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On the Coulomb and Higgs branch formulae for multi-centered black holes and quiver invariants

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ABSTRACT: In previous work we have shown that the equivariant index of multi-centered $\mathcal{N} = 2$ black holes localizes on collinear configurations along a fixed axis. Here we provide a general algorithm for enumerating such collinear configurations and computing their contribution to the index. We apply this machinery to the case of black holes described by quiver quantum mechanics, and give a systematic prescription — the Coulomb branch formula — for computing the cohomology of the moduli space of quiver representations. For quivers without oriented loops, the Coulomb branch formula is shown to agree with the Higgs branch formula based on Reineke's result for stack invariants, even when the dimension vector is not primitive. For quivers with oriented loops, the Coulomb branch formula parametrizes the Poincaré polynomial of the quiver moduli space in terms of single-centered (or pure-Higgs) BPS invariants, which are conjecturally independent of the stability condition (i.e. the choice of Fayet-Iliopoulos parameters) and angular-momentum free. To facilitate further investigation we provide a MATHEMATICA package "CoulombHiggs.m" implementing the Coulomb and Higgs branch formulae.

Contents

1 Introduction

BPS states in $\mathcal{N} = 2$ supersymmetric string vacua offer a rich playground for exploring the microscopic properties of black holes in quantum gravity. Indeed, such states have dual descriptions as black hole solutions in supergravity at strong coupling, and brane configurations at weak coupling. The Witten index (more precisely, the second helicity supertrace [1, 2]) remains unchanged upon varying the string coupling. Computing the index $\Omega(\gamma)$ in both regimes as function of the electromagnetic charges γ can provide non-trivial tests of the equivalence between the microscopic and gravitational descriptions expected in any theory of quantum gravity [3].

This comparison however is complicated by the fact that on the macroscopic side, contributions to the index $\Omega(\gamma)$ originate not only from single centered black holes with charge γ, but also from multi-centered black holes with constituents carrying charges $\{\alpha_i\}$ such that $\gamma = \sum \alpha_i$ [4–7]. The total index carried by single and multi-centered black holes is the quantity that should be compared with the index on the microscopic side. In contrast, the gravitational path integral in the near horizon geometry of each black hole gives information only about the index $\Omega_{\mathrm{S}}(\alpha)$ associated with single centered black holes [8, 9]. Thus it is necessary to express the contribution from multi-centered black holes in terms of the index $\Omega_{\mathrm{S}}(\alpha_i)$ associated to each center. For this it suffices to compute the index[1] $g_{\mathrm{Coulomb}}(\{\alpha_i\}, \{c_i\})$ of the supersymmetric quantum mechanics of n centers carrying charges $\{\alpha_i\}$ interacting by Coulomb and Lorentz forces (and other forces related by supersymmetry). This 'Coulomb index' was computed in [10, 11], leading to a general prescription — the Coulomb branch formula — for expressing the total index $\Omega(\gamma)$ in terms of the single-centered BPS invariants $\Omega_{\mathrm{S}}(\alpha_i)$. Unlike these single-centered BPS invariants, the Coulomb index $g_{\mathrm{Coulomb}}(\{\alpha_i\}, \{c_i\})$ depends on the moduli at infinity — indeed, this dependence is responsible for jumps of the total index $\Omega(\gamma)$ across walls of marginal stability, providing a physically transparent derivation of the wall-crossing formulae known in the mathematical literature [10, 12, 13].

An important technical tool in [10, 11] was to consider a 'refined' version of these indices, which computes the trace $\mathrm{Tr}'(-1)^F y^{2J_3}$, where J_3 is the angular momentum generator with respect to a fixed z-axis, and the prime denotes the removal of fermion zero mode contributions before taking the trace. Unlike the helicity supertrace $\mathrm{Tr}'(-1)^F$, the refined index is not a protected quantity in full-fledged string theory and cannot be directly compared with the microscopic results. This generalization was nonetheless necessary, as one could use localization with respect to rotations along the z axis to compute the refined Coulomb index $g_{\mathrm{Coulomb}}(\{\alpha_i\}, \{c_i\}; y)$ for any number of centers n, and then take the limit $y \to 1$ at the end to recover the result for $\mathrm{Tr}'(-1)^F$. Although the description of the fixed points of this action is straightforward, their enumeration requires solving a set of $n-1$ algebraic equations in $n-1$ real variables, which quickly becomes unpractical as the number of centers increases. The first goal of this paper is to remove this bottleneck and give a completely algorithmic way of computing the Coulomb index $g_{\mathrm{Coulomb}}(\{\alpha_i\}; \{c_i\}; y)$, and hence the total refined index $\Omega(\gamma; y)$.[2] Using the connection between multi-centered black hole quantum mechanics and quiver quantum mechanics described in [14] this leads to an explicit expression for the Poincaré polynomial of quiver moduli spaces. The second goal is to establish the equivalence of this Coulomb branch formula with Reineke's formula [15] and its generalizations [12, 16] for the Poincaré polynomial of quiver moduli spaces for general quivers with no oriented loop.

Coulomb index from localization. Before explaining our new algorithm, let us briefly review the prescription of [11] for computing the refined Coulomb index

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[1]The Fayet-Iliopoulos (FI) parameters $\{c_i\}$ depend on α_i and on the values of the moduli at infinity, and will be treated as independent real parameters subject to the constraint $\sum_i c_i = 0$.

[2]In [10, 11, 14] we used a subscript $_{\mathrm{ref}}$ to denote a refined index. In this paper we shall drop this subscript to avoid cluttering up the formulæ, but it should be understood that whenever the index carries the argument y, it corresponds to a refined index.

$g_{\text{Coulomb}}(\{\alpha_i\}, \{c_i\}; y)$. Let Γ be the charge lattice of electromagnetic charges, equipped with the Dirac-Schwinger-Zwanziger (DSZ) symplectic product $\langle \cdot, \cdot \rangle \in \mathbb{Z}$. Consider a multi-centered black hole configuration where each center carries charge α_i ($i = 1 \ldots n$), with DSZ products $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$ between the charges. For fixed values of the moduli at infinity, encoded in the FI parameters c_i, n-centered configurations are parametrized by a $2n - 2$-dimensional phase space $\mathcal{M}_n(\{\alpha_{ij}\}, \{c_i\})$ [4, 5]. The coordinates of $\mathcal{M}_n$ are the locations of n centers $\vec{r}_i$ up to overall translations, subject to $n - 1$ constraints[3]

$$\forall i = 1 \ldots n, \quad \sum_{\substack{j \\ j \neq i}} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i . \tag{1.1}$$

The space $\mathcal{M}_n$ admits a symplectic form [5, 17, 18], such that the action of SO(3) rotations on $\mathcal{M}_n$ is generated by the moment map

$$\vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_i - \vec{r}_j|} , \tag{1.2}$$

equal to the angular momentum carried by the configuration. We denote by

$$g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y) = \text{Tr}'(-y)^{2J_3} \tag{1.3}$$

the refined index associated to this multi-centered configuration *assuming that all the centers are distinguishible from each other, and that each center carries no intrinsic degeneracy.* Mathematically, this is the equivariant Dirac index of the symplectic space $\mathcal{M}_n(\{\alpha_{ij}\}, \{c_i\})$ [11, 19]. We refer to (1.3) as the Coulomb index of multi-centered black holes with charges $\{\alpha_i\}$. This in turn can be used to compute the index associated with a general multi-centered black hole configurations in terms of indices $\Omega_S(\alpha_i)$ carried by individual centers following the procedure described in [11, 14].

When there exists an ordering of the charges $\{\alpha_i\}$ such that $i \leq j$ if and only if $\alpha_{ij} \geq 0$, and for generic values of the parameters c_i away from walls of marginal stability, the symplectic space $\mathcal{M}_n$ is compact, and the Coulomb index can be computed by localization with respect to rotations around a fixed axis, using the Atiyah-Bott Lefschetz fixed point formula [10, 11]. The configurations which stay invariant under such rotations are collinear configurations, corresponding to solutions of (1.1) lying along the z axis. The result is expressed as

$$g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y) = (-1)^{n-1+\sum_{i<j}\alpha_{ij}} (y-y^{-1})^{-n+1} \sum_{\text{extrema}} \pm y^{\sum_{i<j}\alpha_{ij}\text{sign}(z_j-z_i)}, \tag{1.4}$$

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[3] The constraint (1.1) for $i = n$ follows from the sum of others using the fact that $\sum_{i=1}^{n} c_i = 0$. For a general multi-centered black hole system there are additional constraints besides (1.1) coming from the requirement of the regularity of the metric and other fields. However when the central charges of the constituents nearly align, which is the the limit in which the quiver quantum mechanics becomes a good description, these additional constraints are expected to be satisfied automatically [14]. Throughout this paper we shall be working in this limit.

where the sum runs over solutions to the equations

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i , \quad \text{for } 1 \le i \le n - 1 , \quad z_1 = 0 . \tag{1.5}$$

The $z_1 = 0$ condition fixes the translational zero-mode. The sign $\pm$ in (1.4) is given by the sign of the Hessian $\det(\partial^2 W / \partial z_i \partial z_j)$ of the superpotential

$$W(\{z_i\}) = -\sum_{i<j} \alpha_{ij} \operatorname{sign}(z_j - z_i) \log |z_i - z_j| - \sum_{i=1}^{n} c_i \, z_i , \tag{1.6}$$

whose critical points reproduce the conditions (1.5).

For charges $\{\alpha_i\}$ such that no such ordering exists, the space $\mathcal{M}_n$ may be non-compact, due to the possibility of 'scaling solutions', i.e. a subset of the centers approaching each other at arbitrary short distances [7, 20]. In that case, we continue to define the Coulomb index $g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y)$ by the localization formula (1.4), although the result can no longer be interpreted directly as the refined index associated to the multi-centered black hole configuration (in particular, it may not be a symmetric Laurent polynomial). Nevertheless it can be used to construct such a refined index following the procedure described in [11, 14] and reviewed in section 3.1. When the FI parameters c_i sit on a wall of marginal stability, the space $\mathcal{M}_n$ is also non-compact due to the possibility of separating the centers into two or more clusters at arbitrarily large distances, and we do not assign a value to $g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y)$ in such cases.

A new algorithm for computing the Coulomb index. Except in very special cases, it is usually impossible to find all solutions of (1.5) explicitly. This is also unnecessary, since the contribution of a given solution of (1.5) to the total index (1.4) depends only on the ordering of the centers, via the angular momentum $J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij} \operatorname{sign}(z_j - z_i)$ along the z-axis and the sign of the Hessian W''. For a small number of centers, it is possible to find approximate solutions numerically, and determine both J_3 and the sign of W'', however this becomes quickly unpractical as the number of centers grows. Moreover, the brute force enumeration of solutions of (1.5) does not take into account the fact that there can be cancellations between different solutions with the same ordering. To exploit this fact, it is useful to associate a permutation σ to each solution to (1.5), such that $i < j$ iff $z_{\sigma(i)} < z_{\sigma(j)}$. Defining $\tilde{\alpha}_i = \alpha_{\sigma(i)}$, $x_i = z_{\sigma(i)}$ and $\tilde{c}_i = c_{\sigma(i)}$, solutions of (1.5) correspond to critical points of

$$W(\{x_i\}) = -\sum_{i<j} \tilde{\alpha}_{ij} \log |x_i - x_j| - \sum_{i=1}^{n} \tilde{c}_i \, x_i , \quad \sum_{i=1}^{n} \tilde{c}_i = 0 \tag{1.7}$$

in the physical region

$$0 \equiv x_1 < x_2 < x_3 \cdots < x_n . \tag{1.8}$$

Reorganizing (1.4) as a sum over all permutations σ of $1, 2, \ldots n$, we obtain

$$g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y) = (-1)^{n-1+\sum_{i<j} \alpha_{ij}} (y - y^{-1})^{-n+1} \sum_{\sigma} s(\sigma) y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}} , \tag{1.9}$$

where $s(\sigma)$ is the sum of the sign of the Hessian of (1.7) over each critical point in the physical region. In particular, $s(\sigma)$ is insensitive to pairs of solutions of (1.5) with the same ordering, which may appear under small perturbations of the parameters α_{ij} and c_i, as long as we stay away from walls of marginal stability in the space of FI parameters $\{c_i\}$ and away from certain 'scaling walls' in the space of DSZ products $\{\alpha_{ij}\}$ described in more detail below.

The first aim of this paper is to develop an explicit algorithm for computing $s(\sigma)$ and hence the Coulomb index (1.9) for generic DSZ products $\{\alpha_{ij}\}$ and FI parameters $\{c_i\}$. This is achieved in section 2, where we prove an inductive formula (2.20) for the indexed number $s(\sigma) = F(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\})$ of critical points of the superpotential (1.7), by exploiting its robustness under changes of the DSZ products $\tilde{\alpha}_{ij}$. For quivers without oriented loops and generic products, we arrive at the following completely explicit result:

$$
\begin{aligned}
g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y) = {} & (-1)^{n-1+\sum_{i<j}\alpha_{ij}}(y - y^{-1})^{-n+1} \\
\times \sum_\sigma \prod_{k=1}^{n-1} \Theta & \left(\alpha_{\sigma(k),\sigma(k+1)} \sum_{i=1}^{k} c_{\sigma(i)} \right) (-1)^{\sum_{k=1}^{n-1}\Theta(-\alpha_{\sigma(k),\sigma(k+1)})} \, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}} \,,
\end{aligned}
\tag{1.10}
$$

where Θ is the Heaviside function and the sum runs over all permutations σ of $1, 2, \cdots n$. If some of the α_{ij}'s vanish then we need to deform them away from zero such that the deformed quiver continues to satisfy the no loop condition, compute the result using (1.10) and then take the limit where the deformations are taken back to zero.

As described in (2.20), in the presence of loops the expression for $s(\sigma)$ picks up additional contributions ΔF_A given in (2.32), which depend on the index F with fewer centers and another auxiliary quantity $G(\tilde{\alpha}_1, \cdots \tilde{\alpha}_n)$. The latter counts the (indexed) number[4] of collinear scaling solutions, i.e. solutions of (1.5) with $c_i = 0$ which may arise when the total angular momentum $\frac{1}{2} \sum_{i<j} \tilde{\alpha}_{ij}$ vanishes. We find that this 'scaling index' can itself be computed inductively using (2.38). These formulæ hold when the DSZ products $\tilde{\alpha}_{ij}$ are generic, but we show that even when this is not the case, all the relevant physical quantities can be computed in terms of limits of these formulæ.

Quiver quantum mechanics and pure-Higgs invariants. While the refined index is not protected in full-fledged string theory, it is protected in the context of $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics, which describes the dynamics of certain multi-black hole bound states around special loci in moduli space where the central charges of the constituents become nearly aligned [5, 7]. Thus, by considering a black hole whose dynamics in some region of the moduli space is described by a specific quiver quantum mechanics, we can use our general Coulomb branch formula for multi-black hole bound states to parametrize the refined index in the corresponding quiver quantum mechanics in terms of single-centered BPS invariants [14]. For quivers without oriented loops, the single-centered BPS invariants are trivial, and the Coulomb branch formula is completely explicit. We can then use this to establish the equivalence of the Coulomb branch formula for such quivers with explicit formulae for the cohomology of the moduli space of stable quiver representations known in the mathematical literature [15, 16, 21, 22].

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[4]Due to the scaling symmetry, each solution arises as a one-parameter family, the number of which is counted by G.

Before explaining our results, let us briefly review the relation between quiver quantum mechanics and multi-centered black holes [14]. $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics can be obtained by dimensionally reducing an $\mathcal{N} = 1$ supersymmetric gauge theory in 3+1 dimensions[5] — containing vector multiplets in the adjoint representation of the gauge group $G = \prod_{i=1...K} U(N_i)$ and chiral multiplets in bi-fundamental representations of $U(N_i) \times U(N_j)$ — down to 0+1 dimensions. The scalars coming from the chiral multiplets are called the Higgs variables and those from the vector multiplets are called the Coulomb variables. The vacuum moduli space of the Higgs variables at zero values of the Coulomb variables is equivalent to the moduli space $\mathcal{M}$ of stable quiver representations (in short quiver moduli space), where the stability condition is determined by the FI parameters $\zeta_1, \ldots \zeta_K$ for each $U(N_i)$ factor. BPS states are in one-to-one correspondence with cohomology classes in $H^*(\mathcal{M}; \mathbb{Z})$, and the angular momentum is identified with the component $J_3 = (p - d)/2$ of the Lefschetz SU(2) action on the total cohomology $H^*(\mathcal{M}; \mathbb{Z})$. Thus the refined index is given by the 'Poincaré-Laurent polynomial'[6]

$$Q(\mathcal{M}; y) \equiv \sum_p b_p(\mathcal{M}) (-y)^{p-d}, \tag{1.11}$$

where $b_p(\mathcal{M})$ are the Betti numbers and d is the complex dimension of $\mathcal{M}$. But the same spectrum may also be calculated by first integrating out all the Higgs variables and considering the effective theory for the Coulomb variables. The latter turns out to be given by the same quantum mechanical system as that of multi-centered black holes in $\mathcal{N} = 2$ supersymmetric string theory, upon identifying the charge vector γ with the dimension vector $(N_1, \ldots N_K)$ [5], and the DSZ product γ_{ij} between the basis vectors $\gamma_i = (0, \ldots, 1, \ldots, 0)$, where the only non-vanishing entry occurs in position i, with the number of arrows from the i-th to the j-th node of the quiver. The Coulomb branch formula of [11] can thus be used to express the Poincaré-Laurent polynomial (1.11) in terms of certain 'single centered BPS invariants' $\Omega_S(\alpha)$, which are (conjecturally) independent of the FI parameters and of the fugacity parameter y [14]. In simplest cases, $\Omega_S(\gamma)$ enters just as an additive constant in $Q(\mathcal{M}; y) = \Omega(\gamma)$, and can be identified as the Lefschetz singlet contribution to the total cohomology $H^*(\mathcal{M}; \mathbb{Z})$ [14, 25–27]. In general however, the single centered BPS invariants $\Omega_S(\alpha)$ enter in $Q(\mathcal{M}; y)$ in a more complicated fashion. It is a very interesting open problem to identify the corresponding 'absolutely stable' classes in $H^*(\mathcal{M}; \mathbb{Z})$.

For quivers without oriented loops, the only non-vanishing single centered BPS invariants are those associated to the basis vectors γ_i, and for such vectors $\Omega_S(\gamma_i)$ takes the value 1. In that case, the Coulomb branch formula gives a completely explicit result for $Q(\mathcal{M}; y)$, which can be compared to known results in the mathematical literature. In particular, for

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[5]As an aside it should be noted that $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics is also useful in computing the spectrum of BPS states in $\mathcal{N} = 2$ supersymmetric gauge theories in 3+1 dimensions [19, 23]. Indeed, the BPS spectrum of many gauge theories can be understood in the language of quiver representations [24]. For example in the context of $\mathcal{N} = 2$ supersymmetric pure SU(2) gauge theory, the role of single centered 'black holes' is played by the monopole and the dyon in Seiberg-Witten theory, which are stable for all values of the moduli and whose bound states generate the complete BPS spectrum.

[6]We use this terminology since $Q(\mathcal{M}; y)$ differs from the usual Poincaré polynomial $\sum_p b_p y^p$ by a $y \to -y$ transformation and a multiplicative factor of $(-y)^{-d}$.

primitive dimension vector γ (i.e. such that all N_i are coprime), Reineke's formula [15] gives another completely explicit result for $Q(\mathcal{M}; y)$. For Abelian quivers (i.e such that all N_i are one), the Coulomb branch formula equates the Poincaré-Laurent polynomial $Q(\mathcal{M}; y)$ with the Coulomb index $g_{\text{Coulomb}}(\{\gamma_1, \cdots \gamma_n\}; \{\zeta_1, \cdots \zeta_n\}; y)$ given in (1.10). In section 4.2 we show the equivalence of the Reineke's formula and Coulomb branch results for Abelian quivers, generalizing previous arguments given in the context of wall-crossing [10, 28]. For non-Abelian quivers with primitive dimension vector, the equivalence of the Reineke's formula and Coulomb branch results can be reduced to the Abelian case, by using the Abelianization property satisfied by Reineke's formula [21, 28] (see (4.18) below).

For non-Abelian quivers with non-primitive vector, Reineke's formula no longer computes the Poincaré-Laurent polynomial $Q(\mathcal{M}; y)$ of the quiver moduli space $\mathcal{M}$ (which is singular due to marginal bound states), but rather the 'stack invariant' G_{Higgs}. It is conjectured in the mathematical literature that a bone-fide Poincaré-Laurent polynomial $Q(\mathcal{M}; y)$ may be reconstructed from the stack invariants G_{Higgs} [12, 16] (see eq. (4.1) below), but it is unclear in general how to construct a smooth moduli space $\mathcal{M}$ whose cohomology would agree with G_{Higgs}. At any rate, using a further property of Abelian stack invariants g_{Higgs} established in [22] (see (4.22) below), we prove that the Coulomb branch formula agrees with the Poincaré-Laurent polynomial $Q(\mathcal{M}; y)$ computed from Reineke's formula for the stack invariants. The result can be written as

$$Q(\mathcal{M}; y) = \sum_{m | N_i \ \forall \ i} \frac{\mu(m)}{m} \frac{y - y^{-1}}{y^m - y^{-m}} \sum_{\substack{\{k_j^{(\ell)}\} \\ \sum_\ell \ell k_j^{(\ell)} = N_j/m}} g_{\text{Coulomb}}(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y)$$
$$\times \prod_{j=1}^{K} \prod_\ell \frac{1}{k_j^{(\ell)}!} \left(\frac{y - y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_j^{(\ell)}}, \tag{1.12}$$

where $\mu(m)$ is the Möbius function, $\{(\ell\gamma_j)^{k_j^{(\ell)}}\}$ denotes that we have $k_j^{(\ell)}$ nodes each carrying charge $\ell\gamma_j$ for $\ell \geq 1$ and $1 \leq j \leq n$, and it is understood that in computing g_{Coulomb} whenever some α_{ij} vanishes we need to deform it away from that value to produce an Abelian quiver without loop, and then use (1.10) for computing it.

For brevity we shall henceforth refer to Reineke's formula and its generalizations as the Higgs branch formulæ, since they compute directly the cohomology of the Higgs branch moduli space. For quivers with loops, the Higgs branch formula will refer to the result of any computation of the cohomology of the Higgs branch moduli space, although no general formula is available in such cases.

A MATHEMATICA package for quiver invariants. It should be clear from the above that the Coulomb branch formula produces a parametrization of the Poincaré-Laurent polynomial of any quiver in terms of single-centered BPS invariants in a completely combinatoric way. However, even for moderately complicated quivers the necessary computations quickly become tiresome, and are best implemented on a computer. We have implemented the Coulomb branch formula as well as Reineke's formula for stack invariants (and many related other routines) in a mathematica package "CoulombHiggs"

available from arXiv and described in appendix A, which we hope will facilitate studies of single-centered BPS invariants. This package has been successfully tested on the examples investigated in [14] and many more.

2 A formula for the Coulomb index of multi-centered black holes

In this section, we establish a recursive algorithm for computing the Coulomb index $g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y)$ for general charge configurations α_i, away from the walls of marginal stability in the space of FI parameters c_i. We start in section 2.1 with charge configurations for which there exists a possible ordering of the α_i's such that

$$\alpha_{ij} \geq 0 \quad \text{for} \quad i \leq j, \tag{2.1}$$

and with all DSZ products α_{ij} non-zero. It follows from the discussion in section 1 that this corresponds to an Abelian quiver without any oriented loop. For such systems we obtain the simple formula (2.9) for the indexed number $s(\sigma)$ that enters (1.9). In section 2.2 we show that for the purpose of computing the total Coulomb index, the same result can be used even when some of the α_{ij}'s vanish. In section 2.3, we turn to general multi-centered black hole configurations for which the charges do not admit an ordering satisfying (2.1), and establish an inductive formulae for computing $s(\sigma)$. This is given by $F(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\})$ in (2.20) with $\tilde{\alpha}_i = \alpha_{\sigma(i)}$. During this analysis we also derive a similar formula for the coefficient $s(\sigma)$ for collinear scaling solutions [7, 20] for which all the FI parameters vanish and the α_i's satisfy $\sum_{i<j} \alpha_{\sigma(i)\sigma(j)} = 0$. The corresponding inductive formula for $s(\sigma)$, called $G(\alpha_{\sigma(1)}, \cdots \alpha_{\sigma(n)})$, is given in (2.38). Combining these results and summing over all permutations yields a general algorithm for $g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y)$ with non vanishing DSZ products, as summarized in (2.33).

2.1 Abelian quivers with no oriented loops and all $\alpha_{ij} \neq 0$

We start by considering the case of charge configurations which admit an ordering such that $\alpha_{ij} > 0$ for $i < j$. A special case arises if all α_i are positive linear combinations of two charge vectors γ_1, γ_2 with $\langle \gamma_1, \gamma_2 \rangle > 0$, as is the case for multi-centered configurations relevant for wall-crossing [10].

Defining

$$y_i = x_{i+1} - x_i, \qquad \tilde{d}_i = -\sum_{j=i+1}^{n} \tilde{c}_i = \sum_{j=1}^{i} \tilde{c}_i, \quad \text{for } 1 \leq i \leq n-1, \tag{2.2}$$

we can express (1.7) as

$$W(\{y_i\}) = -\sum_{i<j} \tilde{\alpha}_{ij} \ln \left(\sum_{k=i}^{j-1} y_k \right) + \sum_{i=1}^{n} \tilde{d}_i \, y_i. \tag{2.3}$$

This gives

$$\frac{\partial W}{\partial y_k} = -\sum_{j=1}^{k} \sum_{\ell=k+1}^{n} \tilde{\alpha}_{j\ell} \frac{1}{\sum_{i=j}^{\ell-1} y_i} + \tilde{d}_k \quad \text{for } 1 \leq k \leq n-1. \tag{2.4}$$

Each y_i takes value from 0 to ∞, but by an appropriate coordinate transformation $u_i = f(y_i)$ we can bring the range to $0 \le u_i < 1$ for each i. We can regard the space spanned by the u_i's a unit box. Our goal is to examine the condition under which W has an extremum with respect to the y_i's in the interior of this box, i.e. there is a solution to the equation $\partial W / \partial y_k = 0$ for every k.

To this aim, let us now consider the following deformation of the $\tilde{\alpha}_{ij}$'s:

$$\tilde{\alpha}_{i,i+1} \to \tilde{\alpha}_{i,i+1} \ \forall \, i, \qquad \tilde{\alpha}_{ij} \to \lambda \, \tilde{\alpha}_{ij} \ \text{ for } |i-j| \ge 2, \qquad 0 \le \lambda \le 1 \,. \tag{2.5}$$

In the limit $\lambda \to 0$, only $\tilde{\alpha}_{i,i+1}$'s remain non-zero and (2.4) takes the simple form

$$\frac{\partial W}{\partial y_k} = -\frac{\tilde{\alpha}_{k,k+1}}{y_k} + \tilde{d}_k \,. \tag{2.6}$$

Thus the set of equations $\partial W / \partial y_k = 0$ has a solution in the range $0 < y_k < \infty$ if and only if

$$\text{sign}(\tilde{\alpha}_{k,k+1}) = \text{sign}(\tilde{d}_k) \quad \text{for } 1 \le k \le n-1 \,. \tag{2.7}$$

Furthermore the sign of the Hessian of W at this solution is easily determined to be

$$\prod_{k=1}^{n-1} \text{sign}(\tilde{\alpha}_{k,k+1}) \,. \tag{2.8}$$

These results can be summarized by saying that for quivers without loops the coefficient $s(\sigma)$ of $y^{\sum_{i<j} \tilde{\alpha}_{ij}}$ associated with a given permutation $\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}$ is given, for $\lambda = 0$, by

$$F_0(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\}) = \prod_{k=1}^{n-1} \Theta(\tilde{\alpha}_{k,k+1} \, \tilde{d}_k)(-1)^{\sum_{k=1}^{n-1} \Theta(-\tilde{\alpha}_{k,k+1})} \,, \tag{2.9}$$

where $\Theta(x)$ is the Heaviside function.

Now consider changing λ from 0 to 1. During this deformation new extrema can appear and disappear in pairs in the interior of the box, but since they are weighted by the sign of the Hessian, the result is unaffected. The other possibility is that new extrema can emerge from (or disappear into) the boundary of the box. This happens when a subset of the y_i's approach 0 and/or infinity. Since $\partial W / \partial y_k$ approaches the constant $\tilde{d}_k$ as $y_k \to \infty$ irrespective of the values of the other y_i's, it is clear that away from the marginal stability walls $\tilde{d}_k = 0$, none of the y_i's can approach infinity. Thus the only possible boundary component where extrema of W can appear or disappear is where a subset of the y_i's vanish.

Let us suppose that as λ is increased from 0 to 1, such a phenomenon takes place at some value $\lambda = \lambda_c$. If a new extremum appears at λ_c then for λ slightly above λ_c, there will be an extremum of W where a subset of the y_i's are small, corresponding to a subset of the centers being close to each other. If on the other hand an extremum disappears as λ approaches λ_c from below, then such a configuration exists for λ slightly below λ_c. These correspond to onset or disappearance of scaling solutions [7, 20], with $\lambda = \lambda_c$ being the point at which the scaling solution becomes collinear.

Now if the quiver corresponding to the original α_{ij}'s had no oriented loop then neither does the quiver associated with the deformed α_{ij}'s, since the signs of all the α_{ij}'s are

preserved under the deformation (2.5). This implies that the deformed quiver cannot have a scaling configuration, and hence, as we deform λ from 0 to 1, no extremum of W can emerge from or disappear into the boundary. Thus (2.9) gives the correct contribution to $s(\sigma)$ even at $\lambda = 1$. Using (1.9) we now get

$$
\begin{aligned}
g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y) &= (-1)^{n-1+\sum_{i<j} \alpha_{ij}}(y - y^{-1})^{-n+1} \\
&\times \sum_\sigma \prod_{k=1}^{n-1} \Theta\left(\alpha_{\sigma(k),\sigma(k+1)} \sum_{i=1}^k c_{\sigma(k)}\right)(-1)^{\sum_{k=1}^{n-1}\Theta(-\alpha_{\sigma(k),\sigma(k+1)})} y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}.
\end{aligned}
\tag{2.10}
$$

2.2 Abelian quivers with no oriented loops but some $\alpha_{ij} = 0$

We now turn to the case of a quiver without oriented loops, but for which some of the α_{ij}'s vanish. For this we first deform all the vanishing α_{ij}'s to non-zero values in such a way that the deformed quiver does not have any oriented loop. To see that this is always possible, let us carry out the deformation one link at a time. We begin with the original quiver without oriented loop and make one of the vanishing α_{ij}'s non-zero. If this leads to a quiver with an oriented loop, then there exists some component C of the original quiver, which, together with the new link, gives rise to a quiver with oriented loop. If so let us flip the sign of α_{ij} of the deformed link. In this case C together with the added link no longer forms an oriented loop. Suppose there were another component C' of the original quiver, which combined with the new link would now form an oriented loop. Then $C + C'$ would form an oriented loop in the original quiver, which contradicts our assumption. Thus, by choosing the sign of α_{ij} of the deformed link we can ensure that the new quiver also does not have any oriented loop. We can now repeat the argument and show that all the vanishing α_{ij}'s can be made non-zero and for appropriate choice of sign of the deformed α_{ij}'s the new quiver does not have any oriented loop. Thus we can compute its index by our earlier formula (2.10).

We shall now argue that the index of the original quiver can be obtained by taking the limit of the index of the deformed quiver in which the deformation parameters go to zero. For this we shall work with the total index (1.9) rather than a given permutation. We shall use the original charges α_i and use their locations z_i — related to the x_i by $x_i = z_{\sigma(i)}$ — as the independent variables. The z_i's satisfy (1.5).

Let us now consider the effect of taking α_{pq} to 0 for some specific p, q. For any extremum of W at which the locations of α_p and α_q remain at finite separation, this limit has no drastic effect and the contribution to the index from such extrema at $\alpha_{pq} = 0$ is the same as what we get by taking the $\alpha_{pq} \to 0$ limit. Thus we only have to examine the fate of the critical points for which the locations of α_p and α_q approach each other in the $\alpha_{pq} \to 0$ limit, as generically such critical points will disappear for $\alpha_{pq} = 0$. For such solutions we can replace z_q by z_p in (1.5) except in the $\alpha_{pq}/|z_p - z_q|$ terms, and express (1.5) as

$$
\begin{aligned}
-\sum_{\substack{i=1 \\ i \neq k,p,q}}^n \frac{\alpha_{ik}}{|z_i - z_k|} - \frac{\alpha_{pk} + \alpha_{qk}}{|z_p - z_k|} - c_k = 0 &\quad \text{for } 1 \leq k \leq n, \, k \neq p, q \\
-\sum_{\substack{i=1 \\ i \neq p,q}}^n \frac{\alpha_{ip}}{|z_i - z_p|} - \frac{\alpha_{qp}}{|z_q - z_p|} - c_p = 0, &\qquad -\sum_{\substack{i=1 \\ i \neq p,q}}^n \frac{\alpha_{iq}}{|z_i - z_p|} - \frac{\alpha_{pq}}{|z_q - z_p|} - c_q = 0.
\end{aligned}
\tag{2.11}
$$

By adding and subtracting the last two equations we get

$$-\sum_{\substack{i=1 \\ i\neq k,p,q}}^{n} \frac{\alpha_{ik}}{|z_i-z_k|} - \frac{\alpha_{pk}+\alpha_{qk}}{|z_p-z_k|} - c_k = 0 \quad \text{for } 1 \leq k \leq n, \, k \neq p,q$$

$$-\sum_{\substack{i=1 \\ i\neq p,q}}^{n} \frac{\alpha_{ip}+\alpha_{iq}}{|z_i-z_p|} - c_p - c_q = 0, \qquad -\sum_{\substack{i=1 \\ i\neq p,q}}^{n} \frac{\alpha_{ip}-\alpha_{iq}}{|z_i-z_p|} + 2\frac{\alpha_{pq}}{|z_q-z_p|} - c_p + c_q = 0. \tag{2.12}$$

The first set of equations and the second equation together correspond to the equilibrium configuration of $n-1$ charges in which the charges α_p and α_q have merged to form a charge $\alpha_p + \alpha_q$ and the corresponding FI parameters have been added. The last equation can be interpreted as an equation for $z_q - z_p$. The existence of a solution to this equation requires

$$\text{sign}\left(c_q - c_p - \sum_{\substack{i=1 \\ i\neq p,q}}^{n} \frac{\alpha_{ip}-\alpha_{iq}}{|z_i-z_p|} \right) = \text{sign}\,\alpha_{qp}. \tag{2.13}$$

When this condition is satisfied then $z_q - z_p$ is of order α_{pq} for small α_{pq}. On the other hand when $\alpha_{pq} = 0$, the last equation in (2.12) generically has no solution since the left hand side of the equation becomes independent of z_q and all the z_i's for $i \neq q$ are already fixed by the other equations. This shows that the critical points of W associated with solutions to (2.12) disappear at $\alpha_{pq} = 0$. This could give rise to a discontinuity in the index at $\alpha_{pq} = 0$. The important point to note however is that even if (2.13) is satisfied, the solutions to the last equation in (2.12) always occur in pairs, related by a reversal of the sign of $z_p - z_q$. The exponent of y in (1.4) remains unchanged under this exchange in the $\alpha_{pq} \to 0$ limit since this exchange only flips the sign of the coefficient of α_{pq} in the exponent. Finally it is easy to check that $s(\sigma)$ changes sign under this exchange. Thus the contribution from this pair of solutions cancel and we get a smooth $\alpha_{pq} \to 0$ limit. Repeating this analysis for the other α_{pq}'s we see that the index associated with the original quiver can be obtained as the limit of the index associated with the deformed quiver.

Note that the above argument breaks down if the solution to the equations in the first two lines of (2.12) automatically satisfy

$$c_q - c_p - \sum_{\substack{i=1 \\ i\neq p,q}}^{n} \frac{\alpha_{ip}-\alpha_{iq}}{|z_i-z_p|} = 0. \tag{2.14}$$

This happens for example when α_p and α_q are parallel so that $\alpha_{ip}/\alpha_{iq} = |\alpha_p|/|\alpha_q| = c_p/c_q$. In this case at $\alpha_{pq} = 0$ there is a solution to (1.5) at $z_p = z_q$, obtained by solving the equations in the first two lines of (2.12). Now consider the case when α_{pq} is deformed away from 0. In this case in order to look for a solution to the last equation in (2.12) where z_p and z_q are close to each other we can no longer set $z_p = z_q$ in the regular terms from the beginning, but must keep terms of order $(z_p - z_q)$ in the last equation. If we call this term $A(z_p - z_q)$ for some constant A then we can express this equation as

$$A(z_p - z_q) + 2\frac{\alpha_{pq}}{|z_q - z_p|} = 0. \tag{2.15}$$

This equation is no longer invariant under a change of sign of $z_p - z_q$, and in fact has a solution only for one sign of $z_p - z_q$ irrespective of the sign of α_{pq}. In the $\alpha_{pq} \to 0$ limit this solution smoothly continues to the solution with $z_p = z_q$ at $\alpha_{pq} = 0$. Thus we again see that the $\alpha_{pq} \to 0$ limit is smooth, and agrees with the result for $\alpha_{pq} = 0$.

2.3 Generic Abelian quivers with all α_{ij} non-zero

We shall now consider a generic multi-centered black hole configuration with all α_{ij} non-zero, but whose associated quiver may possess oriented loops. In this case we need to take into account possible contributions from scaling solutions. Our goal in this section will be to compute g_{Coulomb} for such configurations.

2.3.1 An inductive formula for the index of collinear solutions

We proceed as in section 2.1 and consider the deformation (2.5). It is clear that at $\lambda = 0$ the contribution from a given permutation σ will be given by (2.9). Thus we need to investigate the total change in this contribution as λ changes from 0 to 1. As discussed in section 2.1 these changes could come from values of λ at which a set A of neighbouring centers come close to each other. This can happen if the total angular momentum carried by this set of centers vanish,

$$\sum_{i,j \in A; i < j} \tilde{\alpha}'_{ij} = 0 \,, \tag{2.16}$$

where $\tilde{\alpha}'_{ij}$ denotes the deformed $\tilde{\alpha}_{ij}$. To see this, note that in the limit where all y_i for $i, i+1 \in A$ approach zero, the part of the superpotential (2.3) involving the y_i's becomes quasi-homogeneous,

$$W(\{\lambda y_i\}) \sim W(\{y_i\}) - \sum_{i,j \in A; i < j} \tilde{\alpha}'_{ij} \log \lambda \,. \tag{2.17}$$

Differentiating with respect to λ and using $\partial W / \partial y_i = 0$ implies (2.16). Since the set A must contain at least three elements there are at most $(n-2)(n-1)/2$ possible sets A, given by the $(n-2)(n-1)/2$ possible ways of choosing the beginning and the end of the set. Correspondingly there are at most $(n-2)(n-1)/2$ possible values λ_A of the deformation parameter λ where such collinear scaling configurations can arise. Using (2.5), the condition (2.16) becomes a linear equation in λ_A,

$$\lambda_A \sum_{i,j \in A; i \leq j-2} \tilde{\alpha}_{ij} + \sum_{i \in A, i+1 \in A} \tilde{\alpha}_{i,i+1} = 0 \,. \tag{2.18}$$

The index can jump across $\lambda = \lambda_A$ if λ_A lies between 0 and 1. This is so if and only if the left hand side of (2.18) has opposite signs at $\lambda_A = 0$ and 1, i.e.

$$\left(\sum_{i \in A, i+1 \in A} \tilde{\alpha}_{i,i+1} \right) \left(\sum_{i,j \in A, i < j} \tilde{\alpha}_{ij} \right) < 0 \,. \tag{2.19}$$

If $F(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\})$ denotes the coefficient $s(\sigma)$ of $y^{\sum_{i<j} \tilde{\alpha}_{ij}}$ associated with a given permutation, then we have

$$F(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\}) = F_0(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\}) + \sum_A \Delta F_A \,, \tag{2.20}$$

where F_0 is given by (2.9), and ΔF_A is the jump across the critical point λ_A. Our goal will be to compute the expression for ΔF_A.

Let us suppose that set A consists of the integers $k, k+1, \cdots \ell$. We shall examine the configuration close to the critical point by taking

$$\sum_{\substack{r,s \\ k \le r < s \le \ell}} \tilde{\alpha}'_{rs} = \epsilon \tag{2.21}$$

for some small number ϵ. We now define z_s via

$$x_s = x_k + y\, z_s \quad \text{for } k \le s \le \ell, \quad z_k \equiv 0, \quad z_\ell \equiv 1, \tag{2.22}$$

and use $x_1, \cdots x_k, x_{\ell+1}, \cdots x_n, y, z_{k+1}, \cdots z_{\ell-1}$ as independent variables. Then the relevant equations are given by extremizing

$$\begin{aligned}
W = {}&- \sum_{\substack{i,j \\ i<j;\, i,j\, <k\, or\, \ge \ell+1}} \tilde{\alpha}'_{ij} \ln(x_j - x_i) - \sum_{i=1}^{k-1}\sum_{s=k}^{\ell} \tilde{\alpha}'_{is} \ln(x_k + y z_s - x_i) \\
&- \sum_{i=\ell+1}^{n}\sum_{s=k}^{\ell} \tilde{\alpha}'_{si} \ln(x_i - x_k - y z_s) - \sum_{\substack{s,r \\ k \le s < r \le \ell}} \tilde{\alpha}'_{sr} \ln(y(z_r - z_s)) \\
&- \sum_{\substack{i \\ i<k\, or\, i\ge \ell+1}} \tilde{c}_i x_i - \left(\sum_{s=k}^{\ell} \tilde{c}_s\right) x_k - y \sum_{s=k+1}^{\ell} \tilde{c}_s z_s\,.
\end{aligned} \tag{2.23}$$

We shall be examining an extremum of W for which y is small, of order ϵ. In this case the extrema of W with respect to $x_1, \cdots x_k$, $x_{\ell+1}, \cdots x_n$ can be obtained by extremizing

$$\begin{aligned}
W_1 = {}&- \sum_{\substack{i,j \\ i<j;\, i,j\, <k\, or\, \ge \ell+1}} \tilde{\alpha}'_{ij} \ln(x_j - x_i) - \sum_{i=1}^{k-1} \left(\sum_{s=k}^{\ell} \tilde{\alpha}'_{is}\right) \ln(x_k - x_i) \\
&- \sum_{i=\ell+1}^{n} \left(\sum_{s=k}^{\ell} \tilde{\alpha}'_{si}\right) \ln(x_i - x_k) - \sum_{\substack{i \\ i<k\, or\, i\ge \ell+1}} \tilde{c}_i x_i - \left(\sum_{s=k}^{\ell} \tilde{c}_s\right) x_k.
\end{aligned} \tag{2.24}$$

The existence of an extremum of W_1 is equivalent to the existence of a collinear configuration with $n - \ell + k$ centers with charges $\tilde{\alpha}'_1, \cdots \tilde{\alpha}'_{k-1}, \sum_{s=k}^{\ell} \tilde{\alpha}'_s, \tilde{\alpha}'_{\ell+1}, \cdots \tilde{\alpha}'_n$ and FI parameters $\tilde{c}_1, \cdots \tilde{c}_{k-1}, \sum_{s=k}^{\ell} \tilde{c}_s, \tilde{c}_{\ell+1}, \cdots \tilde{c}_n$, situated at $x_1, \cdots x_{k-1}, x_k, x_{\ell+1}, \cdots x_n$. Extremization of (2.23) with respect to the parameters z_s for $k+1 \le s \le \ell-1$ can be obtained by extremizing

$$W_2 = - \sum_{\substack{s,r \\ k \le s < r \le \ell}} \tilde{\alpha}'_{sr} \ln(z_r - z_s)\,. \tag{2.25}$$

The existence of an extremum of W_2 is equivalent to the existence of a collinear scaling configuration of $\ell - k + 1$ centers with charges $\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell$ and zero FI parameters, with the locations of the centers being at $z_k = 0$, z_{k+1}, $\cdots$, $z_{\ell-1}$ and $z_\ell = 1$. Finally y can be

obtained by extremizing[7]

$$W_3 = - \sum_{\substack{s,r \\ k \le s < r \le \ell}} \tilde{\alpha}'_{sr} \ln y - y \sum_{i=1}^{k-1} \sum_{s=k+1}^{\ell} \tilde{\alpha}'_{is} \frac{z_s}{x_k - x_i} + y \sum_{i=\ell+1}^{n} \sum_{s=k+1}^{\ell} \tilde{\alpha}'_{si} \frac{z_s}{x_i - x_k} - y \sum_{s=k+1}^{\ell} \tilde{c}_s z_s \,. \tag{2.26}$$

Using (2.21) this gives

$$- \frac{\epsilon}{y} - \sum_{i=1}^{k-1} \sum_{s=k+1}^{\ell} \tilde{\alpha}'_{is} \frac{z_s}{x_k - x_i} + \sum_{i=\ell+1}^{n} \sum_{s=k+1}^{\ell} \tilde{\alpha}'_{si} \frac{z_s}{x_i - x_k} - \sum_{s=k+1}^{\ell} \tilde{c}_s z_s = 0 \,. \tag{2.27}$$

For generic $\tilde{\alpha}_{ij}$, a solution to this equation with positive y exists only for one choice of sign of ϵ. We shall assume that we have taken the sign of ϵ to be such that the solution exists. Let η_A denote a quantity which takes value 1 (-1) if the solution exists for λ above (below) the critical value λ_A given in (2.18).

Let us now compute the Hessian at this critical point. From (2.24)–(2.26) it is clear that the second derivative of W with respect to all the variables except y remain finite, and we have $\partial^2 W / \partial y^2 \sim \sum_{\substack{s,r \\ k \le s < r \le \ell}} \tilde{\alpha}'_{sr} / y^2 \sim \epsilon / y^2$. Since this is large for $y \sim \epsilon$ and all other second derivatives of W remain finite, the full determinant will be given by the product of $\partial^2 W / \partial y^2$ and the determinant of the Hessian involving the rest of the variables. Furthermore we see from (2.23) that $\partial^2 W / \partial x^i \partial z_s$ goes to zero as $y \to 0$. Thus the Hessian of W with respect to x_i's and z_s's factorizes into the product of the Hessian of W_1 with respect to $x_1, \cdots x_k, x_{\ell+1}, \cdots x_n$ and the Hessian of W_2 with respect to $z_{k+1}, \cdots z_{\ell-1}$. In our notation the sign of the Hessian of W_1 with respect to $x_1, \cdots x_k, x_{\ell+1}, \cdots x_n$ is given by $F(\{\tilde{\alpha}'_1, \cdots \tilde{\alpha}'_{k-1}, \tilde{\alpha}'_k + \cdots \tilde{\alpha}'_\ell, \tilde{\alpha}'_{\ell+1}, \cdots \tilde{\alpha}'_n\}, \{\tilde{c}_1, \cdots \tilde{c}_{k-1}, \tilde{c}_k + \cdots \tilde{c}_\ell, \tilde{c}_{\ell+1}, \cdots \tilde{c}_n\})$. Let

$$G(\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell) = \text{sign} \left(\det_{k+1 \le i, j \le \ell-1} (\partial_{z_i} \partial_{z_j} W_2) \right) \tag{2.28}$$

be the sign of the Hessian of W_2 with respect to $z_{k+1}, \cdots z_{\ell-1}$ when the corresponding scaling solution exists; otherwise we take $G(\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell) = 0$.[8] Then we can write

$$\Delta F_A = F(\{\tilde{\alpha}'_1, \cdots \tilde{\alpha}'_{k-1}, \tilde{\alpha}'_k + \cdots \tilde{\alpha}'_\ell, \tilde{\alpha}'_{\ell+1}, \cdots \tilde{\alpha}'_n\}, \{\tilde{c}_1, \cdots \tilde{c}_{k-1}, \tilde{c}_k + \cdots \tilde{c}_\ell, \tilde{c}_{\ell+1}, \cdots \tilde{c}_n\})$$
$$\times \eta_A \, \text{sign}(\epsilon) \, G(\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell) \, \Theta \left(- \left(\sum_{i=k}^{\ell-1} \tilde{\alpha}_{i,i+1} \right) \left(\sum_{\substack{i,j \\ k \le i < j \le \ell}} \tilde{\alpha}_{ij} \right) \right), \tag{2.29}$$

where the last factor imposes the constraint (2.19). Now it follows from (2.5) that at $\lambda = \lambda_A + \delta\lambda$,

$$\epsilon = \sum_{\substack{r,s \\ k \le r < s \le \ell}} \tilde{\alpha}'_{rs} = \delta\lambda \sum_{\substack{i,j=k \\ i \le j-2}}^{\ell} \tilde{\alpha}_{ij} \,. \tag{2.30}$$

---

[7]Since y is small, we have expanded the terms in W which are non-singular in the $y \to 0$ limit to first order in y.

[8]As usual if there are more than one solutions then we add their contributions.

Suppose the solution exists for $\delta\lambda > 0$. Then we have $\eta_A = 1$ and we see from (2.30) that sign(ϵ)=sign$(\sum_{\substack{i,j=k \\ i\le j-2}}^{\ell} \tilde{\alpha}_{ij})$. On the other hand if the solution exists for $\delta\lambda < 0$ then we have $\eta_A = -1$ and sign(ϵ)=$-$sign$(\sum_{\substack{i,j=k \\ i\le j-2}}^{\ell} \tilde{\alpha}_{ij})$. Thus in either case

$$\eta_A \, \text{sign}(\epsilon) = \text{sign}\left(\sum_{\substack{i,j=k \\ i\le j-2}}^{\ell} \tilde{\alpha}_{ij} \right). \tag{2.31}$$

Substituting this into (2.29) we get

$$\Delta F_A = F(\{\tilde{\alpha}'_1, \cdots \tilde{\alpha}'_{k-1}, \tilde{\alpha}'_k + \cdots \tilde{\alpha}'_\ell, \tilde{\alpha}'_{\ell+1}, \cdots \tilde{\alpha}'_n\}, \{\tilde{c}_1, \cdots \tilde{c}_{k-1}, \tilde{c}_k + \cdots \tilde{c}_\ell, \tilde{c}_{\ell+1}, \cdots \tilde{c}_n\})$$
$$\times \, G(\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell) \, \text{sign}\left(\sum_{\substack{i,j=k \\ i\le j-2}}^{\ell} \tilde{\alpha}_{ij} \right) \Theta\left(-\left(\sum_{i=k}^{\ell-1} \tilde{\alpha}_{i,i+1} \right) \left(\sum_{\substack{i,j \\ k\le i < j \le \ell}} \tilde{\alpha}_{ij} \right) \right). \tag{2.32}$$

Note that a special case of (2.32) is $k = 1$, $\ell = n$ in which case the F on the right hand side of this equation is $F(\tilde{\alpha}'_1 + \cdots \tilde{\alpha}'_n; \tilde{c}_1 + \cdots \tilde{c}_n) = 1$. If we can compute $G(\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell)$, then we can use (2.20) and (2.32) to compute the function F recursively. Once we know how to compute F, the Coulomb index can be computed as

$$g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y) \tag{2.33}$$
$$= (-1)^{n-1+\sum_{i<j} \alpha_{ij}} (y - y^{-1})^{-n+1} \sum_\sigma F\big(\{\alpha_{\sigma(1)}, \cdots \alpha_{\sigma(n)}\}; \{c_{\sigma(1)}, \cdots c_{\sigma(n)}\}\big) y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}},$$

where the sum runs over all permutations σ.

2.3.2 An inductive formula for the index of scaling collinear solutions

We now turn to the computation of $G(\hat{\alpha}_1, \cdots \hat{\alpha}_m)$, the indexed number of critical points of the superpotential

$$\widehat{W} = - \sum_{\substack{i,j \\ 1\le i < j \le m}} \hat{\alpha}_{ij} \, \ln(z_j - z_i)\,, \qquad \sum_{\substack{i,j \\ 1\le i < j \le m}} \hat{\alpha}_{ij} = 0\,, \tag{2.34}$$

in the range $z_j > z_i$ for $j > i$. This coincides with (2.25) under the identifications $\{\hat{\alpha}_1, \cdots \hat{\alpha}_m\} = \{\tilde{\alpha}'_k, \cdots \tilde{\alpha}'_\ell\}$, and obvious redefinitions of z_i. The invariance of the superpotential (2.34) under both translation and rescaling of the z_i's, must be 'gauge fixed' before counting critical points. These invariances were fixed by the conditions $z_1 = 0, z_m = 1$ in (2.22); however in order to compute $G(\hat{\alpha}_1, \cdots \hat{\alpha}_m)$ inductively, it will be more convenient to choose a different gauge $z_1 = 0, z_{m-1} = 1$.

Let us now consider the deformation

$$\hat{\alpha}_{im} \to \mu \, \hat{\alpha}_{im} \quad \text{for } i = 1, 2, \cdots m-1, \quad \hat{\alpha}_{m-3,m-1} \to \hat{\alpha}_{m-3,m-1} + (1-\mu) \sum_{i=1}^{m-1} \hat{\alpha}_{im}, \tag{2.35}$$

so that the deformed $\hat{\alpha}_{ij}$'s (called $\hat{\alpha}'_{ij}$) continue to satisfy $\sum_{i<j} \hat{\alpha}'_{ij} = 0$. In the limit $\mu \to 0$, we can treat the m-th center as a probe in the background of other centers. From

the behaviour of $\widehat{W}$ as a function of z_m in the two limits, $z_m \to z_{m-1}$ and $z_m \to \infty$, we conclude that the solution exists if and only if the $m-1$ centered scaling solution with $\hat{\alpha}_{ij}$ for $1 \le i, j \le m-1$ given by (2.35) for $\mu = 0$ exists, and furthermore

$$\text{sign}(\hat{\alpha}_{m-1,m}) = -\text{sign}\left(\sum_{i=1}^{m-1} \hat{\alpha}_{im}\right). \tag{2.36}$$

Finally the sign of the Hessian associated with the configuration in the $\mu \to 0$ limit, after adding up the contribution from all critical points in the range $z_j > z_i$ for $j > i$, is

$$\text{sign}(\hat{\alpha}_{m-1,m}) \, G(\check{\alpha}_1, \cdots \check{\alpha}_{m-1}), \tag{2.37}$$

where $\check{\alpha}_i$ for $1 \le i \le m-1$ denote the deformed charges at $\mu = 0$. Using (2.35)–(2.37) we get

$$G(\hat{\alpha}_1, \cdots \hat{\alpha}_m) = (-1)^{1+\Theta(\hat{\alpha}_{m-1,m})} \Theta\left(-\hat{\alpha}_{m-1,m} \sum_{i=1}^{m-1} \hat{\alpha}_{im}\right) G(\check{\alpha}_1, \cdots \check{\alpha}_{m-1}) + \sum_B \Delta G_B, \tag{2.38}$$

where ΔG_B denotes the jump in G during the deformation from $\mu = 0$ to $\mu = 1$ across the various critical points μ_B where a subset B of the charges can form scaling solutions.[9] Since the deformations involve the $\hat{\alpha}_{im}$ and $\hat{\alpha}_{m-3,m-1}$, new scaling solutions must involve either the mth center, or both the $(m-1)$'th and $(m-3)$'th center. Three kinds of scaling solutions can be encountered during the deformation:

1. The scaling configuration involves the charges $\hat{\alpha}_{m-2}$, $\hat{\alpha}_{m-1}$ and $\hat{\alpha}_m$. In this case we need

$$\hat{\alpha}'_{m-2,m-1} + \hat{\alpha}'_{m-1,m} + \hat{\alpha}'_{m-2,m} = 0, \tag{2.39}$$

which requires

$$\hat{\alpha}_{m-2,m-1} + \mu_B(\hat{\alpha}_{m-1,m} + \hat{\alpha}_{m-2,m}) = 0. \tag{2.40}$$

2. The scaling configuration involves charges $\hat{\alpha}_k, \cdots \hat{\alpha}_m$ for $2 \le k \le (m-3)$. In this case we require

$$\sum_{\substack{i,j \\ k \le i < j \le m}} \hat{\alpha}'_{ij} = 0 \tag{2.41}$$

which translates to

$$\mu_B \sum_{i=k}^{m-1} \hat{\alpha}_{im} + \sum_{\substack{i,j \\ k \le i < j \le m-1}} \hat{\alpha}_{ij} + (1 - \mu_B) \sum_{i=1}^{m-1} \hat{\alpha}_{im} = 0. \tag{2.42}$$

3. The scaling configuration involves charges $\hat{\alpha}_k, \cdots \hat{\alpha}_{m-1}$ for $2 \le k \le (m-3)$. In this case we require

$$\sum_{\substack{i,j \\ k \le i < j \le m-1}} \hat{\alpha}'_{ij} = 0 \tag{2.43}$$

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[9]Due to scale invariance a configuration where a subset of the centers get infinitely separated from the others is equivalent to a configuration where a subset of the centers come together.

which translates to

$$\sum_{\substack{i,j \\ k \le i < j \le m-1}} \hat{\alpha}_{ij} + (1 - \mu_B) \sum_{i=1}^{m-1} \hat{\alpha}_{im} = 0 \,. \tag{2.44}$$

For each of these cases the computation of ΔG_B follows the procedure used for computing ΔF_A earlier. We shall quote the final results generalizing (2.32):

1. For μ satisfying (2.40) for $m > 4$ we have

$$\Delta G_B = G(\hat{\alpha}'_1, \cdots \hat{\alpha}'_{m-3}, \hat{\alpha}'_{m-2} + \hat{\alpha}'_{m-1} + \hat{\alpha}'_m) \times G(\hat{\alpha}'_{m-2}, \hat{\alpha}'_{m-1}, \hat{\alpha}'_m)$$
$$\times \text{sign}(\hat{\alpha}_{m-1,m} + \hat{\alpha}_{m-2,m}) \Theta(-(\hat{\alpha}_{m-2,m-1} + \hat{\alpha}_{m-1,m} + \hat{\alpha}_{m-2,m})\hat{\alpha}_{m-2,m-1}) \,. \tag{2.45}$$

   The case $m = 4$ requires special attention and will be discussed later.

2. For μ satisfying (2.42) with $k > 2$ we have

$$\Delta G_B = G\left(\hat{\alpha}'_1, \cdots \hat{\alpha}'_{k-1}, \sum_{i=k}^{m} \hat{\alpha}'_i\right) \times G(\hat{\alpha}'_k, \hat{\alpha}'_{k+1}, \cdots, \hat{\alpha}'_m) \, \text{sign}\left(-\sum_{i=1}^{k-1} \hat{\alpha}_{im}\right)$$
$$\times \Theta\left(-\left(\sum_{\substack{i,j \\ k \le i < j \le m}} \hat{\alpha}_{ij}\right)\left(\sum_{\substack{i,j \\ k \le i < j \le m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im}\right)\right) \,. \tag{2.46}$$

   The case $k = 2$ requires special treatment and will be discussed below.

3. For μ satisfying (2.44) we have

$$\Delta G_B = G\left(\hat{\alpha}'_1, \cdots \hat{\alpha}'_{k-1}, \sum_{i=k}^{m-1} \hat{\alpha}'_i, \hat{\alpha}'_m\right) \times G(\hat{\alpha}'_k, \hat{\alpha}'_{k+1}, \cdots, \hat{\alpha}'_{m-1}) \, \text{sign}\left(-\sum_{i=1}^{m-1} \hat{\alpha}_{im}\right)$$
$$\times \Theta\left(-\left(\sum_{\substack{i,j \\ k \le i < j \le m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im}\right)\left(\sum_{\substack{i,j \\ k \le i < j \le m-1}} \hat{\alpha}_{ij}\right)\right) \,. \tag{2.47}$$

The case where the scaling configuration involves charges $\hat{\alpha}'_2, \cdots \hat{\alpha}'_m$ requires a special treatment. If we naively consider this as a special case of (2.46) above with $k = 2$ or of (2.45) for $m = 4$, we would conclude that the jump vanishes since there are no scaling solution with two centers and hence $G(\hat{\alpha}'_1, \hat{\alpha}'_2 + \cdots \hat{\alpha}'_m)$ vanishes. However notice that in this case (2.41) with $k = 2$ or (2.39) for $m = 4$ implies

$$\sum_{j=2}^{m} \hat{\alpha}'_{1j} = 0 \tag{2.48}$$

and hence the two centered configuration with one center with charge $\hat{\alpha}'_1$ and the other center with charge $\hat{\alpha}'_2 + \cdots \hat{\alpha}'_m$ is on a wall of threshold stability.[10] As such configurations

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[10]Recall that a wall of threshold stability is one on which the bound state can separate into two components with vanishing DSZ product. Across this wall the topology of the bound state changes but the index does not jump.

exist for vanishing FI parameters, we need to analyze the situation more carefully by working at $\mu = \mu_B + \delta\mu$ where μ_B is the critical value of μ at which eq.(2.41) is satisfied for $k = 2$. At this point we have

$$\sum_{j=2}^{m} \hat{\alpha}'_{1j} = \begin{cases} \delta\mu\,\hat{\alpha}_{1m} & \text{for } m > 4 \\ -\delta\mu\,(\hat{\alpha}_{24} + \hat{\alpha}_{34}) & \text{for } m = 4 \end{cases}$$

$$\sum_{\substack{i,j \\ 2 \le i < j \le m}} \hat{\alpha}'_{ij} = \begin{cases} -\delta\mu\,\hat{\alpha}_{1m} & \text{for } m > 4 \\ \delta\mu\,(\hat{\alpha}_{24} + \hat{\alpha}_{34}) & \text{for } m = 4 \end{cases}. \tag{2.49}$$

In either case we can proceed to analyze the system following a similar kind of analysis used in computing ΔF_A. We denote the locations of the centers as z_1, $z_i = z_2 + y\,w_i$ for $2 \le i \le m$ with $w_2 \equiv 0$, $w_m \equiv 1$, and look for solutions with $y \sim \delta\mu$. The solution for z_2 and y are found by extremizing

$$\widehat{W_1} \equiv -\sum_{i=2}^{m} \hat{\alpha}'_{1i} \ln(z_2 - z_1) - \frac{y}{z_2 - z_1} \sum_{i=3}^{m} \hat{\alpha}'_{1i} w_i - \ln y \sum_{\substack{i,j \\ 2 \le i < j \le m}} \hat{\alpha}'_{ij}, \tag{2.50}$$

with respect to z_2 and y, and the w_i's are given by extremizing

$$\widehat{W_2} = - \sum_{\substack{s,r \\ 2 \le s < r \le m}} \hat{\alpha}'_{sr} \ln(w_r - w_s), \tag{2.51}$$

with respect to $w_3, \cdots w_{m-1}$. The extremization of $\widehat{W_2}$ with respect to all the w_k and the sign of the corresponding Hessian gives $G(\hat{\alpha}'_2, \cdots \hat{\alpha}'_m)$. On the other hand the extremization with respect to z_2 and y gives identical conditions[11]

$$-\frac{1}{z_2 - z_1} \sum_{i=3}^{m} \hat{\alpha}'_{1i} w_i - \frac{1}{y} \sum_{\substack{i,j \\ 2 \le i < j \le m}} \hat{\alpha}'_{ij} = 0. \tag{2.52}$$

For small $\sum_{\substack{i,j \\ 2 \le i < j \le m}} \tilde{\alpha}'_{ij}$, a solution for small positive y exists for only one sign of $\sum_{\substack{i,j \\ 2 \le i < j \le m}} \tilde{\alpha}'_{ij}$. The corresponding contribution to the sign of the Hessian can be found by taking the second derivative of W either with respect to y or z_2 keeping the other variable fixed, and is given by a multiplicative factor of

$$\text{sign}\left(\sum_{\substack{i,j \\ 2 \le i < j \le m}} \hat{\alpha}'_{ij} \right). \tag{2.53}$$

Following a logic similar to that for ΔF_A and using (2.49) we find that at this critical value, the scaling index G jumps by

$$\Delta G_B = \Theta\left(-\left(\sum_{\substack{i,j \\ 2 \le i < j \le m}} \hat{\alpha}_{ij} \right) \left(\sum_{\substack{i,j \\ 2 \le i < j \le m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im} \right) \right)$$

$$\times \text{sign}(-\hat{\alpha}_{1m})\,G(\hat{\alpha}'_2, \cdots \hat{\alpha}'_m) \quad \text{for } m > 4$$

$$= \Theta\left(-\hat{\alpha}_{23}\,(\hat{\alpha}_{23} + \hat{\alpha}_{34} + \hat{\alpha}_{24}) \right) \times \text{sign}(\hat{\alpha}_{24} + \hat{\alpha}_{34})\,G(\hat{\alpha}'_2, \cdots \hat{\alpha}'_4) \quad \text{for } m = 4. \tag{2.54}$$

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[11]This can be traced to the fact that using scale invariance we can fix either z_2 or y.

This gives a recursive procedure for calculating the scaling index $G(\hat{\alpha}_1, \cdots \hat{\alpha}_m)$, and therefore the total index F using (2.20), (2.32). The recursion is initialized by the result for three centers, given below.

2.3.3 Coulomb index for 3 and 4 centers

As a simple application of the procedure described above we shall calculate the Coulomb index for 3 and 4 centers. For 3 centers, collinear scaling solutions exist for

$$\text{sign}(\hat{\alpha}_{12}) = \text{sign}(\hat{\alpha}_{23}), \quad \hat{\alpha}_{13} = -\hat{\alpha}_{12} - \hat{\alpha}_{23}, \tag{2.55}$$

and the sign of the Hessian of $\widehat{W}$ is $(-1)^{\Theta(\hat{\alpha}_{23})+1}$. Thus

$$G(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \Theta(\hat{\alpha}_{12}\hat{\alpha}_{23})\,(-1)^{\Theta(\hat{\alpha}_{23})+1}. \tag{2.56}$$

The total index given by (2.20), (2.32), is

$$\begin{aligned}
F(\{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3\}; \{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\}) =\;& (-1)^{\Theta(-\tilde{\alpha}_{12})+\Theta(-\tilde{\alpha}_{23})}\,\Theta(\tilde{\alpha}_{12}\,\tilde{c}_1)\,\Theta(\tilde{\alpha}_{23}(\tilde{c}_1 + \tilde{c}_2)) \\
&+ (-1)^{\Theta(\tilde{\alpha}_{12})+\Theta(\tilde{\alpha}_{13})}\Theta(\tilde{\alpha}_{12}\tilde{\alpha}_{23})\Theta(-(\tilde{\alpha}_{12}+\tilde{\alpha}_{23})(\tilde{\alpha}_{12}+\tilde{\alpha}_{23}+\tilde{\alpha}_{13})),
\end{aligned} \tag{2.57}$$

where the first line corresponds to F_0 in (2.9) and the second line to the contribution of the scaling solution occurring at $\lambda = -(\tilde{\alpha}_{12}+\tilde{\alpha}_{23})/\tilde{\alpha}_{13}$. It is straightforward, if tedious, to check that (2.57) agrees with the result given in [14] in a particular chamber. The result (2.57) can be succintly summarized by saying that F vanishes unless the sign of the 5-periodic sequence

$$\Sigma_{123} = \{\tilde{c}_1 + \tilde{c}_2, \tilde{c}_1, \tilde{\alpha}_{23}, \tilde{\alpha}_{12} + \tilde{\alpha}_{23} + \tilde{\alpha}_{13}, \tilde{\alpha}_{12}\} \tag{2.58}$$

is either constant (in which case $F = 1$), or flips 4 times around the sequence (in which case $F = -1$). These signs correspond to the behavior of the superpotential W at the 5 boundaries $y_2 = \infty, y_1 = \infty, y_2 = 0, y_1 = y_2 = 0, y_1 = 0$ of the domain in which the variables y_1, y_2 take values. The rule (2.58) is in agreement with the existence of a gradient flow emanating from a critical point of W inside this domain (see figure 1, left).

For $n = 4$, we first need to compute the scaling index $G(\hat{\alpha}_1, \cdots \hat{\alpha}_4)$. In this case the only contribution to ΔG_B in (2.38) comes from the configuration where the centers 2, 3 and 4 come together during the deformation. Using (2.54), (2.56) we can then express (2.38) as

$$\begin{aligned}
G(\hat{\alpha}_1, \cdots \hat{\alpha}_4) =\;& (-1)^{\Theta(\hat{\alpha}_{23})+\Theta(\hat{\alpha}_{34})}\,\Theta(\hat{\alpha}_{12}\hat{\alpha}_{23})\,\Theta(-\hat{\alpha}_{34}(\hat{\alpha}_{14}+\hat{\alpha}_{24}+\hat{\alpha}_{34})) \\
&+ (-1)^{\Theta(\hat{\alpha}_{23})+\Theta(\hat{\alpha}_{24}+\hat{\alpha}_{34})}\,\Theta(\hat{\alpha}_{23}\hat{\alpha}_{34})\,\Theta(-\hat{\alpha}_{23}(\hat{\alpha}_{23}+\hat{\alpha}_{24}+\hat{\alpha}_{34})).
\end{aligned} \tag{2.59}$$

The rule (2.59) can be summarized by saying that the sign of the 5-periodic sequence

$$\Sigma_{1234} = \{\hat{\alpha}_{12}, \hat{\alpha}_{34}, \hat{\alpha}_{23} + \hat{\alpha}_{34} + \hat{\alpha}_{24}, \hat{\alpha}_{23}, \hat{\alpha}_{12} + \hat{\alpha}_{23} + \hat{\alpha}_{13}\} \tag{2.60}$$

is either constant (in which case $G = 1$) or alternates 4 times (in which case $G = -1$). These signs correspond to the behavior of the superpotential $\hat{W}$ at the 5 boundaries $z_2 = 0$, $z_3 = 1$, $z_2 = z_3 = 1$, $z_2 = z_3$, $z_2 = z_3 = 0$ of the domain in which the variables z_2, z_3 are valued (in the gauge $z_1 = 0, z_4 = 1$). The rule (2.59) is in agreement with the existence of a gradient flow emanating from a critical point of $\hat{W}$ inside this domain (see figure 1, right).
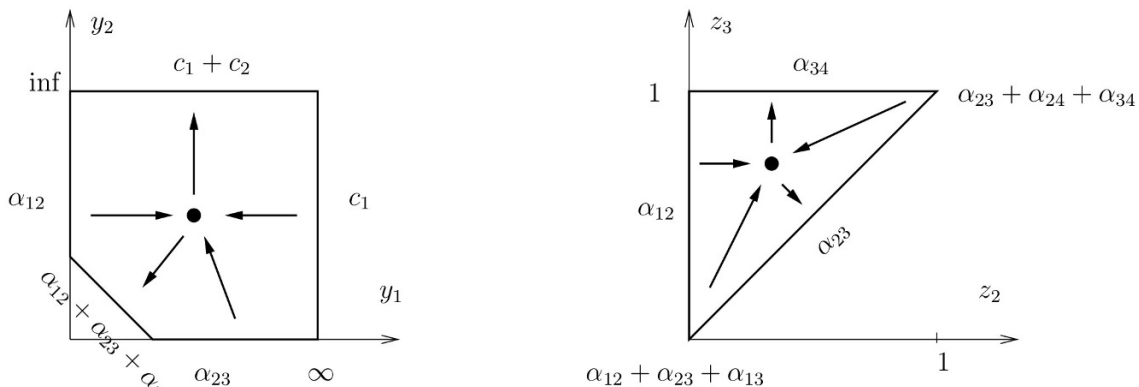
Figure 1. Left: the physical domain for 3-center collinear solutions has 5 boundary components at which the superpotential W diverges. The sign of W on each component is that of the quantity indicated on the corresponding edge. Right: the physical domain for 4-center collinear scaling solutions also has 5 boundary components, at which the superpotential $\hat{W}$ diverges. The sign of W on each component is that of the linear combination of $\hat{\alpha}_{ij}$ indicated on the corresponding edge or vertex. In both cases, by considering the topology of the gradient flow (indicated by the arrows for some suitable choice of signs on the boundary components), it is easy to convince oneself that a critical point exists in the physical domain if and only if the signs on the 5 boundary components are identical, or flip 4 times around the boundary.

Using (2.20) we can now compute $F(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_4\}, \{\tilde{c}_1, \cdots \tilde{c}_4\}))$ as a sum of four terms: $F_0(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_4\}, \{\tilde{c}_1, \cdots \tilde{c}_4\}))$ given in (2.9), and the jumps across the values of μ where all the centers come together, where the centers 2,3,4 come together, and where the centers 1,2,3 come together. The final result takes the form

$$F\left(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_4\}, \{\tilde{c}_1, \cdots \tilde{c}_4\}\right) = \prod_{k=1}^{3} \Theta(\tilde{\alpha}_{k,k+1}\,\tilde{d}_k)(-1)^{\sum_{k=1}^{n-1} \Theta(-\tilde{\alpha}_{k,k+1})}$$

$$+ (-1)^{\Theta(\tilde{\alpha}_{13}+\tilde{\alpha}_{14}+\tilde{\alpha}_{24})+1} G\left(\tilde{\alpha}_1^{(1)}, \cdots \tilde{\alpha}_4^{(1)}\right) \Theta\left(-\left(\sum_{1 \leq i < j \leq 4} \tilde{\alpha}_{ij}\right)(\tilde{\alpha}_{12}+\tilde{\alpha}_{23}+\tilde{\alpha}_{34})\right)$$

$$+ (-1)^{\Theta(\tilde{\alpha}_{24})+1} F\left(\{\tilde{\alpha}_1^{(2)}, \tilde{\alpha}_2^{(2)}+\tilde{\alpha}_3^{(2)}+\tilde{\alpha}_4^{(2)}\}, \{\tilde{c}_1, \tilde{c}_2+\tilde{c}_3+\tilde{c}_4\}\right) G\left(\tilde{\alpha}_2^{(2)}, \tilde{\alpha}_3^{(2)}, \tilde{\alpha}_4^{(2)}\right)$$

$$\times \Theta(-(\tilde{\alpha}_{23}+\tilde{\alpha}_{34})(\tilde{\alpha}_{23}+\tilde{\alpha}_{34}+\tilde{\alpha}_{24}))$$

$$+ (-1)^{\Theta(\tilde{\alpha}_{13})+1} F\left(\{\tilde{\alpha}_1^{(3)}+\tilde{\alpha}_2^{(3)}+\tilde{\alpha}_3^{(3)}, \tilde{\alpha}_4^{(3)}\}, \{\tilde{c}_1+\tilde{c}_2+\tilde{c}_3, \tilde{c}_4\}\right) G\left(\tilde{\alpha}_1^{(3)}, \tilde{\alpha}_2^{(3)}, \tilde{\alpha}_3^{(3)}\right)$$

$$\times \Theta\left(-(\tilde{\alpha}_{12}+\tilde{\alpha}_{23})(\tilde{\alpha}_{12}+\tilde{\alpha}_{23}+\tilde{\alpha}_{13})\right), \tag{2.61}$$

where

$$\tilde{\alpha}_{ij}^{(1)} = \lambda_1 \tilde{\alpha}_{ij} \quad \text{for } |i-j| \leq 2, \, 1 \leq i,j \leq 4, \quad \tilde{\alpha}_{ij}^{(1)} = \tilde{\alpha}_{ij} \quad \text{otherwise},$$

$$\tilde{\alpha}_{24}^{(2)} = \lambda_2 \tilde{\alpha}_{24}, \quad \tilde{\alpha}_{ij}^{(2)} = \tilde{\alpha}_{ij} \quad \text{otherwise},$$

$$\tilde{\alpha}_{13}^{(3)} = \lambda_3 \tilde{\alpha}_{13}, \quad \tilde{\alpha}_{ij}^{(3)} = \tilde{\alpha}_{ij} \quad \text{otherwise}, \tag{2.62}$$

$$\lambda_1 = -\frac{\tilde{\alpha}_{12}+\tilde{\alpha}_{23}+\tilde{\alpha}_{34}}{\tilde{\alpha}_{14}+\tilde{\alpha}_{24}+\tilde{\alpha}_{13}}, \quad \lambda_2 = -\frac{\tilde{\alpha}_{23}+\tilde{\alpha}_{34}}{\tilde{\alpha}_{24}}, \quad \lambda_3 = -\frac{\tilde{\alpha}_{12}+\tilde{\alpha}_{23}}{\tilde{\alpha}_{13}}. \tag{2.63}$$

This can be easily generalized to higher number of centers.

3 Quiver invariants

In this section we shall describe how the results of the previous sections can be used to give a complete prescription for computing the Poincaré-Laurent polynomial of quiver moduli spaces. For this we need to briefly review the prescription given in [14].

3.1 Quiver Poincaré-Laurent polynomial from Coulomb index: a review

We shall consider a quiver with K nodes with a $\mathrm{U}(N_\ell)$ factor at the ℓ-th node, $\gamma_{\ell k}$ arrows from the ℓ-th node to the k-th node representing $\gamma_{\ell k}$ number of $(N_\ell, \bar{N}_k)$ representations of $\mathrm{U}(N_\ell) \times \mathrm{U}(N_k)$ and FI parameters $\zeta_1, \cdots \zeta_K$ satisfying $\sum_\ell N_\ell \zeta_\ell = 0$. A negative $\gamma_{\ell k}$ indicates $\gamma_{k\ell} \equiv -\gamma_{\ell k}$ number of $(\bar{N}_\ell, N_k)$ representations of $\mathrm{U}(N_\ell) \times \mathrm{U}(N_k)$. Instead of considering one specific quiver at a time it turns out to be more convenient to consider the family of quivers labelled by different ranks $\{N_\ell\}$ and different values of FI parameters $\{\zeta_\ell\}$. For this we assign to each node ℓ a basis vector $\gamma_\ell = (0, \ldots, 0, 1, 0, \ldots 0)$ — with 1 inserted at the ℓ-th position — in an abstract vector space $\mathbb{Z}^K$, denote by $\Gamma \subset \mathbb{Z}^K$ the collection of vectors $\gamma = \sum_{\ell=1}^K N_\ell \gamma_\ell$ where N_ℓ are non-negative integers, and by $\mathcal{C}_\gamma$ the hyperplane $\sum_{\ell=1}^K N_\ell \zeta_\ell = 0$ in the space of real vectors $\zeta = \sum_{\ell=1}^K \zeta_\ell \gamma_\ell \in \mathbb{R}^K$. We also introduce a symplectic inner product

$$\langle \gamma, \gamma' \rangle \equiv \sum_{\ell,k=1}^K N_\ell N_k' \, \gamma_{\ell k}, \tag{3.1}$$

between the elements $\gamma = \sum_{\ell=1}^K N_\ell \gamma_\ell$ of Γ. To any vector $\gamma \in \Gamma$ and $\zeta \in \mathcal{C}_\gamma$, we associate a quiver $\mathcal{Q}(\gamma, \zeta)$ with K nodes, $\gamma_{\ell k}$ arrows connecting the node ℓ to the node k, gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \mathrm{U}(N_K)$, and FI parameters $\{\zeta_1, \cdots \zeta_K\}$. If some of the N_ℓ's vanish we just drop the corresponding nodes.

Let $Q(\gamma; \zeta; y)$ be the Poincaré-Laurent polynomial

$$Q(\gamma; \zeta; y) = \sum_{p=1}^{2d} b_p(\mathcal{M}) \, (-y)^{p-d} \tag{3.2}$$

where d is the complex dimension of the moduli space $\mathcal{M}$ of the quiver $\mathcal{Q}(\gamma; \zeta)$ and the b_p's are the topological Betti numbers of $\mathcal{M}$. The Coulomb branch formula for $Q(\gamma; \zeta; y)$, which we denote by $Q_{\mathrm{Coulomb}}(\gamma; \zeta; y)$, takes the form:

$$Q_{\mathrm{Coulomb}}(\gamma; \zeta; y) = \sum_{m|\gamma} \frac{\mu(m)}{m} \frac{y - y^{-1}}{y^m - y^{-m}} \bar{Q}_{\mathrm{Coulomb}}(\gamma/m; \zeta; y^m)$$

$$\bar{Q}_{\mathrm{Coulomb}}(\gamma; \zeta; y) = \sum_{n \geq 1} \sum_{\substack{\{\alpha_i \in \Gamma\} \\ \sum_{i=1}^n \alpha_i = \gamma}} \frac{g_{\mathrm{Coulomb}}\left(\{\alpha_1, \cdots, \alpha_n\}, \{c_1, \cdots c_n\}; y\right)}{|\mathrm{Aut}(\{\alpha_1, \cdots, \alpha_n\})|}$$

$$\times \prod_{i=1}^n \left(\sum_{\substack{m_i \in \mathbb{Z} \\ m_i | \alpha_i}} \frac{1}{m_i} \frac{y - y^{-1}}{y^{m_i} - y^{-m_i}} \Omega_{\mathrm{tot}}(\alpha_i/m_i; y^{m_i}) \right). \tag{3.3}$$

The first line is the standard relation between integer and rational BPS invariants which has appeared in a variety of contexts [10, 19, 29, 30]. $\mu(m)$ is the Möbius function, which is 1 (-1) if m is a square-free positive integer with an even (odd) number of

prime factors, and 0 if m is not square-free. In the second line, $|\text{Aut}(\{\alpha_1, \cdots \alpha_n\})|$ is a symmetry factor given by $\prod_k s_k!$ if among the set $\{\alpha_i\}$ there are s_1 identical vectors $\tilde{\alpha}_1$, s_2 identical vectors $\tilde{\alpha}_2$ etc., and $m|\alpha$ means that m is a common divisor of $(n_1, \cdots, n_K)$ if $\alpha = \sum_\ell n_\ell \gamma_\ell$. The sums over n and $\{\alpha_1, \cdots \alpha_n\}$ in the second equation label all possible ways of expressing γ as (unordered) sums of elements α_i of Γ. The coefficients c_i are determined in terms of the FI parameters ζ_i by $c_i = \sum_\ell A_{i\ell} \zeta_\ell$ whenever $\alpha_i = \sum_\ell A_{i\ell} \gamma_\ell$. From the restrictions $\sum_i \alpha_i = \gamma$ and $\sum_\ell N_\ell \zeta_\ell = 0$ it follows that $\sum_i c_i = 0$. The Coulomb indices $g_{\text{Coulomb}}(\{\alpha_1, \cdots, \alpha_n\}; \{c_1, \cdots c_n\}; y)$ can be computed from (2.33). The functions $\Omega_{\text{tot}}(\alpha; y)$ are expressed in terms of the single-centered BPS invariants Ω_S through

$$\Omega_{\text{tot}}(\alpha; y) = \Omega_S(\alpha; y) + \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i \Omega_S(\beta_i; y^{m_i}). \tag{3.4}$$

Finally, the functions $H(\{\beta_i\}; \{k_i\}; y)$ and $\Omega_S(\gamma; y)$ are determined as follows.

1. When the number of β_i's is less that three, $H(\{\beta_i\}; \{k_i\}; y)$ vanishes.

2. For three or more number of β_i's, observe that the expression for $Q_{\text{Coulomb}}(\sum_i k_i \beta_i; \zeta; y)$ given in (3.3) contains a term proportional to $H(\{\beta_i\}; \{k_i\}; y) \prod_i \Omega_S(\beta_i; y^{k_i})$ arising from the choice $m = 1$ in the first equation in (3.3), $n = 1$, $\alpha_1 = \sum_i k_i \beta_i$, $m_1 = 1$ in the second equation in (3.3), and $m_i = k_i$ in the expression for $\Omega_{\text{tot}}(\sum_i k_i \beta_i; y)$ in eq.(3.4). We fix $H(\{\beta_i\}; \{k_i\}; y)$ by demanding that the net coefficient of the product $\prod_i \Omega_S(\beta_i; y^{k_i})$ in the expression for $Q_{\text{Coulomb}}(\sum_i k_i \beta_i; y)$ is a Laurent polynomial in y. This of course leaves open the possibility of adding to H a Laurent polynomial. This is resolved by using the *minimal modification hypothesis*, which requires that H must be symmetric under $y \to y^{-1}$ and vanish as $y \to \infty$ [11]. We determine $H(\{\beta_i\}; \{m_i\}; y)$ iteratively by beginning with the H's with three β_i's and then determining successively the H's with more β_i's.

3. H is expected to be independent of the FI parameters and hence can be calculated for any value of these parameters.

4. *After determining* $H(\{\beta_i\}; \{k_i\}; y)$ *in this way, we set* $\Omega_S(\gamma_\ell; y) = 1$ *for* $1 \leq \ell \leq K$. For all other charge vectors β, $\Omega_S(\beta; y)$ are fixed integers, independent of y and of the FI parameters, which are left undetermined by the Coulomb branch analysis. Since these unknown constants, as well as the quivers, are labelled by the vectors $\alpha \in \Gamma$, there is one[12] unknown constant for each quiver. This can be fixed e.g. by computing the Euler character of the quiver moduli space for any convenient value of the FI parameters.

As a special case of our result we can consider the case of a general Abelian quiver. This corresponds to $\gamma = \sum_\ell N_\ell \gamma_\ell$ with $N_\ell = 0$ or 1. As a result γ, as well as the α_i's

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[12]Actually the number of unknown constants is less than that of the number of quivers since $\Omega_S(\gamma)$ is non-trivial only if there exists a set of α_i's in Γ such that $\sum_i \alpha_i = \gamma$ and it is possible to find three dimensional vectors $\vec{r}_i$ such that $\sum_{i,j} \alpha_{ij} (\vec{r}_i - \vec{r}_j)/|\vec{r}_i - \vec{r}_j| = 0$.

appearing on the right hand side of (3.3), are primitive vectors and the α_i's are all distinct. Thus (3.3), (3.4) simplifies to

$$Q_{\text{Coulomb}}(\gamma;\zeta;y) = \sum_{\substack{n \geq 1 \\ \sum_{i=1}^n \alpha_i = \gamma}} \sum_{\{\alpha_i \in \Gamma\}} g_{\text{Coulomb}}\left(\{\alpha_1,\cdots,\alpha_n\},\{c_1,\cdots c_n\};y\right) \prod_{i=1}^n \Omega_{\text{tot}}(\alpha_i;y), \quad (3.5)$$

$$\Omega_{\text{tot}}(\alpha;y) = \Omega_{\text{S}}(\alpha;y) + \sum_{\{\beta_i \in \Gamma\},\sum_i \beta_i = \alpha} H(\{\beta_i\};y) \prod_i \Omega_{\text{S}}(\beta_i;y), \quad (3.6)$$

where $H(\{\beta_i\};y) \equiv H(\{\beta_i\};\{1,1,\cdots 1\};y)$. The functions $H(\{\beta_i\};y)$ are determined by requiring that they vanish as $y \to 0, \infty$, are invariant under $y \to y^{-1}$, and that the coefficient of $\prod_i \Omega_{\text{S}}(\alpha_i;y)$ in the expression for $Q_{\text{Coulomb}}(\gamma;\zeta;y)$ is a positive integer for each set $\{\alpha_i\}$.

3.2 Coulomb index for non-generic charges

The formulæ (3.3), (3.4) are completely explicit provided we have an explicit algorithm for computing $g_{\text{Coulomb}}(\{\alpha_1,\cdots\alpha_n\};\{c_1,\cdots c_n\};y)$. We have given such an algorithm in the previous sections for generic α_{ij}'s, e.g. all α_{ij}'s non-zero and no ordered subset $\bar{\alpha}_1,\cdots\bar{\alpha}_s$ of the α_i's satisfying $\sum_{1 \leq k < \ell \leq s} \bar{\alpha}_{k\ell} = 0$. We have also assumed that the FI parameters stay away from the walls of marginal and threshold stability so that e.g. the quantities $\sum_{i=1}^k c_{\sigma(i)}$ appearing in (2.10) never vanish. However we need g_{Coulomb} for non-generic α_{ij}'s and c_i's as well. These come from two sources. First of all the $\gamma_{\ell k}$'s of the original quiver themselves may be non-generic with some $\gamma_{\ell k}$'s vanishing or satisfying special relations. Second, even if the original $\gamma_{\ell k}$'s are generic, in the argument of g_{Coulomb} we may have parallel α_i's. For these the corresponding α_{ij}'s will vanish. Also when the total dimension vector $\{N_1,\cdots N_K\}$ is non-primitive, we shall encounter g_{Coulomb} in (3.3) for which the FI parameters sit on the threshold stability walls. In all such cases we need to evaluate g_{Coulomb} by first deforming the α_{ij}'s and/or c_i's to generic values and then taking the limit back to the original configuration. The goal of this section will be to determine a prescription for such deformations. We shall first describe the prescription and then justify it.

1. To deal with the first problem we deform the $\gamma_{\ell k}$'s to

$$\gamma_{\ell k} \to \gamma_{\ell k} + \epsilon_1 \xi_{\ell k} \quad (3.7)$$

   where ϵ_1 is a small positive number and $\xi_{\ell k}$'s are random numbers between -1 and 1 satisfying $\xi_{\ell k} = -\xi_{k\ell}$. This will make all the $\gamma_{\ell k}$ generic.[13]

2. At this stage in any given term in the sum in (3.3), all the α_{ij}'s are generic except for subsets of α_i's which are all parallel and/or equal. In computing $g_{\text{Coulomb}}(\{\alpha_i\},\{c_i\};y)$ for a set of α_i's we consider an arbitrary ordering[14] of all the

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[13]In order to increase the efficiency of the procedure, we can arrange the nodes in some fixed order and then choose the $\xi_{\ell k}$'s such that $\xi_{\ell k} > 0$ for $\ell < k$. This will minimize the introduction of new oriented loops and hence scaling configurations during this deformation. For example if we have three nodes i, j and k such that $\alpha_{ij} = \alpha_{jk} = \alpha_{ik} = 0$, then under the deformation (3.7) the subquiver containing the nodes i, j and k will not have any oriented loop.

[14]Here we are considering the ordering as a set and not an ordering of the locations of the centers. The same deformation must be used for all possible arrangements of the centers.

α_i's and deform them by

$$\alpha_{ij} \to \alpha_{ij} + \epsilon_2 \, \beta_{ij}, \qquad c_i \to c_i + \epsilon_2 f_i \,, \tag{3.8}$$

where ϵ_2 is a small positive number that is parametrically smaller than the previous paramater ϵ_1, β_{ij}'s for $i < j$ are randomly chosen *positive* numbers between 0 and 1 with $\beta_{ji} = -\beta_{ij}$, and f_i's are random numbers satisfying $\sum_i f_i = 0$. Under such a deformation any subset of the $\{\alpha_i\}$'s which are parallel and/or equal to each other get deformed in such a way that the corresponding subquiver does not contain any oriented loop. Also the FI parameters move away from threshold stability walls even if the undeformed configuration sits on such a wall.

3. At the end of the second step the α_{ij}'s and c_i's are generic and can be used to compute g_{Coulomb}. In particular the $s(\sigma)$ factors in g_{Coulomb} are computed using the deformed α_{ij}'s and c_i's. However in computing the $y^{\sum_{i<j} \tilde{\alpha}_{ij}}$ factors in g_{Coulomb} we use the undeformed α_{ij}'s, since at the end of the computation we are in any case supposed to take the α_{ij}'s to their undeformed values. This g_{Coulomb} is then used to compute $Q_{\mathrm{Coulomb}}(\gamma; \zeta; y)$ via (3.3).

In order to prove the validity of the procedure we need to argue that the deformed result reduces to the undeformed one in the limit when the deformations are switched off. First note that there is a qualitative difference between the deformations generated by ϵ_1 and those generated by ϵ_2. For the latter the deformation of the α_{ij}'s and c_i's we use is specific to the α_i's and c_i's which appear in the argument of a given g_{Coulomb}. As a result we need to establish that each g_{Coulomb} returns to its undeformed value upon switching off this deformation. On the other hand, the deformation generated by ϵ_1 can be carried out for the full index $Q_{\mathrm{Coulomb}}(\gamma; \zeta; y)$ as it applies to the whole family of quivers and does not refer to any specific multi-centered black hole configuration. We shall indeed argue that while the individual g_{Coulomb}'s in the ϵ_1 deformed system do not necessarily reduce to the undeformed result, the total index does.

Let us begin with the ϵ_2 deformation. This contains two parts: deformation of the α_{ij}'s and deformation of the c_i's. First consider the effect of deforming the c_i's. If the initial configuration is away from the walls of marginal and threshold stability then this deformation has no effect. However when $\gamma = \sum_i \alpha_i$ is not primitve, it could happen that the set $\{\alpha_1, \cdots \alpha_n\}$ can be divided into two or more sets such that the sum of the α_i's in each set is parallel to the total charge γ. In this case the sum of the c_i's in each set vanishes and the FI parameters sit on the wall of threshold stability. The deformation of the c_i's given in (3.8) is needed to move away from this wall and make g_{Coulomb} well defined, but the result does not depend on how we deform the c_i's.

Next we turn the effect of the ϵ_2 deformation on the α_{ij}'s. We begin with the deformed system and take the α_{ij}'s one by one back to their values after the first deformation. During this process g_{Coulomb} can jump if two or more centers come together during the deformation. Using the analysis of section 2.2 we know that the possible jumps in g_{Coulomb} could arise if during the deformation a subset A of the centers can come in the collinear scaling configuration by having $\sum_{i<j;i,j\in A} \alpha'_{ij} = 0$. Since at this stage we have already

carried out the ϵ_1 deformation making the $\gamma_{\ell k}$'s generic; the possible subsets where this could happen will only involve the centers carrying equal or parallel charges at the end of the first deformation. However this is ruled out by the fact that the second deformation has been choosen so that any subquiver, containing equal or parallel charges at the end of the first deformation, remains free from oriented loops. This shows that we do not encounter any collinear scaling solutions during the second deformation and hence there is no jump in the index g_{Coulomb} during this deformation.

Finally we turn to the ϵ_1 deformation. To deal with this case we note that the analysis of section 2.2, showing that the refined index of a quiver changes continuously under deformations of the α_{ij}'s, breaks down on a subspace on which $\sum_{i,j \in A; i<j} \tilde\alpha_{ij} = 0$ for some subset A. Around this subspace the Coulomb index computed for a given set of charges will depend on the sign of the deformation parameters since a particular collinear solution may exist for one sign of the deformation, but as the deformation parameter approaches zero the centers in the subset A come close together and for the opposite sign of the deformation parameter the solution ceases to exist. However we shall now argue that $Q_{\mathrm{Coulomb}}(\gamma; \zeta; y)$ computed from (3.4) remains independent of the sign of the deformation. For this suppose that we have three centers carrying charges α_1, α_2 and α_3 such that $\alpha_{12} + \alpha_{23} + \alpha_{13} = 0$, $\alpha_{12}, \alpha_{23} > 0$. In that case for the permutation (123) we can have a collinear scaling solution. Now in the deformed system whether this permutation contributes or not depends on whether α_{12}, α_{23} and α_{31} form an oriented triangle or not, and this in turn will depend on the details of the deformation. The difference in $g_{\mathrm{Coulomb}}(\{\alpha_1, \alpha_2, \alpha_3\}; \{c_1, c_2, c_3\}; y)$ that we shall get between these two cases is, up to a sign, given by $(y - y^{-1})^{-2}$ since $y^{\sum_{i<j} \alpha_{ij}} = 1$. In the expression for $Q_{\mathrm{Coulomb}}(\alpha_1 + \alpha_2 + \alpha_3; \zeta; y)$ there will be a term proportional to $g_{\mathrm{Coulomb}}(\{\alpha_1, \alpha_2, \alpha_3\}; \{c_1, c_2, c_3\}; y)\Omega_{\mathrm{S}}(\alpha_1)\Omega_{\mathrm{S}}(\alpha_2)\Omega_{\mathrm{S}}(\alpha_3)$ and the ambiguity in g_{Coulomb} described above will lead to an ambiguity proportional to $(y - y^{-1})^{-2}\Omega_{\mathrm{S}}(\alpha_1)\Omega_{\mathrm{S}}(\alpha_2)\Omega_{\mathrm{S}}(\alpha_3)$ in the expression for $Q_{\mathrm{Coulomb}}(\alpha_1 + \alpha_2 + \alpha_3; \zeta; y)$. Now the expression for $Q_{\mathrm{Coulomb}}(\alpha_1 + \alpha_2 + \alpha_3; \zeta; y)$ will also contain a term proportional to $H(\{\alpha_1, \alpha_2, \alpha_3\}; y)\Omega_{\mathrm{S}}(\alpha_1)\Omega_{\mathrm{S}}(\alpha_2)\Omega_{\mathrm{S}}(\alpha_3)$, and the function $H(\{\alpha_1, \alpha_2, \alpha_3\}; y)$ is determined by requiring that H vanishes as $y \to 0, \infty$ and that the net coefficient of $\Omega_{\mathrm{S}}(\alpha_1)\Omega_{\mathrm{S}}(\alpha_2)\Omega_{\mathrm{S}}(\alpha_3)$ is a polynomial in y, y^{-1}. Since $(y - y^{-1})^{-2} \to 0$ as $y \to 0, \infty$ we see that the ambiguity in g_{Coulomb} introduced above is absorbed completely into the function H and does not lead to any ambiguity in the expression for $Q_{\mathrm{Coulomb}}(\alpha_1, \alpha_2, \alpha_3; \zeta; y)$. This argument can be easily generalized to argue that all the ambiguities in g_{Coulomb} can be absorbed into the functions H and the expression for $Q_{\mathrm{Coulomb}}(\gamma; \zeta; y)$ is independent of the choice of the deformation of the $\gamma_{\ell k}$'s for general vector γ.

Using the analysis above we can also answer the question: how small should ϵ_1 and ϵ_2 be for the procedure described above to hold? The general rule is that if α'_{ij} denotes the deformed α_{ij}, then during the deformation we should not have $\sum_{i<j; i,j \in A} \alpha'_{ij} = 0$ for any ordered subset A of the centers. This means that we should not encounter any collinear scaling configurations during the deformation except possibly at the beginning.[15]

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[15]This would happen if the initial configuration had $\sum_{i<j; i,j \in A} \alpha_{ij} = 0$ for some ordered subset A of the centers.

4 Coulomb/Higgs equivalence for quivers without loops

For quivers without oriented loops, quiver invariants can be computed using the Harder-Narasimhan recursion, or equivalently using Reineke's solution to this recursion [15]. In this section we shall show that for such quivers the Coulomb branch formula (3.3) agrees with Reineke's formula, both for Abelian and non-Abelian quivers.

4.1 Reineke's formula for quivers without loops

As reviewed in [14], the Poincaré-Laurent polynomial of a quiver $\mathcal{Q}$ without oriented loops can be computed using the Harder-Narasimhan recursion. Henceforth we shall denote the Poincaré-Laurent polynomial computed by this method by $Q_{\text{Higgs}}(\sum_\ell N_\ell \gamma_\ell; \zeta; y)$, to distinguish it from the Coulomb branch formula (3.3). The expression for Q_{Higgs} takes the form [12], [16, Theorem 6.8]

$$Q_{\text{Higgs}}(\gamma; \zeta; y) = \sum_{m|\gamma} \frac{\mu(m)}{m} \frac{y - y^{-1}}{y^m - y^{-m}} \bar{Q}_{\text{Higgs}}(\gamma/m; \zeta; y^m)$$

$$\bar{Q}_{\text{Higgs}}\left(\sum_\ell N_\ell \gamma_\ell; \zeta; y\right) = \sum_\ell \sum_{\substack{\{\vec{M}^{(i)}\} \\ \sum_{i=1}^\ell \vec{M}^{(i)} = \vec{N}; \vec{M}^{(i)} \| \vec{N} \text{ for } i=1,\ldots,\ell}} \frac{1}{\ell\,(y - 1/y)^{\ell-1}} \tag{4.1}$$

$$\times \prod_{i=1}^\ell G_{\text{Higgs}}(\{M_1^{(i)}, \cdots M_K^{(i)}\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y)\,.$$

The sum over $\{\vec{M}^{(i)}\}$ runs over all *ordered* partitions of $\vec{N} \equiv (N_1, \cdots N_K)$ into parallel vectors $\vec{M}^{(1)}, \cdots \vec{M}^{(\ell)}$, and $G_{\text{Higgs}}(\{M_1^{(i)}, \cdots M_K^{(i)}\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y)$ is the 'stack invariant' associated to the quiver with dimension vector $\vec{M}^{(i)}$. If $\vec{M}^{(i)}$ is primitive then the quiver moduli space is smooth and G_{Higgs} is just its Poincaré-Laurent polynomial, however in general it is a rational function of y, not necessarily invariant under $y \to 1/y$. In all cases however, we assume that it is given by Reineke's formula [15],[16]

$$G_{\text{Higgs}}(\{M_1, \cdots M_L\}; \{\alpha_1, \cdots \alpha_L\}; \{c_1, \cdots c_L\}; y) = (-y)^{-\sum_{i,j} M_i M_j \max(\alpha_{ij}, 0) - 1 + \sum_i M_i}$$

$$\times (y^2 - 1)^{1 - \sum_i M_i} \sum_{\text{partitions}} (-1)^{s-1} y^{2\sum_{a \le b} \sum_{i,j} \max(\alpha_{ij}, 0) N_i^b N_j^a} \prod_{a,i} ([N_i^a, y]!)^{-1}, \tag{4.2}$$

where the sum over partitions in (4.2) runs over all *ordered* partitions of the vector $(M_1, \cdots M_L)$ into non-zero vectors $\{(N_1^a, \cdots N_L^a), \quad a = 1, \ldots, s\}$ for $s = 1, \ldots, \sum_i M_i$, satisfying $N_i^a \ge 0$, $\sum_a N_i^a = M_i$ and (assuming that $\sum_i M_i c_i = 0$)

$$\sum_{a=1}^b \sum_{i=1}^L N_i^a c_i > 0 \tag{4.3}$$

for all b between 1 and $s - 1$. $[N, y]!$ denotes the q-deformed factorial,

$$[N, y]! \equiv [1, y][2, y] \ldots [N, y], \qquad [N, y] \equiv \frac{y^{2N} - 1}{y^2 - 1}\,. \tag{4.4}$$

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[16]It is related to the quantity $\mathcal{I}(\gamma, w)$ defined in [14], eq. (2.39) via $G_{\text{Higgs}}(\gamma; \zeta; y) = (1/y - y)\,\mathcal{I}(\gamma; -y)$.

It is worth noting that (4.1) can be inverted to express the stack invariants in terms of the rational Poincaré-Laurent polynomials [16, Theorem 6.8]:

$$G_{\text{Higgs}}(\{N_1, \cdots N_K\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y)$$

$$= \sum_k \sum_{\substack{\{\alpha_i\} \\ \sum_{i=1}^k \alpha_i = \sum_\ell N_\ell \gamma_\ell \\ \alpha_i \| \sum N_\ell \gamma_\ell \text{ for } i=1,\ldots,k}} \frac{(-1)^{k-1}}{k!(y-y^{-1})^{k-1}} \prod_{i=1}^k \bar{Q}_{\text{Higgs}}(\alpha_i; \zeta; y), \qquad (4.5)$$

where again the sum over $\{\alpha_i\}$ runs over all *ordered* partition of $\sum_\ell N_\ell \gamma_\ell$ into parallel vectors $\alpha_1, \alpha_2, \cdots \alpha_k$. We shall refer to the formula (4.1) (or equivalently (4.5)) and (4.2) as the Higgs branch formula for the Poincaré-Laurent polynomial $Q(\gamma; \zeta; y)$. Our goal will be to show the equality of this Higgs branch result with the Coulomb branch formula (3.3).

If $\sum_\ell N_\ell \gamma_\ell$ is a primitive vector, ı.e. the N_ℓ's have no common factor other than unity, then (4.1) reduces to a simple form

$$Q_{\text{Higgs}}\left(\sum_{i=1}^K N_i \gamma_i; \zeta; y\right) = G_{\text{Higgs}}(\{N_1, \cdots N_K\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y). \qquad (4.6)$$

In this case G_{Higgs} is a symmetric Laurent polynomial since Q_{Higgs} is.

4.2 Abelian quivers without loops

For quivers with dimension vector $N_i = 1$, the stack invariant G_{Higgs} appearing in (4.6) is given by (4.2) where the integers N_i^a can only be equal to 0 or 1. We shall use a special symbol g_{Higgs} for labelling the corresponding G_{Higgs}:

$$g_{\text{Higgs}}(\{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y) \equiv G_{\text{Higgs}}(\{1, \cdots 1\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y). \quad (4.7)$$

Assuming for the moment that all γ_{ij}'s are non-vanishing, we shall choose a strict ordering convention for the γ_i's such that $\gamma_{ij} > 0$ iff $i < j$. Thus (4.6), (4.2) can be expressed as

$$Q_{\text{Higgs}}(\gamma_1 + \cdots + \gamma_K; \zeta; y) = g_{\text{Higgs}}(\{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y) = \qquad (4.8)$$

$$= (-1)^{-K+1+\sum_{i<j} \gamma_{ij}} (y-y^{-1})^{1-K} \sum_{\text{partitions}} (-1)^{s-1} y^{2\sum_{a \le b} \sum_{j<i} \gamma_{ji} N_i^a N_j^b - \sum_{1 \le i < j \le K} \gamma_{ij}}.$$

where the sum runs over all ordered partitions of $\gamma = \gamma_1 + \cdots + \gamma_K$ into vectors $\beta^{(a)} = \sum_i N_i \gamma_i$ with $N_i^a = 0, 1$, $a = 1, \cdots s$, satisfying

$$\sum_{a=1}^b \sum_{i=1}^n N_i^a \zeta_i > 0. \qquad (4.9)$$

Since (4.8) is continuous at $\gamma_{ij} = 0$, the result when some of the γ_{ij}'s vanish can be obtained as a limit of (4.8). Furthermore, throughout this subsection we shall work with generic FI parameters for which the left hand side of (4.9) never vanishes. In that case (4.8) is also invariant under small deformations of the FI parameters.

On the Coulomb branch side, the Poincaré-Laurent polynomial $Q_{\text{Coulomb}}(\gamma; \zeta; y)$ coincides with its rational counterpart $\bar{Q}_{\text{Coulomb}}(\gamma; \zeta; y)$. Since the quiver has no oriented loop there are no contributions from scaling solutions, therefore we can set $\Omega_{\text{tot}}(\alpha, y) = \Omega_{\text{S}}(\alpha, y)$ in (3.5). We can further set $\Omega_{\text{S}}(\alpha; y)$ to 0 except when α is equal to one of the basis vectors γ_ℓ in which case its value is 1. The Coulomb branch formula (3.5) thus reduces to

$$Q_{\text{Coulomb}}(\gamma_1 + \cdots + \gamma_K; \zeta; y) = g_{\text{Coulomb}}(\{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y). \qquad (4.10)$$

It follows from the discussion in section 2.2 that the result when some of the γ_{ij}'s vanish can be obtained as a limit of (4.10) for generic non-vanishing γ_{ij}'s. Comparing (4.10) and (4.8) we see that the proof of equivalence of Q_{Coulomb} and Q_{Higgs} for Abelian quivers reduces to the proof of equivalence of g_{Coulomb} and g_{Higgs}. Furthermore since for both g_{Coulomb} and g_{Higgs} the result when some of the γ_{ij}'s vanish can be obtained as limits of the result for generic γ_{ij}'s, it is enough to prove the equivalence for generic non-vanishing γ_{ij}'s.

This equivalence can now be proved by following exactly the same analysis as section 3.3 of ref. [10]. We summarize the main points, refering the reader to [10] for more details. First, to each permutation σ, we associate a family of ordered partitions of $\gamma_1 + \cdots \gamma_K$ into vectors $\{\beta^{(a)}\}$ as follows:

- Break the sequence $\{\sigma(1), \sigma(2), \cdots \sigma(n)\}$ into subsequences $\{\sigma(1), \sigma(2), \cdots \sigma(i_1)\}$, $\{\sigma(i_1 + 1), \cdots \sigma(i_2)\}$, $\cdots$, $\{\sigma(i_{a-1} + 1), \ldots, \sigma(i_a)\}$, $\cdots$, $0 = i_0 < i_1 < \cdots < i_s = K$, such that each subsequence represents an increasing subsequence: $\sigma(i_{a-1} + 1) < \sigma(i_{a-1} + 2) < \cdots < \sigma(i_a)$ for each a. s gives the number of such increasing subsequences.

- For each such breakup of $\{\sigma(1), \cdots \sigma(K)\}$ into increasing subsequences, we can associate a partition of $\gamma_1 + \cdots \gamma_K$ into vectors $\beta^{(a)}$ as follows:

$$\beta^{(s+1-a)} = \sum_{i=i_{a-1}+1}^{i=i_a} \gamma_{\sigma(i)} . \qquad (4.11)$$

This generates an ordered partition of γ but does not necessarily satisfy (4.9). It is easy to show that with this choice the power of y in $y^{\sum_{i<j} \gamma_{\sigma(i)\sigma(j)}}$ in (1.9) matches the power of y inside the sum in (4.8)

For illustration we can consider the permutation $\sigma(1234) = (3412)$. The increasing subsequences are $\{\{34\}, \{12\}\}$, $\{\{3\}, \{4\}, \{12\}\}$, $\{\{34\}, \{1\}, \{2\}\}$, and $\{\{3\}, \{4\}, \{1\}, \{2\}\}$. Associated partitions are $\{\gamma_1 + \gamma_2, \gamma_3 + \gamma_4\}$, $\{\gamma_1 + \gamma_2, \gamma_4, \gamma_3\}$, $\{\gamma_2, \gamma_1, \gamma_3 + \gamma_4\}$ and $\{\gamma_2, \gamma_1, \gamma_4, \gamma_3\}$, respectively. For each of these partitions we have

$$2 \sum_{a \leq b} \sum_{j < i} \gamma_{ji} N_i^a N_j^b - \sum_{1 \leq i < j \leq 4} \gamma_{ij} = \sum_{1 \leq i < j \leq 4} \gamma_{\sigma(i)\sigma(j)} = \gamma_{12} + \gamma_{34} - \gamma_{13} - \gamma_{14} - \gamma_{23} - \gamma_{24} . \quad (4.12)$$

The above analysis shows how to each permutation σ we can associate a set of ordered partitions of $\gamma_1, \cdots \gamma_K$ into vectors $\beta^{(1)}, \cdots \beta^{(s)}$. The converse is also true — all ordered partitions of the vectors $(\gamma_1, \ldots, \gamma_K)$, before imposing (4.9), are in one to one correspondence with the set of all increasing subsequences of all the permutations of $(12 \ldots n)$ via

the rule (4.11). For a given permutation we shall call an increasing subsequence maximal if it is not possible to build bigger increasing subsequences involving the elements of the subsequence. In the example described above the maximal increasing subsequences are $\{\{34\}, \{12\}\}$. The complete contribution to the coefficient of $y^{\sum_{i<j} \gamma_{\sigma(i)\sigma(j)}}$ in the Reineke formula can be generated by beginning with the maximal increasing subsequences associated with the permutation σ and combining them with the contribution from other increasing subsequences associated with the same permutation, but we must be careful to pick only those partitions which satisfy (4.9). As shown in [10], a given permutation contributes if and only if its maximal increasing subsequences generate a partition via (4.11) satisfying (4.9), and none of the other (non-maximal) increasing subsequences generate an allowed partition.[17] The sign of the contribution is given by $(-1)^{s-1}$ where s is the number of maximal increasing subsequences.

It is now easy to see that the condition that the maximal increasing subsequences generate a partition satisfying (4.9) translates to

$$\sum_{i=k+1}^{K} c_{\sigma(i)} > 0 \quad \text{for} \quad \gamma_{\sigma(k),\sigma(k+1)} < 0 \,, \tag{4.13}$$

and the condition that none of the non-maximal increasing subsequences satisfy (4.9) translates to

$$\sum_{i=k+1}^{K} c_{\sigma(i)} < 0 \quad \text{for} \quad \gamma_{\sigma(k),\sigma(k+1)} > 0 \,. \tag{4.14}$$

These precisely agree with (2.7). Finally the number of maximal increasing subsequences is one more than the number of negative $\gamma_{\sigma(k),\sigma(k+1)}$. Thus we have

$$(-1)^{s-1} = \prod_{k=1}^{K-1} \text{sign}(\gamma_{\sigma(k),\sigma(k+1)}) \,, \tag{4.15}$$

in agreement with (2.8). This proves that the Reinecke formula (4.8) agrees with the Coulomb index (1.4), and hence also the Coulomb branch formula (4.10) for the Poincaré-Laurent polynomial,[18]

$$g_{\text{Higgs}}(\{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y) = g_{\text{Coulomb}}(\{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y) \,,$$

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[17]In particular, if some non-maximal increasing subsequence satisfies (4.9), then the net contribution from the partitions associated with the maximal and non-maximal increasing subsequences of a given permutation cancel pairwise.

[18]This relation holds away from walls of marginal and threshold stability. On such a wall, the Higgs branch index on the l.h.s. given by (4.8), (4.9), is still well defined but the derivation of the Coulomb index given in (2.7), (2.8) clearly breaks down. Furthermore even the Higgs branch index given by (4.8), (4.9) does not have the form of a symmetric Laurent polynomial, and hence cannot be interpreted as $\text{Tr}'(-y)^{2J_3}$ associated with a quantum system. On a threshold stability wall we shall define g_{Coulomb} by deforming the FI parameters away from the wall since there is no independent definition of g_{Coulomb} in this case, and the result is independent of the deformation. However for g_{Higgs} we shall continue to use the definition given in (4.8), (4.9) for reasons which will become clear in section 4.4. The price we pay is that g_{Coulomb} and g_{Higgs} are no longer equal on the thereshold stability wall.

$$Q_{\text{Higgs}}(\gamma_1 + \cdots + \gamma_K; \zeta; y) = Q_{\text{Coulomb}}(\gamma_1 + \cdots + \gamma_K; \zeta; y). \qquad (4.16)$$

For charge configurations relevant for wall-crossing, where all the γ_ℓ's lie in a plane and the ζ_ℓ's are determined in terms of γ_{ij}, this equivalence was proved by induction in [28]. Here we have shown that it holds for any Abelian quiver without oriented loop.

4.3 Non-Abelian quivers with primitive dimension vector and without loops

For non-Abelian quivers without loops we still can set $\Omega_{\text{tot}}(\alpha; y) = \Omega_{\text{S}}(\alpha; y)$. We also have $\Omega_{\text{S}}(\alpha; y) = 0$ unless $\alpha = \gamma_\ell$, and $\Omega_{\text{S}}(\gamma_\ell; y) = 1$. Thus $\Omega_{\text{S}}(\alpha_i/m_i; y^{m_i})$ is non-vanishing (and equal to 1) only when α_i/m_i is a basis vector. Furthermore for primitive charge vector, $Q_{\text{Coulomb}}(\gamma; \zeta; y)$ is equal to $\bar{Q}_{\text{Coulomb}}(\gamma; \zeta; y)$. In that case we can express (3.3) as

$$Q_{\text{Coulomb}}\left(\sum_i N_i \gamma_i; \zeta; y\right) = \sum_{\substack{\{k_j^{(\ell)}\} \\ \sum_\ell \ell k_j^{(\ell)} = N_j}} g_{\text{Coulomb}}(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y)$$
$$\times \left\{ \prod_{j=1}^K \prod_\ell \frac{1}{k_j^{(\ell)}!} \left(\frac{y - y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_j^{(\ell)}} \right\}. \ (4.17)$$

The argument of g_{Coulomb} given above corresponds to a choice of $\{\alpha_1, \cdots \alpha_n\}$ where we have $k_j^{(\ell)}$ copies of the vector $\ell\gamma_j$ for $\ell = 1, 2, 3, \cdots$, $j = 1, \cdots K$, and the FI parameter associated with each such copy is $\ell\zeta_j$. It is understood that if some of the α_{ij}'s appearing in the argument of g_{Coulomb} vanish then we must deform them in a way so that the deformed quiver does not contain any oriented loop, compute g_{Coulomb} and then take the deformations back to zero. The $\prod_i \prod_\ell k_i^{(\ell)}!$ factor in the denominator represents $|\text{Aut}(\{\alpha_1, \alpha_2, \cdots, \alpha_n\})|$.

On the other hand, it was shown in [10, 21] that the stack invariants (4.2) satisfy the Abelianization formula[19]

$$G_{\text{Higgs}}(\{N_j\}; \{\gamma_j\}; \{\zeta_j\}; y) = \sum_{\substack{\{k_\ell\} \\ \sum_\ell \ell k_\ell = N_i}} G_{\text{Higgs}}(\{\hat{N}_j\}; \{\hat{\gamma}_j\}; \{\hat{\zeta}_j\}; y) \prod_\ell \left[\frac{1}{k_\ell!} \left(\frac{(y - 1/y)}{\ell(y^\ell - y^{-\ell})} \right)^{k_\ell} \right]$$
$$(4.18)$$

where, on the r.h.s, $G_{\text{Higgs}}(\{\hat{N}_j\}; \{\hat{\gamma}_j\}; \{\hat{\zeta}_j\}; y)$ is the stack invariant of a quiver $\hat{\mathcal{Q}}$ defined as follows:

- The vertices of $\hat{\mathcal{Q}}$ are obtained by replacing the vertex i of $\mathcal{Q}$ by a collection $i_{\ell,k}$ of vertices with $k = 1, \ldots k_\ell$ for any ℓ in the partition $N_i = \sum_\ell \ell k_\ell$, and keeping all the other vertices $j \neq i$ of $\mathcal{Q}$;

- Each arrow $j \to k$ in $\mathcal{Q}$ with $j, k \neq i$ induces an arrow $j \to k$ in $\hat{\mathcal{Q}}$;

---

[19]In [10], appendix D we proved this formula for a special choice of FI parameters relevant for wall crossing. However this assumption was inessential and the result easily generalizes to the case of general FI parameters [21]. The formula (4.18) also holds in the case of quivers with loops, but we shall not make use of this fact here.

- The arrows $i \to j$ in $\mathcal{Q}$ (resp. $j \to i$) with $j \neq i$ induce ℓ arrows $i_{\ell,k} \to j$ (resp. $j \to i_{\ell,k}$) in $\hat{\mathcal{Q}}$ for each ℓ, k;

- *The dimension at each node $i_{\ell,k}$ in $\hat{\mathcal{Q}}$ is equal to one, hence justifying the name*; the dimensions at the other nodes $j \neq i$ are the same as the dimensions N_j in the original quiver $\mathcal{Q}$;

- The FI parameter at each node $i_{\ell,k}$ in $\hat{\mathcal{Q}}$ is equal to $\ell\zeta_i$; the FI parameters at the other nodes $j \neq i$ are the same as the FI parameters ζ_j in the original quiver $\mathcal{Q}$.

Thus for example on the right hand side of (4.18) the set $\{\hat{N}_j\}$ will include all the N_j's except N_i and $\sum_\ell k_\ell$ number of 1's, the set $\{\hat{\gamma}_j\}$ will include all the γ_j's except γ_i, k_1 copy of γ_i, k_2 copy of $2\gamma_i$, etc., and the set $\{\hat{\zeta}_i\}$ will include all the ζ_j's except ζ_i, k_1 copies of ζ_i, k_2 copies of $2\zeta_i$ etc. By successive application of this formula we can express the stack invariant of any non-Abelian quiver in terms of that of a family of Abelian quivers:

$$G_{\text{Higgs}}(\{N_i\}; \{\gamma_i\}; \{\zeta_i\}; y) = \sum_{\substack{\{k_j^{(\ell)}\} \\ \sum_\ell \ell k_j^{(\ell)} = N_j}} g_{\text{Higgs}}(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y) \prod_{i=1}^{K} \prod_\ell \frac{1}{k_i^{(\ell)}!} \left(\frac{y - y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_i^{(\ell)}}.$$

(4.19)

Using (4.6) we can replace the left hand side by $Q_{\text{Higgs}}\left(\sum_{i=1}^{K} N_i \gamma_i; \zeta; y \right)$. The resulting equation is identical to (4.17) with Q_{Coulomb} replaced by Q_{Higgs} and g_{Coulomb} replaced by g_{Higgs}. Using the equivalence of g_{Coulomb} and g_{Higgs} proven in (4.16), we get

$$Q_{\text{Higgs}}\left(\sum_\ell N_\ell \gamma_\ell; \zeta; y \right) = Q_{\text{Coulomb}}\left(\sum_\ell N_\ell \gamma_\ell; \zeta; y \right).$$

(4.20)

Thus the Higgs and Coulomb branch formulae are equivalent for non-Abelian quivers as well, so long as the dimension vector $\{N_i\}$ is primitive.

4.4 Non-primitive dimension vector

We now turn to the case of a non-Abelian quiver with non-primitive dimension vector. In this case the Coulomb branch formula is still given by (3.3) although Q_{Coulomb} and $\bar{Q}_{\text{Coulomb}}$ are no longer identical. On the other hand the Higgs branch formula is given by (4.1) (or equivalently (4.5)) and (4.2). Our goal in this section will be to prove the equality of Q_{Coulomb} and Q_{Higgs}.

When the dimension vector $\{N_i\}_{i=1...K} \equiv \vec{N}$ is not primitive, the Abelianization formula (4.19) is still known to hold, and can be used to express G_{Higgs} in terms of $g_{\text{Higgs}}(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y)$ for integers $k_j^{(\ell)}$ satisfying $\sum_\ell \ell k_j^{(\ell)} = N_j$. However some of the Abelian stack invariants $g_{\text{Higgs}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}; y)$ appearing on the r.h.s. have stability conditions $\{\hat{\zeta}\}$ lying on walls of threshold stability. This happens when the set $\{\hat{\gamma}_i\}$ can be divided into several subsets $A_1, A_2, \cdots$ such that $\sum_{i \in A_s} \hat{\gamma}_i \parallel \sum_\ell N_\ell \gamma_\ell$ for each s. As a result $g_{\text{Higgs}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}; y)$ can no longer be equated to $g_{\text{Coulomb}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}; y)$ (see footnote 18). Let $\{\hat{\zeta}_i\}'$ be a sufficiently generic perturbation of the FI parameters away from the

wall and sufficiently small so that no other walls of marginal stability are crossed. In this case $g_{\text{Higgs}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}'; y)$ can again be equated to $g_{\text{Coulomb}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}'; y)$ and furthermore will be independent of the deformation. Now the HN recursion method [15, 31] relates $g_{\text{Higgs}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}'; y)$ and $g_{\text{Higgs}}(\{\hat{\gamma}_i\}; \{\hat{\zeta}_i\}; y)$ as follows:

$$
\begin{aligned}
& g_{\text{Higgs}}(\alpha_1, \cdots \alpha_n; c_1, \cdots c_n; y) \\
& = \sum_{k \geq 1} (y^{-1} - y)^{1-k} \sum_{\substack{\{A_s\} \\ \sum_{i \in A_s} c_i = 0 \; \forall s; \; \cup_{s=1}^k A_s = \{1,2,\cdots n\}}} \prod_{s=1}^{k} g_{\text{Higgs}}(\{\alpha_i; \; i \in A_s\}; \{c_i'; \; i \in A_s\}; y) \quad (4.21)
\end{aligned}
$$

where the sum over $\{A_s\}$ runs over all *unordered partitions* of the integers $1, 2, \cdots n$ into sets $A_1, A_2, \cdots A_k$ subject to the conditions indicated above. The symbol $\{c_i'\}$ denotes a generic deformation in which the FI parameters associated with each of the n nodes are deformed independently so that the configuration moves away from the wall of threshold stability, subject to the constraint that the sum of the FI parameters carried by all the nodes must vanish. It was shown in ref. [22] that the combinatorics of the summation in (4.21) can be summarized by an equality of generating functions, which in our notation reads

$$
\begin{aligned}
& 1 + (y^{-1} - y)^{-1} \sum_{\{k_i^{(\ell)}\}, \sum_{\ell} \ell k_i^{(\ell)} \propto N_i} g_{\text{Higgs}}\left(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y\right) \prod_{i,\ell} \frac{(t_{i,\ell})^{k_i^{(\ell)}}}{k_i^{(\ell)}!} \\
& = \exp\left[(y^{-1} - y)^{-1} \sum_{\{k_i^{(\ell)}\}, \sum_{\ell} \ell k_i^{(\ell)} \propto N_i} g_{\text{Higgs}}\left(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}'; y\right) \prod_{i,\ell} \frac{(t_{i,\ell})^{k_i^{(\ell)}}}{k_i^{(\ell)}!}\right],
\end{aligned} \quad (4.22)
$$

where $t_{i,\ell}$ are formal parameters and the sum over $\{k_i^{(\ell)}\}$ on either side runs over all integers $k_i^{(\ell)}$ for which $\sum_{\ell} k_i^{(\ell)}$ is of the form cN_i for some fixed vector N_i and arbitrary constant c. Note that for the deformed stability conditions ζ', g_{Higgs} can be equated to g_{Coulomb}, which is by definition equal to the value of g_{Coulomb} for the undeformed stability condition ζ (see footnote 18). As a result, we get for g_{Higgs} either from (4.21) directly, or after equating coefficients of $\prod_{i,\ell}(t_{i,\ell})^{k_i^{(\ell)}}$ on either side,

$$
\begin{aligned}
& g_{\text{Higgs}}\left(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y\right) = \left(\prod_{i,\ell} k_\ell^{(i)}!\right) \sum_k \frac{(y^{-1} - y)^{1-k}}{k!} \\
& \times \sum_{\substack{\{k_i^{(\ell,s)}, \; s=1,\cdots k\} \\ \sum_{\ell} \ell k_i^{(\ell,s)} \propto \sum_{\ell} \ell k_i^{(\ell)} \; \forall s, \; \sum_s k_i^{(\ell,s)} = k_i^{(\ell)}}} \prod_s \left\{ g_{\text{Coulomb}}\left(\{(\ell\gamma_j)^{k_j^{(\ell,s)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell,s)}}\}; y\right) \prod_{i,\ell} \frac{1}{k_i^{(\ell,s)}!} \right\},
\end{aligned} \quad (4.23)
$$

where the sum over $\{k_i^{(\ell,s)}\}$ runs over all *ordered* partitions of $\{k_i^{(\ell)}\}$. Using this we can

express the Abelianization formula (4.19) as

$$G_{\text{Higgs}}(\{N_i\}; \{\gamma_i\}; \{\zeta_i\}; y) \tag{4.24}$$

$$= \sum_{\substack{\{k_j^{(\ell)}\} \\ \sum_\ell \ell k_j^{(\ell)} = N_j}} \left(\prod_{i,\ell} k_\ell^{(i)}! \right) \sum_k \frac{(y^{-1} - y)^{1-k}}{k!} \sum_{\substack{\{k_i^{(\ell,s)}, 1 \le s \le k\} \\ \sum_\ell \ell k_i^{(\ell,s)} \propto \sum_\ell \ell k_i^{(\ell)} \, \forall \, s, \, \sum_s k_i^{(\ell,s)} = k_i^{(\ell)}}} \prod_{s=1}^k \Bigg\{$$

$$\times g_{\text{Coulomb}}\left(\{(\ell\gamma_j)^{k_j^{(\ell,s)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell,s)}}\}; y \right) \prod_{i,\ell} \frac{1}{k_i^{(\ell,s)}!} \Bigg\} \times \left\{ \prod_{i=1}^K \prod_\ell \frac{1}{k_i^{(\ell)}!} \left(\frac{y - y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_i^{(\ell)}} \right\}.$$

We can now remove the sum over $\{k_j^{(\ell)}\}$ by relaxing the constraint $\sum_s k_i^{(\ell,s)} = k_i^{(\ell)}$, replacing $k_i^{(\ell)}$ by $\sum_s k_i^{(\ell,s)}$ everywhere, and imposing the constraints $\sum_{\ell,s} \ell k_i^{(\ell,s)} = N_i$ and $\sum_\ell \ell k_i^{(\ell,s)} \propto N_i$. This gives

$$G_{\text{Higgs}}(\{N_i\}; \{\gamma_i\}; \{\zeta_i\}; y)$$

$$= \sum_k \frac{(y^{-1} - y)^{1-k}}{k!} \sum_{\substack{\{k_i^{(\ell,s)}, 1 \le s \le k\} \\ \sum_\ell \ell k_i^{(\ell,s)} \propto N_i \, \forall \, s, \, \sum_{s,\ell} \ell k_i^{(\ell,s)} = N_i}} \prod_{s=1}^k \Bigg[g_{\text{Coulomb}}\left(\{(\ell\gamma_j)^{k_j^{(\ell,s)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell,s)}}\}; y \right) \tag{4.25}$$

$$\times \left\{ \prod_{i=1}^K \prod_\ell \frac{1}{k_i^{(\ell,s)}!} \left(\frac{y - y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_i^{(\ell,s)}} \right\} \Bigg].$$

Following the discussion at the beginning of section 4.3 but without imposing the primitivity condition on the dimension vector $\gamma \equiv \sum_\ell N_\ell \gamma_\ell$, we see that the Coulomb branch formula (3.3) takes the form

$$Q_{\text{Coulomb}}(\gamma; \zeta; y) = \sum_{m|\gamma} \frac{\mu(m)}{m} \frac{y - y^{-1}}{y^m - y^{-m}} \bar{Q}_{\text{Coulomb}}(\gamma/m; \zeta; y^m) \tag{4.26}$$

$$\bar{Q}_{\text{Coulomb}}(\gamma; \zeta; y) = \sum_{\substack{\{k_j^{(\ell)}\} \\ \sum_\ell \ell k_j^{(\ell)} = N_j}} g_{\text{Coulomb}}(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}; y) \prod_{j=1}^K \prod_\ell \frac{1}{k_j^{(\ell)}!} \left(\frac{y - y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_j^{(\ell)}}.$$

Let us now define G_{Coulomb} by a formula analogous to (4.5) with $\bar{Q}_{\text{Higgs}}$ replaced by $\bar{Q}_{\text{Coulomb}}$ on the right hand side:

$$G_{\text{Coulomb}}(\{N_1, \cdots N_K\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y)$$

$$= \sum_k \sum_{\substack{\{\alpha_i\} \\ \sum_{i=1}^k \alpha_i = \sum_\ell N_\ell \gamma_\ell \\ \alpha_i \| \sum N_\ell \gamma_\ell \text{ for } i=1,\dots,k}} \frac{(-1)^{k-1}}{k!(y - y^{-1})^{k-1}} \prod_{i=1}^k \bar{Q}_{\text{Coulomb}}(\alpha_i; \zeta; y). \tag{4.27}$$

Proving the equivalence of Q_{Higgs} and Q_{Coulomb} is then equivalent to proving the equivalence of G_{Higgs} and G_{Coulomb}. Now using (4.26) we get

$$
\begin{aligned}
&G_{\text{Coulomb}}(\{N_1, \cdots N_K\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y) \\
&= \sum_{\substack{k \\ \sum_{s=1}^{k} \alpha_s = \sum_i N_i \gamma_i \\ \alpha_s \| \sum N_i \gamma_i \text{ for } s=1,\ldots,k}} \sum_{\{\alpha_s\}} \frac{(-1)^{k-1}}{k!(y-y^{-1})^{k-1}} \prod_{s=1}^{k} \sum_{\substack{\{k_j^{(\ell,s)}\} \\ \sum_\ell \ell k_j^{(\ell,s)} \gamma_j = \alpha_s}} g_{\text{Coulomb}}(\{(\ell\gamma_j)^{k_j^{(\ell,s)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell,s)}}\}; y) \\
&\quad \times \left\{ \prod_{i=1}^{K} \prod_\ell \frac{1}{k_i^{(\ell,s)}!} \left(\frac{y-y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_i^{(\ell,s)}} \right\},
\end{aligned} \tag{4.28}
$$

where the sum over $\{k_i^{(\ell,s)}\}$ runs over all *ordered* partitions of $\{k_i^{(\ell)}\}$. We can now remove the sum over α_s's by relaxing the constraint that $\sum_\ell \ell k_j^{(\ell,s)} \gamma_j = \alpha_s$, but imposing the constraints $\sum_{\ell,s} \ell k_i^{(\ell,s)} = N_i$ and $\sum_\ell \ell k_i^{(\ell,s)} \propto N_i$. This gives

$$
\begin{aligned}
&G_{\text{Coulomb}}(\{N_1, \cdots N_K\}; \{\gamma_1, \cdots \gamma_K\}; \{\zeta_1, \cdots \zeta_K\}; y) \\
&= \sum_k \frac{1}{k!} (y^{-1}-y)^{1-k} \sum_{\substack{\{k_i^{(\ell,s)}, 1 \le s \le k\} \\ \sum_\ell \ell k_i^{(\ell,s)} \propto N_i \, \forall \, s, \, \sum_{s,\ell} \ell k_i^{(\ell,s)} = N_i}} \prod_{s=1}^{k} \left[g_{\text{Coulomb}} \left(\{(\ell\gamma_j)^{k_j^{(\ell,s)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell,s)}}\}; y \right) \right. \\
&\quad \left. \times \left\{ \prod_{i=1}^{K} \prod_\ell \frac{1}{k_i^{(\ell,s)}!} \left(\frac{y-y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_i^{(\ell,s)}} \right\} \right].
\end{aligned} \tag{4.29}
$$

Since the right hand side matches the right hand side of (4.25), it follows that G_{Higgs} and G_{Coulomb} are identical, and therefore the Higgs and Coulomb branch computations Q_{Higgs} and Q_{Coulomb} are equivalent, even for non-primitive dimension vector.

Given the equivalence of Q_{Higgs} (defined by the Harder-Narasimhan recursion (4.1), (4.2)) and Q_{Coulomb} (defined by the Abelianization formula (4.26)), and given the equivalence of the Abelian indices g_{Coulomb} and g_{Higgs} for deformed FI parameters, we conclude that the Poincaré-Laurent polynomial Q_{Higgs} of a quiver without loop satisfies, for arbitrary dimension vector, the Abelianization formula:

$$
Q_{\text{Higgs}}(\gamma; \zeta; y) = \sum_{m|\gamma} \frac{\mu(m)}{m} \frac{y-y^{-1}}{y^m - y^{-m}} \bar{Q}_{\text{Higgs}}(\gamma/m; \zeta; y^m) \tag{4.30}
$$

$$
\bar{Q}_{\text{Higgs}}(\gamma; \zeta; y) = \sum_{\substack{\{k_j^{(\ell)}\} \\ \sum_\ell \ell k_j^{(\ell)} = N_j}} g_{\text{Higgs}}(\{(\ell\gamma_j)^{k_j^{(\ell)}}\}; \{(\ell\zeta_j)^{k_j^{(\ell)}}\}'; y) \prod_{j=1}^{K} \prod_\ell \frac{1}{k_j^{(\ell)}!} \left(\frac{y-y^{-1}}{\ell(y^\ell - y^{-\ell})} \right)^{k_j^{(\ell)}}.
$$

This can be regarded as the main result of this subsection.

Acknowledgments

A The MATHEMATICA package "CoulombHiggs"

To facilitate further investigation, we provide a MATHEMATICA package allowing to compute the Coulomb index g_{Coulomb} for multi-centered black holes and the Poincaré-Laurent polynomial $Q_{\text{Coulomb}}(\gamma; \zeta; y)$ and $Q_{\text{Higgs}}(\gamma; \zeta; y)$ of quiver moduli spaces using the Coulomb branch and Higgs branch formulae. We also provide three example files where this package is used to evaluate quiver invariants for the Kronecker quiver (`Kronecker.nb`), for non-Abelian 3-node quivers (`Threenode.nb`) and for several 4 and 5-node Abelian quivers considered in [14] (`Multinode.nb`). The validity of the algorithm for the F and G indices is tested in a fourth file `CoulombIndexCheck.nb`. All these files are included in the "source" of this paper available from arXiv and can be obtained from the second name authors' webpage.

Assuming that the file `CoulombHiggs.m` is present in the user's MATHEMATICA Application directory, the package is loaded by entering

```
In[1]:=   <<CoulombHiggs`
Out[1]:=  CoulombHiggs v 1.0 - A package for evaluating quiver
          invariants using the Coulomb and Higgs branch formulae.
```

If the file `CoulombHiggs.m` has not yet been copied in the user's MATHEMATICA Application directory but is in the same directory as the notebook, evaluate instead

```
In[1]:=   SetDirectory[NotebookDirectory[]]; <<CoulombHiggs`
Out[1]:=  CoulombHiggs v 1.0 - A package for evaluating quiver
          invariants using the Coulomb and Higgs branch formulae.
```

The first main routine is CoulombBranchFormula, whose basic usage is illustrated below:

```
In[1]:=   Simplify[CoulombBranchFormula[4{{0, 1, -1},{-1, 0, 1}, {1,
          -1, 0}}, {1/2, 1/6, -2/3}, {1, 1, 1}, y]]
Out[1]:=  2 + 1/y² + y² + OmS({1,1,1})
```

The first argument corresponds to the matrix of DSZ products α_{ij} (an antisymmetric matrix of integers), the second to the FI parameters ζ_i (a vector of rational numbers), the third to the dimension vector N_i (a vector of integers), and the last to the angular momentum fugacity (a formal variable). In this example, the routine expresses the Poincaré-Laurent polynomial of the moduli space of a three-node Abelian cyclic quiver with 4 arrows between each subsequent node, in terms of the single centered degeneracy $\Omega_S(\gamma_1 + \gamma_2 + \gamma_3)$.

The second main routine is `HiggsBranchFormula`, which computes the Poincaré-Laurent polynomial using the Higgs branch formula (4.1). The arguments are the same as for `CoulombBranchFormula`, but it only applies to quivers without oriented loop (although no test is made to ensure that this is the case):

```
In[1]:=   Simplify[HiggsBranchFormula[{{0, 3},{-3, 0}}, {1/2,-1/2},
          {2, 2}, y]]
Out[1]:=   -((y^2+1)(y^8+y^4+1))/y^5
```

The above command computes the Poincaré-Laurent polynomial for the Kronecker quiver with 3 arrows, FI parameters $(1/2, -1/2)$, dimension vector $(2, 2)$. The package allows for much more, as documented below. Inline documentation can be obtained by typing e.g.

```
In[1]:=   ?CoulombBranchFormula
Out[1]:=
```

A.1   Symbols

- y: Angular momentum fugacity;

- `Om[`*charge vector_,y_*`]`: denotes the refined index $\Omega(\gamma, y)$;

- `Omb[`*charge vector_,y_*`]`: denotes the rational refined index $\bar{\Omega}(\gamma, y)$;

- `OmS[`*charge vector_,y_*`]`: denotes the single centered degeneracy $\Omega_\mathrm{S}(\gamma, y)$. The dependence on y can be omitted, but only at the end of the computation;

- `OmT[`*charge vector_,y_*`]`: denotes the (unevaluated) function $\Omega_\mathrm{tot}(\gamma, y)$ defined in (3.4);

- `Coulombg[`*list of charge vectors_,y_*`]`: denotes the (unevaluated) Coulomb index $g_\mathrm{Coulomb}(\{\alpha_i\}, \{c_i\}, y)$, leaving the FI parameters unspecified;

- `HiggsG[`*charge vector_,y_*`]`: denotes the (unevaluated) stack invariant $G_\mathrm{Higgs}(\gamma, y)$ defined in (4.2);

- `CoulombH[`*list of charge vectors_,multiplicity vector_,y_*`]`: denotes the (unevaluated) factor $H(\{\alpha_i\}, \{n_i\}, y)$ appearing in the formula (3.4) for $\Omega_\mathrm{tot}(\sum n_i \alpha_i, y)$ in terms of $\Omega_\mathrm{S}(\alpha_i, y)$.

- `QFact[`*n_,y_*`]`: represents the (non-evaluated) q-deformed factorial $[n, y]!$

A.2 Environment variables

- $QuiverPerturb1: Sets the size of the perturbation $\epsilon_1 = 1/$QuiverPerturb$ of the DSZ products in (3.7), set to 1000 by default.

- $QuiverPerturb2: Sets the size of the perturbation $\epsilon_2 = 1/$DSZPerturb$ of the DSZ products in (3.8), set to 100000 by default.

- $QuiverNoLoop: If set to True, the quiver will be assumed to have no oriented loop, hence all H factors and all $\Omega_S(\alpha)$ will be set to zero (unless α is a basis vector). Set to False by default.

- $QuiverTestLoop: If set to True, all H factors and $\Omega_S(\alpha)$ corresponding to subquivers without loops will be set to zero (unless α is a basis vector). Set to True by default. Determining whether a subquiver has loops is time-consuming, so for large quivers it may be advisable to disable this feature. Note that $QuiverNoLoop takes precedence over this variable.

- $QuiverMultiplier: Overall scaling factor of the DSZ matrix in any evaluation of Coulombg or HiggsG. Set to 1 by default, could be a formal variable.

- $QuiverVerbose: If set to False, all consistency tests on data and corresponding error messages will be skipped. Set to True by default.

- $QuiverDisplayCoulombH: If set to True, the routine CoulombBranchFormula will return a list {Q, R} where Q is the Poincaré-Laurent polynomial and R is a list of replacement rules for the *CoulombH* factors. Set to False by default.

- $QuiverPrecision: Sets the numerical precision with which all consistency tests are carried out. This is set to 0 by default since all data are assumed to be rational numbers. This can be set to a small real number when using real data, however the user is warned that rounding errors tend to grow quickly.

A.3 Coulomb index

- CoulombF[*Mat_*,*Cvec_*]: returns the index of collinear solutions $F(\{\tilde{\alpha}_1, \cdots \tilde{\alpha}_n\}, \{\tilde{c}_1, \cdots \tilde{c}_n\})$ with DSZ products $\tilde{\alpha}_{ij} = $ Mat$[[i,j]]$, FI terms $\tilde{c}_i = $ Cvec$[[i]]$ and trivial ordering.

- CoulombG[*Mat_*]: returns the index of scaling collinear solutions $G(\{\hat{\alpha}_1, \cdots \hat{\alpha}_n\})$ with DSZ products $\hat{\alpha}_{ij} = $ Mat$[[i,j]]$ and trivial ordering. The total angular momentum $\sum_{i<j} Mat[[i,j]]$ must vanish;

- CoulombIndex[*Mat_*,*PMat_*,*Cvec_*,*y_*]: evaluates the Coulomb index $g_{\text{Coulomb}}(\{\alpha_1, \cdots \alpha_n\}; \{c_1, \cdots c_n\}; y)$ with DSZ products $\alpha_{ij} = $ Mat$[[i,j]]$, perturbed to *PMat*[[i,j]] so as to lift accidental degeneracies, possibly rescaled by an overall factor of $QuiverMultiplier, FI terms $c_i = $ Cvec$[[i]]$, angular momentum fugacity y;

- `CoulombFNum[`*Mat_*`]`: computes numerically the index $F(\{\tilde{\alpha}_1, \ldots \tilde{\alpha}_n\}, \{\tilde{c}_1, \ldots \tilde{c}_n\})$ with DSZ matrix $\tilde{\alpha}_{ij} = $ `Mat[[`i,j`]]` and FI parameters $\tilde{c}_i = $ `Cvec[[`i`]]`. For testing purposes only, works for up to 5 centers.

- `CoulombGNum[`*Mat_*`]`: computes numerically the scaling index $G(\hat{\alpha}_1, \ldots \hat{\alpha}_n)$ with DSZ matrix $\hat{\alpha}_{ij} = $ `Mat[[`i,j`]]`. For testing purposes only, works for up to 6 centers.

- `CoulombIndexNum[`*Mat_*`,`*PMat_*`,`*Cvec_*`,`*k_*`,`*y_*`]`: returns the Coulomb index $g_{\text{Coulomb}}(\{\alpha_1, \cdots \ \alpha_n\}; \{c_1, \cdots c_n\}; y)$ with DSZ products $\alpha_{ij} = $ `Mat[[`i,j`]]`, possibly rescaled by an overall factor of `$QuiverMultiplier`, FI terms $c_i = $ `Cvec[[`i`]]`, angular momentum fugacity y, by searching collinear solutions numerically; for testing purposes only, works for up to 5 centers.

A.4 Coulomb branch formula

- `CoulombBranchFormula[`*Mat_*`,`*Cvec_*`,`*Nvec_*`,`*y_*`]`: computes the Poincaré-Laurent polynomial of a quiver with DSZ products $\alpha_{ij} = $ `Mat[[`i,j`]]`, dimension vector $N_i = $ `Nvec[[`i`]]`, FI parameters $\zeta_i = $ `Cvec[[`i`]]`, in terms of single-centered invariants Ω_{S}. This standalone routine first constructs the Poincaré-Laurent polynomial using (3.3), evaluates the Coulomb indices g_{Coulomb}, and determines the H factors recursively using the minimal modification hypothesis. If `$QuiverDisplayCoulombH` is set to True, the routine returns a list $\{$`Q`,`R`$\}$, where `Q` is the Poincaré polynomial and R is a list of replacement rules for the *CoulombH* factors. For quivers without loops, the process can be sped up greatly by setting `$QuiverNoLoop` to True. For complicated quivers it is advisable to implement the Coulomb branch formula step by step, using the more elementary routines described below.

- `CoulombBranchFormulaFromH[`*Mat_*`,`*Cvec_*`,`*Nvec_*`,`*R_*`,`*y_*`]`: returns the Poincaré-Laurent polynomial of a quiver with DSZ products $\alpha_{ij} = $ `Mat[[`i,j`]]`, dimension vector $N_i = $ `Nvec[[`i`]]`, FI parameters $\zeta_i = $ `Cvec[[`i`]]`, using the rule R to replace all *CoulombH* factors.

- `QuiverPoincarePolynomial[`*Nvec_*`,`*y_*`]`: constructs the Poincaré-Laurent polynomial of a quiver according to (3.3). Coincides with `QuiverPoincarePolynomialRat` for primitive dimension vector;

- `QuiverPoincarePolynomialRat[`*Nvec_*`,`*y_*`]`: constructs the rational Poincaré-Laurent polynomial $\bar{Q}_{\text{Coulomb}}(\gamma; \zeta; y)$ according to (3.3);

- `QuiverPoincarePolynomialExpand[`*Mat_*`,`*PMat_*`,`*Cvec_*`,` *Nvec_*`,` *Q_*`]`: evaluates the Cou-lomb indices g_{Coulomb} and total single center degeneracies $\Omega_{\text{tot}}(\alpha_i, y)$ appearing in the Poincaré-Laurent polynomial Q of a quiver with DSZ products $\alpha_{ij} = $ `Mat[[`i,j`]]`, perturbed to `PMat[[`i,j`]]`, dimension vector $N_i = $ `Nvec[[`i`]]`, FI parameters $\zeta_i = $ `Cvec[[`i`]]`, using (1.9) and (3.4);

- `CoulombHSubQuivers[`*Mat_*`,`*PMat_*`,`*Nvec_*`,`*y_*`]`: computes recursively all *CoulombH* factors for DSZ matrix *Mat*, perturbed to *PMat*, and any dimension vector strictly less than *Nvec*; relies on the next two routines:

- ListCoulombH[*Nvec_*,*Q_*]: returns a pair {ListH, ListC} where *ListH* is a list of *CoulombH* factors possibly appearing in the Poincaré-Laurent polynomial *Q* of a quiver with dimension vector *Nvec*, and *ListC* is the list of coefficients which multiply the monomials in $\Omega_{\mathrm{S}}(\alpha_i, y)$ canonically associated to the *H* factors in *Q*.

- SolveCoulombH[*ListH_*,*ListC_*, *R_*]: returns a list of replacement rules for the *CoulombH* factors listed in *ListH*, by applying the minimal modification hypothesis to the coefficients listed in *ListC*. The last argument is a replacement rule for *CoulombH* factors associated to subquivers.

- MinimalModif[*f_*]: returns the symmetric Laurent polynomial which coincides with the Laurent expansion expansion of the symmetric rational function f at $y = 0$, up to strictly positive powers of y. Here symmetric means invariant under $y \to 1/y$. In practice, MinimalModif[*f*] evaluates the contour integral in [14], eq. 2.9

$$\oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{(1/u - u)\, f(u)}{(1 - uy)(1 - u/y)} \tag{A.1}$$

by deforming the contour around 0 into a sum of counters over all poles of $f(u)$ and zeros of $(1 - uy)(1 - u/y)$. This trick allows to compute (A.1) even if the order of the pole of $f(y)$ at $y = 0$ is unknown, which happens if $QuiverMultiplier is a formal variable.

- SymplifyOmSbasis[*f_*]: replaces $\Omega_{\mathrm{S}}(\gamma, y) \to 1$ when γ is a basis vector;

- SymplifyOmSbasismult[*f_*]: replaces $\Omega_{\mathrm{S}}(\gamma, y) \to 0$ when γ is a non-trivial multiple of a basis vector;

- CoulombHNoLoopToZero[*Mat_*,*f_*]: sets to zero any H factor in f corresponding to subquivers without loop, assuming DSZ products $\alpha_{ij} = $ Mat[[*i, j*]] ; active only on 2-node subquivers if $QuiverTestLoop is set to False

- OmTNoLoopToZero[*Mat_*,*f_*]: sets to zero any Ω_{tot} factor in f corresponding to subquivers without loop, assuming DSZ products $\alpha_{ij} = $ Mat[[*i, j*]] ; active only on 2-node subquivers if $QuiverTestLoop is set to False

- OmSNoLoopToZero[*Mat_*,*f_*]: sets to zero any Ω_{S} factor in f corresponding to subquivers without loop, assuming DSZ products $\alpha_{ij} = $ Mat[[*i, j*]] ; active only on 2-node subquivers if $QuiverTestLoop is set to False

- EvalCoulombH3[*Mat_*,*f_*]: evaluates any 3-center H factor with multiplicity vector $\{1, 1, 1\}$ appearing in f. Not used in any routine so far.

- DropFugacity[*f_*]: drops the second argument in any $\Omega_{\mathrm{S}}(\gamma, y)$ appearing in *f*, to be used at the end of computation only;

A.5 Higgs branch formula

- HiggsBranchFormula[*Mat_*,*Cvec_*,*Nvec_*,*y_*]: computes the Poincaré-Laurent polynomial of a quiver with DSZ products $\alpha_{ij} = \text{Mat}[[i,j]]$ (possibly rescaled by \$QuiverMultiplier), dimension vector $N_i = \text{Nvec}[[i]]$, FI parameters $\zeta_i = \text{Cvec}[[i]]$, using the Higgs branch formula (4.1). This formula is only expected to produce the correct Poincaré-Laurent polynomial for quivers without oriented loops, but no check is made on the quiver to ensure that this is the case;
- StackInvariant[*Mat_*,*Cvec_*,*Nvec_*,*y_*]: gives the stack invariant (4.2) of a quiver with DSZ matrix $\alpha_{ij} = \text{Mat}[[i,j]]$, possibly rescaled by an overall factor of \$QuiverMultiplier, FI parameters $\zeta_i = \text{Cvec}[[i]]$, dimension vector $N_i = \text{Nvec}[[i]]$, using Reineke's formula (4.2); the answer is written in terms of unevaluated q-deformed factorials *QFact[n,y]*;
- AbelianStackInvariant[*Mat_*,*Cvec_*,*y_*]: gives the Abelian stack invariant (4.7) of a quiver with DSZ matrix $\alpha_{ij} = \text{Mat}[[i,j]]$, possibly rescaled by an overall factor of \$QuiverMultiplier, FI parameters $\zeta_i = \text{Cvec}[[i]]$, using Reineke's formula (4.2); coincides with StackInvariant with *Nvec*= $\{1,\dots 1\}$ except that tests of marginal or threshold stability are performed (unless \$QuiverVerbose is set to False);
- QDeformedFactorial[*n_*,*y_*]: gives the q-deformed factorial $[n,y]!$
- EvalQFact[*f_*]: evaluates any *QFact[n,y]* appearing in *f*

A.6 Utilities

- ListAllPartitions[*charge vector_*]: returns the list of unordered partitions $\{\alpha_i\}$ of the positive integer vector γ as a sum of positive, non-zero integer vectors α_i;
- ListAllPartitionsMult[*charge vector_*]: returns the list of unordered partitions $\{\alpha_i, m_i\}$ of the positive integer vector γ as a sum of positive, non-zero integer vectors α_i with multiplicity m_i;
- ListSubQuivers[*Nvec_*]: gives a list of all dimension vectors less or equal to *Nvec*;
- SubDSZ[*Mat_*,*Cvec_*,*Li_*]: gives the DSZ matrix of the subquiver made of vectors in list *Li*;
- SymmetryFactor[*Li_*]: gives the symmetry factor $1/|\text{Aut}(\{\alpha_1, \alpha_2, \cdots, \alpha_n\}|$ for the list of charge vectors *Li*;
- OmTRat[*Nvec_*,*y_*]: gives the rational total invariant $\bar{\Omega}_{\text{tot}}(\gamma; y)$ in terms of $\Omega_{\text{tot}}(\gamma; y)$. Coincides with the latter if γ is primitive.
- OmTToOmS[*f_*]: expands out any $\Omega_{\text{tot}}(\gamma; y)$ in *f* into H factors and Ω_S's using (3.4);
- OmToOmb[*f_*]: expresses any $\Omega(\gamma; y)$ in *f* in terms of $\bar{\Omega}(\gamma; y)$'s;
- OmbToOm[*f_*]: expresses any $\bar{\Omega}(\gamma; y)$ in *f* in terms of $\Omega(\gamma; y)$'s;
- HiggsGToOmb[*Nvec_*,*y_*]: returns the (unevaluated) HN invariant $G_{\text{Higgs}}(\gamma, y)$ in terms of the rational refined indices $\Omega(\gamma; y)$ using (4.5);
- OmbToHiggsG[*Nvec_*,*y_*]: returns the (unevaluated) rational refined index $\Omega(\gamma; y)$ in terms of the (unevaluated) stack invariants $G_{\text{Higgs}}(\gamma, y)$ using (4.1);
- RandomCvec[*Nvec_*]: generates a random set of FI parameters ζ_i between -1 and 1, such that $\sum \zeta_i \text{Nvec}[[i]] = 0$;
- UnitStepWarn[*x_*]: gives 1 for $x > 0$, 0 for $x < 0$, and 1/2 if $x = 0$. Produces a warning in this latter case, irrespective of the value of \$QuiverVerbose. If so, the user is advised run the computation again with a different random perturbation.

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