systems are obtained from (9) and (10):

I. $\{\underline{y}'(x) + t(\overline{y}'(x) - \underline{y}'(x)) \mid t \in [0,1]\} = \{\underline{f}(x, \underline{y}, \overline{y}) + t(\overline{f}(x, \underline{y}, \overline{y}) - \underline{f}(x, \underline{y}, \overline{y})) \mid t \in [0,1]\}$,
$\{\underline{y}(x_0) + t(\overline{y}(x_0) - \underline{y}(x_0)) \mid t \in [0,1]\} = \{\underline{y}_0 + t(\overline{y}_0 - \underline{y}_0) \mid t \in [0,1]\}$,

II. $\{\overline{y}'(x) + t(\underline{y}'(x) - \overline{y}'(x)) \mid t \in [0,1]\} = \{\underline{f}(x, \underline{y}, \overline{y}) + t(\overline{f}(x, \underline{y}, \overline{y}) - \underline{f}(x, \underline{y}, \overline{y})) \mid t \in [0,1]\}$,
$\{\underline{y}(x_0) + t(\overline{y}(x_0) - \underline{y}(x_0)) \mid t \in [0,1]\} = \{\underline{y}_0 + t(\overline{y}_0 - \underline{y}_0) \mid t \in [0,1]\}$.

Hence, for fixed x, two systems can be deduced as follows:

$$\begin{cases} \underline{y}'(x) = \underline{f}(x,\underline{y}(x),\overline{y}(x)), & \underline{y}(x_0) = \underline{y}_0, \\ \overline{y}'(x) = \overline{f}(x,\underline{y}(x),\overline{y}(x)), & \overline{y}(x_0) = \overline{y}_0, \end{cases} \tag{17}$$

$$\begin{cases} \overline{y}'(x) = \underline{f}(x,\underline{y}(x),\overline{y}(x)), & \underline{y}(x_0) = \underline{y}_0, \\ \underline{y}'(x) = \overline{f}(x,\underline{y}(x),\overline{y}(x)), & \overline{y}(x_0) = \overline{y}_0. \end{cases} \tag{18}$$

Theorem 4.2 *Let $F : [x_0, x_0 + p] \times \overline{B}(Y_0, r) \to K_c$ be nontrivial and continuous. If F satisfies the Lipschitz condition $D(F(x,Y), F(x,Z)) \leq LD(Y,Z)$ for all $(x,Y),(x,Z) \in [x_0, x_0 + p] \times \overline{B}(Y_0, r)$, then the interval differential equation*

$$Y'(x) = F\bigl(x, Y(x)\bigr), \tag{19}$$
$$Y(x_0) = Y_0,$$

is equivalent to the union of the systems (17) *and* (18) *on some interval $[x_0, x_0 + p]$. Here, the equivalence means that $Y = [\underline{y}, \overline{y}] : [x_0, x_0 + p] \to K_c$ is a solution of* (19) *if and only if $(\underline{y}, \overline{y}) : [x_0, x_0 + p] \to \mathbb{R}^2$ is a solution of one of the problems* (17) *or* (18).

Proof See [6]. □

Example 4.2 Consider interval differential equation (14), for $x \in [0, 6]$. The corresponding systems are

$$\begin{cases} \underline{y}'(x) = \begin{cases} -\overline{y}(x) + \sin(x), & 0 \leq x \leq \pi, \\ -\overline{y}(x) + 2\sin(x), & \pi \leq x \leq 6, \end{cases} \\ \overline{y}'(x) = \begin{cases} -\underline{y}(x) + 2\sin(x), & 0 \leq x \leq \pi, \\ -\underline{y}(x) + \sin(x), & \pi \leq x \leq 6, \end{cases} \\ \underline{y}(0) = 1, \qquad \overline{y}(0) = 3, \end{cases}$$

$$\begin{cases} \overline{y}'(x) = \begin{cases} -\overline{y}(x) + \sin(x), & 0 \leq x \leq \pi, \\ -\overline{y}(x) + 2\sin(x), & \pi \leq x \leq 6, \end{cases} \\ \underline{y}'(x) = \begin{cases} -\underline{y}(x) + 2\sin(x), & 0 \leq x \leq \pi, \\ -\underline{y}(x) + \sin(x), & \pi \leq x \leq 6, \end{cases} \\ \underline{y}(0) = 1, \qquad \overline{y}(0) = 3. \end{cases}$$

By considering (i)-p-differentiability the following solution is obtained for the interval differential equation (14):

$$Y_1(x) = \begin{cases} [\frac{11}{4}e^{-x} - \frac{1}{2}\cos(x) - \frac{5}{4}e^{x} + \sin(x), \frac{11}{4}e^{-x} - \cos(x) + \frac{5}{4}e^{x} + \frac{1}{2}\sin(x)], & 0 \leq x \leq \pi, \\ [\frac{1}{2}\sin(x) - \cos(x) + 0.1188e^{\pi - x} - 29.4259e^{x-\pi}, & \\ \quad \sin(x) - \frac{1}{2}\cos(x) + 0.1188e^{\pi - x} + 29.4259e^{x-\pi}], & \pi \leq x \leq 6, \end{cases}$$

which has no switching point. The second solution is achieved by starting with (d)-p-differentiability, which has a switching point at 1.3607 for $x \in [0, \pi]$ and

$$Y_2(x) = \begin{cases} [2e^{-x} - \cos(x) + \sin(x), \frac{7}{2}e^{-x} - \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x)], & x \in [0, 1.3607], \\ [\frac{11}{4}e^{-x} + \sin(x) - \frac{1}{2}\cos(x) + 0.0761e^{x}, & \\ \quad \frac{11}{4}e^{-x} + \frac{1}{2}\sin(x) - \cos(x) + 0.0761e^{x}], & x \in [1.3607, \pi]. \end{cases}$$

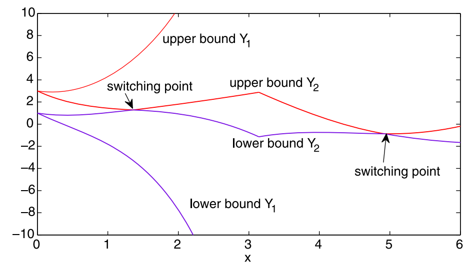


Figure 3 Solutions of interval differential equation (14).

Furthermore,

$$Y_2(x) = \begin{cases} [\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) - 1.6418e^{\pi - x}, \\ \quad \sin(x) - \cos(x) + 1.8795e^{\pi - x}], & x \in [\pi, 4.9225], \\ [\frac{1}{2}\sin(x) - \cos(x) + 2.7500e^{-x} + 0.0014e^{x}, \\ \quad \sin(x) - \frac{1}{2}\cos(x) + 2.7500e^{-x} + 0.0014e^{x}], & x \in [4.9225, 6], \end{cases}$$

for $x \in [\pi, 6]$ and has one switching point at 4.9225. The solutions are shown in Figure 3.

The two proposed approaches are applied to the following examples.

Example 4.3 Consider the interval differential equation

$$\begin{cases} Y'_{\mathbf{D}_\nu^n}(x) = -Y_{\mathbf{D}_\nu^n}(x) \oplus [1,2] \odot x, \\ Y_{\mathbf{D}_\nu^n}(0) = [0,1], \quad x \in [0,4]. \end{cases} \tag{20}$$

By the first approach, the corresponding ordinary differential equation is obtained as

$$\begin{aligned} &y'_{\mathbf{d(t)}}(x) = -y_{\mathbf{d(t)}}(x) + (1 + t')x, \\ &y_{\mathbf{d(t)}}(0) = t'', \quad t', t'' \in [0,1], x \in [0,4]. \end{aligned} \tag{21}$$

The particular solution would be

$$y_{\mathbf{d(t)}}(x) = (1 + t')x - (1 + t') + (1 + t' + t'')e^{-x} = (1 + t')(x - 1 + e^{-x}) + t''e^{-x}.$$

Hence, the unique solution of interval equation (20) is concluded as

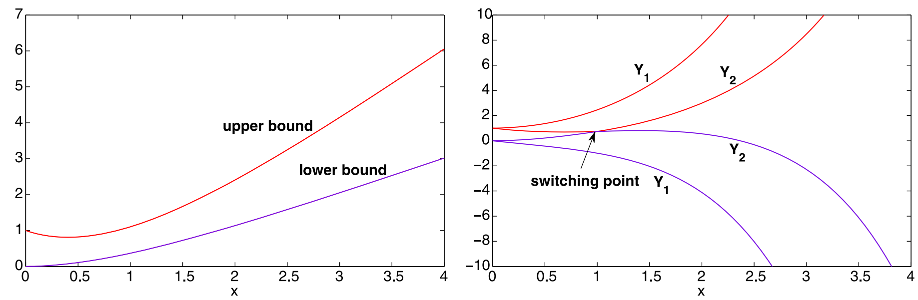
$$Y_{\mathbf{D}_\nu^2}(x) = [1,2] \odot (x - 1 + e^{-x}) \oplus [0,1] \odot e^{-x},$$

which is represented in Figure 4(a). Next, let us to use the second approach for the interval differential equation (20). Thus, the systems (17) and (18) are

$$\begin{cases} \underline{y}'(x) = -\overline{y}(x) + x, & \underline{y}(0) = 0, \\ \overline{y}'(x) = -\underline{y}(x) + 2x, & \overline{y}(0) = 1 \end{cases} \tag{22}$$

and

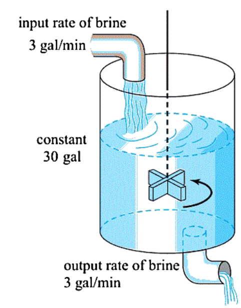
$$\begin{cases} \underline{y}'(x) = -\underline{y}(x) + 2x, & \underline{y}(0) = 0, \\ \overline{y}'(x) = -\overline{y}(x) + x, & \overline{y}(0) = 1, \end{cases} \tag{23}$$



(a) (b)

Figure 4 The solutions of (19) in (a) first approach, (b) second approach.

Figure 5 A tank of brine with a mixture.



respectively. As stated previously, (20) has exactly two solutions; one of them is

$$\begin{aligned} Y_1(x) &= \{\underline{y} + t(\overline{y} - \underline{y}) = 2x + 2e^{-x} - e^x - 1 + t(2e^x - x - 1), \mid t \in [0,1]\} \\ &= [2x + 2e^{-x} - e^x - 1, x + e^x + 2e^{-x} - 2], \end{aligned}$$

which starts with (i)-p-differentiability and there is no switching point on its trajectory. The second one starts with (d)-p-differentiability and has a switching point at $x = 1$ so it has to switch to the case (i)-p-differentiability. Therefore

$$\begin{aligned} Y_2(x) &= \{\underline{y} + t(\overline{y} - \underline{y}) = 2x + 2e^{-x} - 2 + t(1 - x) \mid t \in [0,1]\} \\ &= [2x + 2e^{-x} - 2, x + 2e^{-x} - 1], \end{aligned}$$

for $0 \le x \le 1$ and

$$\begin{aligned} Y_2(x) &= \{\underline{y} + t(\overline{y} - \underline{y}) = 2x - e^{x-1} + 2e^{-x} - 1 + t(2e^{x-1} - x - 1) \mid t \in [0,1]\} \\ &= [2x - e^{x-1} + 2e^{-x} - 1, x + e^{x-1} + 2e^{-x} - 2], \end{aligned}$$

when $1 \le x \le 4$. The solutions are presented in Figure 4(b).

Example 4.4 As an application in engineering, assume the following problem. A tank initially contains 30 gal of brine with c lb of salt. Brine that contains k lb of salt per gallon enters the tank at the rate of 3 gal/min, and the well-mixed brine in the tank flows out at the rate of 3 gal/min (see Figure 5). The problem is the amount of salt that will be present in the tank at a time in the future; in other words, we would like to determinate a function $y(x)$ that gives us the amount of salt in the tank at time $x \geq 0$. We have

$$\begin{cases} y'(x) + \frac{1}{10}y(x) = 3k, \\ y(0) = c, \quad x \in [0,3]. \end{cases}$$

According to uncertainty in measurements, the coefficients c, k are considered as interval numbers $c = [1,3]$, $k = [1,4]$. Then

$$\begin{cases} Y'_{\mathbf{D}_\nu^n}(x) = -\frac{1}{10} Y_{\mathbf{D}_\nu^n}(x) \oplus [3,12], \\ Y_{\mathbf{D}_\nu^n}(0) = [1,3], \quad x \in [0,3]. \end{cases} \tag{24}$$

Using the first approach and without having to calculate the switching points, we obtain a solution that is responsive to the nature of the problem. More precisely, the corresponding ordinary differential equation is obtained as

$$y'_{\mathbf{d(t)}}(x) = \frac{-1}{10} y_{\mathbf{d(t)}}(x) + \left(3 + 9t'\right),$$
$$y_{\mathbf{d(t)}}(0) = 1 + 2t'', \quad t', t'' \in [0,1], x \in [0,3].$$

Hence, the solution of interval equation (24) is

$$Y_{\mathbf{D}_\nu^2}(x) = [30,120] \odot \left(1 - e^{\frac{-x}{10}}\right) \oplus [1,3] \odot \left(e^{\frac{-x}{10}}\right)$$

and its upper bound and lower bounds are shown in Figure 6(a). In the second approach, two systems (17), (18) are

$$\begin{cases} \underline{y}'(x) = -\frac{1}{10}\overline{y}(x) + 3, & \underline{y}(0) = 1, \\ \overline{y}'(x) = -\frac{1}{10}\underline{y}(x) + 12, & \overline{y}(0) = 3, \end{cases}$$

$$\begin{cases} \overline{y}'(x) = -\frac{1}{10}\overline{y}(x) + 3, & \underline{y}(0) = 1, \\ \underline{y}'(x) = -\frac{1}{10}\underline{y}(x) + 12, & \overline{y}(0) = 3. \end{cases}$$

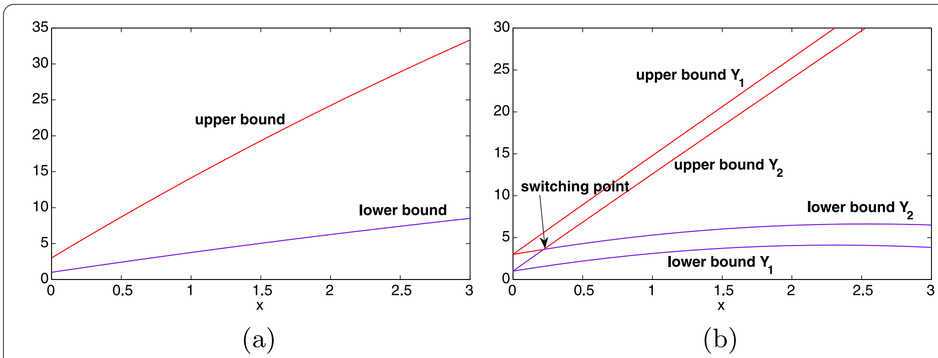


Figure 6 The solutions of (20) in (a) first approach, (b) second approach.

First, by starting with (i)-p-differentiability the following solution for interval differential equation (24) is obtained:

$$Y_1(x) = \left[120 - 46e^{\frac{x}{10}} - 73e^{\frac{-x}{10}}, 46e^{\frac{x}{10}} - 73e^{\frac{-x}{10}} + 30\right],$$

which has no switching point. The second solution is achieved by considering (d)-p-differentiability, that has a switching point at 0.2198 and

$$Y_2(x) = \begin{cases} [120 - 119e^{\frac{-x}{10}}, 30 - 27e^{\frac{-x}{10}}], & x \in [0, 0.2198], \\ [120 - 44.0217e^{\frac{x}{10}} - 73e^{\frac{-x}{10}}, 44.0217e^{\frac{x}{10}} - 73e^{\frac{-x}{10}} + 30], & x \in [0.2198, 3]. \end{cases}$$

The solutions are shown in Figure 6(b).

5 Conclusion

In this study, two different new parametric representations for interval numbers were investigated. The representations had the advantage of following flexible and easy to control shapes of the interval numbers, and they were applicable in practice. Additionally, computational procedures to determine the derivatives and integrals of interval-valued functions were presented.

An interesting line of work is the study of interval differential equations with two approaches. In the first approach, a unique solution with a decreasing length of support was obtained. This interpretation presented an advantage that allowed us to characterize the main properties of ordinary differential equations in a natural way. In the second approach, the solution of an interval differential equation might have decreasing length of support, and more than one solution exists. The existence of several solutions can be an advantage when a decision-maker search is performed for solutions that have specific properties, such as periodic, almost periodic, or asymptotically stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final version of the manuscript.

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