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Iterative algorithms for finding a common solution of system of the set of variational inclusion problems and the set of fixed point problems

Atid Kangtunyakarn

Correspondence:
beawrock@hotmail.com
Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

In this article, we introduce a new mapping generated by infinite family of nonexpansive mapping and infinite real numbers. By means of the new mapping, we prove a strong convergence theorem for finding a common element of the set of fixed point problems of infinite family of nonexpansive mappings and the set of a finite family of variational inclusion problems in Hilbert space. In the last section, we apply our main result to prove a strong convergence theorem for finding a common element of the set of fixed point problems of infinite family of strictly pseudo-contractive mappings and the set of finite family of variational inclusion problems.

Keywords: nonexpansive mapping, strict pseudo contraction, strongly positive operator, variational inclusion problem, fixed point

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping and let $F : C \times C \to \mathbb{R}$ be a bifunction. A mapping T of H into itself is called *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory.

The problem for finding a common fixed point of a family of nonexpansive mappings has been studied by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see, e.g., [2,3]).

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma} \parallel x\parallel^2.$$

A mapping A of C into H is called *inverse-strongly monotone*, see [4], if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha \parallel Ax-Ay\parallel^2$$

for all $x, y \in C$.

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0 \quad \text{for all} v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(C, A)$. Many authors have studied methods for finding solution of variational inequality problems (see, e.g., [5-8]).

In 2008, Qin et al. [9] introduced the following iterative scheme:

$$\begin{cases} y_n = P_C(I - s_nA)x_n \\ x_{n+1} = \alpha_n\gamma f(W_nx_n) + (I - \alpha_nB)W_nP_C(I - r_nA)y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where W_n is the W-mapping generated by a finite family of nonexpansive mappings and real numbers, $A : C \to H$ is relaxed (u,v) cocoercive and μ-Lipschitz continuous, and P_C is a metric projection H onto C. Under suitable conditions of $\{s_n\}$, $\{r_n\}\{\alpha_n\}$, γ, they proved that $\{x_n\}$ converges strongly to an element of the set of variational inequality problem and the set of a common fixed point of a finite family of nonexpansive mappings.

In 2006, Marino and Xu [10] introduced the iterative scheme as follows:

$$x_0 \in H, x_{n+1} = (I - \alpha_nA)Sx_n + \alpha_n\gamma f(x_n), \quad \forall n \geq 0, \tag{1.3}$$

where S is a nonexpansive mapping, f is a contraction with the coefficient $a \in (0, 1)$, A is a strongly positive bounded linear self-adjoint operator with the coefficient $\bar{\gamma}$, and γ is a constant such that $0 < \gamma < \frac{\bar{\gamma}}{a}$. They proved that $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, x \in F(S).$$

We know that a mapping $B : H \to H$ is said to be *monotone*, if for each $x, y \in H$, we have

$$\langle Bx - By, x - y \rangle \geq 0.$$

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is *maximal* if the graph of $Graph(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in Graph(M)$ implies $f \in Mx$.

Next, we consider the following so-called *variational inclusion problem*:

Find a $u \in H$ such that

$$\theta \in Bu + Mu \tag{1.4}$$

where $B : H \to H$, $M : H \to 2^H$ are two nonlinear mappings, and θ is zero vector in H (see, for instance, [11-16]). The set of the solution of (1.4) is denoted by $VI(H, B, M)$.

Let C be a nonempty closed convex subset of Banach space X. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself, and let $\lambda_1, \lambda_2, \ldots,$ be real numbers in [0, 1]; then we define the mapping $K_n : C \to C$ as follows:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \lambda_1T_1U_{n,0} + (1 - \lambda_1)U_{n,0}, \\ U_{n,2} &= \lambda_2T_2U_{n,1} + (1 - \lambda_2)U_{n,1}, \\ U_{n,3} &= \lambda_3T_3U_{n,2} + (1 - \lambda_3)U_{n,2}, \\ &\vdots \\ U_{n,k} &= \lambda_kT_kU_{n,k-1} + (1 - \lambda_k)U_{n,k-1} \\ U_{n,k+1} &= \lambda_{k+1}T_{k+1}U_{n,k} + (1 - \lambda_{k+1})U_{n,k} \\ &\vdots \\ U_{n,n-1} &= \lambda_{n-1}T_{n-1}U_{n,n-2} + (1 - \lambda_{n-1})U_{n,n-2} \\ K_n = U_{n,n} &= \lambda_nT_nU_{n,n-1} + (1 - \lambda_n)U_{n,n-1}. \end{aligned}$$

Such a mapping K_n is called the *K-mapping* generated by $T_1, T_2,\ldots, T_n$ and $\lambda_1, \lambda_2,\ldots, \lambda_n$.

Let $x_1 \in H$ and $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)(\gamma_n K_n x_n + (1-\gamma_n)Sx_n), \quad (1.5)$$

where A is a strongly positive linear-bounded self-adjoint operator with the coefficient $0 < \overline{\gamma} < 1$, $S : C \to C$ is the S - mapping generated by $G_1, G_2,\ldots, G_N$ and $v_1, v_2,\ldots, v_N$, where $G_i : H \to H$ is a mapping defined by $J_{M_{i,\eta}}(I - \eta B_i)x = G_i x$ for every $x \in H$, and $\eta \in (0, 2\delta_i)$ for every $i = 1, 2,\ldots, N$, $f : H \to H$ is contractive mapping with coefficient $\theta \in (0, 1)$ and $0 < \gamma < \frac{\overline{\gamma}}{\theta}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$.

In this article, by motivation of (1.3), we prove a strong convergence theorem of the proposed algorithm scheme (1.5) to an element $z \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} V(H, B_i, M_i)$, under suitable conditions of $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$.

2 Preliminaries

In this section, we provide some useful lemmas that will be used for our main result in the next section.

Let C be a closed convex subset of a real Hilbert space H, and let P_C be the metric projection of H onto C, i.e., for $x \in H$, $P_C x$ satisfies the property:

$$\| x - P_C x \| = \min_{y \in C} \| x - y \| .$$

The following characterizes the projection P_C.

Lemma 2.1. (see [17]) *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2. (see [18]) *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions:

(1) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. (see [19]) *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\Sigma_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \Sigma_{n=1}^{\infty} \lambda_n T_n x_n$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ hold.

Lemma 2.4. (see [20]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be a nonexpansive mapping. Then I - S is demi-closed at zero.*

Lemma 2.5. (see [21]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* [0,1] *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ *for all integer* $n \ge 0$ *and* $\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \le 0$. *Then* $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

In 2009, Kangtunykarn and Suantai [5] introduced the *S*-mapping generated by a finite family of nonexpansive mappings and real numbers as follows:

Definition 2.1. *Let C be a nonempty convex subset of real Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpanxive mappings of C into itself. For each* $j = 1, 2,..., N$, *let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ *where* $I \in [0, 1]$ *and* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, *define the mapping* $S : C \to C$ *as follows:*

$$\begin{aligned} U_0 &= I \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\ &\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned} \tag{2.1}$$

This mapping is called the S-mapping generated by $T_1,..., T_N$ *and* $\alpha_1, \alpha_2,..., \alpha_N$.

Lemma 2.6. (see [5]) *Let C be a nonempty closed convex subset of strictly convex. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpanxive mappings of C into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, j = 1,2,3,..., N, *where* $I = [0, 1]$, $\alpha_1^j \in (0, 1)$, $\alpha_1^j \in (0, 1)$ for all j = 1,2,..., N-1, $\alpha_1^N \in (0, 1]$ $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all j = 1,2,..., N. *Let S be the mapping generated by* $T_1,..., T_N$ *and* $\alpha_1, \alpha_2,..., \alpha_N$. *Then* $F(S) = \bigcap_{i=1}^N F(T_i)$.

Lemma 2.7. (see [5]) *Let C be a nonempty closed convex subset of Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of C into itself and* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ *and* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ *such that* $\alpha_i^{n,j} \to \alpha_i^j \in [0, 1]$ *as* $n \to \infty$ *for i = 1,3 and j = 1,2,3,..., N. Moreover, for every* $n \in \mathbb{N}$, *let S and* S_n *be the S-mappings generated by* $T_1, T_2,..., T_N$ *and* $\alpha_1, \alpha_2,..., \alpha_N$ *and* $T_1, T_2,..., T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_N^{(n)}$, *respectively. Then* $\lim_{n\to\infty} ||S_n x - Sx|| = 0$ *for every* $x \in C$.

Definition 2.2. (see [11]) *Let* $M : H \to 2^H$ *be a multi-valued maximal monotone mapping, then the single-valued mapping* $J_{M,\lambda} : H \to H$ *defined by*

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the resolvent operator associated with M, where λ *is any positive number and I is identity mapping.*

Lemma 2.8. (see [11]) $u \in H$ *is a solution of variational inclusion (1.4) if and only if* $u = J_{M,\lambda}(u - \lambda Bu)$, $\forall \lambda > 0$, *i.e.*,

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, *then* $VI(H, B, M)$ *is closed convex subset in H.*

Lemma 2.9. (see [22]) *The resolvent operator* $J_{M,\lambda}$ *associated with M is single-valued, nonexpansive for all* $\lambda > 0$ *and 1-inverse-strongly monotone.*

Lemma 2.10. *In a strictly convex Banach space E, if*

$$||x|| = ||y|| = ||\lambda x + (1-\lambda)y||$$

for all $x, y \in E$ *and* $\lambda \in (0, 1)$, *then* $x = y$.

Lemma 2.11. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinite family of nonexpanxive mappings of C into itself with* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and let* $\lambda_1, \lambda_2, \ldots,$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots,$ *and* $\sum_{i=1}^{\infty} \lambda_i < \infty$. *For every* $n \in \mathbb{N}$, *let* K_n *be the K-mapping generated by* $T_1, T_2, \ldots, T_n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$. *Then for every* $x \in C$ *and* $k \in \mathbb{N}$, $\lim_{n\to\infty} K_n x$ *exits.*

Proof. Let $x \in C$. Then for $k, n \in \mathbb{N}$, we have

$$\begin{aligned}
\| U_{n+1,k}x - U_{n,k}x \| &= \| \lambda_k T_k U_{n+1,k-1}x + (1-\lambda_k)U_{n+1,k-1}x - \lambda_k T_k U_{n,k-1}x - (1-\lambda_k)U_{n,k-1}x \| \\
&= \| \lambda_k (T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (1-\lambda_k)(U_{n+1,k-1}x - U_{n,k-1}x) \| \\
&\le \lambda_k \| T_k U_{n+1,k-1}x - T_k U_{n,k-1}x \| + (1-\lambda_k) \| U_{n+1,k-1}x - U_{n,k-1}x \| \\
&\le \lambda_k \| U_{n+1,k-1}x - U_{n,k-1}x \| + (1-\lambda_k) \| U_{n+1,k-1}x - U_{n,k-1}x \| \\
&= \| U_{n+1,k-1}x - U_{n,k-1}x \| \\
&= \| \lambda_{k-1} T_{k-1} U_{n+1,k-2}x + (1-\lambda_{k-1})U_{n+1,k-2}x - \lambda_{k-1} T_{k-1} U_{n,k-2}x - (1-\lambda_{k-1})U_{n,k-2}x \| \\
&\le \lambda_{k-1} \| T_{k-1} U_{n+1,k-2}x - T_{k-1} U_{n,k-2}x \| + (1-\lambda_{k-1}) \| U_{n+1,k-2}x - U_{n,k-2}x \| \\
&\le \| U_{n+1,k-2}x - U_{n,k-2}x \| \\
&\vdots \\
&\le \| U_{n+1,1}x - U_{n,1}x \| \\
&= \| \lambda_1 T_1 U_{n+1,0}x + (1-\lambda_1)U_{n+1,0}x - \lambda_1 T_1 U_{n,0}x - (1-\lambda_1)U_{n,0}x \| \\
&= \| \lambda_1 T_1 x + (1-\lambda_1)x - \lambda_1 T_1 x - (1-\lambda_1)x \| \\
&= 0,
\end{aligned} \tag{2.2}$$

which implies that $U_{n+1,k} = U_{n,k}$ for every $k, n \in \mathbb{N}$. Hence, $K_n = U_{n,n} = U_{n+1,n}$. Since $K_{n+1}x = U_{n+1,n+1}x = \lambda_{n+1}T_{n+1}K_n x + (1 - \lambda_{n+1})K_n x$, we have

$$K_{n+1}x - K_n x = \lambda_{n+1}(T_{n+1}K_n x - K_n x). \tag{2.3}$$

Let $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ and $x \in C$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\| K_n x - x^* \| &= \| \lambda_n T_n U_{n,n-1}x + (1-\lambda_n)U_{n,n-1}x - x^* \| \\
&\le \lambda_n \| T_n U_{n,n-1}x - x^* \| + (1-\lambda_n) \| U_{n,n-1}x - x^* \| \\
&\le \| U_{n,n-1}x - x^* \| \\
&= \| \lambda_{n-1} T_{n-1} U_{n,n-2}x + (1-\lambda_{n-1})U_{n,n-2}x - x^* \| \\
&\le \lambda_{n-1} \| T_{n-1} U_{n,n-2}x - x^* \| + (1-\lambda_{n-1}) \| U_{n,n-2}x - x^* \| \\
&\le \| U_{n,n-2}x - x^* \| \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\le \| U_{n,1}x - x^* \| \\
&= \| \lambda_1 T_1 U_{n,0}x + (1-\lambda_1)U_{n,0}x - x^* \| \\
&\le \lambda_1 \| T_1 x - x^* \| + (1-\lambda_1) \| x - x^* \| \\
&= \| x - x^* \|,
\end{aligned} \tag{2.4}$$

which implies that $\{K_n x\}$ is bounded, and so is $\{T_n K_n x\}$. For $m \geq n$, by (2.3) we have

$$\begin{aligned}\| K_m x - K_n x \| &= \| K_m x - K_{m-1}x + K_{m-1}x - K_{m-2}x + K_{m-2}x - + \cdots \\ &\quad - K_{n+1}x + K_{n+1}x - K_n x \| \\ &\leq \| K_m x - K_{m-1}x \| + \| K_{m-1}x - K_{m-2}x \| + \| K_{m-2}x - K_{m-3}x \| + \cdots \\ &\quad + \| K_{n+2} - K_{n+1}x \| + \| K_{n+1}x - K_n x \| \\ &= \lambda_m \| T_m K_{m-1}x - K_{m-1}x \| + \lambda_{m-1} \| T_{m-1}K_{m-2}x - K_{m-2}x \| + \cdots \\ &\quad + \lambda_{n+1} \| T_{n+1}K_n x - K_n x \| \\ &\leq M \sum_{k=n+1}^{m} \lambda_k, \end{aligned} \tag{2.5}$$

where $M = \sup_{n\in \mathbb{N}}\{||T_{n+1}K_n x - K_n x||\}$. This implies that $\{K_n x\}$ is Cauchy sequence. Hence $\lim_{n\to\infty} K_n x$ exists.

From Lemma 2.11, we can define a mapping $K : C \to C$ as follows:

$$Kx = \lim_{n\to\infty} K_n x, \quad x \in C.$$

Such a mapping K is called the *K-mapping* generated by $T_1, T_2,\ldots$, and $\lambda_1, \lambda_2,\ldots$.

Remark 2.12. It is easy to see that for each $n \in \mathbb{N}$, K_n is nonexpansive mappings. Let $x, y \in C$, then we have

$$\| Kx - Ky \| = \lim_{n\Rightarrow\infty} \| K_n x - K_n y \| \leq \| x - y \| . \tag{2.6}$$

By (2.6), we have $K : C \to C$ is nonexpansive mapping. Next, we will show that $\lim_{n\to\infty} sup_{x\in D}||K_n x - Kx|| = 0$ for every bounded subset D of C. To show this, let $x, y \in C$ and D be a bounded subset of C. By (2.5), for $m \geq n$, we have

$$\| K_m x - K_n x \| \leq M \sum_{k=n+1}^{m} \lambda_k.$$

By letting $m \to \infty$, for any $x \in D$, we have

$$\| Kx - K_n x \| \leq M \sum_{k=n+1}^{\infty} \lambda_k.$$

Since $\sum_{n=1}^{\infty} \lambda_n < \infty$, we have

$$\lim_{n\to\infty} \sup_{x\in D} \| Kx - K_n x \| = 0.$$

By the next lemma, we will show that $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$

Lemma 2.13. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2,\ldots$, be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2,\ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n and K be the K-mapping generated by $T_1, T_2,\ldots T_n$ and $\lambda_1, \lambda_2,\ldots \lambda_n$ and $T_1, T_2,\ldots$ and $\lambda_1, \lambda_2,\ldots$, respectively. Then $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. It is easy to see that $\bigcap_{i=1}^{\infty} F(T_i) \subseteq F(K)$. Next, we show that $F(K) \subseteq \bigcap_{i=1}^{\infty} F(T_i)$. Let $x_0 \in F(K)$ and $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$. Let $k \in \mathbb{N}$ be fixed. Since

$$\begin{aligned}
\| K_n x_0 - x^* \| &= \| \lambda_n T_n U_{n,n-1} x_0 + (1-\lambda_n) U_{n,n-1} x_0 - x^* \| \\
&= \| \lambda_n (T_n U_{n,n-1} x_0 - x^*) + (1-\lambda_n)(U_{n,n-1} x_0 - x^*) \| \\
&\leq \lambda_n \| T_n U_{n,n-1} x_0 - x^* \| + (1-\lambda_n) \| U_{n,n-1} x_0 - x^* \| \\
&\leq \| U_{n,n-1} x_0 - x^* \| \\
&= \| \lambda_{n-1} (T_{n-1} U_{n,n-2} x_0 - x^*) + (1-\lambda_{n-1}) U_{n,n-2} (x_0 - x^*) \| \\
&\leq \lambda_{n-1} \| T_{n-1} U_{n,n-2} x_0 - x^* \| + (1-\lambda_{n-1}) \| U_{n,n-2} x_0 - x^* \| \\
&\leq \| U_{n,n-2} x_0 - x^* \| \\
&\vdots \\
&\cdot \\
&\cdot \\
&\leq \| U_{n,k} x_0 - x^* \| \\
&= \| \lambda_k (T_k U_{n,k-1} x_0 - x^*) + (1-\lambda_k)(U_{n,k-1} x_0 - x^*) \| \\
&\leq \lambda_k \| T_k U_{n,k-1} x_0 - x^* \| + (1-\lambda_k) \| U_{n,k-1} x_0 - x^* \| \\
&\leq \| U_{n,k-1} x_0 - x^* \| \\
&\cdot \\
&\cdot \\
&\cdot \\
&\leq \| U_{n,1} x_0 - x^* \| \\
&= \| \lambda_1 (T_1 x_0 - x^*) + (1-\lambda_1)(x_0 - x^*) \| \\
&\leq \lambda_1 \| T_1 x_0 - x^* \| + (1-\lambda_1) \| x_0 - x^* \| \\
&\leq \| x_0 - x^* \|,
\end{aligned} \tag{2.7}$$

we have

$$\begin{aligned}
\| x_0 - x^* \| = \lim_{n\to\infty} \| K_n x_0 - x^* \| &\leq \| \lambda_1 (T_1 x_0 - x^*) + (1-\lambda_1)(x_0 - x^*) \| \\
&\leq \lambda_1 \| T_1 x_0 - x^* \| + (1-\lambda_1) \| x_0 - x^* \| \\
&\leq \| x_0 - x^* \|,
\end{aligned}$$

this implies that

$$\| x_0 - x^* \| = \| T_1 x_0 - x^* \| = \| \lambda_1 (T_1 x_0 - x^*) + (1-\lambda_1)(x_0 - x^*) \| .$$

By Lemma 2.10, we have $T_1 x_0 = x_0$, that is $x_0 \in F(T_1)$. It follows that $U_{n,1} x_0 = x_0$. By (2.7), we have

$$\begin{aligned}
\| K_n x_0 - x^* \| \leq \| U_{n,2} x_0 - x^* \| &= \| \lambda_2 (T_2 U_{n,1} x_0 - x^*) + (1-\lambda_2)(U_{n,1} x_0 - x^*) \| \\
&= \| \lambda_2 (T_2 x_0 - x^*) + (1-\lambda_2)(x_0 - x^*) \| \\
&\leq \lambda_2 \| T_2 x_0 - x^* \| + (1-\lambda_2) \| x_0 - x^* \| \\
&\leq \| x_0 - x^* \| .
\end{aligned}$$

It follows that

$$\begin{aligned}
\| x_0 - x^* \| &= \lim_{n\to\infty} \| K_n x_0 - x^* \| \\
&\leq \| \lambda_2 (T_2 x_0 - x^*) + (1-\lambda_2)(x_0 - x^*) \| \\
&\leq \lambda_2 \| T_2 x_0 - x^* \| + (1-\lambda_2) \| x_0 - x^* \| \\
&\leq \| x_0 - x^* \|,
\end{aligned}$$

which implies

$$\| x_0 - x^* \| = \| T_2 x_0 - x^* \| = \| \lambda_2 (T_2 x_0 - x^*) + (1 - \lambda_2)(x_0 - x^*) \| .$$

By Lemma 2.10, we obtain that $T_2 x_0 = x_0$, that is $x_0 \in F(T_2)$. It follows that $U_{n,2} x_0 = x_0$. By using the same argument, we can conclude that $T_i x_0 = x_0$ and $U_i x_0 = x_0$ for $i = 1, 2, \ldots, k - 1$. By (2.7), we have

$$\begin{aligned} \| K_n x_0 - x^* \| &\le \| U_{n,k} x_0 - x^* \| \\ &= \| \lambda_k (T_k U_{n,k-1} x_0 - x^*) + (1 - \lambda_k)(U_{n,k-1} x_0 - x^*) \| \\ &= \| \lambda_k (T_k x_0 - x^*) + (1 - \lambda_k)(x_0 - x^*) \| \\ &\le \lambda_k \| T_k x_0 - x^* \| + (1 - \lambda_k) \| x_0 - x^* \| \\ &\le \| x_0 - x^* \| . \end{aligned}$$

It follows that

$$\begin{aligned} \| x_0 - x^* \| &= \lim_{n \to \infty} \| K_n x_0 - x^* \| \\ &= \| \lambda_k (T_k x_0 - x^*) + (1 - \lambda_k)(x_0 - x^*) \| \\ &\le \lambda_k \| T_k x_0 - x^* \| + (1 - \lambda_k) \| x_0 - x^* \| \\ &\le \| x_0 - x^* \|, \end{aligned} \tag{2.8}$$

which implies

$$\| x_0 - x^* \| = \| T_k x_0 - x^* \| = \| \lambda_k (T_k x_0 - x^*) + (1 - \lambda_k)(x_0 - x^*) \| . \tag{2.9}$$

By Lemma 2.10, we have $T_k x_0 = x_0$, that is $x_0 \in F(T_k)$. This implies that $x_0 \in \bigcap_{i=1}^{\infty} F(T_i)$.

3 Main result

Theorem 3.1. *Let H be a real Hilbert space, and let* $M_i : H \to 2^H$ *be maximal monotone mappings for every* $i = 1, 2, \ldots, N$. *Let* $B_i : H \to H$ *be a* δ_i *- inverse strongly monotone mapping for every* $i = 1, 2, \ldots, N$ *and* $\{T_i\}_{i=1}^{\infty}$ *an infinite family of nonexpansive mappings from H into itself. Let A be a strongly positive linear-bounded self-adjoint operator with the coefficient* $0 < \overline{\gamma} < 1$. *Let* $G_i : H \to H$ *be defined by* $J_{M_{i,\eta}}(I - \eta B_i)x = G_i x$ *for every* $x \in H$ *and* $\eta \in (0, 2\delta_i)$ *for every* $i = 1, 2, \ldots, N$ *and let* $v_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \ldots, N$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ *for all* $j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0, 1]$ $\alpha_2^j, \alpha_3^j \in [0, 1)$ *for all* $j = 1, 2, \ldots, N$.. *Let* $S : C \to C$ *be the S-mapping generated by* $G_1, G_2, \ldots, G_N$ *and* $v_1, v_2, \ldots, v_N$. *Let* $\lambda_1, \lambda_2, \ldots,$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots,$ *with* $\sum_{i=1}^{\infty} \lambda_i < \infty$, *and let* K_n *be the K-mapping generated by* $T_1, T_2, \ldots, T_n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *and let K be the K-mapping generated by* $T_1, T_2, \ldots,$ *and* $\lambda_1, \lambda_2, \ldots,$ *i.e.,*

$$Kx = \lim_{n \to \infty} K_n x$$

for every $x \in C$. *Assume that* $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} V(H, B_i, M_i) \neq \emptyset$. *For every* $n \in \mathbb{N}$, $i = 1, 2, \ldots, N$, *let* $x_1 \in H$ *and* $\{x_n\}$ *be the sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)(\gamma_n K_n x_n + (1 - \gamma_n) S x_n), \tag{3.1}$$

where $f : H \to H$ *is contractive mapping with coefficient* $\theta \in (0, 1)$ *and* $0 < \gamma < \frac{\overline{\gamma}}{\theta}$. *Let* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *be sequences in* $[0, 1]$, *satisfying the following conditions:*

(*i*) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$

(*ii*) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(*iii*) $\lim_{n\to\infty}\gamma_n = c \in (0,1)$

Then $\{x_n\}$ *converges strongly to* $z \in \mathfrak{F}$, *which solves uniquely the following variational inequality:*

$$\langle (A-\gamma f)z, z-x^*\rangle \le 0, \quad \forall x^* \in \mathfrak{F}. \tag{3.2}$$

Equivalently, we have $P_{\mathfrak{F}}(I-A+\gamma f)z = z$.

Proof. Let z be the unique solution of (3.2). First, we will show that the mapping G_i is a nonexpansive mapping for every i = 1, 2,..., N. Let $x, y \in H$, since B_i is δ_i - inverse strongly monotone mapping and $0 < \eta < 2\delta_i$, for every i = 1, 2,..., N, we have

$$\begin{aligned}\|(I-\eta B_i)x-(I-\eta B_i)y\|^2 &= \|x-y-\eta(B_ix-B_iy)\|^2\\ &= \|x-y\|^2 - 2\eta\langle x-y, B_ix-B_iy\rangle + \eta^2\|B_ix-B_iy\|^2\\ &\le \|x-y\|^2 - 2\delta_i\eta\|B_ix-B_iy\|^2 + \eta^2\|B_ix-B_iy\|^2\\ &= \|x-y\|^2 + \eta(\eta-2\delta_i)\|B_ix-B_iy\|^2\\ &\le \|x-y\|^2.\end{aligned} \tag{3.3}$$

Thus, $(I - \eta B_i)$ is a nonexpansive mapping for every i = 1, 2,..., N . By Lemma 2.9, we have $G_i = J_{M_{i,\eta}}(I-\eta B_i)$ is a nonexpansive mappings for every i = 1, 2,..., N. Let $x^* \in \mathfrak{F}$; by Lemma 2.8, we have

$$x^* = G_ix^* = J_{M_i,\eta}(I-\eta B_i)x^*, \forall i = 1,2,\ldots N. \tag{3.4}$$

Let $e_n = \gamma_nK_nx_n + (1 - \gamma_n)Sx_n$. Since G_i is a nonexpansive mapping for every i = 1, 2,..., N, we have that S is a nonexpansive mapping. By nonexpansiveness of K_n we have

$$\begin{aligned}\|e_n-x^*\| &= \|\gamma_n(K_nx_n-x^*)+(1-\gamma_n)(Sx_n-x^*)\|\\ &\le \gamma_n\|K_nx_n-x^*\| + (1-\gamma_n)\|Sx_n-x^*\|\\ &\le \gamma_n\|x_n-x^*\| + (1-\gamma_n)\|x_n-x^*\|\\ &\le \|x_n-x^*\|.\end{aligned} \tag{3.5}$$

Without loss of generality, by conditions (*i*) and (*ii*), we have $\alpha_n \le (1 - \beta_n)||A||^{-1}$. Since A is a strongly positive linear-bounded self-adjoint operator, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.6}$$

For each $x \in C$ with $||x||$ = 1, we have

$$\langle((1-\beta_n)I-\alpha_nA)x, x\rangle = 1-\beta_n-\alpha_n\langle Ax, x\rangle \ge 1-\beta_n-\alpha_n\|A\| \ge 0, \tag{3.7}$$

then $(1 - \beta_n)I - \alpha_nA$ is positive. By (3.6) and (3.7), we have

$$\begin{aligned}\|(1-\beta_n)I-\alpha_nA\| &= \sup\{\langle((1-\beta_n)I-\alpha_nA)x, x\rangle : x \in C, \|x\| = 1\}\\ &= \sup\{\langle 1-\beta_n-\alpha_n\langle Ax, x\rangle : x \in C, \|x\| = 1\}\\ &\le 1-\beta_n-\alpha_n\langle Ax, x\rangle\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned} \tag{3.8}$$

We shall divide our proof into six steps.

Step 1. We will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \mathfrak{F}$, by (3.5) and (3.8), we have

$$\begin{aligned}
\| x_{n+1} - x^* \| &= \| \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)e_n - x^* \| \\
&= \| \alpha_n \gamma f(x_n) - \alpha_n Ax^* + \alpha_n Ax^* - \beta_n x^* + \beta_n x^* + \beta_n x_n \\
&\quad + ((1-\beta_n)I - \alpha_n A)e_n - x^* \| \\
&\leq \alpha_n \| \gamma f(x_n) - Ax^* \| + \beta_n \| x_n - x^* \| + \| ((1-\beta_n)I - \alpha_n A)(e_n - x^*) \| \\
&\leq \alpha_n \| \gamma f(x_n) - Ax^* \| + \beta_n \| x_n - x^* \| + ((1-\beta_n)I - \alpha_n \overline{\gamma}) \| x_n - x^* \| \\
&\leq \alpha_n (\| \gamma f(x_n) - \gamma f(x^*) \| + \| \gamma f(x^*) - Ax^* \|) + (1 - \alpha_n \overline{\gamma}) \| x_n - x^* \| \\
&\leq \alpha_n \gamma \theta \| x_n - x^* \| + \alpha_n \| \gamma f(x^*) - Ax^* \| + (1 - \alpha_n \overline{\gamma}) \| x_n - x^* \| \\
&= \alpha_n \| \gamma f(x^*) - Ax^* \| + (1 - \alpha_n(\overline{\gamma} - \gamma\theta)) \| x_n - x^* \| \\
&\leq \max\{\| x_n - x^* \|, \frac{\| \gamma f(x^*) - Ax^* \|}{\overline{\gamma} - \gamma\theta}\}.
\end{aligned}$$

By induction, we can prove that $\{x_n\}$ is bounded, and so are $\{e_n\}$, $\{K_n x_n\}$, $\{Sx_n\}$ and $\{G_i(x_n)\}$ for every $i = 1, 2,..., N$. Without loss of generality, we can assume that there exists a bounded set $D \subset H$ such that

$$e_n, x_n, Sx_n, K_n x_n, G_i x_n \in D, \forall n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, N. \tag{3.9}$$

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Define sequence $\{z_n\}$ by $z_n = \frac{1}{1-\beta_n}(x_{n+1} - \beta_n x_n)$.

Then $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$.

Since $\{x_n\}$ is bounded, we have

$$\begin{aligned}
||z_{n+1} - z_n|| &= ||\frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \left(\frac{x_{n+1} - \beta_n x_n}{1-\beta_n}\right)|| \\
&= ||\frac{\alpha_{n+1}\gamma f(x_{n+1}) + \left((1-\beta_{n+1})I - \alpha_{n+1}A\right) e_{n+1}}{1-\beta_{n+1}} \\
&\quad - \left(\frac{\alpha_n \gamma f(x_n) + \left((1-\beta_n)I - \alpha_n A\right) e_n}{1-\beta_n}\right)|| \\
&\leq \alpha_{n+1}||\frac{\gamma f(x_{n+1}) - Ae_{n+1}}{1-\beta_{n+1}}|| + ||e_{n+1} - e_n|| \\
&\quad + \alpha_n||\frac{\gamma f(x_n) - Ae_n}{1-\beta_n}||.
\end{aligned} \tag{3.10}$$

By definition of e_n and nonexpansiveness of S, we have

$$\begin{aligned}
\|e_{n+1} - e_n\| &= \left\|\gamma_{n+1}K_{n+1}x_{n+1} + (1-\gamma_{n+1})Sx_{n+1} - \gamma_n K_n x_n - (1-\gamma_n)Sx_n\right\| \\
&= \left\|\gamma_{n+1}K_{n+1}x_{n+1} + (1-\gamma_{n+1})Sx_{n+1} - \gamma_{n+1}K_n x_n + \gamma_{n+1}K_n x_n\right. \\
&\quad \left. -(1-\gamma_{n+1})Sx_n + (1-\gamma_{n+1})Sx_n - \gamma_n K_n x_n - (1-\gamma_n)Sx_n\right\| \\
&= \left\|\gamma_{n+1}(K_{n+1}x_{n+1} - K_n x_n) + (1-\gamma_{n+1})(Sx_{n+1} - Sx_n)\right. \\
&\quad \left. +(\gamma_{n+1} - \gamma_n)K_n x_n + (\gamma_n - \gamma_{n+1})Sx_n\right\| \\
&\leq \gamma_{n+1}\|K_{n+1}x_{n+1} - K_n x_n\| + (1-\gamma_{n+1})\|x_{n+1} - x_n\| \\
&\quad + |\gamma_{n+1} - \gamma_n|\,\|K_n x_n\| + |\gamma_n - \gamma_{n+1}|\,\|Sx_n\| \\
&\leq \gamma_{n+1}\|K_{n+1}x_{n+1} - K_n x_n\| + (1-\gamma_{n+1})\|x_{n+1} - x_n\| \\
&\quad + 2|\gamma_{n+1} - \gamma_n|M,
\end{aligned}$$

where $M = \max_{n\in\mathbb{N}}\{||K_nx_n||, ||Sx_n||\}$. Substituting (3.11) into (3.10), we have

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \alpha_{n+1}\left|\left|\frac{\gamma f(x_{n+1}) - Ae_{n+1}}{1-\beta_{n+1}}\right|\right| + \alpha_n\left|\left|\frac{\gamma f(x_n) - Ae_n}{1-\beta_n}\right|\right| + \|e_{n+1} - e_n\| \\ &\le \alpha_{n+1}\left|\left|\frac{\gamma f(x_{n+1}) - Ae_{n+1}}{1-\beta_{n+1}}\right|\right| + \alpha_n\left|\left|\frac{\gamma f(x_n) - Ae_n}{1-\beta_n}\right|\right| \\ &\quad + \gamma_{n+1}\|K_{n+1}x_{n+1} - K_nx_n\| + (1-\gamma_{n+1})\|x_{n+1} - x_n\| \\ &\quad + 2|\gamma_{n+1} - \gamma_n|M \\ &\le \alpha_{n+1}\left|\left|\frac{\gamma f(x_{n+1}) - Ae_{n+1}}{1-\beta_{n+1}}\right|\right| + \alpha_n\left|\left|\frac{\gamma f(x_n) - Ae_n}{1-\beta_n}\right|\right| \\ &\quad + \gamma_{n+1}(\|K_{n+1}x_{n+1} - K_{n+1}x_n\| + \|K_{n+1}x_n - K_nx_n\|) \\ &\quad + (1-\gamma_{n+1})\|x_{n+1} - x_n\| + 2|\gamma_{n+1} - \gamma_n|M \\ &\le \alpha_{n+1}\left|\left|\frac{\gamma f(x_{n+1}) - Ae_{n+1}}{1-\beta_{n+1}}\right|\right| + \alpha_n\left|\left|\frac{\gamma f(x_n) - Ae_n}{1-\beta_n}\right|\right| + \|x_{n+1} - x_n\| \\ &\quad + \|K_{n+1}x_n - K_nx_n\| + 2|\gamma_{n+1} - \gamma_n|M.\end{aligned} \tag{3.12}$$

It implies that

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \alpha_{n+1}\left|\left|\frac{\gamma f(x_{n+1}) - Ae_{n+1}}{1-\beta_{n+1}}\right|\right| + \alpha_n\left|\left|\frac{\gamma f(x_n) - Ae_n}{1-\beta_n}\right|\right| \\ &\quad + \|K_{n+1}x_n - K_nx_n\| + 2|\gamma_{n+1} - \gamma_n|M.\end{aligned} \tag{3.13}$$

By (2.3), it implies that

$$K_{n+1}x_n - K_nx_n = \lambda_{n+1}(T_{n+1}K_nx_n - K_nx_n),$$

since $\lambda_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|K_{n+1}x_n - K_nx_n\| = 0. \tag{3.14}$$

By (3.13), (3.14) and conditions (*i*), (*iii*), we have

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.15}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.16}$$

By condition (*ii*) and (3.16)

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0. \tag{3.17}$$

Step 3. We will show that

$$\lim_{n\to\infty} \|e_n - x_n\| = 0. \tag{3.18}$$

Since $x_{n+1} = \alpha_n\gamma f + \beta_nx_n + ((1 - \beta_n)I - \alpha_nA)\, e_n$, we have

$$\begin{aligned}\|x_{n+1} - e_n\| &= \left\|\alpha_n(\gamma f(x_n) - Ae_n) + \beta_n(x_n - e_n)\right\| \\ &\le \alpha_n\left\|\gamma f(x_n) - Ae_n\right\| + \beta_n(\|x_n - x_{n+1}\| + \|x_{n+1} - e_n\|),\end{aligned}$$

it implies that

$$(1-\beta_n)\|x_{n+1} - e_n\| \le \alpha_n\left\|\gamma f(x_n) - Ae_n\right\| + \beta_n\|x_n - x_{n+1}\|,$$

and it follows that

$$\|x_{n+1} - e_n\| \le \frac{\alpha_n}{(1-\beta_n)} \left\|\gamma f(x_n) - Ae_n\right\| + \frac{\beta_n}{(1-\beta_n)} \|x_n - x_{n+1}\| .$$

By conditions (*i*), (*ii*) and (3.17), we have

$$\lim_{n\to\infty} \|x_{n+1} - e_n\| = 0. \tag{3.19}$$

Since $||e_n - x_n|| \le ||e_n - x_{n+1}|| + ||x_{n+1} - x_n||$, by (3.17) and (3.19), we have

$$\lim_{n\to\infty} \|e_n - x_n\| = 0.$$

Step 4. Define a mapping $Q : H \to H$ by

$$Qx = cKx + (1-c)Sx, \quad \forall x \in H. \tag{3.20}$$

We will show that

$$\lim_{n\to\infty} ||Qx_n - x_n|| = 0. \tag{3.21}$$

Since

$$\begin{aligned}\|Qx_n - e_n\| &= \left\|cKx_n + (1-c)Sx_n - \gamma_n K_n x_n - (1-\gamma_n)Sx_n\right\| \\ &\le \|cKx_n - \gamma_n K_n x_n\| + |\gamma_n - c| \, \|Sx_n\| \\ &= \|cKx_n - \gamma_n Kx_n + \gamma_n Kx_n - \gamma_n K_n x_n\| + |\gamma_n - c| \, \|Sx_n\| \\ &\le \ |c - \gamma_n| \, \|Kx_n\| + \gamma_n \|Kx_n - K_n x_n\| + |\gamma_n - c| \, \|Sx_n\| \\ &\le \ |c - \gamma_n| \, \|Kx_n\| + \sup_{x\in D} \{\|Kx - K_n x\|\} + |\gamma_n - c| \, \|Sx_n\| .\end{aligned}$$

By remark 2.12 and condition (*iii*), we have

$$\lim_{n\to\infty} \|Qx_n - e_n\| = 0. \tag{3.22}$$

Since $||Qx_n - x_n|| \le ||Qx_n - e_n|| + ||e_n - x_n||$, from (3.22) and (3.18), we have

$$\lim_{n\to\infty} \ ||Qx_n - x_n|| \ = 0.$$

Step 5. We will show that

$$\limsup_{n\to\infty} \langle(\gamma f - A)z, x_n - z\rangle \le 0, \tag{3.23}$$

where $z = P_{\mathfrak{F}}(I - (A - \gamma f))z$. Let $\{x_{n_j}\}$ be subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle(\gamma f - A)z, x_n - z\rangle = \lim_{j\to\infty} \langle(\gamma f - A)z, x_{n_j} - z\rangle. \tag{3.24}$$

Without loss of generality, we may assume that $\{x_{n_j}\}$ converses weakly to some $q \in H$. By nonexpansiveness of S and K, (3.20) and Lemma 2.3, we have that Q is nonexpansive mapping and

$$F(Q) = F(K) \bigcap F(S). \tag{3.25}$$

Since $J_{M_{i,\eta}}(I-\eta B_i)x = G_i x$ for every $x \in H$ and $i = 1, 2,\dots N$, by Lemma 2.8, we have

$$VI(H, B_i, M_i) = F(J_{M_{i,\eta}}(I-\eta B_i)) = F(G_i), \quad \forall i = 1, 2, \dots N. \tag{3.26}$$

By Lemma 2.6 and Lemma 2.9, we have

$$F(S) = \bigcap_{i=1}^{N} F(G_i) = \bigcap_{i=1}^{N} VI(H, B_i, M_i) \tag{3.27}$$

By Lemma 2.13, we have

$$F(K) = \bigcap_{i=1}^{\infty} F(T_i). \tag{3.28}$$

By (3.25), (3.27), and (3.28), we have

$$F(Q) = F(K) \bigcap F(S) = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} VI(H, B_i, M_i). \tag{3.29}$$

Since $x_{n_j} \rightharpoonup q$ as $j \to \infty$, nonexpansiveness of Q, (3.21) and Lemma 2.4, we have

$$q \in F(Q) = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} VI(H, B_i, M_i) = \mathfrak{F}. \tag{3.30}$$

By (3.24) and (3.30), we have

$$\limsup_{n\to\infty} \langle (\gamma f - A)z, x_n - z\rangle = \lim_{n\to\infty} \langle (\gamma f - A)z, x_{n_j} - z\rangle = \langle (\gamma f - A)z, q - z\rangle \le 0.$$

Step 6. Finally, we will show that $x_n \to z$ as $n \to \infty$, where $z = P_{\mathfrak{F}}(I - (A - \gamma f))z$. Since

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right) e_n - z\right\|^2 \\
&= \left\|\alpha_n(\gamma f(x_n) - Az) + \beta_n(x_n - z) + \left((1-\beta_n)I - \alpha_n A\right)(e_n - z)\right\|^2 \\
&\le \left\|\beta_n(x_n - z) + \left((1-\beta_n)I - \alpha_n A\right)(e_n - z)\right\|^2 + 2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\
&= \left(\left\|\beta_n(x_n - z) + \left((1-\beta_n)I - \alpha_n A\right)(e_n - z)\right\|\right)^2 + 2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\
&\le \left(\left\|\beta_n(x_n - z)\right\| + \left\|(1-\beta_n)I - \alpha_n A\right\| \|e_n - z\|\right)^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - \gamma f(z), x_{n+1} - z\rangle + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\
&\le \left(\left\|\beta_n(x_n - z)\right\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|e_n - z\|\right)^2 \\
&\quad + 2\alpha_n\gamma\theta\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\
&\le \left(\left\|\beta_n(x_n - z)\right\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - z\|\right)^2 \\
&\quad + 2\alpha_n\gamma\theta\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\
&\le \left((1 - \alpha_n\bar{\gamma})\|x_n - z\|\right)^2 + \alpha_n\gamma\theta\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\
&\le (1 - 2\alpha_n\bar{\gamma} + \alpha_n\gamma\theta)\|x_n - z\|^2 + \alpha_n^2\bar{\gamma}^2\|x_n - z\|^2 + \alpha_n\gamma\theta\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle,
\end{aligned}$$

it implies that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\gamma\theta)}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\theta}\|x_n-z\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle\gamma f(z)-Az,x_{n+1}-z\rangle \\
&=\frac{(1-\alpha_n\gamma\theta+\alpha_n\gamma\theta-2\alpha_n\bar{\gamma}+\alpha_n\gamma\theta)}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\theta}\|x_n-z\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle\gamma f(z)-Az,x_{n+1}-z\rangle \\
&=\frac{(1-\alpha_n\gamma\theta-2\alpha_n(\bar{\gamma}-\gamma\theta))}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\theta}\|x_n-z\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle\gamma f(z)-Az,x_{n+1}-z\rangle \\
&=(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\theta))}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\theta}\|x_n-z\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle\gamma f(z)-Az,x_{n+1}-z\rangle \\
&=(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\theta))}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{\alpha_n}{1-\alpha_n\gamma\theta}\left(\alpha_n\bar{\gamma}^2\|x_n-z\|^2\right. \\
&\quad\left.+2\langle\gamma f(z)-Az,x_{n+1}-z\rangle\right) \\
&=(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\theta))}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{2(\bar{\gamma}-\gamma\theta)}{2(\bar{\gamma}-\gamma\theta)}\frac{\alpha_n}{1-\alpha_n\gamma\theta}\left(\alpha_n\bar{\gamma}^2\|x_n-z\|^2\right. \\
&\quad\left.+2\langle\gamma f(z)-Az,x_{n+1}-z\rangle\right) \\
&=(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\theta))}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\theta)}{1-\alpha_n\gamma\theta}\left(\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\theta)}\|x_n-z\|^2\right. \\
&\quad\left.+\frac{2}{2(\bar{\gamma}-\gamma\theta)}\langle\gamma f(z)-Az,x_{n+1}-z\rangle\right),
\end{aligned}
$$

from condition *i*, step 5 and Lemma 2.2, we can conclude that $x_n \to z$ as $n \to \infty$, where $z = P_F(I-(A-\gamma f))z$. This completes the proof.

By means of our main result, we have the following results in the framework of Hilbert space. To prove these results, we need definition and lemma as follows:

Definition 3.1. *A mapping $T: C \to C$ is said to be a κ-strict pseudo-contraction mapping, if there exists $\kappa \in [0, 1)$ such that*

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+\kappa\|(I-T)x-(I-T)y\|^2, \quad \forall x,y \in C.$$

Lemma 3.2. (see [23]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T: C \to C$ a κ-strict pseudo-contraction. Define $S: C \to C$ by $Sx = \alpha x + (1-\alpha)Tx$, for each $x \in C$. Then, as $\alpha \in [\kappa, 1)$, S is nonexpansive such that $F(S) = F(T)$.*

Corollary 3.3. *Let H be a real Hilbert space and let $M_i : H \to 2^H$ be maximal monotone mappings for every $i = 1, 2,\ldots N$. Let $B_i : H \to H$ be a δ_i - inverse strongly monotone mapping for every $i = 1, 2,\ldots N$ and $\{T_i\}_{i=1}^{\infty}$an infinite family of κ_i - strictly pseudo-contractive mappings from H into itself. Define a mapping $T_{\kappa_i}$by $T_{\kappa_i} = \kappa_i x + (1-\kappa_i)T_i x$, $\forall x \in H$, $i \in \mathbb{N}$. Let A be a strongly positive linear-bounded self-adjoint operator with the coefficient $0 < \overline{\gamma} < 1$. Let $G_i : H \to H$ be defined by $J_{M_{i,\eta}}(I-\eta B_i)x = G_i x$for every $x \in H$ and $\eta \in (0, 2\delta_i)$ for every $i = 1, 2,\ldots, N$ and let $v_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3,\ldots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$for all $j = 1, 2,\ldots, N-1$, $\alpha_1^N \in (0,1]\alpha_2^j, \alpha_3^j \in [0,1)$* for all j = 1,2,..., N.. *Let $S: C \to C$ be the S - mapping generated by $G_1, G_2,\ldots, G_N$ and $v_1, v_2,\ldots, v_N$. Let*

λ_1, λ_2,..., be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2,...,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$, and let K_n be the K-mapping generated by $T_{\kappa_1}, T_{\kappa_2}, \ldots, T_{\kappa_n}$, and $\lambda_1, \lambda_2,..., \lambda_n$, and let K be the K-mapping generated by $T_{\kappa_1}, T_{\kappa_2}, \ldots,$ and $\lambda_1, \lambda_2,...$, i.e.,

$$Kx = \lim_{n\to\infty} K_n x$$

for every $x \in C$. Assume that $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} V(H, B_i, M_i) \neq \emptyset$. For every $n \in \mathbb{N}$, $i = 1, 2,..., N$, let $x_1 \in H$ and $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)(\gamma_n K_n x_n + (1-\gamma_n)Sx_n), \tag{3.31}$$

where $f : H \to H$ is contractive mapping with coefficient $\theta \in (0, 1)$ and $0 < \gamma < \frac{\overline{\gamma}}{\theta}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iii) $\lim_{n\to\infty} \gamma_n = c \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $z \in \mathfrak{F}$, which solves uniquely the following variational inequality:

$$\langle (A - \gamma f)z, z - x^* \rangle \leq 0, \quad \forall x^* \in \mathfrak{F}. \tag{3.32}$$

Equivalently, we have $P_{\mathfrak{F}}(I - A + \gamma f)z = z$.

Proof. For every $i \in \mathbb{N}$, by Lemma 3.2, we have that T_{κ_i} is a nonexpansive mapping and $\bigcap_{i=1}^{\infty} F(T_{\kappa_i}) = \bigcap_{i=1}^{\infty} F(T_i)$. From Theorem 3.1 and Lemma 2.13, we can reach the desired conclusion.

Corollary 3.4. *Let H be a real Hilbert space and let $M : H \to 2^H$ be maximal monotone mappings. Let $B : H \to H$ be a δ - inverse strongly monotone mapping and $\{T_i\}_{i=1}^{\infty}$ an infinite family of κ_i - strictly pseudo-contractive map-pings from H into itself. Define a mapping $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x$ by $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x$, $\forall x \in H$, $i \in \mathbb{N}$. Let A be a strongly positive linear-bounded self-adjoint operator with the coefficient $0 < \overline{\gamma} < 1$. Let $\lambda_1, \lambda_2,...,$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2,...,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$, and let K_n be the K-mapping generated by $T_{\kappa_1}, T_{\kappa_2}, \ldots, T_{\kappa_n}$ and $\lambda_1, \lambda_2,..., \lambda_n$, and let K be the K-mapping generated by $T_{\kappa_1}, T_{\kappa_2}, \ldots,$ and $\lambda_1, \lambda_2,...$, i.e.,*

$$Kx = \lim_{n\to\infty} K_n x$$

for every $x \in C$. Assume that $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap V(H, B, M) \neq \emptyset$. For every $n \in \mathbb{N}$, let $x_1 \in H$ and $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)(\gamma_n K_n x_n + (1-\gamma_n)J_{M,\eta}(I - \eta B)x_n), \tag{3.33}$$

where $f : H \to H$ is contractive mapping with coefficient $\theta \in (0, 1)$ and $0 < \gamma < \frac{\overline{\gamma}}{\theta}$, $\eta \in (0, 2\delta)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(*iii*) $\lim_{n\to\infty} \gamma_n = c \in (0, 1)$.

Then $\{x_n\}$ *converges strongly to* $z \in \mathfrak{F}$, *which solves uniquely the following variational inequality*

$$\langle (A - \gamma f)z, z - x^* \rangle \le 0, \quad \forall x^* \in \mathfrak{F}. \tag{3.34}$$

Equivalently, we have $P_{\mathfrak{F}}(I - A + \gamma f)z = z$.

Proof. Putting $N = 1$ in Corollary 3.3, we can reach the desired conclusion.

Competing interests

The authors declare that they have no competing interests.

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