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On some Hadamard-type inequalities for (h_1, h_2) -preinvex functions on the co-ordinates

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Abstract

We introduce the class of (h_1, h_2) -preinvex functions on the co-ordinates, and we prove some new inequalities of Hermite-Hadamard and Fejér type for such mappings.

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1 Introduction

A function $f : I \rightarrow R$, $I \subseteq R$ is an interval, is said to be a convex function on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.1)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1.1) holds, then f is concave.

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite-Hadamard inequality. This double inequality is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (1.2)$$

where $f : [a, b] \rightarrow R$ is a convex function. The above inequalities are in reversed order if f is a concave function.

In 1978, Breckner introduced an s -convex function as a generalization of a convex function [1].

Such a function is defined in the following way: a function $f : [0, \infty) \rightarrow R$ is said to be s -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (1.3)$$

holds for all $x, y \in \infty$, $t \in [0, 1]$ and for fixed $s \in (0, 1]$.

Of course, s -convexity means just convexity when $s = 1$.

In [2], Dragomir and Fitzpatrick proved the following variant of the Hermite-Hadamard inequality, which holds for s -convex functions in the second sense:

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

In the paper [3] a large class of non-negative functions, the so-called h -convex functions, is considered. This class contains several well-known classes of functions such as non-negative convex functions and s -convex in the second sense functions. This class is defined in the following way: a non-negative function $f : I \rightarrow R$, $I \subseteq R$ is an interval, is called h -convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.5}$$

holds for all $x, y \in I$, $t \in (0, 1)$, where $h : J \rightarrow R$ is a non-negative function, $h \neq 0$ and J is an interval, $(0, 1) \subseteq J$.

In the further text, functions h and f are considered without assumption of non-negativity.

In [4] Sarikaya, Saglam and Yildirim proved that for an h -convex function the following variant of the Hadamard inequality is fulfilled:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a)+f(b)] \cdot \int_0^1 h(t) dt. \tag{1.6}$$

In [5] Bombardelli and Varošanec proved that for an h -convex function the following variant of the Hermite-Hadamard-Fejér inequality holds:

$$\begin{aligned} \frac{\int_a^b w(x) dx}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) &\leq \int_a^b f(x)w(x) dx \\ &\leq (b-a)(f(a)+f(b)) \int_0^1 h(t)w(ta+(1-t)b) dt, \end{aligned} \tag{1.7}$$

where $w : [a, b] \rightarrow R$, $w \geq 0$ and symmetric with respect to $\frac{a+b}{2}$.

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir [6] as follows.

Let us consider a bidimensional $\Delta = [a, b] \times [c, d]$ in R^2 with $a < b$ and $c < d$. A mapping $f : \Delta \rightarrow R$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \rightarrow R$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow R$, $f_x(v) = f(x, v)$ are convex for all $x \in [a, b]$ and $y \in [c, d]$.

In the same article, Dragomir established the following Hadamard-type inequalities for convex functions on the co-ordinates:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \\ &\leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \end{aligned} \tag{1.8}$$

The concept of s -convex functions on the co-ordinates was introduced by Alomari and Darus [7]. Such a function is defined in following way: the mapping $f : \Delta \rightarrow R$ is s -convex

in the second sense if the partial mappings $f_y : [a, b] \rightarrow R$ and $f_x : [c, d] \rightarrow R$ are s -convex in the second sense.

In the same paper, they proved the following inequality for an s -convex function:

$$4^{s-1}f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dx dy \leq \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{(s+1)^2}. \tag{1.9}$$

For refinements and counterparts of convex and s -convex functions on the co-ordinates, see [6–10].

The main purpose of this paper is to introduce the class of (h_1, h_2) -preinvex functions on the co-ordinates and establish new inequalities like those given by Dragomir in [6] and Bombardelli and Varošanec in [5].

Throughout this paper, we assume that considered integrals exist.

2 Main results

Let $f : X \rightarrow R$ and $\eta : X \times X \rightarrow R^n$, where X is a nonempty closed set in R^n , be continuous functions. First, we recall the following well-known results and concepts; see [11–16] and the references therein.

Definition 2.1 Let $u \in X$. Then the set X is said to be invex at u with respect to η if

$$u + t\eta(v, u) \in X$$

for all $v \in X$ and $t \in [0, 1]$.

X is said to be an invex set with respect to η if X is invex at each $u \in X$.

Definition 2.2 The function f on the invex set X is said to be preinvex with respect to η if

$$f(u + t\eta(v, u)) \leq (1-t)f(u) + tf(v)$$

for all $u, v \in X$ and $t \in [0, 1]$.

We also need the following assumption regarding the function η which is due to Mohan and Neogy [11].

Condition C Let $X \subseteq R$ be an open invex subset with respect to η . For any $x, y \in X$ and any $t \in [0, 1]$,

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y),$$

$$\eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y).$$

Note that for every $x, y \in X$ and every $t_1, t_2 \in [0, 1]$ from Condition C, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

In [12], Noor proved the Hermite-Hadamard inequality for preinvex functions

$$f\left(a + \frac{1}{2}\eta(b, a)\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq \frac{f(a) + f(b)}{2}. \tag{2.1}$$

Definition 2.3 Let $h : [0, 1] \rightarrow R$ be a non-negative function, $h \not\equiv 0$. The non-negative function f on the invex set X is said to be h -preinvex with respect to η if

$$f(u + t\eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v)$$

for each $u, v \in X$ and $t \in [0, 1]$.

Let us note that:

- if $\eta(v, u) = v - u$, then we get the definition of an h -convex function introduced by Varošanec in [3];
- if $h(t) = t$, then our definition reduces to the definition of a preinvex function;
- if $\eta(v, u) = v - u$ and $h(t) = t$, then we obtain the definition of a convex function.

Now let X_1 and X_2 be nonempty subsets of R^n , let $\eta_1 : X_1 \times X_1 \rightarrow R^n$ and $\eta_2 : X_2 \times X_2 \rightarrow R^n$.

Definition 2.4 Let $(u, v) \in X_1 \times X_2$. We say $X_1 \times X_2$ is invex at (u, v) with respect to η_1 and η_2 if for each $(x, y) \in X_1 \times X_2$ and $t_1, t_2 \in [0, 1]$,

$$(u + t_1\eta_1(x, u), v + t_2\eta_2(y, v)) \in X_1 \times X_2.$$

$X_1 \times X_2$ is said to be an invex set with respect to η_1 and η_2 if $X_1 \times X_2$ is invex at each $(u, v) \in X_1 \times X_2$.

Definition 2.5 Let h_1 and h_2 be non-negative functions on $[0, 1]$, $h_1 \not\equiv 0$, $h_2 \not\equiv 0$. The non-negative function f on the invex set $X_1 \times X_2$ is said to be co-ordinated (h_1, h_2) -preinvex with respect to η_1 and η_2 if the partial mappings $f_y : X_1 \rightarrow R, f_y(x) = f(x, y)$ and $f_x : X_2 \rightarrow R, f_x(y) = f(x, y)$ are h_1 -preinvex with respect to η_1 and h_2 -preinvex with respect to η_2 , respectively, for all $y \in X_2$ and $x \in X_1$.

If $\eta_1(x, u) = x - u$ and $\eta_2(y, v) = y - v$, then the function f is called (h_1, h_2) -convex on the co-ordinates.

Remark 1 From the above definition it follows that if f is a co-ordinated (h_1, h_2) -preinvex function, then

$$\begin{aligned} & f(x + t_1\eta_1(b, x), y + t_2\eta_2(d, y)) \\ & \leq h_1(1 - t_1)f(x, y + t_2\eta_2(d, y)) + h_1(t_1)f(b, y + t_2\eta_2(d, y)) \\ & \leq h_1(1 - t_1)h_2(1 - t_2)f(x, y) + h_1(1 - t_1)h_2(t_2)f(x, d) \\ & \quad + h_1(t_1)h_2(1 - t_2)f(b, y) + h_1(t_1)h_2(t_2)f(b, d). \end{aligned}$$

Remark 2 Let us note that if $\eta_1(x, u) = x - u$, $\eta_2(y, v) = y - v$, $t_1 = t_2$ and $h_1(t) = h_2(t) = t$, then our definition of a co-ordinated (h_1, h_2) -preinvex function reduces to the definition

of a convex function on the co-ordinates proposed by Dragomir [6]. Moreover, if $h_1(t) = h_2(t) = t^s$, then our definition reduces to the definition of an s -convex function on the co-ordinates proposed by Alomari and Darus [7].

Now, we will prove the Hadamard inequality for the new class functions.

Theorem 2.1 *Suppose that $f : [a, a + \eta(b, a)] \rightarrow R$ is an h -preinvex function, Condition C for η holds and $a < a + \eta(b, a)$, $h(\frac{1}{2}) > 0$. Then the following inequalities hold:*

$$\frac{1}{2h(\frac{1}{2})} f\left(a + \frac{1}{2}\eta(b, a)\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq [f(a) + f(b)] \cdot \int_0^1 h(t) dt. \quad (2.2)$$

Proof From the definition of an h -preinvex function, we have that

$$f(a + t\eta(b, a)) \leq h(1 - t)f(a) + h(t)f(b).$$

Thus, by integrating, we obtain

$$\int_0^1 f(a + t\eta(b, a)) dt \leq f(a) \int_0^1 h(1 - t) dt + f(b) \int_0^1 h(t) dt = [f(a) + f(b)] \int_0^1 h(t) dt.$$

But

$$\int_0^1 f(a + t\eta(b, a)) dt = \frac{1}{\eta(b, a)} \cdot \int_a^{a+\eta(b, a)} f(x) dx.$$

So,

$$\frac{1}{\eta(b, a)} \cdot \int_a^{a+\eta(b, a)} f(x) dx \leq [f(a) + f(b)] \int_0^1 h(t) dt.$$

The proof of the second inequality follows by using the definition of an h -preinvex function, Condition C for η and integrating over $[0, 1]$.

That is,

$$\begin{aligned} f\left(a + \frac{1}{2}\eta(b, a)\right) &= f(a + t\eta(b, a) + \frac{1}{2}\eta(a + (1 - t)\eta(b, a), a + t\eta(b, a))) \\ &\leq h\left(\frac{1}{2}\right) [f(a + t\eta(b, a)) + f(a + (1 - t)\eta(b, a))], \\ f\left(a + \frac{1}{2}\eta(b, a)\right) &\leq h\left(\frac{1}{2}\right) \left[\int_0^1 f(a + t\eta(b, a)) dt + \int_0^1 f(a + (1 - t)\eta(b, a)) dt \right], \\ f\left(a + \frac{1}{2}\eta(b, a)\right) &\leq 2 \cdot h\left(\frac{1}{2}\right) \frac{1}{\eta(b, a)} \cdot \int_a^{a+\eta(b, a)} f(x) dx. \end{aligned}$$

The proof is complete. □

Theorem 2.2 *Suppose that $f : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$ is an (h_1, h_2) -preinvex function on the co-ordinates with respect to η_1 and η_2 , Condition C for η_1 and η_2 is fulfilled,*

and $a < a + \eta_1(b, a)$, $c < c + \eta_2(d, c)$, and $h_1(\frac{1}{2}) > 0$, $h_2(\frac{1}{2}) > 0$. Then one has the following inequalities:

$$\begin{aligned}
 & \frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \\
 & \leq \frac{1}{4 \cdot h_1(\frac{1}{2})\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2}\eta_1(b, a), y\right) dy \\
 & \quad + \frac{1}{4 \cdot h_2(\frac{1}{2})\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) dx \\
 & \leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy \\
 & \leq \frac{1}{2\eta_1(b, a)} \int_0^1 h_2(t_2) dt_2 \left[\int_a^{a+\eta_1(b, a)} f(x, c) dx + \int_a^{a+\eta_1(b, a)} f(x, d) dx \right] \\
 & \quad + \frac{1}{2\eta_2(d, c)} \int_0^1 h_1(t_1) dt_1 \left[\int_c^{c+\eta_2(d, c)} f(a, y) dy + \int_c^{c+\eta_2(d, c)} f(b, y) dy \right] \\
 & \leq [f(a, c) + f(b, c) + f(a, d) + f(b, d)] \int_0^1 h_1(t_1) dt_1 \cdot \int_0^1 h_2(t_2) dt_2. \tag{2.3}
 \end{aligned}$$

Proof Since f is (h_1, h_2) -preinvex on the co-ordinates, it follows that the mapping f_x is h_2 -preinvex and the mapping f_y is h_1 -preinvex. Then, by the inequality (2.2), one has

$$\begin{aligned}
 \frac{1}{2h_2(\frac{1}{2})}f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) & \leq \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(x, y) dy \\
 & \leq [f(x, c) + f(x, d)] \int_0^1 h_2(t) dt
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{1}{2h_1(\frac{1}{2})}f\left(a + \frac{1}{2}\eta_1(b, a), y\right) & \leq \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x, y) dx \\
 & \leq [f(a, y) + f(b, y)] \int_0^1 h_1(t) dt.
 \end{aligned}$$

Dividing the above inequalities for $\eta_1(b, a)$ and $\eta_2(d, c)$ and then integrating the resulting inequalities on $[a, a + \eta_1(b, a)]$ and $[c, c + \eta_2(d, c)]$, respectively, we have

$$\begin{aligned}
 & \frac{1}{\eta_1(b, a) \cdot 2h_2(\frac{1}{2})} \int_a^{a+\eta_1(b, a)} f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) dx \\
 & \leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy \\
 & \leq \frac{1}{\eta_1(b, a)} \int_0^1 h_2(t) dt \left[\int_a^{a+\eta_1(b, a)} f(x, c) dx + \int_a^{a+\eta_1(b, a)} f(x, d) dx \right]
 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\eta_2(b, a) \cdot 2h_1(\frac{1}{2})} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2}\eta_1(b, a), y\right) dy \\ & \leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy \\ & \leq \frac{1}{\eta_2(d, c)} \int_0^1 h_1(t) dt \left[\int_c^{c+\eta_2(d, c)} f(a, y) dy + \int_c^{c+\eta_2(d, c)} f(b, y) dy \right]. \end{aligned}$$

Summing the above inequalities, we get the second and the third inequalities in (2.3).

By the inequality (2.2), we also have

$$\frac{1}{2h_2(\frac{1}{2})} f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \leq \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2}\eta_1(b, a), y\right) dy$$

and

$$\frac{1}{2h_1(\frac{1}{2})} f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \leq \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) dx,$$

which give, by addition, the first inequality in (2.3).

Finally, by the same inequality (2.2), we can also state

$$\begin{aligned} & \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(a, y) dy \leq [f(a, c) + f(a, d)] \int_0^1 h_2(t) dt, \\ & \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(b, y) dy \leq [f(b, c) + f(b, d)] \int_0^1 h_2(t) dt, \\ & \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x, c) dx \leq [f(a, c) + f(b, c)] \int_0^1 h_1(t) dt, \\ & \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x, d) dx \leq [f(a, d) + f(b, d)] \int_0^1 h_1(t) dt, \end{aligned}$$

which give, by addition, the last inequality in (2.3). □

Remark 3 In particular, for $\eta_1(b, a) = b - a$, $\eta_2(d, c) = d - c$, $h_1(t_1) = h_2(t_2) = t$, we get the inequalities obtained by Dragomir [6] for functions convex on the co-ordinates on the rectangle from the plane R^2 .

Remark 4 If $\eta_1(b, a) = b - a$, $\eta_2(d, c) = d - c$, and $h_1(t_1) = h_2(t_2) = t^s$, then we get the inequalities obtained by Alomari and Darus in [7] for s -convex functions on the co-ordinates on the rectangle from the plane R^2 .

Theorem 2.3 Let $f, g : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$ with $a < a + \eta_1(b, a)$, $c < c + \eta_2(d, c)$. If f is (h_1, h_2) -preinvex on the co-ordinates and g is (k_1, k_2) -preinvex on the co-

ordinates with respect to η_1 and η_2 , then

$$\begin{aligned} & \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)g(x, y) \, dx \, dy \\ & \leq M_1(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(t_1)k_2(t_2) \, dt_1 \, dt_2 \\ & \quad + M_2(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(t_1)k_2(1-t_2) \, dt_1 \, dt_2 \\ & \quad + M_3(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(1-t_1)k_2(t_2) \, dt_1 \, dt_2 \\ & \quad + M_4(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(1-t_1)k_2(1-t_2) \, dt_1 \, dt_2, \end{aligned}$$

where

$$\begin{aligned} M_1(a, b, c, d) &= f(a, c)g(a, c) + f(a, d)g(a, d) + f(b, c)g(b, c) + f(b, d)g(b, d), \\ M_2(a, b, c, d) &= f(a, c)g(a, d) + f(a, d)g(a, c) + f(b, c)g(b, d) + f(b, d)g(b, c), \\ M_3(a, b, c, d) &= f(a, c)g(b, c) + f(a, d)g(b, d) + f(b, c)g(a, c) + f(b, d)g(a, d), \\ M_4(a, b, c, d) &= f(a, c)g(b, d) + f(a, d)g(b, c) + f(b, c)g(a, d) + f(b, d)g(a, c). \end{aligned}$$

Proof Since f is (h_1, h_2) -preinvex on the co-ordinates and g is (k_1, k_2) -preinvex on the co-ordinates with respect to η_1 and η_2 , it follows that

$$\begin{aligned} & f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq h_1(1-t_1)h_2(1-t_2)f(a, c) + h_1(1-t_1)h_2(t_2)f(a, d) \\ & \quad + h_1(t_1)h_2(1-t_2)f(b, c) + h_1(t_1)h_2(t_2)f(b, d) \end{aligned}$$

and

$$\begin{aligned} & g(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq k_1(1-t_1)k_2(1-t_2)g(a, c) + k_1(1-t_1)k_2(t_2)g(a, d) \\ & \quad + k_1(t_1)k_2(1-t_2)g(b, c) + k_1(t_1)k_2(t_2)g(b, d). \end{aligned}$$

Multiplying the above inequalities and integrating over $[0, 1]^2$ and using the fact that

$$\begin{aligned} & \int_0^1 \int_0^1 f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \cdot g(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \, dt_1 \, dt_2 \\ & = \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)g(x, y) \, dx \, dy, \end{aligned}$$

we obtain our inequality. □

In the next two theorems, we will prove the so-called Hermite-Hadamard-Fejér inequalities for an (h_1, h_2) -preinvex function.

Theorem 2.4 Let $f : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$ be (h_1, h_2) -preinvex on the co-ordinates with respect to η_1 and η_2 , $a < a + \eta_1(b, a)$, $c < c + \eta_2(d, c)$, and $w : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$, $w \geq 0$, symmetric with respect to

$$\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c) \right).$$

Then

$$\begin{aligned} & \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) dx dy \\ & \leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \cdot \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) dt_1 dt_2. \end{aligned} \tag{2.4}$$

Proof From the definition of (h_1, h_2) -preinvex on the co-ordinates with respect to η_1 and η_2 , we have

(a)

$$\begin{aligned} & f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq h_1(1 - t_1)h_2(1 - t_2)f(a, c) + h_1(1 - t_1)h_2(t_2)f(a, d) \\ & \quad + h_1(t_1)h_2(1 - t_2)f(b, c) + h_1(t_1)h_2(t_2)f(b, d), \end{aligned}$$

(b)

$$\begin{aligned} & f(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) \\ & \leq h_1(t_1)h_2(t_2)f(a, c) + h_1(t_1)h_2(1 - t_2)f(a, d) \\ & \quad + h_1(1 - t_1)h_2(t_2)f(b, c) + h_1(1 - t_1)h_2(1 - t_2)f(b, d), \end{aligned}$$

(c)

$$\begin{aligned} & f(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) \\ & \leq h_1(1 - t_1)h_2(t_2)f(a, c) + h_1(1 - t_1)h_2(1 - t_2)f(a, d) \\ & \quad + h_1(t_1)h_2(t_2)f(b, c) + h_1(t_1)h_2(1 - t_2)f(b, d), \end{aligned}$$

(d)

$$\begin{aligned} & f(a + (1 - t_1)\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq h_1(t_1)h_2(1 - t_2)f(a, c) + h_1(t_1)h_2(t_2)f(a, d) \\ & \quad + h_1(1 - t_1)h_2(1 - t_2)f(b, c) + h_1(1 - t_1)h_2(t_2)f(b, d). \end{aligned}$$

Multiplying both sides of the above inequalities by $w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c))$, $w(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))$, $w(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))$, $w(a + (1 - t_1)\eta_1(b, a), c +$

$t_2\eta_2(d, c)$), respectively, adding and integrating over $[0, 1]^2$, we obtain

$$\begin{aligned} & \frac{4}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) \, dx \, dy \\ & \leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \quad \cdot 4 \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \, dt_1 \, dt_2, \end{aligned}$$

where we use the symmetricity of the w with respect to $(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c))$, which completes the proof. \square

Theorem 2.5 *Let $f : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$ be (h_1, h_2) -preinvex on the co-ordinates with respect to η_1 and η_2 , and $a < a + \eta_1(b, a)$, $c < c + \eta_2(d, c)$, $w : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$, $w \geq 0$, symmetric with respect to $(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c))$. Then, if Condition C for η_1 and η_2 is fulfilled, we have*

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \cdot \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} w(x, y) \, dx \, dy \\ & \leq 4 \cdot h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \cdot \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) \, dx \, dy. \end{aligned} \tag{2.5}$$

Proof Using the definition of an (h_1, h_2) -preinvex function on the co-ordinates and Condition C for η_1 and η_2 , we obtain

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \\ & \leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \cdot [f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \quad + f(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) + f(a + (1 - t_1)\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \quad + f(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))]. \end{aligned}$$

Now, we multiply it by $w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) = w(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) = w(a + (1 - t_1)\eta_1(b, a), c + t_2\eta_2(d, c)) = w(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))$ and integrate over $[0, 1]^2$ to obtain the inequality

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \int_0^1 \int_0^1 w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \, dt_1 \, dt_2 \\ & = f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} w(x, y) \, dx \, dy \\ & \leq 4 \cdot h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) \, dx \, dy, \end{aligned}$$

which completes the proof. \square

Now, for a mapping $f : [a, b] \times [c, d] \rightarrow R$, let us define a mapping $H : [0, 1]^2 \rightarrow R$ in the following way:

$$H(t, r) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy. \tag{2.6}$$

Some properties of this mapping for a convex on the co-ordinates function and an s -convex on the co-ordinates function are given in [6, 7], respectively. Here we investigate which of these properties can be generalized for (h_1, h_2) -convex on the co-ordinates functions.

Theorem 2.6 *Suppose that $f : [a, b] \times [c, d]$ is (h_1, h_2) -convex on the co-ordinates. Then:*

- (i) *The mapping H is (h_1, h_2) -convex on the co-ordinates on $[0, 1]^2$,*
- (ii) *$4h_1(\frac{1}{2})h_2(\frac{1}{2})H(t, r) \geq H(0, 0)$ for any $(t, r) \in [0, 1]^2$.*

Proof (i) The (h_1, h_2) -convexity on the co-ordinates of the mapping H is a consequence of the (h_1, h_2) -convexity on the co-ordinates of the function f . Namely, for $r \in [0, 1]$ and for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0, 1]$, we have:

$$\begin{aligned} & H(\alpha t_1 + \beta t_2, r) \\ &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(\alpha\left(t_1x + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2x + (1-t_2)\frac{a+b}{2}\right), ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &\leq h_1(\alpha) \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_1x + (1-t_1)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &\quad + h_1(\beta) \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_2x + (1-t_2)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &= h_1(\alpha)H(t_1, r) + h_1(\beta)H(t_2, r). \end{aligned}$$

Similarly, if $t \in [0, 1]$ is fixed, then for all $r_1, r_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we also have

$$H(t, \alpha r_1 + \beta r_2) \leq h_2(\alpha)H(t, r_1) + h_2(\beta)H(t, r_2),$$

which means that H is (h_1, h_2) -convex on the co-ordinates.

- (ii) After changing the variables $u = tx + (1-t)\frac{a+b}{2}$ and $v = ry + (1-r)\frac{c+d}{2}$, we have

$$\begin{aligned} H(t, r) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &= \frac{1}{(b-a)(d-c)} \int_{u_L}^{u_U} \int_{v_L}^{v_U} f(u, v) \frac{b-a}{u_U - u_L} \cdot \frac{d-c}{v_U - v_L} du dv \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(u_U - u_L)(v_U - v_L)} \int_{u_L}^{u_U} \int_{v_L}^{v_U} f(u, v) \, du \, dv \\
 &\geq \frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right),
 \end{aligned}$$

where $u_L = ta + (1-t)\frac{a+b}{2}$, $u_U = tb + (1-t)\frac{a+b}{2}$, $v_L = rc + (1-r)\frac{c+d}{2}$ and $v_U = rd + (1-r)\frac{c+d}{2}$, which completes the proof. \square

Remark 5 If f is convex on the co-ordinates, then we get $H(t, r) \geq H(0, 0)$. If f is s -convex on the co-ordinates in the second sense, then we have the inequality $H(t, r) \geq 4^{s-1}H(0, 0)$.

Competing interests

The author declares that he has no competing interests.

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