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Characters of the W_3 algebra

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ABSTRACT: Traces of powers of the zero mode in the W_3 Algebra have recently been found to be of interest, for example in relation to Black Hole thermodynamics, and arise as the terms in an expansion of the full characters of the algebra. We calculate the first few such powers in two cases. Firstly, we find the traces in the 3-state Potts model by using null vectors to derive modular differential equations for the traces. Secondly, we calculate the exact results for Verma module representations. We compare our two methods with each other and the result of brute-force diagonalisation for low levels and find complete agreement.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence, Conformal and W Symmetry

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1 Introduction

W-algebras are generalisations of the Virasoro algebra which appear in a number of different situations. They all include the Virasoro algebra as a sub-algebra and so for any highest-weight representation V one can define a *reduced* character as

$$\chi_V^{red.}(q) = \text{Tr}_V(q^{L_0 - c/24}) . \tag{1.1}$$

However, a W-algebra will typically have a larger set of commuting zero-modes for which one can try to extend the reduced character to a full character (or simply, character). Here we will only consider the W_3 algebra which has one extra commuting mode¹ W_0 and for which one can consider the character

$$\chi_V(q, y) = \text{Tr}_V(q^{L_0 - c/24} y^{W_0}) . \tag{1.2}$$

These full characters are of particular interest in holography — see [1] for a review of the duality between W_N CFTs and AdS₃ gravity theories. In [2] for example, we see that the AdS₃/CFT₂ correspondence gives us a relation between the character of a Virasoro algebra module on the CFT side, and the partition function on the AdS side. Generalising the character to the W_3 algebra may allow the correspondence to be investigated for a wider range of gravity models, those with spin greater than 2 (see [3] for a review).

Traces over W-algebra zero modes were calculated in [4] for the specific cases of free bosons and free fermions. Similarly, in [5] the authors calculate $\text{Tr}(W_0^n q^{L_0 - c/24})$ for $n = 2, 4, 6$ in the limits $q \rightarrow 0$ and large central charge. An exact calculation of the generalised character can both verify and extend their results.

1.1 Content of the paper

The ultimate aim is to calculate the full character of both the Verma module and irreducible representations,

$$\text{Tr}_V(q^{L_0 - c/24} y^{W_0}) = \text{Tr}_V(q^{L_0 - c/24} e^{2\pi i z W_0}) . \tag{1.3}$$

In this paper we calculate the first few terms in the expansion of this second form for several classes of expressions. To make the results simpler, we shall drop the extra term $(-c/24)$ from the character which is usually included to improve modular transformation properties. It is a simple matter to reinstate it if necessary.

To summarise the content of the rest of the paper:

In section 3, we obtain explicit results up to level 6 for any highest weight Verma module for the traces $\text{Tr}_V(W_0 q^{L_0})$ through $\text{Tr}_V(W_0^5 q^{L_0})$ by direct calculation.

In section 4, we obtain exact expressions for $\text{Tr}_L(W_0^2 q^{L_0})$, and a differential equation whose solutions are exact expressions for $\text{Tr}_L(W_0 q^{L_0})$ for the irreducible minimal representations L at $c = 4/5$, i.e. those relevant to the 3-state Potts model.

In section 5 we find an exact (i.e. to all levels) expression for the traces over any highest weight Verma module, $\text{Tr}_V(W_0 q^{L_0})$ and $\text{Tr}_V(W_0^2 q^{L_0})$.

We compare our results from these three methods of computation in section 6 and show that they agree. Finally, we conclude with some open questions and directions for future research.

¹As the Virasoro algebra has an infinite set of mutually commuting modes $\{L_0, \Lambda_0, \dots\}$, so the W_3 algebra has $\{L_0, W_0, (LW)_0 + \text{corrections}, \dots\}$. Here we are looking at zero modes of fundamental fields L and W only.

2 The W_3 algebra and its representations

2.1 The W_3 algebra

The W_3 algebra is an extension of the Virasoro algebra introduced by A. Zamolodchikov in [6] in which the usual spin-2 operators L_n are augmented by spin-3 operators W_n . We will use Zamolodchikov's normalisation so that the commutation relations are

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \quad (2.1a)$$

$$[L_m, W_n] = (2m - n)W_{m+n}, \quad (2.1b)$$

$$[W_m, W_n] = (m - n) \left[\frac{1}{15} (m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2) \right] L_{m+n} \\ + \beta(m-n)\Lambda_{m+n} + \frac{c}{360}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}, \quad (2.1c)$$

where

$$\beta = \frac{16}{22 + 5c}. \quad (2.2)$$

The operators Λ_n are given by

$$\Lambda_n = \sum_{p \leq -2} L_p L_{n-p} + \sum_{p \geq -1} L_{n-p} L_p - \frac{3}{10} (n+2)(n+3)L_n \\ = \sum_{p=-\infty}^{\lfloor (n-1)/2 \rfloor} L_p L_{n-p} + \sum_{p=\lceil n/2 \rceil}^{\infty} L_{n-p} L_p + \gamma(n)L_n, \quad (2.3)$$

with

$$\gamma(n) = \begin{cases} -\frac{1}{20}(n^2 - 4) & n \text{ even} \\ -\frac{1}{20}(n^2 - 9) & n \text{ odd} \end{cases} \quad (2.4)$$

Their commutation relations with L_m and W_m are

$$[L_m, \Lambda_n] = (3m - n)\Lambda_{m+n} + \frac{22 + 5c}{30}m(m^2 - 1)L_{m+n}, \quad (2.5)$$

$$[W_m, \Lambda_n] = \frac{n+3}{5} [3(m+n+3)(m+n+4) - 8(m+1)(m+2)]W_{m+n} \\ + \sum_{p \leq -2} (2m - 4n + 4p)L_p W_{m+n-p} + \sum_{p \geq -1} (2m - 4n + 4p)W_{m+n-p}L_p. \quad (2.6)$$

2.2 Representations of the W_3 algebra

As for the Virasoro algebra, the W_3 algebra can have highest weight representations. Since we have two commuting zero modes, highest weight representations have two labels: the L_0 eigenvalue h and the W_0 eigenvalue w . As such, highest-weight states are labelled $|h, w\rangle$. One can consider a wide range of different highest weight representations, but we will only consider Verma modules and irreducible modules.

The Verma module $V_{h,w;c}$ is generated by the action of the negative modes W_{-m}, L_{-m} on the highest weight state $|h, w\rangle$. One can show that a basis is taken by deciding on a fixed ordering of these negative modes — any ordering will do — but we will choose a particularly useful ordering in section 5 when we calculate the full character of the Verma module.

The irreducible module with highest weight $|h, w\rangle$, $L_{h,w;c}$, is given as the quotient of the Verma module with the same highest weight state by its maximal submodule. Detailed (and apparently correct) conjectures for the structure of the Verma modules of the W_3 algebra and their relation to the irreducible modules are given in [7].

The reduced characters of the Verma module representations are trivial to find,

$$\chi_{V_{h,w;c}}^{red.}(q) = \text{Tr}_{V_{h,w;c}}(q^{L_0 - c/24}) = q^{h-c/24} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}; \quad (2.7)$$

the reduced characters of *any* irreducible representation can then be calculated using the results in [7]. In many cases the reduced characters have alternative or closed-form expressions, for example for the minimal models for which the characters are given in [8]. We now summarise the results of [7] that will be useful to us.

For any module M , one can form a *local composition series*, a sequence of submodules wherein each is itself a submodule of the preceding one. The number of times a given module M' appears in the sequence is called the multiplicity of M' in M . The multiplicity is denoted $[M : M']$ and is independent of the particular sequence chosen. For the particular case of $M = V$ a Verma module and $M' = L$ an irreducible module, the multiplicities $[V : L]$ are invertible and we denote these inverse multiplicities $(L : V)$. The *Kazhdan-Lusztig conjecture* says that $[V : L]$ (resp. $(L : V)$) are given by the Kazhdan-Lusztig polynomials (resp. inverse KL polynomials).

In order to capture all contributions to χ_M without over-counting, we can sum over characters of irreducible modules, weighted by their multiplicities in M : $\chi_M = \sum_L [M : L] \chi_L$, which for our purposes becomes $\chi_L = \sum_V (L : V) \chi_V$. The inverse KL polynomials and the sum over Verma modules for W_N minimal models (such as the 3-state Potts model considered in section 4) are very simple: noting that W_3 is associated to the Lie algebra a_2 , the Verma modules that contribute are those in representations related by the action of the affine Weyl group of a_2 to the particular L representation of interest, and the inverse KL polynomials are ± 1 , where the sign depends on the affine Weyl group element (as explained below).

To make this concrete, let us introduce a new notation. Instead of h, w , we can label representations by two pairs of integers $mn; m'n'$, where each pair of integers is a weight of a_2 , and a parameter t that labels the minimal model $M_{pp'}$ by $t = p/p'$. This notation can label both Verma modules $V_{mn;m'n'}$ and irreducible modules $L_{mn;m'n'}$ — for the latter, we have $m + n < p$, $m' + n' < p'$, and $m, n, m', n' > 0$. These new parameters are related to h, w and c by [9]

$$h(mn; m'n'; t) = \frac{1}{3t} \left[(m - m't)^2 + (m - m't)(n - n't) + (n - n't)^2 - 3(1 - t)^2 \right] \quad (2.8)$$

$$w(mn; m'n'; t) = \frac{\sqrt{2}}{9t\sqrt{3(5t-3)(5-3t)}} [m - n - (m' - n')t] \times [2m + n - (2m' + n')t] [m + 2n - (m' + 2n')t] \quad (2.9)$$

$$c(t) = 50 - 24t - \frac{24}{t} = 2 \left(1 - \frac{12(p-p')^2}{pp'} \right) \quad (2.10)$$

The affine Weyl group is the semi-direct product of the finite Weyl group with translations in the root lattice (the simple roots of a_2 are $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$), and for $l(\omega)$ the length of a finite Weyl group element ω , the inverse KL polynomials are simply $(-1)^{l(\omega)}$. Writing $\chi_{mn;m'n'}^L = \text{Tr}_{L_{mn;m'n'}}(Zq^{L_0})$ for compactness, and similarly for $\chi_{mn;m'n'}^V$, we therefore find

$$\chi_{mn;m'n'}^L = \sum_{r,s=-\infty}^{\infty} \sum_{\omega} (-1)^{l(\omega)} \chi_{\omega(m,n)+rp\alpha_1+sp\alpha_2;m'n'}^V \quad (2.11)$$

$$\begin{aligned} &= \sum_{r,s=-\infty}^{\infty} \left(\chi_{m+2rp-sp,n+2sp-rp;m'n'}^V - \chi_{-m+2rp-sp,m+n+2sp-rp;m'n'}^V \right. \\ &\quad - \chi_{m+n+2rp-sp,-n+2sp-rp;m'n'}^V + \chi_{n+2rp-sp,-m-n+2sp-rp;m'n'}^V \\ &\quad \left. + \chi_{-m-n+2rp-sp,m+2sp-rp;m'n'}^V - \chi_{-n+2rp-sp,-m+2sp-rp;m'n'}^V \right). \quad (2.12) \end{aligned}$$

For representations with $c > 2$ there are only a finite number of null states and so these sums are even simpler. For example, the vacuum representation $[1, 1; 1, 1]$ has character

$$\chi_{h=0,w=0,c}^L = \chi_{1,1;1,1}^V - \chi_{-1,2;1,1}^V - \chi_{2,-1;1,1}^V + \chi_{1,-2;1,1}^V + \chi_{-2,1;1,1}^V - \chi_{-1,-1;1,1}^V \quad (2.13)$$

$$\begin{aligned} &= \chi_{h=0,w=0,c}^V - \chi_{h=1,w=w_-,c}^V - \chi_{h=1,w=-w_-,c}^V \\ &\quad + \chi_{h=3,w=w_+,c}^V + \chi_{h=3,w=-w_+,c}^V - \chi_{h=4,w=0,c}^V \quad (2.14) \end{aligned}$$

where (2.9) gives

$$w_{\pm} = \frac{3\sqrt{2}(1 \pm t)}{\sqrt{3}(5t-3)(5-3t)}. \quad (2.15)$$

3 Series-expansion solutions by direct calculation

Of course, once the commutation relations (2.1) are known, it is a simple (in principle) matter to “brute force” calculate $\text{Tr}_V(Zq^{L_0})$ or $\text{Tr}_L(Zq^{L_0})$ for any operator Z .

For each state $|S_l\rangle$ in V or L , where l denotes the level, commute Z to the right (through the creation operators for that state), and then commute the annihilation operators within Z to the right until they annihilate the highest-weight state $|h, w\rangle$. All that remains will be q^{h+l} times some linear combination of all the states at level l . So, the contribution of $|S_l\rangle$ to the trace is just $\alpha_S q^{h+l}$, where α_S is the coefficient of $|S_l\rangle$ in the linear combination.

Clearly though, this procedure will not give the complete answer as there are infinitely many states in V or L . However, we can use it level-by-level to obtain a series expansion in q . This is more easily done for a Verma module V since for an irreducible module L , there is the additional complication of first deducing which states are present (i.e. by finding the null states and removing them).

Using this method, we found the results given in table 1, i.e.

$$\begin{aligned} \text{Tr}_V(W_0 q^{L_0}) &= wq^h + 2wq^{h+1} + 5wq^{h+2} + 10wq^{h+3} \\ &\quad + 20wq^{h+4} + 36wq^{h+5} + 65wq^{h+6} + \dots \quad (3.1) \end{aligned}$$

$$\text{Tr}_V(W_0^2 q^{L_0}) = w^2 q^h + \left(2w^2 + \frac{4}{22+5c} (32h - c + 2) \right) q^{h+1} + \dots \quad (3.2)$$

$$\text{Tr}_V(W_0^3 q^{L_0}) = w^3 q^h + \left(2w^3 + \frac{12w}{22+5c}(32h-c+2)\right) q^{h+1} + \dots \quad (3.3)$$

etc.

4 Exact results for irreducible modules in the 3-state Potts model

In section 3, we found series-expansion expressions for $\text{Tr}_V(W_0^n q^{L_0})$ up to $O(q^6)$. In this section, we use a different method to find an exact solution for $\text{Tr}_L(W_0 q^{L_0})$ and $\text{Tr}_L(W_0^2 q^{L_0})$. This method has the benefits of a) being exact, and b) giving a result for the irreducible module L , but the drawback of requiring one to choose a value of c . Despite this caveat, the method itself is applicable to any minimal model (i.e. any choice of c) for which one can find null vectors, and can be used to calculate $\text{Tr}_L(W_0^n q^{L_0})$ for any value of n .

For this method, we use the fact that²

$$\text{Tr}_L(N_0 q^{L_0}) = 0 \quad \text{for any null field } N. \quad (4.1)$$

This is immediately apparent from the definition of L , a module which has had all null states removed. The method is demonstrated below; here we simply give an explanation.

For every term in $\text{Tr}_L(N_0 q^{L_0})$, we can use the commutation relations (2.1) and the cyclicity of the trace to move one of the operators (L_p or W_p , say) in that term all the way through the other operators and back to its starting position. In doing so, we pick up an overall factor of q^p from moving the operator through q^{L_0} , and introduce new terms from the commutation relations. If these newly-introduced terms are comprised entirely of L_0 or W_0 operators, we are finished — if not, we repeat this procedure on the remaining non- L_0/W_0 parts until only L_0 or W_0 remain. We then rearrange the result to get an expression for our original term solely in terms of traces of L_0 and W_0 operators, with some coefficients that can depend on q , p and c .

Since L_0 can be considered to be a differential operator via

$$L_0 q^{L_0} |h, w\rangle = L_0 q^h |h, w\rangle = h q^h |h, w\rangle = q \frac{\partial}{\partial q} q^h |h, w\rangle = q \frac{\partial}{\partial q} q^{L_0} |h, w\rangle, \quad (4.2)$$

this can have one of two effects:

- if we have no mixed $\text{Tr}_L(L_0^m W_0^n q^{L_0})$ terms, it will turn every term in $\text{Tr}_L(N_0 q^{L_0})$ into some differential operator acting on the reduced character $\text{Tr}_L(q^{L_0})$ — except for terms like $\text{Tr}_L(W_0^n q^{L_0})$. Since $\text{Tr}_L(N_0 q^{L_0}) = 0$, we can thus write $\text{Tr}_L(W_0^n q^{L_0})$ in terms of known quantities: differential operators acting on the reduced character.
- if we do have mixed $\text{Tr}_L(L_0^m W_0^n q^{L_0})$ terms, we instead get differential operators acting on $\text{Tr}_L(W_0^n q^{L_0})$, and then $\text{Tr}_L(N_0 q^{L_0}) = 0$ gives a differential equation for $\text{Tr}_L(W_0^n q^{L_0})$ which we can solve to find $\text{Tr}_L(W_0^n q^{L_0})$.

²We use the field-state correspondence $|F\rangle = \lim_{z \rightarrow 0} F(z) |0\rangle$ and the Laurent expansion of the field $F(z) = \sum_m F_m z^{-m-h_F}$, where h_F is the weight of F , to relate states, fields and operators/modes.

n	0	1	2	3
W_0	w	$2w$	$5w$	$10w$
W_0^2	w^2	$2w^2 + \frac{1}{4}\beta(32h-c+2)$	$5w^2 + \beta(64h-c+18)$	$10w^2 + \frac{1}{4}\beta(928h+3c+634)$
W_0^3	w^3	$2w^3 + \frac{3}{4}w\beta(32h-c+2)$	$5w^3 + 3w\beta(64h-c+18)$	$10w^3 + \frac{3}{4}w\beta(928h+3c+634)$
W_0^4	w^4	$2w^4 + \frac{3}{2}w^2\beta(32h-c+2) + \frac{1}{32}\beta^2(32h-c+2)^2$	$5w^4 + 6w^2\beta(64h-c+18) + \frac{1}{2}\beta^2((64h-c+18)^2 - 64h(32h-c+2) - 128(4h+1))$	$10w^4 + \frac{3}{2}w^2\beta(928h+3c+634) + \frac{1}{32}\beta^2(250880h^2+1728ch+512640h + 309c^2 - 5076c + 127700)$
W_0^5	w^5	$2w^5 + \frac{5}{2}w^3\beta(32h-c+2) + \frac{5}{32}w\beta^2(32h-c+2)^2$	$5w^5 + 10w^3\beta(64h-c+18) + \frac{5}{2}w\beta^2((64h-c+18)^2 - 64h(32h-c+2) - 128(4h-5))$	$10w^5 + \frac{5}{2}w^3\beta(928h+3c+634) + \frac{5}{32}w\beta^2(250880h^2+1728ch+512640h + 309c^2 - 5076c + 447188)$

n	4	5
W_0	$20w$	$36w$
W_0^2	$20w^2 + 16\beta(46h+c+49)$	$36w^2 + \frac{1}{2}\beta(3616h+155c+5770)$
W_0^3	$20w^3 + 48w\beta(46h+c+49)$	$36w^3 + \frac{3}{2}w\beta(3616h+155c+5770)$
W_0^4	$20w^4 + 96w^2\beta(46h+c+49) + 2\beta^2(21248h^2+928ch+66112h + 73c^2+420c+28340)$	$36w^4 + 3w^2\beta(3616h+155c+5770) + \frac{1}{16}\beta^2(2450432h^2+212672ch + 11408000h+19681c^2+269820c+7354500)$
W_0^5	$20w^5 + 160w^3\beta(46h+c+49) + 10w\beta^2(21248h^2+928ch+66112h + 73c^2+420c+79988)$	$36w^5 + 5w^3\beta(3616h+155c+5770) + \frac{5}{16}w\beta^2(2450432h^2+212672ch + 11408000h+19681c^2+269820c+17492100)$

n	6
W_0	$65w$
W_0^2	$65w^2 + \beta(4384h+273c+8894)$
W_0^3	$65w^3 + 3w\beta(4384h+273c+8894)$
W_0^4	$65w^4 + 6w^2\beta(4384h+273c+8894) + \frac{1}{2}\beta^2(1037312h^2+129600ch+5983104h+14351c^2+254660c+5266492)$
W_0^5	$65w^5 + 10w^3\beta(4384h+273c+8894) + \frac{5}{2}w\beta^2(1037312h^2+129600ch+5983104h+14351c^2+254660c+11256892)$

Table 1. The results of the brute-force calculation of $\text{Tr}_{V_n}(Z)$ described in section 3 for the subspaces V_n of the Verma module at level n and for the insertions $Z = W_0, \dots, W_0^5$, up to $O(q^6)$.

h, w	$0, 0$	$\frac{1}{15}, -\frac{1}{9}\sqrt{\frac{2}{195}}$	$\frac{1}{15}, +\frac{1}{9}\sqrt{\frac{2}{195}}$	$\frac{2}{3}, -\frac{2}{9}\sqrt{\frac{26}{15}}$	$\frac{2}{3}, +\frac{2}{9}\sqrt{\frac{26}{15}}$	$\frac{2}{5}, 0$
$mn; m'n'$	$11; 11$	$12; 11$	$21; 11$	$13; 11$	$31; 11$	$22; 11$

Table 2. W_3 representations of the 3-state Potts model

The first effect appears when we find $\text{Tr}_L(W_0^2 q^{L_0})$, the second when we find $\text{Tr}_L(W_0 q^{L_0})$.

In this section, we choose to work in the 3-state Potts model, as this has null fields at low levels, making calculations especially tractable. This is the W_3 minimal model $M_{5,4}$, i.e. $t = 5/4$ and $c = 4/5$, and it has the representations given in table 2.

4.1 The trace $\text{Tr}_L(W_0 q^{L_0})$

For our calculation of $\text{Tr}_L(W_0 q^{L_0})$, we will need a null field that includes a term that is just $W(z)$ or its derivatives — such a field can be found at level seven in the Potts model:

$$N^7 = -\frac{1}{12}W'''' + \frac{52}{121}(W''L) + \frac{47}{121}(W'L') - \frac{27}{121}(W\Lambda) + \frac{141}{605}(WL'') \quad (4.3)$$

where the brackets (...) denote normal ordering of the enclosed fields by [10]

$$(AB)_m = \sum_{n \leq -h_A} A_n B_{m-n} + \sum_{n > -h_A} B_{m-n} A_n. \quad (4.4)$$

This field has zero mode

$$\begin{aligned} N_0^7 = & \sum_{p \geq 1} \left(\frac{636}{605}p^2 + \frac{54}{121}p + \frac{126}{605} \right) L_{-p} W_p + \sum_{p \geq 1} \left(\frac{636}{605}p^2 - \frac{54}{121}p + \frac{126}{605} \right) W_{-p} L_p \\ & - \frac{27}{121} \sum_{p \geq 1} W_{-p} \Lambda_p - \frac{27}{121} \sum_{p \geq 1} \Lambda_{-p} W_p - \frac{27}{121} \Lambda_0 W_0 + \frac{126}{605} L_0 W_0 - \frac{6}{605} W_0 \end{aligned} \quad (4.5)$$

which we can then substitute into the null field relation (4.1).

Let us demonstrate the method outlined above in detail for the first term in (4.5):

$$\text{Tr}_L(L_{-p} W_p q^{L_0}) = q^p \text{Tr}_L(W_p L_{-p} q^{L_0}) \quad (4.6)$$

$$= q^p \text{Tr}_L(L_{-p} W_p q^{L_0}) + q^p \text{Tr}_L([W_p, L_{-p}] q^{L_0}) \quad (4.7)$$

$$= \frac{q^p}{1 - q^p} \text{Tr}_L(3p W_0 q^{L_0}) \quad (4.8)$$

giving a final expression for the trace over this first term of

$$\begin{aligned} & \text{Tr}_L \left(\sum_{p \geq 1} \left(\frac{636}{605}p^2 + \frac{54}{121}p + \frac{126}{605} \right) L_{-p} W_p q^{L_0} \right) \\ &= \sum_{p \geq 1} \frac{3p q^p}{1 - q^p} \left(\frac{636}{605}p^2 + \frac{54}{121}p + \frac{126}{605} \right) \text{Tr}_L(W_0 q^{L_0}) \\ &= \left(\frac{1908}{605} \sum_{p \geq 1} \frac{p^3 q^p}{1 - q^p} + \frac{162}{121} \sum_{p \geq 1} \frac{p^2 q^p}{1 - q^p} + \frac{378}{605} \sum_{p \geq 1} \frac{p q^p}{1 - q^p} \right) \text{Tr}_L(W_0 q^{L_0}). \end{aligned} \quad (4.9)$$

Repeating this process for the remaining terms in (4.5), we end up with

$$f_2 \text{Tr}_L (L_0^2 W_0 q^{L_0}) + f_1 \text{Tr}_L (L_0 W_0 q^{L_0}) + f_0 \text{Tr}_L (W_0 q^{L_0}) = 0, \quad (4.10)$$

which gives (using (4.2) and writing $\mathcal{W} \equiv \text{Tr}_L (W_0 q^{L_0})$)

$$q^2 f_2 \mathcal{W}'' + q (f_2 + f_1) \mathcal{W}' + f_0 \mathcal{W} = 0. \quad (4.11)$$

In these expressions, the coefficient functions f_i are given by

$$f_0 = -\frac{1944}{121} \left(\sum_{p \geq 1} \frac{pq^p}{1-q^p} \right)^2 - \frac{324}{121} \sum_{p \geq 1} \frac{p^2 q^p}{(1-q^p)^2} + \frac{306}{55} \sum_{p \geq 1} \frac{p^3 q^p}{1-q^p} + \frac{882}{605} \sum_{p \geq 1} \frac{pq^p}{1-q^p} - \frac{6}{605} \quad (4.12)$$

$$f_1 = -\frac{432}{121} \sum_{p \geq 1} \frac{pq^p}{1-q^p} + \frac{9}{55}, \quad (4.13)$$

$$f_2 = -\frac{27}{121}. \quad (4.14)$$

These functions can also be expressed in terms of Eisenstein series $E_{2k}(q)$ as

$$f_0 = -\frac{3}{12100} (1 + 20E_2 + 75(E_2)^2 - 56E_4), \quad f_1 = \frac{9}{605} (1 + 10E_2), \quad (4.15)$$

where here, and later, we use

$$E_2 = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad E_4 = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad E_6 = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \quad (4.16)$$

In principle, we now solve (4.11) to find $\text{Tr}_L (W_0 q^{L_0})$. However, it is clear that this is not a simple ODE to solve exactly, but it does allow us to solve for a series expansion to any desired order. The leading power of q is determined by the indicial equation and can only take the values $1/15$ or $2/3$. The series expansions can then be found order-by-order. The first few terms are

$$\text{Tr}_L (W_0 q^{L_0}) = \pm \frac{1}{9} \sqrt{\frac{2}{195}} q^{\frac{1}{15}} (1 + 46q + 74q^2 + 192q^3 - 121q^4 + 286q^5 - 314q^6 - 166q^7 + O(q^8)) \quad (4.17a)$$

and

$$\text{Tr}_L (W_0 q^{L_0}) = \pm \frac{2}{9} \frac{1}{\sqrt{15 \cdot 26}} q^{\frac{2}{3}} (26 + 143q + 142q^2 + 214q^3 - 22q^4 - 23q^5 - 386q^6 + 26q^7 + O(q^8)). \quad (4.17b)$$

We would like to point out that these are the unique non-trivial series expansion solutions to the differential equation (4.11). Particularly, the leading powers of q , $1/15$ or $2/3$, are exactly the values of h in the Potts model representations that have non-zero w (see table 2).

4.2 The trace $\text{Tr}_L(W_0^2 q^{L_0})$

To calculate $\text{Tr}_L(W_0^2 q^{L_0})$, we will need a null field that includes a term like $(WW)(z)$, which we find at level six in the Potts model:

$$N^6 = (WW) - \frac{85}{78}(L'L') - \frac{95}{117}(L\Lambda) - \frac{25}{234}L'''' + \frac{29}{39}(LL''), \quad (4.18)$$

which has zero mode

$$\begin{aligned} N_0^6 = & 2 \sum_{p \geq 1} W_{-p} W_p - \frac{95}{117} \sum_{p \geq 1} L_{-p} \Lambda_p - \frac{95}{117} \sum_{p \geq 1} \Lambda_{-p} L_p + \sum_{p \geq 1} \left(\frac{11}{3} p^2 + \frac{8}{39} \right) L_{-p} L_p \\ & + W_0^2 - \frac{95}{117} \Lambda_0 L_0 + \frac{4}{39} L_0^2 - \frac{22}{195} L_0 + \frac{4}{9} \Lambda_0. \end{aligned} \quad (4.19)$$

Again, we substitute this into the null field relation (4.1) and perform the manipulations above to find

$$\text{Tr}_L(W_0^2 q^{L_0}) = g_3 \text{Tr}_L(L_0^3 q^{L_0}) + g_2 \text{Tr}_L(L_0^2 q^{L_0}) + g_1 \text{Tr}_L(L_0 q^{L_0}) + g_0 \text{Tr}_L(q^{L_0}) \quad (4.20)$$

$$= \left[q^3 g_3 \frac{\partial^3}{\partial q^3} + q^2 (3g_3 + g_2) \frac{\partial^2}{\partial q^2} + q (g_3 + g_2 + g_1) \frac{\partial}{\partial q} + g_0 \right] \text{Tr}_L(q^{L_0}) \quad (4.21)$$

with

$$g_0 = \frac{1}{1263600} (-38 - 426E_2 - 705(E_2)^2 + 663E_4 + 72E_2E_4 + 434E_6) \quad (4.22)$$

$$g_1 = \frac{1}{14040} (38 + 284E_2 + 235(E_2)^2 - 221E_4) \quad (4.23)$$

$$g_2 = -\frac{1}{234} (19 + 71E_2), \quad g_3 = \frac{95}{117} \quad (4.24)$$

As can be seen, this does not give a differential equation for $\text{Tr}_L(W_0^2 q^{L_0})$, but instead gives $\text{Tr}_L(W_0^2 q^{L_0})$ in terms of a differential operator acting on the reduced character $\text{Tr}_L(q^{L_0})$.

The reduced characters for the irreducible modules, $\text{Tr}_L(q^{L_0})$, are found by applying (2.11) to the reduced Verma module characters (2.7). So, since the coefficient functions g_i and the characters $\text{Tr}_L(q^{L_0})$ are known, we can evaluate the right-hand side of (4.21) to find an expression for $\text{Tr}_L(W_0^2 q^{L_0})$. The solutions, which are composed of products of infinite sums, are not particularly enlightening and so we do not state them here. Instead, we give their series expansions up to $O(q^6)$ in table 3.

5 Exact results for Verma modules

The Verma module has a particularly simple and explicit basis which means we can, at some effort, calculate the trace over a Verma module of any particular power $W_0^n q^{L_0}$ by explicitly calculating the action of W_0^n at each level. We shall take as our canonical basis states of the form

$$\prod_{n=1}^{\infty} (L_{-n}^{a_n} W_{-n}^{b_n}) |h, w\rangle = \cdots W_{-p-1}^{b_{p+1}} L_{-p}^{a_p} W_{-p}^{b_p} L_{-p+1}^{a_{p-1}} \cdots |h, w\rangle, \quad a_n, b_n \geq 0. \quad (5.1)$$

h, w	$q^{-h}\text{Tr}_L(W_0^2 q^{L_0})$ to $O(q^6)$
$0, 0$	$12q^3 + \frac{4352}{65}q^4 + \frac{3064}{13}q^5 + \frac{50864}{65}q^6$
$\frac{1}{15}, \pm \frac{1}{9}\sqrt{\frac{2}{195}}$	$\frac{2}{15795} + \frac{4232}{15795}q + \frac{22868}{3159}q^2 + \frac{227216}{5265}q^3 + \frac{2965402}{15795}q^4 + \frac{9096296}{15795}q^5 + \frac{23727188}{15795}q^6$
$\frac{2}{3}, \pm \frac{2}{9}\sqrt{\frac{26}{15}}$	$\frac{104}{1215} + \frac{3146}{1215}q + \frac{365224}{15795}q^2 + \frac{279764}{3159}q^3 + \frac{944152}{3159}q^4 + \frac{11270746}{15795}q^5 + \frac{5613800}{3159}q^6$
$\frac{2}{5}, 0$	$\frac{28}{13}q + \frac{256}{13}q^2 + \frac{6872}{65}q^3 + \frac{20992}{65}q^4 + \frac{61872}{65}q^5 + \frac{140592}{65}q^6$

Table 3. Series expansions for $\text{Tr}_L(W_0^2 q^{L_0})$ in the three-state Potts model obtained from the exact expressions (not shown). Note that for the $h = 1/15$ and $h = 2/3$ representations, the expansions are the same for either choice of w sign.

commutator	$[L, L]$	$[L, W]$	$[W, W]$
mode	central L	W	central L LL
change in grade	$(0, -2)$ $(0, -1)$	$(0, -1)$	$(-2, -2)$ $(-2, -1)$ $(-2, 0)$

Table 4. Possible changes in grade after a commutator.

There are two simple results we need which will prove very useful. The first is that we can grade the monomials in the W_3 -algebra by the number of W -modes and by the total number of modes. If a monomial has w W -modes and n modes in total, we say it has grade (w, n) , and we can assign it a “total grade” $(w + n)$.

The second is that the effect of commuting two modes changes the grade by one of four possibilities, which are given in table 4. Since each of the commutators decreases the total grade, we see that the result of repeated commutators is guaranteed to terminate. Secondly, we only need consider a finite (and small) number of possible commutation relations between modes in a monomial to calculate the contribution to the trace.

5.1 The trace $\text{Tr}_V(q^{L_0})$

We first restate the result for the reduced character of the Verma module,

$$\text{Tr}_{V_{h,w;c}}(q^{L_0}) = q^h \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^2} = \frac{q^h}{\phi(q)^2}. \tag{5.2}$$

The two factors $(1 - q^n)$ in the denominator correspond to the (equal) contributions from the modes W_{-n} and L_{-n} .

5.2 The trace $\text{Tr}_V(W_0 q^{L_0})$

The next simplest result is for the trace with the insertion of a single W_0 mode. Let us denote a state in the basis (5.1) as $\mathcal{P} |h, w\rangle$ where \mathcal{P} is a monomial in the lowering modes of the algebra. We have

$$W_0 \mathcal{P} |h, w\rangle = [W_0, \mathcal{P}] |h, w\rangle + w \mathcal{P} |h, w\rangle. \tag{5.3}$$

Suppose \mathcal{P} has W-number w . After the action of a single commutator and no further re-arranging, $[W_0, \mathcal{P}]$ has terms with W-number $(w + 1)$ arising from the commutator $[W_0, L_{-p}] = 2pW_{-p}$ (for some p) and terms with W-number $(w - 1)$ arising from the commutator $[W_0, W_{-p}]$ for some p . Since W-number is non-increasing, the terms with W-number $(w - 1)$ cannot contribute to the trace, so we can restrict attention to the terms where $[W_0, L_{-p}] = 2pW_{-p}$. These terms have W-number $w + 1$, and so we need to reduce this by 1 if we are to get a contribution to the trace. The only way to do that is to create a mode W_0 during the rearrangement of the terms which can act on the highest weight state, but such a mode can only be created by the action $[L_p, W_{-p}]$ and the mode L_p can itself only come from the expansion of a term Λ_{-q} , which would itself come from a W-number reducing commutator $[W_r, W_s]$. The result is that such a term cannot arise and the only contribution to the trace of W_0 comes from the action of the mode W_0 on the highest weight state itself. This gives the immediate result

$$\text{Tr}_{V_{h,w;c}}(W_0 q^L) = w q^h \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^2} = \frac{w q^h}{\phi(q)^2}. \tag{5.4}$$

This agrees with the result given in section 3, see table 1 and (3.1), that

$$\text{Tr}_{V_{h,w;c}}(W_0 q^{L_0}) = wq^h + 2wq^{h+1} + 5wq^{h+2} + 10wq^{h+3} + 20wq^{h+4} + 36wq^{h+5} + 65wq^{h+6} + \dots \tag{5.5}$$

5.3 The trace $\text{Tr}_V(W_0^2 q^{L_0})$

We can calculate the trace of W_0^2 on one of the basis states (5.1) by keeping track of the possible ways in which the action of W_0^2 can reproduce the original state. We have

$$W_0^2 \mathcal{P} |h, w\rangle = w^2 \mathcal{P} |h, w\rangle + 2w [W_0, \mathcal{P}] |h, w\rangle + [W_0, [W_0, \mathcal{P}]] |h, w\rangle \tag{5.6}$$

The first term is simply the action of W_0^2 on the highest weight state, giving a term in the character

$$\frac{w^2 q^h}{\phi(q)^2}. \tag{5.7}$$

The second term does not contribute to the trace, as we have already considered exactly this term in the previous section.

The third term is more involved. We calculate this term by splitting the action of the W_0 -mode into several pieces, and by considering the possible action on the modes in the monomial on a case-by-case basis.

It is helpful to split each $[W_0, W_n]$ commutator into various separate terms and analyse the contributions from each of these terms to the trace. We split the commutator up into

(1)	(2)	(3)	(4)	(5)
\Downarrow L_{-r}	\Downarrow W_{-r}	\downarrow $L_{-r} \cdots L_{-r}$	\downarrow $L_{-r} \cdots W_{-r}$	\downarrow $W_{-r} \cdots W_{-r}$
(6)	(7)	(8)	(9)	
\downarrow $L_{-r} \cdots L_{-s}$	\downarrow $L_{-r} \cdots W_{-s}$	\downarrow $W_{-r} \cdots L_{-s}$	\downarrow $W_{-r} \cdots W_{-s}$	

Table 5. possible choices for the action of W_0 ; note that $r > s$.

five terms (a)–(e) as

$$\begin{aligned}
 [W_0, W_{-m}] &= \alpha_r L_{-m} & (a) \\
 &+ m\beta \epsilon_m L_{-m/2} L_{-m/2} & (b) \\
 &+ 2m\beta \sum_{s=\lceil(m+1)/2\rceil}^{m-1} L_{-s} L_{-m+s} & (c) \\
 &+ 2m\beta L_{-m} L_0 & (d) \\
 &+ 2m\beta \sum_{s=m+1}^{\infty} L_{-s} L_{-m+s} & (e),
 \end{aligned} \tag{5.8}$$

where

$$\alpha_r = \left(\frac{r(r^2 - 4)}{15} + r\beta\gamma(r) \right), \quad \epsilon_r = 1 \text{ if } r \text{ is even and } 0 \text{ otherwise.} \tag{5.9}$$

Next, we list the modes in a monomial which are acted upon by the commutators $[W_0, \cdot]$. There are nine possibilities listed in table 5 labelled (1)–(9). The arrows indicate which modes are acted upon by each $[W_0, \cdot]$; a double arrow indicates the same mode is acted upon twice. We consider these nine possibilities in turn.

5.3.1 Contributions from term (1)

We consider all contributions from terms where W_0 acts twice on the same mode L_{-r} . The first action gives

$$[W_0, L_{-r}] = 2r W_{-r} \tag{5.10}$$

The second gives five terms, from the five contributions in equation (5.8). We list these in turn.

(1)(a): $[W_0, [W_0, L_{-r}]] \rightarrow 2r\alpha_r L_{-r}$.

This directly contributes to the trace. For the mode L_{-r} we get a term $2r\alpha_r q^r / (1 - q^r)^2$, and for all the other modes X_{-s} we get $1/(1 - q^s)$, so that the total contribution is

$$T_{1a} = \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{2r \alpha_r q^r}{1 - q^r}. \tag{5.11}$$

(1)(b): $[W_0, [W_0, L_{-r}]] \rightarrow 2r^2 \beta \epsilon_r L_{-r/2} L_{-r/2}$.

The action of the W_0 modes has removed a mode L_{-r} from the monomial and replaced it by two modes $L_{-r/2}$, so that the grade has increased by $(0, 1)$. To recover this lost mode, we can consider at most one commutator of the form $[L, L]$ or $[L, W]$, which change the grade by $(0, -1)$, but no such commutator will produce the desired mode, so this term does not contribute to the trace,

(1)(c): $[W_0, [W_0, L_{-r}]] \rightarrow 4r^2 \beta \sum_s L_{-s} L_{-r+s}$.

We need to reorder the monomial into canonical order, which we do by first moving the mode L_{-r+s} to the right, and then the mode L_{-s} . In the process of the first move, we have to commute L_{-r+s} past possible modes L_{-p} and modes W_{-p} . Only one such commutator can reinstate the missing mode L_{-r} , which is $[L_{-r+s}, L_{-s}] = (2s-r)L_{-r}$. The total contribution from all such terms is thus

$$T_{1c} = 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r^2 \frac{q^r}{1-q^r} \sum_{s>r/2}^{r-1} (2s-r) \frac{q^s}{1-q^s} \tag{5.12}$$

(1)(d): $[W_0, [W_0, L_{-r}]] \rightarrow 4r^2 \beta L_{-r} L_0$.

This again contributes directly to the trace, since L_0 acts on all the remaining terms in the monomial and the highest weight state as well. The action on the highest weight state gives the simple contribution

$$T_{1di} = 4h\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{1-q^r} \tag{5.13}$$

The action of L_0 on the remaining modes gives the more complicated contribution

$$T_{1dii} = q^h \sum_{r=1}^{\infty} 4r^2 \beta \left[\prod_{k=r+1}^{\infty} \frac{1}{(1-q^k)^2} \right] \frac{q^r}{1-q^r} q \frac{d}{dq} \left[\prod_{k=1}^r \frac{1}{(1-q^k)^2} \right] \tag{5.14}$$

$$= 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{1-q^r} \sum_{k=1}^r \frac{2k q^k}{1-q^k} \tag{5.15}$$

(1)(e): $[W_0, [W_0, L_{-r}]] \rightarrow 2r \beta \sum_s L_{-s} L_{-r+s}$.

This has removed a mode L_{-r} from the monomial and (by considering the grade) we are only allowed to use a single commutator to return this to the monomial. This could only arise by moving L_{-r+s} or L_{-s} to their canonical positions, but neither of these could create L_{-r} in this manner and so this term does not contribute to the trace.

5.3.2 Contributions from term (2)

We next consider all contributions from terms where W_0 acts twice on the same mode W_{-r} . The first action gives five terms, one for each of the possible terms in (5.8). We consider the action of W_0 on each of these five terms in turn

(2)(a): $[W_0, [W_0, W_{-r}]] \rightarrow [W_0, \alpha_r L_{-r}] = 2r\alpha_r W_{-r}.$

This directly contributes to the trace, giving the same result as term (1)(b):

$$T_{2a} = \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} 2r \alpha_r \frac{q^r}{1 - q^r}. \tag{5.16}$$

(2)(b): $[W_0, [W_0, W_{-r}]] \rightarrow [W_0, r \beta \epsilon_r L_{-r/2} L_{-r/2}] = r^2 \beta \epsilon_r (W_{-r/2} L_{-r/2} + L_{-r/2} W_{-r/2}).$

The action of the W_0 modes has removed a mode W_{-r} , and inserted modes $W_{-r/2}$ and $L_{-r/2}$ which need to be commuted to the right. In the process, one of the modes $W_{-r/2}$ can commute with a mode $L_{-r/2}$ giving $[W_{-r/2}, L_{-r/2}] = (r/2)W_{-r}$. The two terms $W_{-r/2}L_{-r/2}$ and $L_{-r/2}W_{-r/2}$ differ in their contribution because the first term allows for this extra commutator giving W_{-r} , and together they give

$$T_{2b} = \frac{\beta}{2} \frac{q^h}{\phi(q)^2} \sum_{r=2, \text{ even}}^{\infty} r^3 \frac{1 + q^{r/2}}{1 - q^{r/2}} \frac{q^r}{(1 - q^r)}. \tag{5.17}$$

(2)(c): $[W_0, [W_0, W_{-r}]] \rightarrow \sum_s 2r\beta [W_0, L_{-s} L_{-r+s}] = 2r\beta \sum_s (2sW_{-s}L_{-r+s} + 2(r - s)L_{-s}W_{-r+s}).$

We consider the two terms separately. In the first term, we first move L_{-r+s} to the right; doing so, it can commute with a mode W_{-s} to give $[L_{-r+s}, W_{-s}] = (3s - 2r)W_{-r}$, restoring the mode W_{-r} . Moving the mode W_{-s} to the right cannot give a mode W_{-r} since $(-r + s) > -s$ and so W_{-s} will not pass through any modes L_{-r+s} . Hence the first term gives

$$T_{2ci} = 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{rq^r}{1 - q^r} \sum_{s>r/2}^{r-1} \frac{s(3s - 2r)q^s}{1 - q^s}. \tag{5.18}$$

The second term contributes when W_{-r+s} commutes through a mode L_{-s} giving $(3s - r)W_{-r}$, and a total contribution

$$T_{2cii} = 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{rq^r}{1 - q^r} \sum_{s>r/2}^{r-1} \frac{(r - s)(3s - r)q^s}{1 - q^s}. \tag{5.19}$$

(2)(d): $[W_0, [W_0, W_{-r}]] \rightarrow 2r \beta [W_0, L_{-r} L_0] = 4r^2 \beta W_{-r} L_0.$

This again contributes directly to the trace, since L_0 acts on all the remaining terms in the monomial and the highest weight state as well. The action on the highest weight state gives the simple contribution

$$T_{2di} = 4h\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{1 - q^r} \tag{5.20}$$

The action of L_0 on the remaining modes gives the more complicated contribution

$$T_{2dii} = \sum_{r=1}^{\infty} 4r^2 \beta q^h \left[\prod_{k=r}^{\infty} \frac{1}{(1-q^k)^2} \right] q^r q \frac{d}{dq} \left[\frac{1}{1-q^r} \prod_{k=1}^{r-1} \frac{1}{(1-q^k)^2} \right] \quad (5.21)$$

$$= 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{1-q^r} \left(\frac{r q^r}{1-q^r} + \sum_{k=1}^{r-1} \frac{2k q^k}{1-q^k} \right) \quad (5.22)$$

$$= 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{1-q^r} \left(\sum_{k=1}^r \frac{2k q^k}{1-q^k} - \frac{r q^r}{1-q^r} \right) \quad (5.23)$$

$$(2)(e): [W_0, [W_0, W_{-r}] \rightarrow 2r \beta \sum_s [W_0, L_{-s} L_{-r+s}] = 2r \beta \sum_s (2s W_{-s} L_{-r+s} + 2(s-r) L_{-s} W_{-r+s}).$$

As with term (1)(e), this has extra modes, either W_{-s} or L_{-s} , which cannot be removed by any commutator, and so case (2)(e) does not contribute to the trace.

5.3.3 Contributions from term (3)

In term (3), each W_0 commutes with a mode L_{-r} giving $4r^2 W_{-r} W_{-r}$. This differs by grade (0, 2) so we would need to reduce the number of W -modes by 2 to get a contribution to the trace. This can only happen if one or both of the surplus W_{-r} modes commutes with another W -mode while moving into its canonical ordering. However, it does not pass through any such modes and so no commutators arise and this gives no contribution to the trace.

5.3.4 Contributions from term (4)

In term (4), there are again five contributions from $[W_0, W_{-r}]$ which we again deal with in turn. There is also an overall factor 2 since the W_0 modes can act on the two modes in either order.

$$(4)(a): [W_0, L_{-r}] \cdots [W_0, W_{-r}] \rightarrow 4r \alpha_r W_{-r} \cdots L_{-r}.$$

This has grade (0, 0) and directly contributes to the trace a term

$$T_{4a} = 4 \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r \alpha_r \frac{q^{2r}}{(1-q^r)^2}. \quad (5.24)$$

$$(4)(b): [W_0, L_{-r}] \cdots [W_0, W_{-r}] \rightarrow 4r^2 \beta W_{-r} \cdots L_{-r/2} L_{-r/2}.$$

We need to recover the mode L_{-r} but that is not possible from any of the reorderings and so it gives no contribution to the trace.

$$(4)(c): [W_0, L_{-r}] \cdots [W_0, W_{-r}] \rightarrow 8r^2 \beta \sum_s W_{-r} \cdots L_{-s} L_{-r+s}.$$

The only way this contributes to the trace is from the commutator $[L_{-r+s}, L_{-s}] = (2s-r)L_{-r}$ when moving the mode L_{-r+s} to the right. This leads to the total term

$$T_{4c} = 8\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r^2 \frac{q^{2r}}{(1-q^r)^2} \sum_{s>r/2}^{r-1} (2s-r) \frac{q^s}{1-q^s} \quad (5.25)$$

(4)(d): $[W_0, L_{-r}] \cdots [W_0, W_{-r}] \rightarrow 8r^2\beta W_{-r} \cdots L_{-r}L_0.$

This contributes directly to the trace when the mode L_0 commutes through the modes to the right of it, and from the highest weight state. The action on the highest weight state gives the simple contribution

$$T_{4di} = 8h\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^{2r}}{(1-q^r)^2} \tag{5.26}$$

The action of L_0 on the remaining modes gives the more complicated contribution

$$T_{4dii} = \sum_{r=1}^{\infty} 8r^2\beta q^h \left[\prod_{k=r}^{\infty} \frac{1}{(1-q^k)^2} \right] \frac{q^{2r}}{1-q^r} q \frac{d}{dq} \left[\frac{1}{1-q^r} \prod_{k=1}^{r-1} \frac{1}{(1-q^k)^2} \right] \tag{5.27}$$

$$= 8\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^{2r}}{(1-q^r)^2} \left(\frac{rq^r}{1-q^r} + \sum_{k=1}^{r-1} \frac{2kq^k}{1-q^k} \right) \tag{5.28}$$

$$= 8\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^{2r}}{(1-q^r)^2} \left(\sum_{k=1}^r \frac{2kq^k}{1-q^k} - \frac{rq^r}{1-q^r} \right) \tag{5.29}$$

(4)(e): $[W_0, L_{-r}] \cdots [W_0, W_{-r}] \rightarrow 8r^2\beta \sum_s W_{-r} \cdots L_{-s}L_{-r+s}.$

There is again no way to eliminate the mode L_{-s} with $s > r$ so this does not contribute to the trace.

5.3.5 Contributions from term (5)

The commutator $[W_0, W_{-r}] \cdots [W_0, W_{-r}]$ reduces the w -number of the monomial by two. Since the w -number is non-increasing under commutators, this does not contribute to the trace.

5.3.6 Contributions from terms (6)

In term (3), each W_0 commutes with a mode of the Virasoro algebra giving a change in mode of $(2, 0)$: $[W_0, L_{-r}] \cdots [W_0, L_{-s}] = 4rs W_{-r} \cdots W_{-s}$. We need to reduce the w -number which needs a $[W, W]$ commutator, but these do not arise when moving W_{-r} and W_{-s} into canonical ordering, so there is no contribution to the trace.

5.3.7 Contributions from terms (7)

We have the commutators $[W_0, L_{-r}] \cdots [W_0, W_{-s}]$ with $r \neq s$. This changes the grade of the monomial by $(0, 0)$ or $(0, 1)$ allowing at most one commutator. We have removed the mode W_{-s} from the monomial and added the mode W_{-r} . We cannot remove the extra mode W_{-r} when it is moved into canonical position and so there is no contribution to the trace.

5.3.8 Contributions from terms (8)

We have the commutators $[W_0, W_{-r}] \cdots [W_0, L_{-s}]$ with $r \neq s$. We have again changed the grade of the monomial by $(0, 0)$ or $(0, 1)$ allowing again at most one commutator. This time we have added a mode W_{-s} and we can remove it if a mode L_p included in the $[W_0, W_{-r}] \rightarrow LL$ terms commutes with it when moving to canonical position. We have at the same time removed a W_{-r} mode, which needs to be replaced. This is possible if $p - s = -r$. This can come from either the terms (b) or (c) in the $[W_0, W_{-r}]$ commutator.

$$(8)(b): [W_0, W_{-r}] \cdots [W_0, L_{-r/2}] \rightarrow (r\beta L_{-r/2} L_{-r/2}) \cdots (rW_{-r/2}) \rightarrow r^3 \beta L_{-r/2} \cdots W_{-r}.$$

This gives a contribution

$$T_{8b} = \beta \frac{q^h}{\phi(q)^2} \sum_{r=2, \text{even}}^{\infty} r^3 \frac{q^{2r}}{(1-q^r)(1-q^{r/2})^2} \quad (5.30)$$

$$(8)(c): [W_0, W_{-r}] \cdots [W_0, L_{-s}] \rightarrow \sum_s (2r\beta L_{-s} L_{-r+s}) \cdots (2sW_{-s}) \rightarrow 4r \sum_s s(3s - 2r) \beta L_{-s} \cdots W_{-r}.$$

Remembering to include all the possible $L_{-r/2}$ modes in the monomial as well as those created in $[W_0, W_{-r}]$, this gives a contribution

$$T_{8c} = 8\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r \frac{q^r}{1-q^r} \sum_{s>r/2}^{r-1} s(3s-2r) \frac{q^s}{(1-q^s)^2}. \quad (5.31)$$

5.3.9 Contributions from terms (9)

The change in mode from $[W_0, W_{-r}] \cdots [W_0, W_{-s}]$ is $(-2, 0)$, $(-2, 1)$ or $(-2, 2)$. Since no re-orderings can possibly increase the number of W -modes, there are no contributions to the trace from these terms.

5.3.10 Combining all contributions

This gives the final expression for the trace over the Verma module as

$$\begin{aligned} \text{Tr}_V(W_0^2 q^{L_0}) &= T_{1a} + T_{1c} + T_{1di} + T_{1dii} + T_{2a} + T_{2b} + T_{2ci} + T_{2cii} + T_{2di} + T_{2dii} \\ &\quad + T_{4a} + T_{4c} + T_{4di} + T_{4dii} + T_{8b} + T_{8c} \end{aligned} \quad (5.32)$$

The final result can be simplified by combining some of the terms. The terms with a factor h simplify as

$$T_{1di} + T_{2di} + T_{4di} = 8h\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{(1-q^r)^2}. \quad (5.33)$$

The terms with α_r combine as

$$T_{1a} + T_{2a} + T_{4a} = 4 \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r \alpha_r \frac{q^r}{(1-q^r)^2}. \quad (5.34)$$

The terms coming from the derivative with respect to q combine as

$$T_{1dii} + T_{2dii} + T_{4dii} = 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{(1-q^r)^2} \left(-rq^r \frac{1+q^r}{1-q^r} + \sum_{k=1}^r \frac{4kq^k}{1-q^k} \right). \quad (5.35)$$

The sums over r even can be combined as

$$T_{2b} + T_{8b} = \frac{\beta}{2} \frac{q^h}{\phi(q)^2} \sum_{r=2, \text{even}}^{\infty} r^3 \frac{1+q^r}{1-q^r} \frac{q^r}{(1-q^{r/2})^2} \quad (5.36)$$

$$= 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r^3 \frac{1+q^{2r}}{1-q^{2r}} \frac{q^{2r}}{(1-q^r)^2}. \quad (5.37)$$

These in turn can then be combined with the terms from the derivatives to give

$$T_{1dii} + T_{2dii} + T_{4dii} + T_{2b} + T_{8b} = 4\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} \frac{r^2 q^r}{(1-q^r)^2} \left(-\frac{2rq^r}{1-q^{2r}} + \sum_{k=1}^r \frac{4kq^k}{1-q^k} \right). \quad (5.38)$$

Finally, four of the terms with sums over $r/s < s < r$ can be combined as

$$T_{1c} + T_{2ci} + T_{2cii} + T_{4c} = 8\beta \frac{q^h}{\phi(q)^2} \sum_{r=1}^{\infty} r^2 \frac{q^r}{(1-q^r)^2} \sum_{s>r/2}^{r-1} (2s-r) \frac{q^s}{1-q^s}. \quad (5.39)$$

Putting this all together, we arrive at an expression for the trace over the Verma module as

$$\text{Tr}_V(W_0^2 q^{L_0}) = \frac{q^h}{\phi(q)^2} \left[\begin{aligned} & w^2 + \frac{4}{15} \sum_{r=1}^{\infty} \frac{r^2(r^2-4)q^r}{(1-q^r)^2} \\ & + 4\beta \sum_{r=1}^{\infty} \frac{r^2 q^r}{(1-q^r)^2} \left[2h + \gamma(r) - 2 \frac{rq^{2r}}{1-q^{2r}} + 4 \sum_{k=1}^r \frac{kq^k}{1-q^k} \right] \\ & + 8\beta \sum_{r=1}^{\infty} \frac{rq^r}{1-q^r} \sum_{s>r/2}^{r-1} \frac{q^s}{1-q^s} \left[\frac{r(2s-r)}{1-q^r} + \frac{s(3s-2r)}{1-q^s} \right] \end{aligned} \right]. \quad (5.40)$$

Despite our best efforts, we have not as yet been able to write this result in a neater form. Applying (2.11) to this directly does not yield a tractable result.

6 Comparison of results

To compare the results from section 5 to those in section 4, we need to convert from a trace over the Verma module $V_{h,w;c}$ to a trace over the irreducible module $L_{h,w;c}$ using the methods described in section 2.2: by applying (2.11) to our expressions for $\text{Tr}_V(W_0 q^{L_0})$, given in (5.4), and $\text{Tr}_V(W_0^2 q^{L_0})$ from (5.40), we can find expressions for $\text{Tr}_L(W_0 q^{L_0})$ and $\text{Tr}_L(W_0^2 q^{L_0})$ as desired. We must then series-expand these expressions for the case of the 3-state Potts model, as this is how our irreducible module results are presented in section 4.

For $\text{Tr}_L(W_0 q^{L_0})$ with $h = 1/15$ and $h = 2/3^3$ we find (4.17a) and (4.17b) respectively, and for $\text{Tr}_L(W_0^2 q^{L_0})$, the results we obtain coincide exactly with those given in table 3. The agreements persist as far as we have calculated, $O(q^{30})$.

³Note that as $w = 0$ for $h = 0$ or $h = 2/5$, we have $\text{Tr}_V(W_0 q^{L_0}) = 0$ and hence $\text{Tr}_L(W_0 q^{L_0}) = 0$, which of course trivially satisfies (4.11), for these representations.

In summary then, in section 5 we found exact Verma module expressions at any value of c for $\text{Tr}_V(W_0 q^{L_0})$ and $\text{Tr}_V(W_0^2 q^{L_0})$, and verified that they agreed with the “brute force” expansions from section 3. We then used the techniques described in section 2.2 to convert these into irreducible module expressions and, for the specific example of the Potts model, compared them at the level of series expansions to the results obtained in section 4. Again, they agreed as far as we calculated. Due to the unwieldy form of (5.40), and the complicated form of (4.21), we have not yet been able to show that applying (2.11) to (5.40) is identically equivalent to applying the differential operator (4.21) to the exact expressions for the reduced characters, although it should in principle be possible. Likewise, we have not yet been able to show that the expressions for $\text{Tr}_L(W_0 q^{L_0})$ obtained from (4.11) and from applying (2.11) to (5.4) are equivalent.

7 Future research directions

We conclude our discussion with a mention of possible directions that future related research may take.

We think the results here are interesting in their own right, and can be used in other areas but we would obviously like to extend the results we have here, both to general minimal models and to traces of higher powers of W_0 . Expressions are known for singular vectors in W_3 -minimal models [9] and it may be possible to use these to find traces of powers of W_0 in those representations.

Equations for Virasoro characters have been found from particle-like descriptions for a variety of particle content. In the 3-state Potts model for example, results have been found that can be interpreted as one-particle bosonic, two-particle bosonic, fermionic with two quasi-particles, and fermionic with one quasi-particle and two ghost particles [11]. It would be interesting to know if results could similarly be found for W_3 characters in this way, and if so how they relate to the expressions given in this work.

Calculations of the partition function for $\text{Tr}(q^{L_0})$ and $\text{Tr}(y^Q q^{L_0})$, where Q is a $U(1)$ charge, can be performed using the modular transformations of these traces, for example in [3]. Determining the modular transformation properties of the generalised W_3 characters $\text{Tr}(y^{W_0} q^{L_0})$ may offer an alternative route to results for the character, or indeed vice versa: our results may help to shed light on the transformation properties.

Many of the interesting applications of the results presented in this paper rely on their modular transformation properties: for example, $\hat{q} = \exp(-2\pi i/\tau)$ in [5] below is given by the action of the $SL(2, \mathbb{Z})$ generator $\mathcal{S} : \tau \rightarrow -1/\tau$ on $q = \exp(2\pi i\tau)$. Unfortunately, these properties are not currently known in general, and naïve application of \mathcal{S} to our results does not yield useful expressions. The modular transformation properties may need to be deduced before the research directions below and in the following section become feasible, nevertheless we state them here as we believe them to be interesting.

In [5], the authors calculate the quantities $\text{Tr}(W_0^n \hat{q}^{L_0 - c/24})$ for $n = 2, 4, 6$ in the limits $q \rightarrow 0$, i.e. $\hat{q} \rightarrow 1$, and large central charge, in the $W_\infty[\lambda]$ algebra. Using the relation

$$W_\infty[\lambda]|_{\lambda=N} \cong W_N \tag{7.1}$$

from [13] (see below), we could take the appropriate limits of our results and test them against the results in [5] with $\lambda = 3$.

We can use (5.40) to calculate the leading term in $\text{Tr}(W_0^2 \hat{q}^{L_0 - c/24})$ over a single representation, e.g. a Verma module or the vacuum representation (using (2.13)), as $q \rightarrow 0$, and we do not reproduce the results of [5]. It is not clear at the moment whether this is due to the differences between considering the full spectrum of the theory and a single representation, the continuum representation content in the modular transform of a single representation for $c > 2$, or yet other effects. This is currently the subject of study.

We can find differential equations for the q -dependence of $\text{Tr}_L(W_0 \hat{q}^{L_0 - c/24})$ and $\text{Tr}_L(W_0^2 \hat{q}^{L_0 - c/24})$ in the 3-state Potts model by applying a modular transformation to equations (4.11) and (4.21). Again, the analysis of the resulting equations is currently under study.

7.1 Holography

As mentioned above, the quantities we have investigated in this work are of relevance to holography, see [3] for a review. Interest in $\text{AdS}_3/\text{CFT}_2$ holography was sparked by [12], in which the authors proposed, and gave evidence to support, a duality between higher-spin Vasiliev theories on AdS_3 , which are based on the higher-spin algebras $\text{hs}[\lambda]$, and W_N minimal models in the large N limit. The symmetry algebra of these large- N models is thought to be $W_\infty[\lambda]$, where the 't Hooft coupling $\lambda = N/(N+k)$ is kept fixed as N and the level k are taken to infinity.⁴ Black holes in these bulk theories were found in [4] and checked against the dual CFT results for free fermions and free bosons ($\lambda = 0$ and 1 respectively), and the agreement was extended to general λ in [5], as described above. We may relate $W_\infty[\lambda]$ results to W_N quantities using the algebra isomorphisms [13]

$$W_{N,k} \cong W_\infty \left[\frac{N}{N+k} \right] \cong W_\infty[N]. \tag{7.2}$$

As the works cited above, and others, extended the study of the CFT side of this duality from $\lambda = 0, 1$ to generic λ at large c , so our work further extends the understanding of these theories to include information about the generalised character for all values of c . We hope this will be of interest in a range of holographic applications. For example, in [2] the authors discuss the relation between the generalised character and the partition function Z_{bh} to which a BTZ black hole, generalised to carry spin-3 charge, contributes:

$$Z_{bh}(\tau, \alpha) = \text{Tr}(y^{W_0} q^{L_0}). \tag{7.3}$$

As the contribution of the spin-3 black hole to Z_{bh} is only known at large c (i.e. the classical limit), a calculation of the character appearing on the right-hand side of (7.3) to further orders in c would give quantum corrections to the black hole partition function. This calculation was performed to leading order in c in [2] (cf. the large central charge limit employed in [5]), but our results should hold for all c .

⁴For the unitary minimal models, i.e. those with $p' = p + 1$, we have $p = k + 3$. The Potts model, for example, corresponds to $k = 1$.

Finally, we note that as (7.3) gives a black hole partition function in terms of a CFT quantity, the thermodynamic properties of black holes can be investigated using CFT results — see e.g. [14]. The results we have obtained in this work may also find application in this context, leading to new discoveries regarding the thermodynamic properties of 2+1-dimensional black holes with spin-3 charge.

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