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Solvability of fractional boundary value problem with p-Laplacian via critical point theory

Taiyong Chen and Wenbin Liu*

*Correspondence:
wblium@163.com
Department of Mathematics,
China University of Mining and
Technology, Xuzhou, 221116,
P.R. China

Abstract

In this paper, we discuss the fractional boundary value problem containing left and right fractional derivative operators and p-Laplacian. By using critical point theory we obtain some results on the existence of weak solutions of such a fractional boundary value problem.

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1 Introduction

Fractional-order models can be found to be more adequate than integer-order models in some real-world problems since fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, *et cetera* (see [1–6]). As a consequence, the subject of fractional differential equations is gaining more importance and attention.

In the past decade, many results on the existence and multiplicity of solutions to nonlinear fractional differential equations were obtained by using techniques of nonlinear analysis, such as fixed point theory [7], topological degree theory [8], and comparison method [9]. For more papers on fractional differential equations, see [10–18] and the references therein. Recently, the equations including both left and right fractional derivatives are also discussed [19–21]. Apart from their possible applications, equations with left and right fractional derivatives are an interesting and new field in fractional differential equations theory.

We should note that critical point theory has also turned out to be a very effective tool in determining the existence of solutions of integer-order differential equations with variational structures [22–26]. The idea behind is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. We refer the readers to the books [27, 28] and the references therein.

In recent paper [20], for the first time, the authors showed that critical point theory is an effective approach to tackle the existence of weak solutions of fractional boundary value problems (FBVPs) of the form

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0. \end{cases}$$

We note that it is not easy to use critical point theory to study FBVPs since it is often very difficult to establish suitable function spaces and variational functionals for FBVPs.

Motivated by the above-mentioned classical works, we want to contribute with the development of this new area on fractional differential equations theory. More precisely, we study the existence of weak solutions of the following FBVP:

$$\begin{cases} {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) = f(t,u(t)), & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right Riemann-Liouville fractional derivatives of order $\alpha \in (0,1]$ respectively, $\phi_p : \mathbb{R} \to \mathbb{R}$ is the p-Laplacian defined by

$$\phi_p(s) = |s|^{p-2}s \quad \text{if } s \neq 0, \qquad \phi_p(0) = 0, \quad p > 1,$$

and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

(H_1) $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$.

Note that p-Laplacian ϕ_p introduced by Leibenson [29] often occurs in non-Newtonian fluid theory, nonlinear elastic mechanics, and so forth. Moreover, when $p = 2$, the nonlinear operator ${}_tD_T^\alpha \phi_p({}_0D_t^\alpha)$ reduces to the linear operator ${}_tD_T^\alpha {}_0D_t^\alpha$.

The rest of this paper is organized as follows. Section 2 contains some necessary notation, definitions, and properties of fractional calculus. In Section 3, we introduce the function space and energy functional for FBVP (1.1). In Section 4, based on some critical point theorems due to Mawhin and Willem [27] and Rabinowitz [28], we establish two theorems on the existence of weak solutions of FBVP (1.1).

2 Fractional calculus

For the convenience of the reader, we introduce some basic definitions and properties of the fractional calculus, which can be found, for instance, in [30, 31].

Definition 2.1 (Left and right Riemann-Liouville fractional integrals [30, 31]) Let x be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\gamma > 0$ for x, denoted by ${}_aD_t^{-\gamma}x(t)$ and ${}_tD_b^{-\gamma}x(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{-\gamma}x(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} x(s)\,ds, \\ {}_tD_b^{-\gamma}x(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} x(s)\,ds, \end{aligned}$$

provided that the right-hand side integrals are pointwise defined on $[a,b]$.

Definition 2.2 (Left and right Riemann-Liouville fractional derivatives [30, 31]) Let x be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\gamma > 0$ for x, denoted by ${}_aD_t^\gamma x(t)$ and ${}_tD_b^\gamma x(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^\gamma x(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_aD_t^{\gamma-n}x(t), \\ {}_tD_b^\gamma x(t) &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_tD_b^{\gamma-n}x(t), \end{aligned}$$

where $n-1 \leq \gamma < n$ and $n \in \mathbb{N}$.

Remark 2.1

(i) For $n \in \mathbb{N}$, if γ becomes an integer $n-1$, according to Definition 2.2, we recover the usual definitions, namely

$$\begin{aligned} {}_aD_t^{n-1}x(t) &= x^{(n-1)}(t), \\ {}_tD_b^{n-1}x(t) &= (-1)^{n-1}x^{(n-1)}(t), \end{aligned}$$

where $x^{(n-1)}$ is the usual derivative of order $n-1$.

(ii) If $x \in AC^n([a,b],\mathbb{R})$, following [30], we know that the Riemann-Liouville fractional derivative of order $\gamma \in [n-1,n)$ exists a.e. on $[a,b]$, where $AC([a,b],\mathbb{R})$ is the space of functions that are absolutely continuous on $[a,b]$, and $AC^k([a,b],\mathbb{R})$ ($k = 1,2,\ldots$) is the space of functions x such that $x \in C^{k-1}([a,b],\mathbb{R})$ and $x^{(k-1)} \in AC([a,b],\mathbb{R})$.

Definition 2.3 (Left and right Caputo fractional derivatives [30]) Let $\gamma \geq 0$ and $n \in \mathbb{N}$. If $n-1 < \gamma < n$ and $x \in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order γ for x, denoted by ${}_a^cD_t^\gamma x(t)$ and ${}_t^cD_b^\gamma x(t)$, respectively, exist almost everywhere on $[a,b]$ and are represented by

$$\begin{aligned} {}_a^cD_t^\gamma x(t) &= {}_aD_t^{\gamma-n}x^{(n)}(t), \\ {}_t^cD_b^\gamma x(t) &= (-1)^n{}_tD_b^{\gamma-n}x^{(n)}(t). \end{aligned}$$

If $\gamma = n-1$ and $x \in AC^{n-1}([a,b],\mathbb{R})$, then ${}_a^cD_t^{n-1}x(t)$ and ${}_t^cD_b^{n-1}x(t)$ are represented by

$$\begin{aligned} {}_a^cD_t^{n-1}x(t) &= x^{(n-1)}(t), \\ {}_t^cD_b^{n-1}x(t) &= (-1)^{n-1}x^{(n-1)}(t). \end{aligned}$$

The Riemann-Liouville fractional derivatives and the Caputo fractional derivatives are connected with each other by the following relations.

Property 2.1 ([30, 31]) Let $n \in \mathbb{N}$ and $n-1 < \gamma < n$. If x is a function defined on $[a,b]$ for which the Caputo fractional derivatives ${}_a^cD_t^\gamma x(t)$ and ${}_t^cD_b^\gamma x(t)$ of order γ exist together with the Riemann-Liouville fractional derivatives ${}_aD_t^\gamma x(t)$ and ${}_tD_b^\gamma x(t)$, then

$$ {}_a^cD_t^\gamma x(t) = {}_aD_t^\gamma x(t) - \sum_{j=0}^{n-1}\frac{x^{(j)}(a)}{\Gamma(j-\gamma+1)}(t-a)^{j-\gamma}$$

and

$$ {}_t^cD_b^\gamma x(t) = {}_tD_b^\gamma x(t) - \sum_{j=0}^{n-1} \frac{x^{(j)}(b)}{\Gamma(j-\gamma+1)}(b-t)^{j-\gamma}. $$

In particular, when $0 < \gamma < 1$, we get

$$ {}_a^cD_t^\gamma x(t) = {}_aD_t^\gamma x(t) - \frac{x(a)}{\Gamma(1-\gamma)}(t-a)^{-\gamma}, \tag{2.1} $$

$$ {}_t^cD_b^\gamma x(t) = {}_tD_b^\gamma x(t) - \frac{x(b)}{\Gamma(1-\gamma)}(b-t)^{-\gamma}. \tag{2.2} $$

Now we present the rule for fractional integration by parts.

Property 2.2 ([30, 32]) We have the following property of fractional integration:

$$ \int_a^b \left({}_aD_t^{-\gamma} x(t)\right) y(t)\,dt = \int_a^b x(t)\,{}_tD_b^{-\gamma} y(t)\,dt, \quad \gamma > 0, $$

provided that $x \in L^p([a,b],\mathbb{R})$, $y \in L^q([a,b],\mathbb{R})$, and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \gamma$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \gamma$.

3 Fractional derivative space and variational structure

In order to establish the variational framework that will enable us to reduce the existence of solutions of FBVP (1.1) to the problem of finding critical points of the corresponding functional, it is necessary to introduce an appropriate function space. In the following, we cite some results from [20].

Definition 3.1 ([20]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by

$$ E_0^{\alpha,p} = \left\{ u \in L^p([0,T],\mathbb{R}) \,|\, {}_0^cD_t^\alpha u \in L^p([0,T],\mathbb{R}), u(0) = u(T) = 0 \right\} $$

with the norm

$$ \|u\|_{\alpha,p} = \left(\|u\|_{L^p}^p + \left\|{}_0^cD_t^\alpha u\right\|_{L^p}^p\right)^{\frac{1}{p}}, \quad \forall u \in E_0^{\alpha,p}, \tag{3.1} $$

where $\|u\|_{L^p} = (\int_0^T |u(t)|^p\,dt)^{1/p}$ is the norm of $L^p([0,T],\mathbb{R})$.

Remark 3.1 For any $u \in E_0^{\alpha,p}$, according to (2.1) and (2.2) and in view of $u(0) = u(T) = 0$, we have ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, ${}_t^cD_T^\alpha u(t) = {}_tD_T^\alpha u(t)$ for $t \in [0,T]$.

Lemma 3.1 ([20]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 3.2 ([20]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For $u \in E_0^{\alpha,p}$, we have*

$$ \|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}_0^cD_t^\alpha u\right\|_{L^p}. \tag{3.2} $$

Moreover, if $\alpha > 1/p$ *and* $1/p + 1/q = 1$, *then*

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left\| {}_0^cD_t^\alpha u \right\|_{L^p}, \tag{3.3}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ *is the norm of* $C([0,T],\mathbb{R})$.

Remark 3.2 According to (3.2), we know that the norm (3.1) is equivalent to the norm of the form

$$\|u\|_{\alpha,p} = \left\| {}_0^cD_t^\alpha u \right\|_{L^p}. \tag{3.4}$$

Hence, in what follows, we can consider $E_0^{\alpha,p}$ with the norm (3.4).

Lemma 3.3 ([20]) *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$. *Assume that* $\alpha > 1/p$ *and the sequence* $\{u_k\}$ *converges weakly to* u *in* $E_0^{\alpha,p}$, *that is*, $u_k \rightharpoonup u$. *Then* $u_k \to u$ *in* $C([0,T],\mathbb{R})$, *that is*,

$$\|u_k - u\|_\infty \to 0, \quad k \to \infty.$$

For $v \in E_0^{\alpha,p}$, by Remark 3.1 and Definition 2.2 we have

$$\begin{aligned}
\int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0D_t^\alpha u(t)\right)\right] v(t)\,dt &= \int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0^cD_t^\alpha u(t)\right)\right] v(t)\,dt \\
&= -\int_0^T v(t)\,d\left[{}_tD_T^{\alpha-1} \phi_p\left({}_0^cD_t^\alpha u(t)\right)\right] \\
&= \int_0^T \left[{}_tD_T^{\alpha-1} \phi_p\left({}_0^cD_t^\alpha u(t)\right)\right] v'(t)\,dt.
\end{aligned}$$

Thus, from Property 2.2 and Definition 2.3 we have

$$\begin{aligned}
\int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0D_t^\alpha u(t)\right)\right] v(t)\,dt &= \int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right) {}_0D_t^{\alpha-1} v'(t)\,dt \\
&= \int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right) {}_0^cD_t^\alpha v(t)\,dt.
\end{aligned}$$

So we can define the weak solutions of FBVP (1.1) as follows.

Definition 3.2 By a weak solution to FBVP (1.1) we mean a function $u \in E_0^{\alpha,p}$ such that $f(\cdot, u(\cdot)) \in L^1([0,T],\mathbb{R})$ and the following equation holds:

$$\int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right) {}_0^cD_t^\alpha v(t)\,dt = \int_0^T f\left(t, u(t)\right) v(t)\,dt, \quad \forall v \in E_0^{\alpha,p}.$$

Next, we shall introduce a functional for FBVP (1.1) on $E_0^{\alpha,p}$. Also, we will show that the critical points of that functional are weak solutions of FBVP (1.1).

Define the functional $I : E_0^{\alpha,p} \to \mathbb{R}$ by

$$I(u) = \frac{1}{p}\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^p\,dt - \int_0^T F\bigl(t,u(t)\bigr)\,dt, \tag{3.5}$$

where $F(t,s) = \int_0^s f(t,\tau)\,d\tau$.

Remark 3.3 By Lemma 3.3 we get that the functional $u \to \int_0^T F(t,u(t))\,dt$ is weakly continuous on $E_0^{\alpha,p}$. Hence, as the sum of a convex continuous functional and a weakly continuous one, I is a weakly lower semicontinuous functional on $E_0^{\alpha,p}$ with $\alpha > 1/p$. Moreover, following [28], we can show that $I \in C^1(E_0^{\alpha,p},\mathbb{R})$, and we have

$$\begin{aligned}\bigl\langle I'(u),v\bigr\rangle &= \int_0^T \phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr){}_0^cD_t^\alpha v(t)\,dt \\ &\quad - \int_0^T f\bigl(t,u(t)\bigr)v(t)\,dt, \quad \forall v \in E_0^{\alpha,p}.\end{aligned} \tag{3.6}$$

Remark 3.4 By Definition 3.2 and (3.6), if $u \in E_0^{\alpha,p}$ is a solution of the Euler equation $I'(u) = 0$, then u is a weak solution of FBVP (1.1).

4 Existence of weak solutions of FBVP (1.1)

For finding the critical points of the functional I defined by (3.5), we need to use some critical point theorems, which can be found, for example, in [27, 28]. For the reader's convenience, we present some necessary definitions and theorems of critical point theory.

Let X be a real Banach space, and let $C^1(X,\mathbb{R})$ denote the space of continuously Fréchet-differentiable functionals on X.

Definition A Let $\varphi \in C^1(X,\mathbb{R})$. If any sequence $\{u_k\} \subset X$ for which $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence, then we say that φ satisfies the Palais-Smale condition (P.S. condition for short).

Theorem A ([27]) *Let X be a real reflexive Banach space. If the functional $\varphi : X \to \mathbb{R}$ is weakly lower semicontinuous and coercive, that is, $\lim_{\|z\|\to\infty}\varphi(z) = +\infty$, then there exists $z_0 \in X$ such that $\varphi(z_0) = \inf_{z\in X}\varphi(z)$. Moreover, if φ is also Fréchet differentiable on X, then $\varphi'(z_0) = 0$.*

Theorem B (Mountain pass theorem [28]) *Let X be a real Banach space, and $\varphi \in C^1(X,\mathbb{R})$ satisfy the P.S. condition. Suppose that*

(C_1) $\varphi(0) = 0$,
(C_2) *there exist $\rho > 0$ and $\sigma > 0$ such that $\varphi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$,*
(C_3) *there exists $z_1 \in X$ with $\|z_1\| \geq \rho$ such that $\varphi(z_1) < \sigma$.*

Then φ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{g\in\Omega}\max_{z\in g([0,1])}\varphi(z),$$

where $\Omega = \{g \in C([0,1],X) | g(0) = 0, g(1) = z_1\}$.

First, we use Theorem A to consider the existence of weak solutions of FBVP (1.1).

Theorem 4.1 *Let* $1/p < \alpha \leq 1$ *and* (H_1) *be satisfied. Assume that*

(H_2) *there exist* $a \in (0, (\Gamma(\alpha+1))^p/pT^{\alpha p})$ *and* $b \in L^1([0,T], \mathbb{R}^+)$ *such that*

$$\left|F(t,x)\right| \leq a|x|^p + b(t), \quad \forall t \in [0,T], x \in \mathbb{R}.$$

Then FBVP (1.1) *has at least one weak solution that minimizes* I *on* $E_0^{\alpha,p}$.

Proof According to Lemma 3.1, Remark 3.3, and Theorem A, we only need to prove that I is coercive on $E_0^{\alpha,p}$.

For $u \in E_0^{\alpha,p}$, it follows from (H_2) that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{\alpha,p}^p - \int_0^T F\bigl(t,u(t)\bigr)\,dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - a\int_0^T \bigl|u(t)\bigr|^p\,dt - \int_0^T b(t)\,dt \\ &= \frac{1}{p}\|u\|_{\alpha,p}^p - a\|u\|_{L^p}^p - \|b\|_{L^1}, \end{aligned}$$

which, together with (3.2), implies

$$I(u) \geq \left[\frac{1}{p} - \frac{aT^{\alpha p}}{(\Gamma(\alpha+1))^p}\right]\|u\|_{\alpha,p}^p - \|b\|_{L^1}.$$

Thus, noting that $a \in (0, (\Gamma(\alpha+1))^p/pT^{\alpha p})$, we have

$$\lim_{\|u\|_{\alpha,p}\to\infty} I(u) = +\infty,$$

that is, I is coercive. The proof is complete. □

Next, we use Theorem B to discuss the existence of mountain pass solutions of FBVP (1.1).

Theorem 4.2 *Let* $1/p < \alpha \leq 1$ *and* (H_1) *be satisfied. Assume that*

(H_3) *there exist constants* $\mu \in (0, 1/p)$ *and* $M > 0$ *such that*

$$0 < F(t,x) \leq \mu x f(t,x), \quad \forall t \in [0,T], x \in \mathbb{R} \textit{ with } |x| \geq M,$$

(H_4) *for* $t \in [0,T]$ *and* $x \in \mathbb{R}$, *we have*

$$\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^p} < \frac{(\Gamma(\alpha+1))^p}{pT^{\alpha p}}.$$

Then FBVP (1.1) *has at least one nontrivial weak solution on* $E_0^{\alpha,p}$.

Proof We will verify that I satisfies all the conditions of Theorem B.

First, we show that I satisfies the P.S. condition. Since $F(t,x) - \mu x f(t,x)$ is continuous, there exists $c \in \mathbb{R}^+$ such that

$$F(t,x) \leq \mu x f(t,x) + c, \quad t \in [0,T], |x| \leq M.$$

Thus, from (H_3) we get

$$F(t,x) \leq \mu x f(t,x) + c, \quad t \in [0,T], x \in \mathbb{R}. \tag{4.1}$$

Let $\{u_k\} \subset E_0^{\alpha,p}$ be such that

$$\left|I(u_k)\right| \leq K, \quad I'(u_k) \to 0 \quad \text{as } k \to \infty.$$

According to (3.6), we have

$$\left\langle I'(u_k), u_k\right\rangle = \|u_k\|_{\alpha,p}^p - \int_0^T f\bigl(t, u_k(t)\bigr) u_k(t)\,dt,$$

which, together with (4.1), yields

$$\begin{aligned} K &\geq I(u_k) \\ &= \frac{1}{p}\|u_k\|_{\alpha,p}^p - \int_0^T F\bigl(t,u_k(t)\bigr)\,dt \\ &\geq \frac{1}{p}\|u_k\|_{\alpha,p}^p - \mu\int_0^T f\bigl(t,u_k(t)\bigr)u_k(t)\,dt - cT \\ &= \left(\frac{1}{p} - \mu\right)\|u_k\|_{\alpha,p}^p + \mu\left\langle I'(u_k), u_k\right\rangle - cT \\ &\geq \left(\frac{1}{p} - \mu\right)\|u_k\|_{\alpha,p}^p - \mu\left\|I'(u_k)\right\|_{-\alpha,q}\|u_k\|_{\alpha,p} - cT, \end{aligned}$$

where q is a constant such that $1/p + 1/q = 1$. Since $I'(u_k) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$K \geq \left(\frac{1}{p} - \mu\right)\|u_k\|_{\alpha,p}^p - \|u_k\|_{\alpha,p} - cT, \quad k > N_0.$$

It follows from $\mu \in (0, 1/p)$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive space, up to a subsequence, we can assume that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$. Hence, we have

$$\begin{aligned} &\left\langle I'(u_k) - I'(u), u_k - u\right\rangle \\ &\quad = \left\langle I'(u_k), u_k - u\right\rangle - \left\langle I'(u), u_k - u\right\rangle \\ &\quad \leq \left\|I'(u_k)\right\|_{-\alpha,q}\|u_k - u\|_{\alpha,p} - \left\langle I'(u), u_k - u\right\rangle \\ &\quad \to 0, \quad k \to \infty. \end{aligned} \tag{4.2}$$

Moreover, by (3.3) and Lemma 3.3 we obtain that u_k is bounded in $C([0,T],\mathbb{R})$ and $\|u_k - u\|_\infty \to 0$ as $k \to \infty$. Then we get

$$\int_0^T \bigl(f\bigl(t,u_k(t)\bigr) - f\bigl(t,u(t)\bigr)\bigr)\bigl(u_k(t) - u(t)\bigr)\,dt \to 0, \quad k \to \infty. \tag{4.3}$$

Note that

$$\begin{aligned}
&\bigl\langle I'(u_k) - I'(u), u_k - u\bigr\rangle \\
&\quad = \int_0^T \bigl(\phi_p\bigl({}_0^cD_t^\alpha u_k(t)\bigr) - \phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr)\bigr)\bigl({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr)\,dt \\
&\qquad - \int_0^T \bigl(f\bigl(t,u_k(t)\bigr) - f\bigl(t,u(t)\bigr)\bigr)\bigl(u_k(t) - u(t)\bigr)\,dt.
\end{aligned}$$

Thus, from (4.2) and (4.3) we have

$$\int_0^T \bigl(\phi_p\bigl({}_0^cD_t^\alpha u_k(t)\bigr) - \phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr)\bigr)\bigl({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr)\,dt \to 0 \tag{4.4}$$

as $k \to \infty$.

Following [33], we obtain that there exist $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\int_0^T \bigl(\phi_p\bigl({}_0^cD_t^\alpha u_k(t)\bigr) - \phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr)\bigr)\bigl({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr)\,dt \\
&\quad \geq \begin{cases} c_1 \int_0^T |{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^p\,dt, & p \geq 2, \\ c_2 \int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\,dt, & 1 < p < 2. \end{cases}
\end{aligned} \tag{4.5}$$

When $1 < p < 2$, we have

$$\begin{aligned}
&\int_0^T \bigl|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr|^p\,dt \\
&\quad \leq \biggl(\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\,dt\biggr)^{\frac{p}{2}} \\
&\qquad \cdot \biggl(\int_0^T \bigl(\bigl|{}_0^cD_t^\alpha u_k(t)\bigr| + \bigl|{}_0^cD_t^\alpha u(t)\bigr|\bigr)^p\,dt\biggr)^{\frac{2-p}{2}}.
\end{aligned}$$

Thus, noting that $(s_1 + s_2)^\gamma \leq 2^{\gamma-1}(s_1^\gamma + s_2^\gamma)$ where $s_1, s_2 \geq 0$, $\gamma \geq 1$ (see [34]), we have

$$\begin{aligned}
&\int_0^T \bigl|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr|^p\,dt \\
&\quad \leq c_3\bigl(\|u_k\|_{\alpha,p}^p + \|u\|_{\alpha,p}^p\bigr)^{\frac{2-p}{2}} \\
&\qquad \cdot \biggl(\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\,dt\biggr)^{\frac{p}{2}},
\end{aligned}$$

where $c_3 = 2^{(p-1)(2-p)/2}$, which, together with (4.5), implies

$$\begin{aligned}&\int_0^T \bigl(\phi_p\bigl({}_0^cD_t^\alpha u_k(t)\bigr) - \phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr)\bigr)\bigl({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr)\,dt \\ &\quad \geq c_2 c_3^{-\frac{2}{p}} \bigl(\|u_k\|_{\alpha,p}^p + \|u\|_{\alpha,p}^p\bigr)^{\frac{p-2}{p}} \|u_k - u\|_{\alpha,p}^2, \quad 1 < p < 2. \end{aligned} \tag{4.6}$$

When $p \geq 2$, by (4.5) we get

$$\begin{aligned}&\int_0^T \bigl(\phi_p\bigl({}_0^cD_t^\alpha u_k(t)\bigr) - \phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr)\bigr)\bigl({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\bigr)\,dt \\ &\quad \geq c_1 \|u_k - u\|_{\alpha,p}^p, \quad p \geq 2. \end{aligned} \tag{4.7}$$

It follows from (4.4), (4.6), and (4.7) that

$$\|u_k - u\|_{\alpha,p} \to 0, \quad k \to \infty,$$

that is, $\{u_k\}$ converges strongly to u in $E_0^{\alpha,p}$.

Now we show that I satisfies the geometry conditions of mountain pass theorem.

By (H_4) there exist $\varepsilon \in (0,1)$ and $\delta > 0$ such that

$$F(t,x) \leq \frac{(1-\varepsilon)(\Gamma(\alpha+1))^p}{pT^{\alpha p}}|x|^p, \quad t \in [0,T], x \in \mathbb{R} \text{ with } |x| \leq \delta. \tag{4.8}$$

Let $\rho = \frac{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}{T^{\alpha-1/p}}\delta > 0$ and $\sigma = \varepsilon\rho^p/p > 0$. Then, by (3.3) we have

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|u\|_{\alpha,p} = \delta, \quad u \in E_0^{\alpha,p} \text{ with } \|u\|_{\alpha,p} = \rho,$$

which, together with (3.2) and (4.8), implies

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{\alpha,p}^p - \int_0^T F\bigl(t,u(t)\bigr)\,dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{(1-\varepsilon)(\Gamma(\alpha+1))^p}{pT^{\alpha p}}\int_0^T \bigl|u(t)\bigr|^p\,dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{1-\varepsilon}{p}\|u\|_{\alpha,p}^p \\ &= \frac{\varepsilon}{p}\|u\|_{\alpha,p}^p \\ &= \sigma, \quad \forall u \in E_0^{\alpha,p} \text{ with } \|u\|_{\alpha,p} = \rho. \end{aligned}$$

Hence, condition (C_2) in Theorem B is satisfied.

By (H_3) a simple argument using the very definition of derivative shows that there exist $c_4, c_5 > 0$ such that

$$F(t,x) \geq c_4|x|^{\frac{1}{\mu}} - c_5, \quad t \in [0,T], x \in \mathbb{R}.$$

For any $u \in E_0^{\alpha,p} \setminus \{0\}$, $\xi \in \mathbb{R}^+$, noting that $\mu \in (0, 1/p)$, we get

$$\begin{aligned} I(\xi u) &= \frac{1}{p}\|\xi u\|_{\alpha,p}^p - \int_0^T F\big(t, \xi u(t)\big)\,dt \\ &\leq \frac{\xi^p}{p}\|u\|_{\alpha,p}^p - c_4\int_0^T \big|\xi u(t)\big|^{\frac{1}{\mu}}\,dt + c_5 T \\ &= \frac{\xi^p}{p}\|u\|_{\alpha,p}^p - c_4\xi^{\frac{1}{\mu}}\|u\|_{L^{\frac{1}{\mu}}}^{\frac{1}{\mu}} + c_5 T \\ &\to -\infty, \quad \xi \to \infty. \end{aligned}$$

Thus, taking ξ_0 large enough and letting $e = \xi_0 u$, we have $I(e) \leq 0$. Therefore, condition (C_3) in Theorem B is also satisfied.

Lastly, noting that $I(0) = 0$, we get a critical point u such that $I(u) \geq \sigma > 0$. Hence, u is a nontrivial weak solution of FBVP (1.1). The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. They read and approved the final manuscript.

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