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Research Article

Existence and Asymptotic Stability of Solutions for Hyperbolic Differential Inclusions with a Source Term

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We study the existence of global weak solutions for a hyperbolic differential inclusion with a source term, and then investigate the asymptotic stability of the solutions by using Nakao lemma.

1. Introduction

In this paper, we are concerned with the global existence and the asymptotic stability of weak solutions for a hyperbolic differential inclusion with nonlinear damping and source terms:

$$\begin{gathered} y_{tt} - \Delta y_t - \operatorname{div}\left(|\nabla y|^{p-2}\nabla y\right) + \Xi = \lambda |y|^{m-2} y \quad \text{in } \Omega\times(0,\infty), \\ \Xi(x,t) \in \varphi\left(y_t(x,t)\right) \quad \text{a.e. } (x,t)\in\Omega\times(0,\infty), \\ y = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \quad \text{in } x\in\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $p \geq 2$, $\lambda > 0$, and φ is a discontinuous and nonlinear set-valued mapping by filling in jumps of a locally bounded function b.

Recently, a class of differential inclusion problems is studied by many authors [2, 6, 7, 11, 14–16, 19]. Most of them considered the existence of weak solutions for differential inclusions of various forms. Miettinen [6] Miettinen and Panagiotopoulos [7] proved the existence of weak solutions for some parabolic differential inclusions. J. Y. Park et al. [14] showed the existence of a global weak solution to the hyperbolic differential inclusion

(1.1) with $\lambda = 0$ by making use of the Faedo-Galerkin approximation, and then considered asymptotic stability of the solution by using Nakao lemma [8]. The background of these variational problems are in physics, especially in solid mechanics, where nonconvex, nonmonotone, and multivalued constitutive laws lead to differential inclusions. We refer to [11, 12] to see the applications of differential inclusions.

On the other hand, it is interesting to mention the existence and nonexistence of global solutions for nonlinear wave equations with nonlinear damping and source terms [4, 5, 10, 13, 18] in the past twenty years. Thus, in this paper, we will deal with the existence and the asymptotic behavior of a global weak solution for the hyperbolic differential inclusion (1.1) involving p-Laplacian, a nonlinear, discontinuous, and multivalued damping term and a nonlinear source term. The difficulties come from the interaction between the p-Laplacian and source terms. As far as we are concerned, there is a little literature dealing with asymptotic behavior of solutions for differential inclusions with source terms.

The plan of this paper is as follows. In Section 2, the main results besides notations and assumptions are stated. In Section 3, the existence of global weak solutions to problem (1.1) is proved by using the potential-well method and the Faedo-Galerkin method. In Section 4, the asymptotic stability of the solutions is investigated by using Nakao lemma.

2. Statement of main results

We first introduce the following abbreviations: $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{k,p} = \|\cdot\|_{W^{k,p}(\Omega)}$. For simplicity, we denote $\|\cdot\|_2$ by $\|\cdot\|$. For every $q \in (1,\infty)$, we denote the dual of $W_0^{1,q}$ by $W^{-1,q'}$ with $q' = q/(q-1)$. The notation $(\cdot,\cdot)$ for the L^2-inner product will also be used for the notation of duality pairing between dual spaces.

Throughout this paper, we assume that p and m are positive real numbers satisfying

$$2 \le p < m < \frac{Np}{2(N-p)} + 1 \quad (2 \le p < m < \infty \text{ if } p \ge N). \tag{2.1}$$

Define the potential well

$$\mathcal{W} = \{y \in W_0^{1,p}(\Omega) \mid I(y) = \|\nabla y\|_p^p - \lambda \|y\|_m^m > 0\} \cup \{0\}. \tag{2.2}$$

Then $\mathcal{W}$ is a neighborhood of 0 in $W_0^{1,p}(\Omega)$. Indeed, Sobolev imbedding (see [1])

$$W_0^{1,p}(\Omega) \hookrightarrow L^m(\Omega) \tag{2.3}$$

and Poincare's inequality yield

$$\lambda \|y\|_m^m \le \lambda c_*^m \|\nabla y\|_p^m \le \lambda c_*^m \|\nabla y\|_p^{m-p} \|\nabla y\|_p^p, \quad \forall y \in W_0^{1,p}(\Omega), \tag{2.4}$$

where c_* is an imbedding constant from $W_0^{1,p}(\Omega)$ to $L^m(\Omega)$. From this, we deduce that $I(y) > 0$ (i.e., $y \in \mathcal{W}$) as $\|\nabla y\|_p < (\lambda^{-1} c_*^{-m})^{1/(m-p)}$.

For later purpose, we introduce the functional J defined by

$$J(y) := \frac{1}{p}\|\nabla y\|_p^p - \frac{\lambda}{m}\|y\|_m^m. \tag{2.5}$$

Obviously, we have

$$J(y) = \frac{1}{m}I(y) + \frac{m-p}{mp}\|\nabla y\|_p^p. \tag{2.6}$$

Define the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$Ay = -\operatorname{div}\left(|\nabla y|^{p-2}\nabla y\right), \quad \forall y \in W_0^{1,p}(\Omega), \tag{2.7}$$

then A is bounded, monotone, hemicontinuous (see, e.g., [3]), and

$$(Ay, y) = \|\nabla y\|_p^p, \quad (Ay, y_t) = \frac{1}{p}\frac{d}{dt}\|\nabla y\|_p^p \quad \text{for } y \in W_0^{1,p}(\Omega). \tag{2.8}$$

Now, we formulate the following assumptions.

(H_1) Let $b : \mathbb{R} \to \mathbb{R}$ be a locally bounded function satisfying

$$b(s)s \geq \mu_1 s^2, \quad |b(s)| \leq \mu_2 |s|, \quad \text{for } s \in \mathbb{R}, \tag{2.9}$$

where μ_1 and μ_2 are some positive constants.

(H_2) $y_0 \in \mathcal{W}$, $y_1 \in L^2(\Omega)$, and

$$0 < E(0) = \frac{1}{2}||y_1||^2 + \frac{1}{p}||\nabla y_0||_p^p - \frac{\lambda}{m}||y_0||_m^m < \frac{m-p}{2mp}\left(\frac{m-p}{\lambda c_*^m 2(m-1)p}\right)^{p/(m-p)}. \tag{2.10}$$

The multivalued function $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$ is obtained by filling in jumps of a function $b : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\epsilon$, $\overline{b}_\epsilon$, $\underline{b}$, $\overline{b} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{gathered} \underline{b}_\epsilon(t) = \operatorname{ess}\inf_{|s-t|\leq\epsilon} b(s), \qquad \overline{b}_\epsilon(t) = \operatorname{ess}\sup_{|s-t|\leq\epsilon} b(s), \\ \underline{b}(t) = \lim_{\epsilon\to 0^+} \underline{b}_\epsilon(t), \qquad \overline{b}(t) = \lim_{\epsilon\to 0^+} \overline{b}_\epsilon(t), \\ \varphi(t) = [\underline{b}(t), \overline{b}(t)]. \end{gathered} \tag{2.11}$$

We will need a regularization of b defined by

$$b^n(t) = n\int_{-\infty}^{\infty} b(t-\tau)\rho(n\tau)\,d\tau, \tag{2.12}$$

where $\rho \in C_0^\infty((-1,1))$, $\rho \geq 0$ and $\int_{-1}^1 \rho(\tau)\,d\tau = 1$. It is easy to show that b^n is continuous for all $n \in \mathbb{N}$ and $\underline{b}_\epsilon$, $\overline{b}_\epsilon$, $\underline{b}$, $\overline{b}$, b^n satisfy the same condition (H_1) with a possibly different constant if b satisfies (H_1). So, in the sequel, we denote the different constants by the same symbol as the original constants.

Definition 2.1. A function $y(x,t)$ is *a weak solution* to problem (1.1) if for every $T>0$, y satisfies $y\in L^\infty(0,T;W_0^{1,p}(\Omega))$, $y_t\in L^2(0,T;W^{1,2}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, $y_{tt}\in L^2(0,T;W^{-1,p'}(\Omega))$, there exists $\Xi\in L^\infty(0,T;L^2(\Omega))$ and the following relations hold:

$$\begin{gathered}\int_0^T\{(y_{tt}(t),z)+(\nabla y_t(t),\nabla z)+(|\nabla y(t)|^{p-2}\nabla y(t),\nabla z)+(\Xi(t),z)\}dt\\ =\int_0^T(\lambda|y(t)|^{m-2}y(t),z)dt,\quad\forall z\in W_0^{1,p}(\Omega),\\ \Xi(x,t)\in\varphi(y_t(x,t))\quad\text{a.e. }(x,t)\in Q_T,\\ y(0)=y_0,\qquad y_t(0)=y_1.\end{gathered}\tag{2.13}$$

Theorem 2.2. *Under the assumptions* (H_1) *and* (H_2), *problem (1.1) has a weak solution.*

Theorem 2.3. *Under the same conditions of Theorem 2.2, the solutions of problem (1.1) satisfy the following decay rates.*

If $p=2$, then there exist positive constants C and γ such that

$$E(t)\le C\exp(-\gamma t)\quad a.e.\ t\ge 0,\tag{2.14}$$

and if $p>2$, then there exists a constant $C>0$ such that

$$E(t)\le C(1+t)^{-p/(p-2)}\quad a.e.\ t\ge 0,\tag{2.15}$$

where $E(t)=(1/2)\|y_t(t)\|^2+(1/p)\|\nabla y(t)\|_p^p-(\lambda/m)\|y(t)\|_m^m$.

In order to prove the decay rates of Theorem 2.3, we need the following lemma by Nakao (see [8, 9] for the proof).

Lemma 2.4. *Let $\phi:\mathbb{R}^+\to\mathbb{R}$ be a bounded nonincreasing and nonnegative function for which there exist constants $\alpha>0$ and $\beta\ge 0$ such that*

$$\sup_{t\le s\le t+1}(\phi(s))^{1+\beta}\le\alpha(\phi(t)-\phi(t+1)),\quad\forall t\ge 0.\tag{2.16}$$

Then the following hold.

(1) *If $\beta=0$, there exist positive constants C and γ such that*

$$\phi(t)\le C\exp(-\gamma t),\quad\forall t\ge 0.\tag{2.17}$$

(2) *If $\beta>0$, there exists a positive constant C such that*

$$\phi(t)\le C(1+t)^{-1/\beta},\quad\forall t\ge 0.\tag{2.18}$$

3. Proof of Theorem 2.2

In this section, we are going to show the existence of solutions to problem (1.1) using the Faedo-Galerkin approximation and the potential method. To this end let $\{w_j\}_{j=1}^\infty$ be a basis in $W_0^{1,p}(\Omega)$ which are orthogonal in $L^2(\Omega)$. Let $V_n=\operatorname{Span}\{w_1,w_2,\dots,w_n\}$.

We choose y_0^n and y_1^n in V_n such that

$$y_0^n \longrightarrow y_0 \quad \text{in } W_0^{1,p}, \qquad y_1^n \longrightarrow y_1 \quad \text{in } L^2(\Omega). \tag{3.1}$$

Let $y^n(t) = \sum_{j=1}^n g_{jn}(t)w_j$ be the solution to the approximate equation

$$\begin{gathered}(y_{tt}^n(t), w_j) + (\nabla y_t^n(t), \nabla w_j) + (Ay^n(t), w_j) + (b^n(y_t^n(t)), w_j) = (\lambda |y^n(t)|^{m-2} y^n(t), w_j),\\ y^n(0) = y_0^n, \qquad y_t^n(0) = y_1^n.\end{gathered} \tag{3.2}$$

By standard methods of ordinary differential equations, we can prove the existence of a solution to (3.2) on some interval $[0, t_m)$. Then this solution can be extended to the closed interval $[0, T]$ by using the a priori estimate below.

Step 1 (a priori estimate). Equation (3.1) and the condition $y_0 \in \mathcal{W}$ imply that

$$I(y_0^n) = \|\nabla y_0^n\|_p^p - \lambda \|y_0^n\|_m^m \longrightarrow I(y_0) > 0. \tag{3.3}$$

Hence, without loss of generality, we assume that $I(y_0^n) > 0$ (i.e., $y_0^n \in \mathcal{W}$) for all n. Substituting w_j in (3.2) by $y_t^n(t)$, we obtain

$$\frac{d}{dt}E^n(t) + \|\nabla y_t^n(t)\|^2 + (b^n(y_t^n(t)),\ y_t^n(t)) = 0, \tag{3.4}$$

where

$$\begin{aligned}E^n(t) &= \frac{1}{2}\|y_t^n(t)\|^2 + \frac{1}{p}\|\nabla y^n(t)\|_p^p - \frac{\lambda}{m}\|y^n(t)\|_m^m\\ &= \frac{1}{2}\|y_t^n(t)\|^2 + J(y^n(t)).\end{aligned} \tag{3.5}$$

Integrating (3.4) over $(0, t)$ and using assumption (H_1), we have

$$\frac{1}{2}\|y_t^n(t)\|^2 + J(y^n(t)) + \int_0^t \|\nabla y_t^n(\tau)\|^2 d\tau \le E^n(0). \tag{3.6}$$

Since $E^n(0) \to E(0)$ and $E(0) > 0$, without loss of generality, we assume that $E^n(0) < 2E(0)$ for all n. Now, we claim that

$$y^n(t) \in \mathcal{W}, \quad t > 0. \tag{3.7}$$

Assume that there exists a constant $T > 0$ such that $y^n(t) \in \mathcal{W}$ for $t \in [0, T)$ and $y^n(T) \in \partial\mathcal{W}$, that is, $I(y^n(T)) = 0$. From (2.6), (3.4), and (3.5), we obtain

$$J(y^n(T)) = \frac{m-p}{pm}\|\nabla y^n(T)\|_p^p \le E^n(T) \le E^n(0) < 2E(0), \tag{3.8}$$

and therefore

$$\|\nabla y^n(T)\|_p < \left(\frac{2pm}{m-p}E(0)\right)^{1/p}. \tag{3.9}$$

Combining this with (2.4) and using (2.10), we see that

$$\begin{aligned}\lambda\|y^n(T)\|_m^m &< \lambda c_*^m\left(\frac{2pm}{m-p}E(0)\right)^{(m-p)/p}\|\nabla y^n(T)\|_p^p \\ &< \frac{m-p}{2(m-1)p}\|\nabla y^n(T)\|_p^p < \|\nabla y^n(T)\|_p^p,\end{aligned}\tag{3.10}$$

where we used the fact that $(m-p)/2(m-1)p < 1$. This gives $I(y^n(T)) > 0$, which is a contradiction. Therefore (3.7) is valid. From (2.6), (3.6), and (3.7),

$$\frac{1}{2}\|y_t^n(t)\|^2 + \frac{m-p}{pm}\|\nabla y^n(t)\|_p^p + \int_0^t \|\nabla y_t^n(s)\|^2 ds < 2E(0).\tag{3.11}$$

By (H_1) and (3.11), it follows that

$$\|b^n(y_t^n(t))\|^2 \le \mu_2^2\|y_t^n(t)\|^2 \le cE(0),\tag{3.12}$$

here and in the sequel we denote by c a generic positive constant independent of n and t.

It follows from (3.11) and (3.12) that

$$\begin{gathered}(y^n) \text{ is bounded in } L^\infty(0,T;W_0^{1,p}(\Omega)),\\ (y_t^n) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),\\ (b^n(y_t^n)) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\end{gathered}\tag{3.13}$$

and since $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a bounded operator, it follows from (3.13) that

$$(Ay^n) \text{ is bounded in } L^\infty(0,T;W^{-1,p'}(\Omega)).\tag{3.14}$$

Finally, we will obtain an estimate for y_{tt}^n. Since the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^m(\Omega)$ is continuous, we have

$$\left|\left(|y^n(t)|^{m-2}y^n(t),z\right)\right| \le \|y^n(t)\|_m^{m-1}\|z\|_m \le c\|y^n(t)\|_{1,p}^{m-1}\|z\|_{1,p}.\tag{3.15}$$

From (3.2), it follows that

$$\begin{aligned}\left|\int_0^T (y_{tt}^n(t),z)dt\right| \le \int_0^T \Big| &-(Ay^n(t),z) - (\nabla y_t^n(t),\nabla z)\\ &-(b^n(y_t^n(t)),z) + \lambda(|y^n(t)|^{m-2}y^n(t),z)\Big|dt, \quad \forall z \in V_m,\end{aligned}\tag{3.16}$$

and hence we obtain from (3.13)–(3.15) that

$$\int_0^T \|y_{tt}^n(t)\|_{-1,p'}^2 dt \le c.\tag{3.17}$$

Step 2 (passage to the limit). From (3.13), (3.14), and (3.17), we can extract a subsequence from $\{y^n\}$, still denoted by $\{y^n\}$, such that

$$\begin{aligned} y^n &\longrightarrow y \quad \text{weakly star in } L^\infty(0,T;W_0^{1,p}(\Omega)), \\ y_t^n &\longrightarrow y_t \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \\ y_t^n &\longrightarrow y_t \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \\ y_{tt}^n &\longrightarrow y_{tt} \quad \text{weakly in } L^2(0,T;W^{-1,p'}(\Omega)), \\ Ay^n &\longrightarrow \zeta \quad \text{weakly star in } L^\infty(0,T;W^{-1,p'}(\Omega)), \\ b^n(y^n) &\longrightarrow \Xi \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{3.18}$$

Considering that the imbeddings $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ are compact and using the Aubin-Lions compactness lemma [3], it follows from (3.18) that

$$y^n \longrightarrow y \quad \text{strongly in } L^2(Q_T), \tag{3.19}$$

$$y_t^n \longrightarrow y_t \quad \text{strongly in } L^2(Q_T). \tag{3.20}$$

Using the first convergence result in (3.18) and the fact that the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{2(m-1)}(\Omega)$ ($p < m < Np/2(N-p)+1$ if $N > p$ and $p < m < \infty$ if $p \geq N$) is continuous, we obtain

$$\left\| |y^n|^{m-2} y^n \right\|_{L^2(Q_T)}^2 = \int_0^T \int_\Omega |y^n(x,t)|^{2(m-1)} dx\,dt \leq c. \tag{3.21}$$

This implies that

$$|y^n|^{m-2} y^n \longrightarrow \xi \quad \text{weakly in } L^2(Q_T). \tag{3.22}$$

On the other hand, we have from (3.19) that $y^n(x,t) \to y(x,t)$ a.e. in Q_T, and thus $|y^n(x,t)|^{m-2} y^n(x,t) \to |y(x,t)|^{m-2} y(x,t)$ a.e. in Q_T. Therefore, we conclude from (3.22) that $\xi(x,t) = |y(x,t)|^{m-2} y(x,t)$ a.e. in Q_T.

Letting $n \to \infty$ in (3.2) and using the convergence results above, we have

$$\begin{aligned} &\int_0^T \{(y_{tt}(t),z) + (\nabla y_t(t), \nabla z) + (\zeta(t),z) + (\Xi(t),z)\} dt \\ &\quad = \int_0^T (\lambda |y(t)|^{m-2} y(t), z)\,dt, \quad \forall z \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.23}$$

Step 3 ((y,Ξ) is a solution of (1.1)). Let $\phi \in C^1[0,T]$ with $\phi(T)=0$. By replacing w_j by $\phi(t)w_j$ in (3.2) and integrating by parts the result over $(0,T)$, we have

$$\begin{aligned}(y_t^n(0),\phi(0)w_j)+\int_0^T(y_t^n(t),\phi_t(t)w_j)\,dt &= \int_0^T(\nabla y_t^n(t),\phi(t)\nabla w_j)\,dt\\ &+\int_0^T(Ay^n(t),\phi(t)w_j)\,dt+\int_0^T(b^n(y_t^n(t)),\phi(t)w_j)\,dt\\ &-\int_0^T(\lambda|y^n(t)|^{m-2}y^n(t),\phi(t)w_j).\end{aligned}\tag{3.24}$$

Similarly from (3.23), we get

$$\begin{aligned}(y_t(0),\phi(0)w_j)+\int_0^T(y_t(t),\phi_t(t)w_j)\,dt &= \int_0^T(\nabla y_t(t),\phi(t)\nabla w_j)\,dt\\ &+\int_0^T(\zeta(t),\phi(t)w_j)\,dt+\int_0^T(\Xi(t),\phi(t)w_j)\,ds\\ &-\int_0^T(\lambda|y(t)|^{m-2}y(t),\phi(t)w_j).\end{aligned}\tag{3.25}$$

Comparing between (3.24) and (3.25), we infer that

$$\lim_{n\to\infty}(y_t^n(0)-y_t(0),w_j)=0,\quad j=1,2,\ldots.\tag{3.26}$$

This implies that $y_t^n(0)\to y_t(0)$ weakly in $W^{-1,p'}(\Omega)$. By the uniqueness of limit, $y_t(0)=y_1$. Analogously, taking $\phi\in C^2[0,T]$ with $\phi(T)=\phi'(T)=0$, we can obtain that $y(0)=y_0$.

Now, we show that $\Xi(x,t)\in\varphi(y_t(x,t))$ a.e. in Q_T. Indeed, since $y_t^n\to y_t$ strongly in $L^2(Q_T)$ (see (3.20)), $y_t^n(x,t)\to y_t(x,t)$ a.e. in Q_T. Let $\eta>0$. Using the theorem of Lusin and Egoroff, we can choose a subset $\omega\subset Q_T$ such that $|\omega|<\eta$, $y_t\in L^2(Q_T\setminus\omega)$, and $y_t^n\to y_t$ uniformly on $Q_T\setminus\omega$. Thus, for each $\epsilon>0$, there is an $M>2/\epsilon$ such that

$$|y_t^n(x,t)-y_t(x,t)|<\frac{\epsilon}{2}\quad\text{for } n>M,\ (x,t)\in Q_T\setminus\omega.\tag{3.27}$$

Then, if $|y_t^n(x,t)-s|<1/n$, we have $|y_t(x,t)-s|<\epsilon$ for all $n>M$ and $(x,t)\in Q_T\setminus\omega$. Therefore, we have

$$\underline{b}_\epsilon(y_t(x,t))\le b^n(y_t^n(x,t))\le\overline{b}_\epsilon(y_t(x,t)),\quad\forall n>M,\ (x,t)\in Q_T\setminus\omega.\tag{3.28}$$

Let $\phi\in L^1(0,T;L^2(\Omega))$, $\phi\ge0$. Then

$$\begin{aligned}\int_{Q_T\setminus\omega}\underline{b}_\epsilon(y_t(x,t))\phi(x,t)\,dx\,dt &\le\int_{Q_T\setminus\omega}b^n(y_t^n(x,t))\phi(x,t)\,dx\,dt\\ &\le\int_{Q_T\setminus\omega}\overline{b}_\epsilon(y_t(x,t))\phi(x,t)\,dx\,dt.\end{aligned}\tag{3.29}$$

Letting $n \to \infty$ in this inequality and using the last convergence result in (3.18), we obtain

$$\begin{aligned}\int_{Q_T\setminus\omega} \underline{b}_\epsilon(y_t(x,t))\phi(x,t)\,dx\,dt &\le \int_{Q_T\setminus\omega} \Xi(x,t)\phi(x,t)\,dx\,dt\\ &\le \int_{Q_T\setminus\omega} \overline{b}_\epsilon(y_t(x,t))\phi(x,t)\,dx\,dt.\end{aligned} \tag{3.30}$$

Letting $\epsilon \to 0^+$ in this inequality, we deduce that

$$\Xi(x,t) \in \varphi(y_t(x,t)) \quad \text{a.e. in } Q_T \setminus \omega, \tag{3.31}$$

and letting $\eta \to 0^+$, we get

$$\Xi(x,t) \in \varphi(y_t(x,t)) \quad \text{a.e. in } Q_T. \tag{3.32}$$

It remains to show that $\zeta = Ay$. From the approximated problem and the convergence results (3.18)–(3.22), we see that

$$\begin{aligned}\limsup_{n\to\infty}\int_0^T (Ay^n(t),y^n(t))\,dt \le{}& (y_1,y_0)-(y_t(T),y(T))\\ &+\int_0^T (y_t(t),y_t(t))\,dt-\frac{1}{2}\|\nabla y(T)\|^2\\ &+\frac{1}{2}\|\nabla y_0\|^2-\int_0^T (\Xi(t),y(t))\,dt\\ &+\int_0^T (\lambda|y(t)|^{m-2}y(t),y(t))\,dt.\end{aligned} \tag{3.33}$$

On the other hand, it follows from (3.23) that

$$\begin{aligned}\int_0^T (\zeta(t),y(t))\,dt ={}& (y_1,y_0)-(y_t(T),y(T))+\int_0^T (y_t(t),y_t(t))\,dt\\ &-\frac{1}{2}\|\nabla y(T)\|^2+\frac{1}{2}\|\nabla y_0\|^2-\int_0^T (\Xi(t),y(t))\,dt\\ &+\int_0^T (\lambda|y(t)|^{m-2}y(t),y(t))\,dt.\end{aligned} \tag{3.34}$$

Combining (3.33) and (3.34), we get

$$\limsup_{n\to\infty}\int_0^T (Ay^n(t),y^n(t))\,dt \le \int_0^T (\zeta(t),y(t))\,dt. \tag{3.35}$$

Since A is a monotone operator, we have

$$\begin{aligned}0 &\le \limsup_{n\to\infty}\int_0^T (Ay^n(t)-Az(t),y^n(t)-z(t))\,dt\\ &\le \int_0^T (\zeta(t)-Az(t),y(t)-z(t))\,dt, \quad \forall z\in L^2(0,T;W_0^{1,p}(\Omega)).\end{aligned} \tag{3.36}$$

By Mintiy's monotonicity argument (see, e.g., [17]),

$$\zeta = Ay \quad \text{in } L^2(0,T;W^{-1,p'}(\Omega)). \tag{3.37}$$

Therefore the proof of Theorem 2.2 is completed.

4. Asymptotic behavior of solutions

In this section, we will prove the decay rates (2.14) and (2.15) in Theorem 2.3 by applying Lemma 2.4. To prove the decay property, we first obtain uniform estimates for the approximated energy

$$E^n(t) = \frac{1}{2}\|y_t^n(t)\|^2 + \frac{1}{p}\|\nabla y^n(t)\|_p^p - \frac{\lambda}{m}\|y^n(t)\|_m^m \tag{4.1}$$

and then pass to the limit. Note that $E^n(t)$ is nonnegative and uniformly bounded. Let us fix an arbitary $t > 0$. From the approximated problem (3.2) and $w_j = y_t^n(t)$, we get

$$\frac{d}{dt}E^n(t) + \|\nabla y_t^n(t)\|^2 = -(b^n(y_t^n(t)), y_t^n(t)) \le -\mu_1\|y_t^n(t)\|^2. \tag{4.2}$$

This implies that $E^n(t)$ is a nonincreasing function. Setting $F_n^2(t) = E^n(t) - E^n(t+1)$ and integrating (4.2) over $(t,t+1)$, we have

$$F_n^2(t) \ge \int_t^{t+1} (\|\nabla y_t^n(s)\|^2 + \mu_1\|y_t^n(s)\|^2)\,ds \ge (\lambda_1 + \mu_1)\int_t^{t+1} \|y_t^n(s)\|^2 ds, \tag{4.3}$$

where $\lambda_1 > 0$ is the constant $\lambda_1\|v\|^2 \le \|\nabla v\|^2$, $\forall v \in W_0^{1,2}(\Omega)$. By applying the mean value theorem, there exist $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|y_t^n(t_i)\| \le \frac{2}{\sqrt{\lambda_1 + \mu_1}}F_n(t), \quad i = 1,2. \tag{4.4}$$

Now, replacing w_j by $y^n(t)$ in the approximated problem, we have

$$\begin{aligned}(Ay^n(t), y^n(t)) &- \lambda(|y^n(t)|^{m-2}y^n(t), y^n(t)) \\ &= -(y_{tt}^n(t), y^n(t)) - (\nabla y_t^n(t), \nabla y^n(t)) - (b^n(y_t^n(t)), y^n(t)).\end{aligned} \tag{4.5}$$

Integrating this over (t_1, t_2) and using (4.2) and (H_1), we get

$$
\begin{aligned}
&\int_{t_1}^{t_2} \frac{1}{p} \|\nabla y^n(s)\|_p^p ds - \lambda \int_{t_1}^{t_2} \|y^n(s)\|_m^m ds \\
&\quad \leq \int_{t_1}^{t_2} \|\nabla y^n(s)\|_p^p ds - \lambda \int_{t_1}^{t_2} \|y^n(s)\|_m^m ds \\
&\quad = -\left(y_t^n(t_2), y^n(t_2)\right) + \left(y_t^n(t_1), y^n(t_1)\right) + \int_{t_1}^{t_2} \|y_t^n(s)\|^2 ds \\
&\qquad - \int_{t_1}^{t_2} \left(\nabla y_t^n(s), \nabla y^n(s)\right) ds - \int_{t_1}^{t_2} \left(b^n(y_t^n(s)), y^n(s)\right) ds \\
&\quad \leq \|y_t^n(t_2)\| \|y^n(t_2)\| + \|y_t^n(t_1)\| \|y^n(t_1)\| + \int_{t_1}^{t_2} \|y_t^n(s)\|^2 ds \\
&\qquad + c \int_{t_1}^{t_2} \|\nabla y_t^n(s)\| \left(\sup_{t \leq s \leq t+1} \|\nabla y^n(s)\|_p \right) ds + \mu_2 \int_{t_1}^{t_2} \|y_t^n(s)\| \|y^n(s)\| ds.
\end{aligned}
\tag{4.6}
$$

Using Holder's inequality, Poincaré inequality, and (4.3)–(4.6), we get

$$
\begin{aligned}
\int_{t_1}^{t_2} E^n(s) ds = {}& \frac{1}{2} \int_{t_1}^{t_2} \|y_t^n(s)\|^2 ds \\
&+ \frac{1}{p} \int_{t_1}^{t_2} \|\nabla y^n(s)\|_p^p ds - \frac{\lambda}{m} \int_{t_1}^{t_2} \|y^n(s)\|_m^m ds \leq c F_n^2(t) \\
&+ c F_n(t) \left\{ \|\nabla y^n(t_2)\|_p + \|\nabla y^n(t_1)\|_p + \sup_{t \leq s \leq t+1} \|\nabla y^n(s)\|_p \right\} \\
&+ \lambda \left(1 - \frac{1}{m}\right) \int_{t_1}^{t_2} \|y^n(s)\|_m^m ds,
\end{aligned}
\tag{4.7}
$$

and hence we derive that

$$
\int_{t_1}^{t_2} E^n(s) ds \leq c F_n^2(t) + c F_n(t) E^n(t)^{1/p} + C_1 E^n(t),
\tag{4.8}
$$

where $C_1 = \lambda(1 - (1/m)) c_*^m (2mp/(m-p)E(0))^{(m-p)/p} (mp/(m-p))$ and we used the fact that $\|\nabla y^n(t)\|_p^p \leq (mp/(m-p)) E^n(t)$, $E^n(t)$ is a nonincreasing function, and (2.4).

Young's inequality implies that

$$
\int_{t_1}^{t_2} E^n(s) ds \leq c F_n^2(t) + C_\eta F_n(t)^{p/(p-1)} + \frac{1}{\eta} E^n(t) + C_1 E^n(t).
\tag{4.9}
$$

Noting that $E^n(t+1) \leq 2 \int_{t_1}^{t_2} E^n(s) ds$ and $E^n(t+1) = E^n(t) - F_n^2(t)$, we have from (4.9) that

$$
\left(\frac{1}{2} - C_1 - \frac{1}{\eta}\right) E^n(t) \leq \left(c + \frac{1}{2}\right) F_n^2(t) + C_\eta F_n(t)^{p/(p-1)}.
\tag{4.10}
$$

By assumption (2.10), $1/2 - C_1 > 0$, and hence taking $\eta > 0$ sufficiently small such that $1/2 - C_1 - 1/\eta > 0$, we obtain that

$$E^n(t) \le cF_n^2(t) + cF_n(t)^{p/(p-1)}. \tag{4.11}$$

If $p = 2$ then $E^n(t) \le cF_n^2(t)$, and since $E^n(t)$ is decreasing from Lemma 2.4 there exist positive constants C and γ such that

$$E^n(t) \le C\exp(-\gamma t), \quad \forall t \ge 0. \tag{4.12}$$

If $p > 2$, then (4.11) and the boundedness of $F_n(t)$ imply that

$$E^n(t) \le cF_n(t)^{p/(p-1)}, \tag{4.13}$$

and then

$$E^n(t)^{2(p-1)/p} \le c^{2(p-1)/p}\left(E^n(t) - E^n(t+1)\right). \tag{4.14}$$

Applying Lemma 2.4 to $\beta = (p-2)/p$, we obtain a constant $C > 0$ such that

$$E^n(t) \le C(1+t)^{-p/(p-2)}, \quad \forall t \ge 0. \tag{4.15}$$

Passing to the limit $n \to \infty$ in (4.12) and (4.15), we get (2.14) and (2.15). This completes the proof of Theorem 2.3.

References

[1] R. A. Adams, *Sobolev Spaces*, vol. 65 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1975.
[2] S. Carl and S. Heikkilä, "Existence results for nonlocal and nonsmooth hemivariational inequalities," *Journal of Inequalities and Applications*, vol. 2006, Article ID 79532, 2006.
[3] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
[4] L. Liu and M. Wang, "Global existence and blow-up of solutions for some hyperbolic systems with damping and source terms," *Nonlinear Analysis*, vol. 64, no. 1, pp. 69–91, 2006.
[5] S. A. Messaoudi, "Global existence and nonexistence in a system of Petrovsky," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 296–308, 2002.
[6] M. Miettinen, "A parabolic hemivariational inequality," *Nonlinear Analysis*, vol. 26, no. 4, pp. 725–734, 1996.
[7] M. Miettinen and P. D. Panagiotopoulos, "On parabolic hemivariational inequalities and applications," *Nonlinear Analysis*, vol. 35, no. 7, pp. 885–915, 1999.
[8] M. Nakao, "A difference inequality and its application to nonlinear evolution equations," *Journal of the Mathematical Society of Japan*, vol. 30, no. 4, pp. 747–762, 1978.
[9] M. Nakao, "Energy decay for the quasilinear wave equation with viscosity," *Mathematische Zeitschrift*, vol. 219, no. 2, pp. 289–299, 1995.
[10] K. Ono, "On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation," *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 321–342, 1997.
[11] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications Convex and Nonconvex Energy Functions*, Birkhäuser, Boston, Mass, USA, 1985.

[12] P. D. Panagiotopoulos, "Modelling of nonconvex nonsmooth energy problems. Dynamic hemivariational inequalities with impact effects," *Journal of Computational and Applied Mathematics*, vol. 63, no. 1–3, pp. 123–138, 1995.

[13] J. Y. Park and J. J. Bae, "On existence of solutions of degenerate wave equations with nonlinear damping terms," *Journal of the Korean Mathematical Society*, vol. 35, no. 2, pp. 465–490, 1998.

[14] J. Y. Park, H. M. Kim, and S. H. Park, "On weak solutions for hyperbolic differential inclusion with discontinuous nonlinearities," *Nonlinear Analysis*, vol. 55, no. 1-2, pp. 103–113, 2003.

[15] J. Y. Park and S. H. Park, "On solutions for a hyperbolic system with differential inclusion and memory source term on the boundary," *Nonlinear Analysis*, vol. 57, no. 3, pp. 459–472, 2004.

[16] J. Rauch, "Discontinuous semilinear differential equations and multiple valued maps," *Proceedings of the American Mathematical Society*, vol. 64, no. 2, pp. 277–282, 1977.

[17] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, vol. 49 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1997.

[18] G. Todorova, "Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms," *Journal of Mathematical Analysis and Applications*, vol. 239, no. 2, pp. 213–226, 1999.

[19] C. Varga, "Existence and infinitely many solutions for an abstract class of hemivariational inequalities," *Journal of Inequalities and Applications*, vol. 2005, no. 2, pp. 89–105, 2005.

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