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ORIGINAL PAPER

Note on Group Distance Magic Graphs $G[C_4]$

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Abstract A *group distance magic labeling* or a $\mathcal{G}$-distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection f from V to an Abelian group $\mathcal{G}$ of order n such that the weight $w(x) = \sum_{y \in N_G(x)} f(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \mathcal{G}$, called the *magic constant*. In this paper we will show that if G is a graph of order $n = 2^p(2k + 1)$ for some natural numbers p, k such that $\deg(v) \equiv c \pmod{2^{p+1}}$ for some constant c for any $v \in V(G)$, then there exists a $\mathcal{G}$-distance magic labeling for any Abelian group $\mathcal{G}$ of order $4n$ for the composition $G[C_4]$. Moreover we prove that if $\mathcal{G}$ is an arbitrary Abelian group of order $4n$ such that $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order n, then there exists a $\mathcal{G}$-distance magic labeling for any graph $G[C_4]$, where G is a graph of order n and n is an arbitrary natural number.

Keywords Distance magic labeling · Magic constant · Sigma labeling · Graph labeling · Abelian group · Composition of graphs · Lexicographic product of graphs

Mathematics Subject Classification (2000) 05C76 · 05C78

1 Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph G whose order we denote by $|G| = n$. Write $V(G)$ for the vertex set and $E(G)$ for the

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edge set of the graph G. The *neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the degree $\deg(x)$ of x is $|N(x)|$, the size of the neighborhood of x.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $l\colon V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer μ (called the *magic constant*) such that $\sum_{y \in N_G(x)} l(y) = \mu$ for every $x \in V$. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph* ([14]). The sum $\sum_{y \in N_G(x)} l(y)$ is called the *weight of the vertex x* and denoted by $w(x)$. The concept of distance magic labeling has been motivated by the construction of magic squares.

The following observations were independently proved:

Observation 11 [10–12,14] *Let G be an r-regular distance magic graph on n vertices. Then $\mu = \frac{r(n+1)}{2}$.*

Observation 12 [10–12,14] *No r-regular graph with r odd can be a distance magic graph.*

The problem of distance magic labeling of r-regular graphs has been studied recently (see [2,3,6,11,13]). It is interesting that if you replace each vertex of an r-regular G graph by some specific p-regular graph (like C_4 or $\overline{K}_{2n}$; cf. Definition 13), then the obtained graph H is a distance magic graph. More formally, we have the following definition:

Definition 13 Let G and H be two graphs where $\{x^1, x^2, \ldots, x^p\}$ is the vertex set of G. Based upon the graph G, an isomorphic copy H^j of H replaces every vertex x^j, for $j = 1, 2, \ldots, p$, in such a way that a vertex in H^j is adjacent to a vertex in H^i if and only if $x^j x^i$ was an edge in G. Let $G[H]$ denote the resulting graph. The graph $G[H]$ is called the composition of the graphs G and H [8].

The graph $G[H]$ is also called the *lexicographic product* and denoted by $G \circ H$ (see [9]).

Miller at al. [11] proved the following results.

Theorem 14 [11] *The cycle C_n of length n is a distance magic graph if and only if $n = 4$.*

Theorem 15 [11] *Let $r \geq 1$, $n \geq 3$, G be an r-regular graph and C_n be the cycle of length n. The graph $G[C_n]$ admits a distance magic labeling if and only if $n = 4$.*

Theorem 16 [11] *Let G be an arbitrary regular graph. Then $G[\overline{K}_n]$ is distance magic for any even n.*

The following problem was posted in [2].

Problem 17 [2] *If G is a non-regular graph, determine if there is a distance magic labeling of $G[C_4]$.*

It seems to be very hard to characterize such graphs. For example there were considered all graphs $K_{m,n}[C_4]$ for $1 \leq m < n \leq 2700$ and only $K_{9,21}[C_4]$, $K_{20,32}[C_4]$, $K_{428,548}[C_4]$ are distance magic graphs (see [1]).

Froncek in [5] defined the notion of *group distance magic graphs*, i.e. the graphs allowing the bijective labeling of vertices with elements of an Abelian group resulting in constant sums of neighbor labels.

Definition 18 A group distance magic labeling or a $\mathcal{G}$-distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection f from V to an Abelian group $\mathcal{G}$ of order n such that the weight $w(x) = \sum_{y \in N_G(x)} f(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \mathcal{G}$, called the magic constant.

Let G be a distance magic graph of order n with the magic constant μ'. If we replace the label n in a distance magic labeling for the graph G by the label 0, then we obtain a $\mathbb{Z}_n$-distance magic labeling for the graph G with the magic constant $\mu \equiv \mu' \pmod{n}$. Hence every distance magic graph with n vertices admits a $\mathbb{Z}_n$-distance magic labeling. Although a $\mathbb{Z}_n$-distance magic graph on n vertices is not necessarily a distance magic graph (see [5]), it was proved that Observation 12 also holds for a $\mathbb{Z}_n$-distance magic labeling [4].

Observation 19 [4] *Let r be a positive odd integer. No r-regular graph on n vertices can be a $\mathbb{Z}_n$-distance magic graph.*

In this paper we will prove that if G is a graph of order $n = 2^p(2k+1)$ for some natural numbers p, k such that $\deg(v) \equiv c \pmod{2^{p+1}}$ for some constant c for any $v \in V(G)$, then there exists a $\mathcal{G}$-distance magic labeling for any Abelian group $\mathcal{G}$ of order $4n$ for the graph $G[C_4]$. Moreover we show that if $\mathcal{G}$ is an Abelian group of order $4n$ such that $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order n, then there exists a $\mathcal{G}$-distance magic labeling for any graph $G[C_4]$, where G is a graph of order n and n is an arbitrary natural number.

2 Main Results

We start with the following lemma.

Lemma 21 *Let G be a graph of order n and $\mathcal{G}$ be an arbitrary Abelian group of order $4n$ such that $\mathcal{G} \cong \mathbb{Z}_{2^p} \times \mathcal{A}$ for $p \geq 2$ and some Abelian group $\mathcal{A}$ of order $\frac{n}{2^{p-2}}$. If* $\deg(v) \equiv c \pmod{2^{p-1}}$ *for some constant c and any $v \in V(G)$, then there exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.*

Proof Let G have the vertex set $V(G) = \{x^0, x^1, \ldots, x^{n-1}\}$, and let $C_4 = v_0v_1v_2v_3v_0$ and $H = G[C_4]$.

For $j = 0, 1, 2, 3$ let v_j^i be the vertices of H that replace x^i, $i = 0, 1, \ldots, n-1$, in G.

Using the isomorphism $\phi\colon \mathcal{G} \to \mathbb{Z}_{2^p} \times \mathcal{A}$, we identify every $g \in \mathcal{G}$ with its image $\phi(g) = (w, a_i)$, where $w \in \mathbb{Z}_{2^p}$ and $a_i \in \mathcal{A}$, $i = 0, 1, \ldots, \frac{n}{2^{p-2}} - 1$.

Label the vertices of H in the following way:

$$f(v^i_j) = \begin{cases} \left((2i+j) \mod 2^{p-1}, a_{\lfloor i\cdot 2^{-p+2}\rfloor}\right) & \text{for } j = 0, 1, \\ (2^p - 1, 0) - f(v^i_{j-2}) & \text{for } j = 2, 3 \end{cases}$$

for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, 2, 3$.

Notice that for every i

$$f(v^i_0) + f(v^i_2) = f(v^i_1) + f(v^i_3) = (2^p - 1, 0).$$

So the sum of the labels in the ith copy of C_4 is

$$f(v^i_0) + f(v^i_1) + f(v^i_2) + f(v^i_3) = (2^p - 2, 0),$$

which is independent on i. Since $\deg(v) \equiv c \pmod{2^{p-1}}$ for any $v \in V(G)$, therefore the weight of every $x \in V(H)$ is $w(x) = (-2c - 1, 0)$. □

Theorem 22 *Let G be a graph of order n and $\mathcal{G}$ be an arbitrary Abelian group of order $4n$ such that $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order n. There exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.*

Proof Let $V(G) = \{x^0, x^1, \ldots, x^{n-1}\}$ be the vertex set of G. Let $C_4 = v_0v_1v_2v_3v_0$ and $H = G[C_4]$.

For $j = 0, 1, 2, 3$ let v^i_j be the vertices of H that replace x^i, $i = 0, 1, \ldots, n-1$, in G.

Using the isomorphism $\phi\colon \mathcal{G} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$, we identify every $g \in \mathcal{G}$ with its image $\phi(g) = (j_1, j_2, a_i)$, where $j_1, j_2 \in \mathbb{Z}_2$ and $a_i \in \mathcal{A}$, $i = 0, 1, \ldots, n-1$.

Label the vertices of H in the following way:

$$f(v^i_j) = \begin{cases} (0, 0, a_i) & \text{for } j = 0, \\ (1, 0, a_i) & \text{for } j = 1, \\ (1, 1, -a_i) & \text{for } j = 2, \\ (0, 1, -a_i) & \text{for } j = 3 \end{cases}$$

for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, 2, 3$.

Notice that for every $i = 0, \ldots, n-1$

$$f(v^i_0) + f(v^i_2) = f(v^i_1) + f(v^i_3) = (1, 1, 0).$$

So the sum of the labels in the ith copy of C_4 is

$$f(v^i_0) + f(v^i_1) + f(v^i_2) + f(v^i_3) = (0, 0, 0),$$

which is independent on i. Therefore, for every $x \in V(H)$,

$$w(x) = (1, 1, 0).$$

□

Theorem 23 *Let G be a graph of order n and $\mathcal{G}$ be an Abelian group of order $4n$. If $n = 2^p(2k+1)$ for some natural numbers p, k and $\deg(v) \equiv c \pmod{2^{p+1}}$ for some constant c for any $v \in V(G)$, then there exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.*

Proof The fundamental theorem of finite Abelian groups states that the finite Abelian group $\mathcal{G}$ can be expressed as the direct sum of cyclic subgroups of prime-power order. This implies that $\mathcal{G} \cong \mathbb{Z}_{2^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_m^{\alpha_m}}$ for some $\alpha_0 > 0$, where $4n = 2^{\alpha_0} \prod_{i=1}^{m} p_i^{\alpha_i}$ and p_i for $i = 1, \ldots, m$ are not necessarily distinct primes. Suppose first that $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order n, then we are done by Theorem 22. Observe now that the assumption $\deg(v) \equiv c \pmod{2^{p+1}}$ and the unique (additive) decomposition of any natural number c into powers of 2 imply that there exist constants $c_1, c_2, \ldots, c_p$ such that $\deg(v) \equiv c_i \pmod{2^i}$, for $i = 1, 2, \ldots, p$, for any $v \in V(G)$. Hence if $\mathcal{G} \cong \mathbb{Z}_{2^{\alpha_0}} \times \mathcal{A}$ for some $2 \leq \alpha_0 \leq p+2$ and some Abelian group $\mathcal{A}$ of order $\frac{4n}{2^{\alpha_0}}$, then we obtain by Lemma 21 that there exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$. □

The next observation follows easily from the above Theorem 23, however the below Observation 25 shows an infinite family of Eulerian graphs with odd order such that none of these graphs is distance magic.

Observation 24 *Let G be a graph of odd order n and $\mathcal{G}$ be an Abelian group of order $4n$. If G is an Eulerian graph (i.e. all vertices of the graph G have even degrees), then there exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.*

Let us introduce the following definition. The *Dutch windmill graph* D_m^t is the graph obtained by taking $t > 1$ copies of the cycle C_m with a vertex c in common [7]. Thus for t being even a graph D_4^t is an Eulerian graph of odd order $3t+1$. Let the ith copy of a cycle C_4 in D_4^t be $cy^i x^i z^i c$, for $i = 0, \ldots, t-1$. Let now $C_4 = v_0 v_1 v_2 v_3 v_0$ and $H = D_4^t[C_4]$. For $j = 0, 1, 2, 3$, let x_j^i (y_j^i, z_j^i, c_j resp.) be the vertices of H that replace x^i (y^i, z^i, c, resp.) in D_4^t, $i = 0, 1, \ldots, t-1$.

Observation 25 *There does not exist a distance magic graph $D_4^t[C_4]$.*

Proof Suppose that l is a distance magic labeling of the graph $D_4^t[C_4]$ and $\mu = w(x)$, for all vertices $x \in V(D_4^t[C_4])$. It is easy to observe that there exist natural numbers a_c, a_x^i, a_y^i and a_z^i, $0 \leq i \leq t-1$, such that:

- $l(c_0) + l(c_2) = l(c_1) + l(c_3) = a_c$.
- $l(x_0^i) + l(x_2^i) = l(x_1^i) + l(x_3^i) = a_x^i$, for $0 \leq i \leq t-1$.
- $l(y_0^i) + l(y_2^i) = l(y_1^i) + l(y_3^i) = a_y^i$, for $0 \leq i \leq t-1$.
- $l(z_0^i) + l(z_2^i) = l(z_1^i) + l(z_3^i) = a_z^i$, for $0 \leq i \leq t-1$.

Since $a_z^i + 2a_x^i + 2a_c = w(z_j^i) = \mu = w(y_j^i) = a_y^i + 2a_x^i + 2a_c$, $j = 0, 1, 2, 3$, we obtain that $a_z^i = a_y^i =: a^i$ for $0 \leq i \leq t-1$. Moreover $w(x_j^i) = a_x^i + 4a^i = w(y_j^i) = a^i + 2a_x^i + 2a_c$, $j = 0, 1, 2, 3$, hence $a_x^i = 3a^i - 2a_c$ for $0 \leq i \leq t-1$. Because of $7a^i - 2a_c = w(z_j^i) = \mu = w(z_j^{i'}) = 7a^{i'} - 2a_c$, $j = 0, 1, 2, 3$, we obtain $a^i = a^{i'} =: a$, for all $0 \leq i, i' \leq t-1$.

Since $7a - 2a_c = \mu = w(c_j) = a_c + 4ta$, for $j = 0, 1, 2, 3$, it has to be that $3a_c = (7 - 4t)a$ and therefore $t = 1$ (because $a_c, a > 0$), a contradiction. □

The following observation shows that in the general case the condition on the degrees of the vertices $v \in V(G)$ in Theorem 23 is not necessary for the existence of a $\mathcal{G}$-distance magic labeling of a graph $G[C_4]$:

Observation 26 *Let $K_{p,q}$ be a complete bipartite graph with p even and q odd and let $\mathcal{G}$ be an Abelian group of order $4(p+q)$. There exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.*

Proof If $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order $n = p + q$, then there exists a $\mathcal{G}$-distance magic labeling for the graph $K_{p,q}[C_4]$ by Theorem 22. Suppose now that $\mathcal{G} \cong \mathbb{Z}_4 \times \mathcal{A}$ for some Abelian group $\mathcal{A}$ of order $p + q$. Let $K_{p,q}$ have the partition vertex sets $A = \{x^0, x^1, \ldots, x^{p-1}\}$, $B = \{y^0, y^1, \ldots, y^{q-1}\}$ and let $C_4 = v_0v_1v_2v_3v_0$. For $j = 0, 1, 2, 3$, let x_j^i be the vertices of $K_{p,q}[C_4]$ that replace the vertex $x^i, 0 \leq i \leq p-1$, in $K_{p,q}$, and y_j^l be the vertices that replace $y^l, 0 \leq l \leq q-1$.

Using the isomorphism $\phi\colon \mathcal{G} \to \mathbb{Z}_4 \times \mathcal{A}$, we identify every $g \in \mathcal{G}$ with its image $\phi(g) = (j, a_i)$, where $j \in \mathbb{Z}_4$ and $a_i \in \mathcal{A}$, $i = 0, 1, \ldots, p+q-1$.

Label the vertices of $K_{p,q}[C_4]$ in the following way:

$$f(x_j^i) = \begin{cases} (2j, a_i) & \text{for } j = 0, 1, \\ (1, 0) - f(x_{j-2}^i) & \text{for } j = 2, 3 \end{cases}$$

for $i = 0, 1, \ldots, p-1$ and $j = 0, 1, 2, 3$.

$$f(y_j^l) = \begin{cases} (2j, a_{p+l}) & \text{for } j = 0, 1, \\ (3, 0) - f(y_{j-2}^l) & \text{for } j = 2, 3 \end{cases}$$

for $l = 0, 1, \ldots, q-1$ and $j = 0, 1, 2, 3$.

Notice that

$$\begin{aligned} f(x_0^i) + f(x_2^i) = f(x_1^i) + f(x_3^i) = (1, 0), \\ f(y_0^l) + f(y_2^l) = f(y_1^l) + f(y_3^l) = (3, 0), \end{aligned}$$

for every $i = 0, \ldots, p-1$ and for every $l = 0, \ldots, q-1$.
This implies:

$$\begin{aligned} \sum_{i=0}^{p-1} \left(\sum_{j=0}^{3} f(x_j^i) \right) &= \quad p(2, 0) = (0, 0), \\ \sum_{l=0}^{q-1} \left(\sum_{j=0}^{3} f(y_j^l) \right) &= \quad q(2, 0) = (2, 0). \end{aligned}$$

Hence $w(x) = (3, 0)$ for every $x \in V(K_{p,q}[C_4])$. □

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