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Lifshitz black holes in higher spin gravity

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ABSTRACT: We study asymptotically Lifshitz solutions to three dimensional higher spin gravity in the $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ Chern-Simons formulation. We begin by specifying the most general connections satisfying Lifshitz boundary conditions, and we verify that their algebra of symmetries contains a Lifshitz sub-algebra. We then exhibit connections that can be viewed as higher spin Lifshitz black holes. We show that when suitable holonomy conditions are imposed, these black holes obey sensible thermodynamics and possess a gauge in which the corresponding metric exhibits a regular horizon.

Contents

1 Introduction

In the past few years there has been a renewed interest in higher spin gravity in various dimensions following the work of Vasiliev and collaborators (see [1] for a review). In the present paper we focus on higher spin theories in three spacetime dimensions. Gaberdiel and Gopakumar proposed a duality of the two dimensional W_N minimal model CFTs to three dimensional Vasiliev theory [2]. The original proposal has passed many checks and some refinements in recent years, see e.g. [5–12]. An interesting feature of the three dimensional Vasiliev theory [13] is that while it is a complicated nonlinear theory coupling an infinite tower of higher spin fields to scalar matter, if the scalars are linearized, the

theory can be reformulated in terms of a Chern-Simons theory with an infinite dimensional gauge algebra $hs(\lambda) \times hs(\lambda)$ [14–16]. The deformation parameter λ is associated with the 't Hooft coupling of the dual CFT [2]. The Chern-Simons theory simplifies if $\lambda = \pm N$, where N is an integer and the theory reduces to Chern-Simons theory with gauge group $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$ and is purely topological, corresponding to a theory of massless fields of spin $2, 3, \cdots, N$. Note that Einstein gravity with negative cosmological constant is included by taking $N = 2$ [3, 4].

The simplest solutions of the Chern-Simons theory correspond to AdS_3 vacua. The asymptotic symmetry of the AdS vacuum in $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$ higher spin gravity depends on the embedding of a $\mathrm{SL}(2, \mathbb{R})$ sub-algebra in $\mathrm{SL}(N, \mathbb{R})$. For the principal embedding one obtains W_N symmetry [17, 18], whereas for non-principal embeddings other higher spin algebras such as $W_N^{(2)}$ can occur [19, 20].

The construction of black holes in AdS/CFT is important since (large) black holes describe the dual CFT in thermal equilibrium at finite temperature. The BTZ solution [21] of three dimensional gravity has been a very important part of exploring the AdS/CFT correspondence (see [22] for a review). In higher spin theories the definition of what constitutes a black hole is nontrivial since the metric field transforms under higher spin gauge transformations [17] and hence the standard geometric characterization of a black hole, i.e. the existence of a horizon is not gauge invariant. In [23] a new criterion was proposed which uses the holonomy of the Chern-Simons gauge field around the contractable euclidean time circle to characterize a regular black hole. The holonomy condition has been applied to various black holes in 3 dimensional higher spin theories [24–27] and it has been checked by comparing bulk and CFT calculations of thermal correlation functions [28–30], see [31] for a review and a more extended list of references. Note that there are some puzzles remaining, for example there are two different proposals for the entropy, namely the "holomorphic" [23] and the "canonical" [32, 32] one. See [34–37] for recent work on the two proposals and their possible relation.

In the Chern-Simons formulation of of higher spin gravity, the W_N extension of the Virasoro symmetry of the boundary theory is obtained via the Drinfeld-Sokolov reduction by specifying asymptotic boundary fall off conditions for the gauge fields and considering nontrivial gauge transformations which respect these boundary conditions. If the boundary conditions are consistent then the boundary charges are integrable, finite and conserved and generate the (extended) symmetry algebra.

It is a very interesting question whether the higher spin gravity/CFT duality in three dimensions can be generalized to non-AdS backgrounds. In [38, 39] a general recipe and examples including Lobachevsky ($\mathbb{R} \times AdS_2$), Lifshitz, Schrödinger and warped AdS backgrounds were given. More recently the same philosophy was applied to flat space holography in [40, 41].

In the present paper we are interested in a construction and detailed analysis of higher spin realizations of asymptotically Lifshitz spacetimes. Such spacetimes provide candidates for a holographic description of field theories with Lifshitz scaling invariance. These theories exhibit an anisotropic scaling symmetry with respect to space and time $\vec{x} \to \lambda \vec{x}$ and $t \to \lambda^z t$, with $z \neq 1$ and are important in various condensed matter systems (see [42] for

references). In [42] a holographic Lifshitz spacetime solution of a gravity theory coupled to anti-symmetric tensor fields in four dimension was given. Subsequently Lifshitz space times have been ground in many (super)gravity theories, see e.g. [43–46]. In holographic theories black hole or black brane solutions provide the dual description of field theories at finite temperature (and chemical potential if the black holes are charged). For Lifshitz spacetimes the construction of black holes was initiated in [47–50], but most solutions in the literature are only known numerically.

In the present paper we focus mainly on the simplest three dimensional higher spin theory which is based on $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ Chern-Simons theory and corresponds to gravity coupled to a massless spin three field. For simplicity, most explicit calculations are performed in this theory, but we shall also comment on generalizations to $N > 3$ and $hs(\lambda)$.

The structure of the paper is as follows: in section 2 we give a brief review of the Chern-Simons formulation of higher spin gravity. In section 3 we review some salient features of field theories which enjoy Lifshitz scaling symmetry, and we discuss the holographic realization of such theories. We then review how the Lifshitz spacetime can be obtained as a solution to $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ Chern-Simons theory, and we demonstrate that the algebra generating Lifshitz isometries can be realized in a higher spin context.

In section 4 we construct black hole solutions with Lifshitz scaling, focusing on the simplest case of non-rotating black holes. We discuss the gauge freedom and the holonomy conditions as well as the thermodynamics. When the holonomy conditions are solved to express the temperature and chemical potential in terms of the extensive parameters there are six different branches. Only two of the six have positive temperature and entropy and are hence physically sensible. We consider two additional conditions on the branches, first the local thermodynamic stability and second the existence of a radial gauge where the metric exhibits a regular horizon and find that only one branch satisfies all of these conditions.

In section 5 we discuss generalizations of our work including the possibility of constructing rotating black hole solutions as well as Lifshitz black holes in $hs(\lambda)$ higher spin theory.

We close with a brief discussion of our results in section 6. For completeness we summarize our conventions for $\mathrm{SL}(3,\mathbb{R})$ and $hs(\lambda)$ in an appendix.

2 Chern-Simons formulation of higher spin gravity

The Chern-Simons formulation of three dimensional (higher spin) gravity is based on two copies of the Chern-Simons action at level k and $-k$ and gauge group $\mathrm{SL}(N,\mathbb{R}) \times \mathrm{SL}(N,\mathbb{R})$.

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \tag{2.1}$$

where

$$S_{CS}[A] = \frac{k}{4\pi} \int \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{2.2}$$

The equations of motion are simply flatness conditions,

$$F = dA + A \wedge A = 0, \qquad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = 0. \tag{2.3}$$

Ordinary gravity is given by the case $N = 2$; in the following we will mainly focus on the case $N = 3$. This theory was studied in detail in [17] and it was shown that the CS theory is equivalent to AdS gravity coupled to a massless spin three field. The vielbein and spin connection take values in the $\mathrm{SL}(3, \mathbb{R})$ Lie algebra and are related to the CS gauge fields as follows:

$$e_\mu = \frac{l}{2}(A_\mu - \bar{A}_\mu), \qquad \omega_\mu = \frac{1}{2}(A_\mu + \bar{A}_\mu). \tag{2.4}$$

In the following we set the length scale l to one for notational ease. Using the expression of the vielbein (2.4) in terms of the connection, the metric and spin 3 field can be expressed as

$$g_{\mu\nu} = \frac{1}{2}\operatorname{tr}(e_\mu e_\nu), \qquad \phi_{\mu\nu\rho} = \frac{1}{6}\operatorname{tr}(e_{(\mu} e_\nu e_{\rho)}). \tag{2.5}$$

The gauge transformations act on the Chern-Simons connections as follows

$$\delta A = d\Lambda + [A, \Lambda], \qquad \delta\bar{A} = d\bar{\Lambda} + [A, \bar{\Lambda}]. \tag{2.6}$$

In the construction of asymptotically AdS as well as asymptotically Lifshitz spacetimes, we employ a special choice of coordinates and choice of gauge. We define a radial coordinate ρ, where the holographic boundary will be located at $\rho \to \infty$. In addition we define a timelike coordinate t and a space like coordinate x, which can be either compact or non-compact and hence the boundary has either the topology of $\mathbb{R} \times S^1$ or $\mathbb{R} \times \mathbb{R}$. The "radial gauge" that we will use is constructed by defining $b = \exp(\rho L_0)$ and setting

$$A_\mu = b^{-1} a_\mu b + b^{-1}\partial_\mu b, \qquad \bar{A}_\mu = b\,\bar{a}_\mu b^{-1} + b\,\partial_\mu(b^{-1}). \tag{2.7}$$

where $a_\mu = a_\mu(t, x)$ and $\bar{a}_\mu = \bar{a}_\mu(t, x)$ do not depend on ρ.

3 Lifshitz spacetimes

Quantum field theories which exhibit a scaling symmetry which is anisotropic with respect to space and time

$$t \to \lambda^z t, \qquad x \to \lambda x \tag{3.1}$$

appear in many condensed matter systems. The dynamical scaling coefficient $z \neq 1$ breaks relativistic symmetry. If one augments the symmetry of the theory to include space and time translations, then one obtains a theory that is said to possess Lifshitz symemtry. Lifshitz symmetry can therefore be encoded as a Lie algebra generated by time translations H, spatial translations P and Lifshitz scalings D satisfying the following structure relations:

$$[P, H] = 0 \qquad [D, H] = zH \qquad [D, P] = P. \tag{3.2}$$

In two dimensions, conformal symmetry (with $z = 1$) implies a conserved, traceless and symmetric stress tensor. For theories with Lifshitz scaling the stress tensor does not have to be symmetric, since they do not possess boost invariance. The stress-energy complex for field theories in 1+1 dimensions with Lifshitz scaling exponent z contains the following objects: the energy density $\mathcal{E}$, the energy flux $\mathcal{E}^x$, the momentum density $\mathcal{P}_x$ and the stress energy tensor $\Pi_x{}^x$. These quantities satisfy the following conservation equations (see e.g. [52]):

$$\partial_t \mathcal{E} + \partial_x \mathcal{E}^x = 0, \qquad \partial_t \mathcal{P}_x + \partial_x \Pi_x{}^x = 0. \tag{3.3}$$

In addition, the Lifshitz scaling with exponent z implies a modified tracelessness condition

$$z\mathcal{E} + \Pi_x{}^x = 0. \tag{3.4}$$

The Lifshitz symmetries of a (1+1)-dimensional metric can be realized holographically with the following metric:

$$ds^2 = L^2 \Big(d\rho^2 - e^{2z\rho} dt^2 + e^{2\rho} dx^2 \Big) \tag{3.5}$$

where the Lifshitz scaling transformation corresponds to a translation $\rho \to \rho + \ln \lambda$. This metric is not a solution of Einstein gravity with negative cosmological constant; one has to add matter or higher derivative terms to the action to obtain it as a solution.

One can realize the $z = 2$ Lifshitz metric in the $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$ higher spin theory [39] by choosing the radial gauge as in (2.7) and by choosing the following connections $a = a_\mu \, dx^\mu$ and $\bar{a} = \bar{a}_\mu \, dx^\mu$:

$$a = W_2 \, dt + L_1 \, dx, \qquad \bar{a} = W_{-2} \, dt + L_{-1} \, dx. \tag{3.6}$$

It follows from (2.5) that this connection reproduces the Lifshitz metric (3.5) with scaling exponent $z = 2$. Lifshitz spacetimes with critical exponents $z > 2$ can be obtained using $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$ Chern-Simons theory with $N > 3$.

3.1 Asymptotically Lifshitz connections

Focusing on $N = 3$ and $z = 2$, we explore Chern-Simons connections that behave asymptotically like Lifshitz. In this section, we use primes to denote derivatives with respect to x and overdots to denote derivatives with respect to t. With the gauge connections defined in (2.7), we look for the most general, flat connections with the property that

$$A - A_{\mathrm{Lif}} \sim \mathcal{O}(1), \qquad \text{as } \rho \to \infty \tag{3.7}$$

$$\bar{A} - \bar{A}_{\mathrm{Lif}} \sim \mathcal{O}(1), \qquad \text{as } \rho \to \infty \tag{3.8}$$

where A_{Lif} and $\bar{A}_{\mathrm{Lif}}$ are the Lifshitz connections specified in (3.6). The most general connections that obey these asymptotics are obtained by adding terms to the Lifshitz connections a in (3.6) proprotional to W_0, W_{-1}, W_{-2} and L_0, L_{-1} (and similarly for $\bar{a}$). In particular, we consider the following ansatz:

$$a_t = W_2 + \ell_{t,0} L_0 + \ell_{t,-1} L_{-1} + w_{t,0} W_0 + w_{t,-1} W_{-1} + w_{t,-2} W_{-2}, \tag{3.9}$$

$$a_x = L_1 + \ell_{x,0} L_0 + \ell_{x,-1} L_{-1} + w_{x,0} W_0 + w_{x,-1} W_{-1} + w_{x,-2} W_{-2}. \tag{3.10}$$

Before applying flatness conditions, we allow all coefficients $\ell_{t,i}, \ell_{x,i}, w_{t,m}, w_{x,m}$ to be arbitrary functions of t and x. By suitable gauge transformations, we can set

$$w_{x,0} = 0, \qquad w_{x,-1} = 0, \qquad \ell_{x,0} = 0. \tag{3.11}$$

Employing the same notation as used in the higher spin black holes, we denote

$$\ell_{x,-1} = -\mathcal{L}, \qquad w_{x,-2} = \mathcal{W}, \tag{3.12}$$

and, after applying the flatness conditions,[1] we obtain

$$a_t = W_2 - 2\mathcal{L}W_0 + \frac{2}{3}\mathcal{L}'W_{-1} - 2\mathcal{W}L_{-1} + \left(\mathcal{L}^2 - \frac{1}{6}\mathcal{L}''\right)W_{-2}, \tag{3.13}$$

$$a_x = L_1 - \mathcal{L}L_{-1} + \mathcal{W}W_{-2}, \tag{3.14}$$

where henceforth, an over-dot denotes a t-derivative and a prime denotes an x-derivative. Flatness also results in the following evolution equations for $\mathcal{L}$ and $\mathcal{W}$:

$$\dot{\mathcal{L}} = 2\mathcal{W}' \tag{3.15}$$

$$\dot{\mathcal{W}} = \frac{4}{3}(\mathcal{L}^2)' - \frac{1}{6}\mathcal{L}'''. \tag{3.16}$$

If we follow the same procedure for the barred sector, imposing the condition (3.8), then we find the following asymptotically Lifshitz connections:

$$\bar{a}_t = W_{-2} - 2\bar{\mathcal{L}}W_0 - \frac{2}{3}\bar{\mathcal{L}}'W_1 + 2\bar{\mathcal{W}}L_1 + \left(\bar{\mathcal{L}}^2 - \frac{1}{6}\bar{\mathcal{L}}''\right)W_2, \tag{3.17}$$

$$\bar{a}_x = L_{-1} - \bar{\mathcal{L}}L_1 - \bar{\mathcal{W}}W_2. \tag{3.18}$$

where again the flatness conditions produce evolution equations for the barred quantities

$$\dot{\bar{\mathcal{L}}} = -2\bar{\mathcal{W}}', \tag{3.19}$$

$$\dot{\bar{\mathcal{W}}} = -\frac{4}{3}(\bar{\mathcal{L}}^2)' + \frac{1}{6}\bar{\mathcal{L}}''', \tag{3.20}$$

which can be obtained, from (3.15) and (3.16) by replacing $\mathcal{L}$ and $\mathcal{W}$ by $\bar{\mathcal{L}}$ and $-\bar{\mathcal{W}}$. The signs were chosen so that we can now express the quantities appearing in the energy-momentum complex (3.3) in terms of the parameters appearing in the connection as follows:

$$\begin{aligned}
\mathcal{E} &= \mathcal{W} + \bar{\mathcal{W}}, \\
\mathcal{P}_x &= \mathcal{L} - \bar{\mathcal{L}}, \\
\Pi_x^x &= -2\mathcal{W} - 2\bar{\mathcal{W}}, \\
\mathcal{E}^x &= -\left(\frac{4}{3}\mathcal{L}^2 - \frac{1}{6}\partial_x^2\mathcal{L}\right) + \left(\frac{4}{3}\bar{\mathcal{L}}^2 - \frac{1}{6}\partial_x^2\bar{\mathcal{L}}\right).
\end{aligned} \tag{3.21}$$

It is straightforward to verify that that evolution equations (3.15) and (3.16) imply the equations for the Lifshitz stress-tensor complex (following the terminology of [52]) with $z = 2$, given by (3.3) and (3.4).

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[1]See [51] for discussion of closely related connections and their symmetries.

3.2 Realization of Lifshitz symmetries

We now show that among the gauge transformations that leave the connections (3.13) and (3.14) form-invariant, there exist those that realize the Lifshitz algebra as a Poisson algebra of boundary charges. To begin, recall that for each gauge parameter Λ, the standard definition of the variations of asymptotic symmetry boundary charges in Chern-Simons theory is as follows [17]:

$$\delta Q(\Lambda) = -\frac{k}{2\pi} \int_{-\infty}^{\infty} dx \, \mathrm{tr}(\Lambda \delta A_x). \tag{3.22}$$

We now show that there exist gauge parameters $\Lambda_H, \Lambda_P, \Lambda_D$ that leave the asymptotically Lifshitz connections form-invariant. Moreover, we show that the variations $\delta Q(\Lambda_H), \delta Q(\Lambda_P)$ and $\delta Q(\Lambda_D)$ as defined in (3.22) are integrable and yield charges $Q(\Lambda_H), Q(\Lambda_P)$ and $Q(\Lambda_D)$ that realize the Lifshitz algebra as a Poisson algebra.

As our first step, we determine the most general gauge parameter that results in a gauge transformation that leaves the asymptotically Lifshitz connections form-invariant. The radial gauge (2.7) is preserved under gauge transformations if and only if the gauge parameter is of the form

$$\Lambda(\rho, t, x) = b^{-1}(\rho) \lambda(t, x) b(\rho). \tag{3.23}$$

Given this form, gauge transformations are characterized by the function λ and act on the connections as follows:

$$\delta_\lambda a_\mu = \partial_\mu \lambda + [a_\mu, \lambda]. \tag{3.24}$$

Now consider a general gauge parameter λ;

$$\lambda = \sum_{i=-1}^{1} \epsilon_i L_i + \sum_{m=-2}^{2} \chi_m W_m, \tag{3.25}$$

where $\epsilon_i = \epsilon_i(t, x)$ and $\chi_m = \chi_m(t, x)$. Gauge transformations are now explicitly given by

$$\delta_\lambda a_t = -2\delta\mathcal{L}W_0 + \frac{2}{3}(\delta\mathcal{L})'W_{-1} - 2\delta\mathcal{W}L_{-1} + \left(2\mathcal{L}\delta\mathcal{L} - \frac{1}{6}(\delta\mathcal{L})''\right)W_{-2}, \tag{3.26}$$

$$\delta_\lambda a_x = -\delta\mathcal{L}L_{-1} + \delta\mathcal{W}W_{-2}, \tag{3.27}$$

and enforcing form-invariance of the connections allows one to solve for all parameters ϵ_i and χ_i in terms of the two parameters $\epsilon = \epsilon_1$ and $\chi = \chi_2$.

$$
\begin{aligned}
\epsilon_0 &= -\epsilon', \\
\epsilon_{-1} &= -\mathcal{L}\epsilon + \frac{1}{2}\epsilon'' - 2\mathcal{W}\chi, \\
\chi_1 &= -\chi', \\
\chi_0 &= -2\mathcal{L}\chi + \frac{1}{2}\chi'', \\
\chi_{-1} &= \frac{2}{3}\mathcal{L}'\chi + \frac{5}{3}\mathcal{L}\chi' - \frac{1}{6}\chi''', \\
\chi_{-2} &= \mathcal{W}\epsilon + \mathcal{L}^2\chi - \frac{1}{6}\mathcal{L}''\chi - \frac{7}{12}\mathcal{L}'\chi' - \frac{2}{3}\mathcal{L}\chi'' + \frac{1}{24}\chi''''.
\end{aligned}
\tag{3.28}
$$

Form-invariance also gives evolution equations for ϵ and χ

$$\dot{\epsilon} = \frac{8}{3}\mathcal{L}\chi' - \frac{1}{6}\chi''', \tag{3.29}$$

$$\dot{\chi} = 2\epsilon', \tag{3.30}$$

and it constrains the forms of the variations $\delta\mathcal{L}$ and $\delta\mathcal{W}$

$$\delta\mathcal{L} = \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} + 2\chi\mathcal{W}' + 3\chi'\mathcal{W} - \frac{1}{2}\epsilon''', \tag{3.31}$$

$$\delta\mathcal{W} = \mathcal{W}'\epsilon + 3\mathcal{W}\epsilon' + \left(\frac{4}{3}(\mathcal{L}^2)' - \frac{1}{6}\mathcal{L}'''\right)\chi + \left(\frac{8}{3}\mathcal{L}^2 - \frac{3}{4}\mathcal{L}''\right)\chi' - \frac{5}{4}\mathcal{L}'\chi'' - \frac{5}{6}\mathcal{L}\chi''' + \frac{1}{24}\chi'''''. \tag{3.32}$$

Now that we know the precise form of the most general gauge parameters leaving the connections form-invariant, we attempt to identify which of these parameters Λ_H, Λ_P and Λ_D lead to charges that satisfy a Lifshitz algebra. To find these parameters, we first notice that given the Lifshitz metric (3.5), the Lifshitz algebra is geometrically realized by the following killing vectors:

$$\xi_H = \partial_t, \tag{3.33}$$

$$\xi_P = \partial_x, \tag{3.34}$$

$$\xi_D = \partial_\rho - x\partial_x - zt\partial_t. \tag{3.35}$$

Explicitly, one easily verifies that

$$[\xi_P, \xi_H] = 0, \qquad [\xi_D, \xi_H] = 2\xi_H, \qquad [\xi_D, \xi_P] = \xi_P. \tag{3.36}$$

This is precisely the Lifshitz algebra (3.2) with $z = 2$. These killing vectors generate spacetime diffeomorphisms, and there is a standard realization diffeomorphisms as gauge transformations in Chern-Simons theory via field-dependent gauge parameters [60]

$$\Lambda = -\xi^\mu A_\mu. \tag{3.37}$$

For the asymptotically Lifshitz connections of section 3.1, we expect that there exists a realization of the Lifshitz algebra, but it is not immediately obvious which gauge parameters one should pick that yield charges satisfying this algebra. However, motivated by the method of generating diffeomorphisms via gauge transformations, we try the following:

$$\Lambda_H = -(\xi_H)^\mu A_\mu = b^{-1}(-a_t)b, \tag{3.38}$$

$$\Lambda_P = -(\xi_P)^\mu A_\mu = b^{-1}(-a_x)b, \tag{3.39}$$

$$\Lambda_D = -(\xi_D)^\mu A_\mu = b^{-1}(-L_0 + xa_x + 2ta_t)b. \tag{3.40}$$

These gauge parameters leave the asymptotically Lifshitz connections form-invariant because one can show that there exists choices of the parameters ϵ and χ that lead to these gauge parameters. To see this explicitly, notice that given $\epsilon(t,x)$ and $\chi(t,x)$, if we let

$\widehat{\lambda}(\epsilon(t,x), \chi(t,x))$ denote the gauge parameter $\lambda(t,x)$ of (3.25) obtained after all ϵ_i and χ_m have been substituted for their expressions in terms of ϵ and χ in (3.28), then we have

$$\Lambda_H = b^{-1}\widehat{\lambda}(0,-1)b, \tag{3.41}$$

$$\Lambda_P = b^{-1}\widehat{\lambda}(-1,0)b, \tag{3.42}$$

$$\Lambda_D = b^{-1}\widehat{\lambda}(x,2t)b. \tag{3.43}$$

We now have candidates for gauge parameters from which to construct charges that satisfy the Lifshitz algebra. Using the definition (3.22), we find that the expressions for the variations of the charges corresponding to these gauge parameters are integrable and give

$$Q(\Lambda_H) = \frac{2k}{\pi} \int_{-\infty}^{\infty} dx \mathcal{W}, \tag{3.44}$$

$$Q(\Lambda_P) = \frac{2k}{\pi} \int_{-\infty}^{\infty} dx \mathcal{L}, \tag{3.45}$$

$$Q(\Lambda_D) = -\frac{2k}{\pi} \int_{-\infty}^{\infty} dx(2t\mathcal{W} + x\mathcal{L}). \tag{3.46}$$

To determine the Poisson algebra of these charges, we recall that for any two gauge parameters Λ and Γ, one has [17, 60]

$$\{Q(\Lambda), Q(\Gamma)\} = \delta_\Lambda Q(\Gamma). \tag{3.47}$$

We assume that the fields $\mathcal{L}$ and $\mathcal{W}$ vanish sufficiently rapidly as $x \to \pm\infty$ to ensure that any boundary terms encountered in computing the gauge-variations of the charges vanish. After some tedious but straightforward calculation, we find that

$$\{Q(\Lambda_H), Q(\Lambda_P)\} = 0, \tag{3.48}$$

$$\{Q(\Lambda_D), Q(\Lambda_H)\} = 2Q(\Lambda_H), \tag{3.49}$$

$$\{Q(\Lambda_D), Q(\Lambda_P)\} = Q(\Lambda_P). \tag{3.50}$$

This is precisely the Lifshitz algebra (3.2). In two dimensions we expect that the Lifshitz algebra will be extended to an infinite-dimensional algebra, in analogy with the extension of global conformal symmetry to a Virasoro algebra. A proposal for an infinite-dimensional extension of the Lifshitz symmetry was made in [53] and can be investigated using the Chern-Simons formulation.

4 Non-rotating Lifshitz black hole

The most general solutions of the Chern-Simons theory have connections A and $\bar{A}$ which are independent. We relate the barred and unbarred charges by setting

$$\bar{a}_x = -a_x^T, \qquad \bar{a}_t = a_t^T, \tag{4.1}$$

leaving the solutions to be characterized by only by the unbarred connection a_μ. Consequently, the expression for the metric (2.5) is diagonal, i.e. the g_{tx} component of the metric vanishes.

4.1 Most general non-rotating black hole solutions

Restricting ourselves to $\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(3,\mathbb{R})$ Chern-Simons, we start with a generalization of the ansatz (3.9), (3.10) in which we allow for source terms as coefficients of the generators W_2 and L_1 in the temporal components of the connections. This changes the asymptotics, but as we will see presently, this extra freedom will allow us to interpret the resulting solutions as finite energy excitations above the asymptotic Lifshitz vacuum. We also restrict our attention to coordinate-independent connection coefficients. Our general ansatz for the unbarred sector is

$$a_t = \ell_{t,1}L_1 + w_{t,2}W_2 + \ell_{t,0}L_0 + \ell_{t,-1}L_{-1} + w_{t,1}W_1 + w_{t,0}W_0 + w_{t,-1}W_{-1} + w_{t,-2}W_{-2},$$
(4.2)

$$a_x = L_1 + \ell_{x,0}L_0 + \ell_{x,-1}L_{-1} + w_{x,0}W_0 + w_{x,-1}W_{-1} + w_{x,-2}W_{-2}.$$
(4.3)

Notice that the ansatz (3.9), (3.10) of the last section is a special case of this ansatz obtained by setting $\ell_{t,1} = 0$ and $w_{t,2} = 1$. In order for this ansatz to be a solution of our theory we need to impose the flatness conditions which constrain the connections;

$$
\begin{aligned}
w_{t,1} &= 0, \\
\ell_{x,0} &= 0, \\
w_{t,-1} &= \ell_{t,1}w_{x,-1}, \\
\ell_{t,0} &= -w_{t,2}w_{x,-1}, \\
\ell_{t,-1} &= \ell_{t,1}\ell_{x,-1} - 2w_{t,2}w_{x,-2}, \\
w_{t,0} &= \ell_{t,1}w_{x,0} + 2\ell_{x,-1}w_{t,2}, \\
w_{t,-2} &= \ell_{x,-1}^2 w_{t,2} + \ell_{t,1}w_{x,-2} + w_{x,0}w_{t,2}w_{x,-2} - \frac{1}{4}w_{2,t}w_{x,-1}^2.
\end{aligned}
$$
(4.4)

These conditions seems to indicate that a flat solution is specified by parameters $\ell_{t,1}, \ell_{x,-1}$ and $w_{t,2}, w_{x,0}, w_{x,-1}, w_{x,-2}$. However we have not fixed all the gauge freedom, and some of these parameters are gauge artifacts. In order to see which of these parameters are the charges and sources of the theory and which of them can be gauged away, it suffices to look at the only gauge invariant quantities of the theory: the holonomies. A quick inspection of the holonomies around the thermal and angular cycles shows that the following quantities distinguish different solutions

$$
\begin{aligned}
\mu_2 &= w_{t,2}, \\
\mu_1 &= \ell_{t,1} + \frac{1}{3}w_{x,0}w_{t,2}, \\
\mathcal{L} &= -\ell_{x,-1} + \frac{1}{12}w_{x,0}^2, \\
\mathcal{W} &= w_{x,-2} + \frac{1}{54}\left(18\ell_{x,-1}w_{x,0} - w_{x,0}^3\right).
\end{aligned}
$$
(4.5)

Under these identifications we will interpret μ_1, μ_2 and $4\mathcal{L}, -4\mathcal{W}$ as sources and their conjugate charges. We will expand on this interpretation in section 4.3. Finally, to obtain a generic solution for the barred sector, we take $\bar{A} = -A^T$ replacing μ_i by $\bar{\mu}_i$ and $\mathcal{L}, \mathcal{W}$ by

$\bar{\mathcal{L}}$ and $\bar{\mathcal{W}}$. Limiting out attention to non non-rotating solutions implies setting $\bar{\mu}_i = -\mu_i$, $\bar{\mathcal{L}} = \mathcal{L}$ and $\mathcal{W}$.

Note that for a non-vanishing source μ_1, the connection (4.2) has a nonzero L_1 component and does not satisfy the criterion for an asymptotically Lifshitz connection (3.7). This indicates that the source μ_1 deforms the Lifshitz vacuum just as in the case of the higher spin CFTs. We note that it was shown in [61] that in the case of the asymptotically AdS theory with a deformation by a source still enjoys the full W_3 symmetry. It is quite likely that this is the case for our solution too, but we have not shown it.

4.2 Holonomy conditions

In the context of Chern-Simons higher spin theories, black hole solutions need to satisfy certain holonomy conditions and should have a thermodynamical interpretation [23, 24]. In particular, the requirement of a smooth Euclidean geometry implies that the thermal holonomy of the Chern-Simons connection is trivial;

$$\mathcal{P} \exp \left(\oint_t dt A_t \right) = \mathbf{1}, \tag{4.6}$$

where $\mathbf{1}$ is the SL(3, $\mathbb{R}$) identity, and the thermal cycle is from $t = 0$ to $t = 2\pi i$. This condition can be recast in more than one equivalent way. Diagonalizing a_t, and noting that a_t is constant, we find that the condition of a trivial thermal holonomy is equivalent to the following condition on the eigenvalues λ_1, λ_2, and λ_3 of a_t;

$$e^{2\pi i \lambda_1} = e^{2\pi i \lambda_2} = e^{2\pi i \lambda_3} = 1. \tag{4.7}$$

This means that each eigenvalue of a_t must be an integer. Since A_t is an element of $\mathfrak{sl}(3, \mathbb{R})$, it must be traceless, and this gives a second requirement on the eigenvalues; they must sum to zero.

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{4.8}$$

The simplest nontrivial solution is then $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, -1)$. This solution contains the famous BTZ black hole and its higher spin generalizations studied in [23].

In order to find black hole solutions one demands that the connections (4.2) and (4.3) obey (4.7) and (4.8). These conditions can be cast in a computationally convenient light. Employing the Cayley-Hamilton theorem, we note that every 3-by-3 complex matrix X satisfies its own characteristic polynomial. This means that there exist complex numbers $\Theta_0, \Theta_1, \Theta_2$ for which

$$X^3 = \Theta_0 I + \Theta_1 X + \Theta_2 X^2. \tag{4.9}$$

In particular, this allows one to compute any integer power of X knowing only the coefficients of the characteristic polynomial, and therefore allows for evaluation of the matrix exponential of X in terms of these coefficients. In the special case that X is traceless, which is the case for the argument of the exponential in the thermal holonomy, there are simple

expressions for the coefficients of the characteristic polynomial, which therefore serve to determine the thermal holonomy completely;

$$\Theta_0 = \det(X), \qquad \Theta_1 = \frac{1}{2}\operatorname{tr}(X^2), \qquad \Theta_2 = 0. \tag{4.10}$$

Applying this to the triviality condition (4.6), we find that the eigenvalues of a_t are related to the characteristic polynomial coefficients;

$$\Theta_0 = (2\pi i)^3 \lambda_1 \lambda_2 \lambda_3, \qquad \Theta_1 = -2\pi^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2). \tag{4.11}$$

In the case of, for example, the BTZ black hole, with $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, -1)$ one obtains

$$\Theta_0 = 0, \qquad \Theta_1 = -4\pi^2, \qquad \Theta_2 = 0. \tag{4.12}$$

In the context of finding a higher spin Lifshitz black hole solutions, we see no compelling reason to choose the BTZ holonomy conditions over others, but we do so anyway because they are simple and non-trivial. In principle, however, any conditions on the eigenvalues λ_j satisfying (4.7) and (4.8) should give rise to independent solutions. Applying the conditions (4.12) to our solution, we obtain the following holonomy conditions:

$$0 = 3\mathcal{L}\mu_1^2 + 9\mathcal{W}\mu_1\mu_2 + 4\mathcal{L}^2\mu_2^2 - \frac{3}{4}, \tag{4.13}$$

$$0 = 108\mathcal{W}^2\mu_2^3 + 8\mathcal{L}^2\mu_2\left(9\mu_1^2 - 4\mathcal{L}\mu_2^2\right) + 27\mathcal{W}\left(\mu_1^3 + 4\mathcal{L}\mu_1\mu_2^2\right). \tag{4.14}$$

These two equations can be used to solve for any two of $\mathcal{L}, \mathcal{W}, \mu_1, \mu_2$ in terms of the remaining two. In the next section we shall argue that thermodynamically $\mathcal{L}$ and $\mathcal{W}$ are charges and μ_1, μ_2 are the conjugate potentials.

4.3 Action and entropy

Since the black holes we are studying are gravitational solutions, we need to check that the Chern-Simons theory provides a correct variational principle. Let I_0 denote the euclidean Chern-Simons action. The on-shell, euclidean action I_0^{os}, namely the action in which the equations of motion have been used, is given by a boundary term

$$I_0^{\mathrm{os}} = -\frac{k}{4\pi}\int d\phi\, dt\, \operatorname{tr}(a_t a_x) + \frac{k}{4\pi}\int d\phi\, dt\, \operatorname{tr}(\bar{a}_t \bar{a}_x), \tag{4.15}$$

and evaluating the action on our non-rotating connections gives [54];

$$I_0^{\mathrm{os}} = -4k\left(2\mathcal{L}\mu_1 + 3\mathcal{W}\mu_2\right). \tag{4.16}$$

However I_0^{os} does not obey a thermodynamically sensible variational principle because the on-shell variation of I_0 is

$$(\delta I_0)^{\mathrm{os}} = 8k\left(\mathcal{L}\delta\mu_1 + \mathcal{W}\delta\mu_2\right) + \delta(4k\mu_2\mathcal{W}). \tag{4.17}$$

The third term spoils the identification of μ_1, μ_2 with sources having conjugate charges $\mathcal{L}$ and $\mathcal{W}$. As discussed in [33, 54], in the context of the higher spin black holes, it is possible

to obtain a canonical action I_1 that is thermodynamically sensible by adding a boundary term to I_0. When we evaluate I_1 on our non-rotating solutions, we obtain

$$I_1^{\text{os}} = -8k(\mu_1\mathcal{L} + 2\mu_2\mathcal{W}), \tag{4.18}$$

and it has the corresponding on-shell variaton

$$(\delta I_1)^{\text{os}} = 8k(\mathcal{L}\delta\mu_1 + \mathcal{W}\delta\mu_2). \tag{4.19}$$

This relation follows directly from the holonomy conditions (4.13) and (4.14). Taking derivatives of the conditions with respect to the sources, one can show that

$$\frac{\partial\mathcal{L}}{\partial\mu_1} = -\frac{6\mu_1\mathcal{L} - 18\mu_2\mathcal{W}}{3\mu_1^2 - 16\mu_2^2\mathcal{L}}, \qquad \frac{\partial\mathcal{W}}{\partial\mu_2} = -\frac{8\mathcal{L}(\mu_1\mathcal{L} - 3\mu_2\mathcal{W})}{3\mu_1^2 - 16\mu_2^2\mathcal{L}}, \tag{4.20}$$

$$\frac{\partial\mathcal{L}}{\partial\mu_2} = \frac{16\mu_2\mathcal{L}^2 - 9\mu_1\mathcal{W}}{3\mu_1^2 - 16\mu_2^2\mathcal{L}}, \qquad \frac{\partial\mathcal{W}}{\partial\mu_1} = \frac{16\mu_2\mathcal{L}^2 - 9\mu_1\mathcal{W}}{3\mu_1^2 - 16\mu_2^2\mathcal{L}}. \tag{4.21}$$

Using these expression one can easily show that that μ_1, μ_2 are conjugate to $\mathcal{L}$ and $\mathcal{W}$ respectively;

$$\frac{\partial I_1^{\text{os}}}{\partial\mu_1} = 8k\mathcal{L}, \qquad \frac{\partial I_1^{\text{os}}}{\partial\mu_2} = 8k\mathcal{W}. \tag{4.22}$$

The following integrability relation follows immediately from the equality of mixed partial derivatives:

$$\frac{\partial\mathcal{W}}{\partial\mu_1} = \frac{\partial\mathcal{L}}{\partial\mu_2}. \tag{4.23}$$

The entropy S is naturally a function of the charges $\mathcal{L}, \mathcal{W}$. It can can be obtained by performing a Legendre transform of $I_1^{\text{os}}(\mu_1, \mu_2)$ with respect to the conjugate variables $\mathcal{L}$ and $\mathcal{W}$.

$$\begin{aligned} S(\mathcal{L}, \mathcal{W}) &= \frac{\partial I_1^{\text{os}}}{\partial\mu_1}\mu_1 + \frac{\partial I_1^{\text{os}}}{\partial\mu_2}\mu_2 - I_1^{\text{os}} \\ &= 8k\big(2\mu_1\mathcal{L} + 3\mu_2\mathcal{W}\big). \end{aligned} \tag{4.24}$$

Moreover, using the holonomy conditions one can easily verify that the following inverse thermodynamic relations are:

$$\frac{\partial S}{\partial(8k\mathcal{L})} = \mu_1, \qquad \frac{\partial S}{\partial(8k\mathcal{W})} = \mu_2. \tag{4.25}$$

4.4 Temperature and grand potential

Recall that for any thermodynamic system, the grand potential is defined as follows in terms of the thermal partition function:

$$\Phi = -\frac{1}{\beta}\ln Z. \tag{4.26}$$

Using the saddle point approximation, we identify the on-shell, Euclidean Chern-Simons action with the log of the partition function, so we obtain

$$\Phi = \frac{1}{\beta} I_1^{\text{os}}. \tag{4.27}$$

The thermodynamic potential Φ is associated with the grand canonical ensemble and has as natural variables the temperature T and the chemical potential α. These can be related to μ_1, μ_2 as follows.

In euclidean signature we have chosen the periodicity of the euclidean time circle to be 1. A different euclidean periodicity β is equivalent to keeping the periodicity equal to 1 and rescaling A_t by a factor of β. This leads us to re-express the potentials μ_1, μ_2 in terms of β (or the temperature T) and a higher spin potential α.

$$\mu_1 = \beta\alpha = \frac{1}{T}\alpha, \qquad \mu_2 = \beta = \frac{1}{T}. \tag{4.28}$$

This prescription also ensures that after the rescaling, the connections have Lifshitz asymptotics.

In thermodynamics it is a well known fact that the grand canonical potential has the following differential

$$d\Phi = -SdT - Qd\alpha. \tag{4.29}$$

It follows that the entropy S and charge Q can be computed as appropriate partial derivatives of the grand potential;

$$\left.\frac{\partial\Phi}{\partial T}\right|_\alpha = -S, \qquad \left.\frac{\partial\Phi}{\partial\alpha}\right|_T = -Q. \tag{4.30}$$

For the Lifhsitz black hole, the grand potential Φ is related to the on shell action I_1^{os} via (4.27) which is turn is given by (4.18), giving

$$\Phi = -8k\left(\alpha\mathcal{L} + 2\mathcal{W}\right). \tag{4.31}$$

Using the holonomy conditions (4.13) and (4.14) to eliminate derivatives with respect to α, T one can calculate the entropy

$$S = -\left.\frac{\partial\Phi}{\partial T}\right|_\alpha = \frac{1}{T}8k\left(2\alpha\mathcal{L} + 3\mathcal{W}\right). \tag{4.32}$$

Note that the entropy agrees with (4.24). The charge conjugate to the potential α is given by

$$Q = -\left.\frac{\partial\Phi}{\partial\alpha}\right|_T = -8k\mathcal{L}. \tag{4.33}$$

We can use the thermodynamic relation between grand potential and the internal energy (which we can identify with the mass of the black hole) to obtain a formula for the energy[2] E;

$$E = \Phi + TS + \alpha Q = 8k\mathcal{W}. \tag{4.34}$$

---

[2]Notice that it follows from (4.34) that the Gibbs-Duhem relation $E = TS + \alpha Q$ doesn't hold for our black hole solution, since it would imply $\Phi = 0$. This is a common feature of black hole thermodynamics which has been noticed at various points in the literature (see e.g. [56, 57]).

Note that this result agrees with the identification of $\mathcal{W}$ with the energy in the holographic Lifshitz em-complex given in (3.21). We can perform one last consistency check by solving the holonomy conditions with S and Q as independent variables, it is straightforward to verify that the First Law of thermodynamics is indeed satisfied;

$$dE = TdS + \alpha dQ. \tag{4.35}$$

4.5 Branches

After clarifying the thermodynamical interpretation of the parameters in the connection, we are ready to look for black hole solutions to the holonomy conditions (4.13) and (4.14). In this section we will express the intensive parameters T and α in terms of the extensive parameters $\mathcal{L}$ and $\mathcal{W}$. Note that due to the nonlinear nature of the holonomy conditions, there will be multiple branches which can be interpreted as different phases of the theory.

In order to simplify the calculation it proves useful to replace $\mathcal{W}$ by a parameter θ which is given by

$$\mathcal{W} = \sqrt{\frac{16\mathcal{L}^3}{27}} \sin\theta. \tag{4.36}$$

Using (4.25), we eliminate μ_1, μ_2 in the first holonomy condition (4.13) in favor of derivatives of the entropy with respect to $\mathcal{L}$ and θ.

$$64k^2\mathcal{L} = 9\left(\frac{\partial S}{\partial \theta}\right)^2 + 4\mathcal{L}^2\left(\frac{\partial S}{\partial \mathcal{L}}\right)^2. \tag{4.37}$$

This partial differential equation for S is solved by the following family of solutions parametrized by a constant C.

$$S(\mathcal{L}, \theta) = 8k\sqrt{\mathcal{L}} \cos\left(\frac{\theta}{3} + C\right). \tag{4.38}$$

which indicates that $C = n\pi/3$ for $n = 0, \ldots, 5$. Hence there are six different solutions labelled by n. All of these solutions can be regarded as thermodynamical branches of a Lifshitz black hole. The branches $n = 1, 2, 3, 4$ all show pathologies that make them unphysical. The $n = 1$ case has negative temperature for all values of $\theta, \mathcal{L}$, while $n = 2$ has both negative temperature and entropy. Finally, the $n = 3, 4$ branches have strictly negative entropy.

Consequently, only the branches with $n = 0$ and $n = 5$ seem to be physically sensible. The entropy and temperature of the first brach ($n = 0$) read

$$S_{n=0} = 8k\sqrt{\mathcal{L}} \cos\left(\frac{\theta}{3}\right), \qquad T_{n=0} = -\frac{4\mathcal{L}}{\sqrt{3}}\frac{\cos\theta}{\sin\frac{\theta}{3}}, \qquad \alpha_{n=0} = -2\sqrt{\frac{\mathcal{L}}{3}}\frac{\cos\frac{2\theta}{3}}{\cos\frac{\theta}{3}}. \tag{4.39}$$

This implies that for the temperature to be positive, one needs $-\pi/2 < \theta < 0$, which imposes the constraint $-\sqrt{16\mathcal{L}^3/27} < \mathcal{W} < 0$. In this range, the entropy has its minimum at zero temperature, in accordance with the third law of thermodynamics. Note that under this constraint, the energy (4.34) is negative, but bounded from below. In section 4.8 we
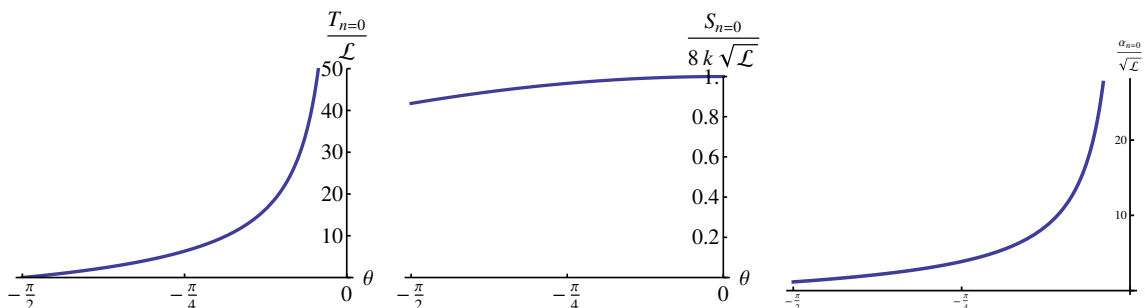
Figure 1. Temperature, entropy and chemical potential of the $n = 0$ branch.
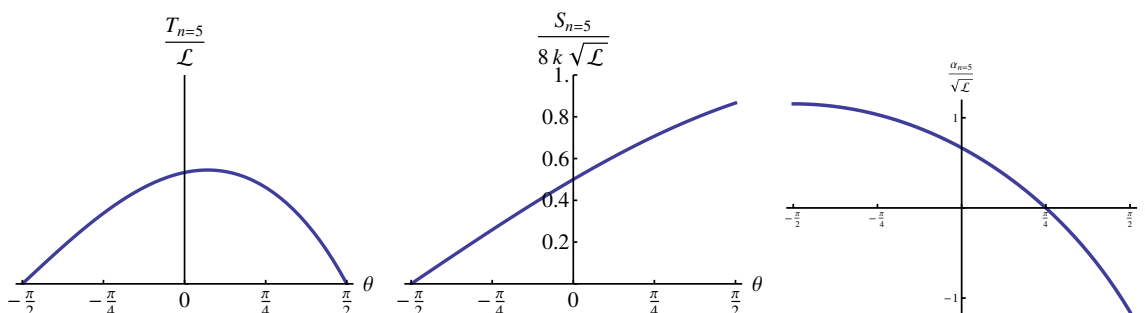

Figure 2. Temperature, entropy and chemical potential of the $n = 5$ branch.

will discuss a simple radial gauge for which this solution looks explicitly like a black hole. Interestingly, this gauge only exists for this branch and $n = 4$, which has exactly the same entropy but with the opposite sign. This does not mean that other branches do not have black hole gauges, as we have not explored non-radial gauges. For now, the plots of the temperature and entropy as a function of θ for a fixed value of $\mathcal{L}$, are shown in figure 1.

Inserting this in the second holonomy condition (4.14) gives us a restriction for C given by

$$\sin(3C) = 0, \tag{4.40}$$

The sixth branch ($n = 5$) shows the following behavior with respect to $\mathcal{L}$ and θ

$$S_{n=5} = 8k\sqrt{\mathcal{L}}\cos\left(\frac{\theta + 5\pi}{3}\right), \qquad T_{n=5} = \frac{4\mathcal{L}}{\sqrt{3}}\frac{\cos\theta}{\cos\frac{2\theta+\pi}{6}}, \qquad \alpha_{n=5} = 2\sqrt{\frac{\mathcal{L}}{3}}\frac{\cos\frac{2\theta+\pi}{3}}{\cos\frac{2\theta+\pi}{6}}. \tag{4.41}$$

This branch has positive values of temperature and entropy for all values of $\theta \in [-\pi/2, \pi/2]$, as shown in figure 2.

4.6 Entropy as a function of intensive parameters

Study of the stability and thermodynamical dominance of the different branches requires an expression for the entropy as a function of intensive parameters. This, in turn, requires us to solve the holonomy conditions for $\mathcal{L}, \mathcal{W}$ in terms of α, T, and then write the entropy

(a) Entropies.

(b) Grand potentials.

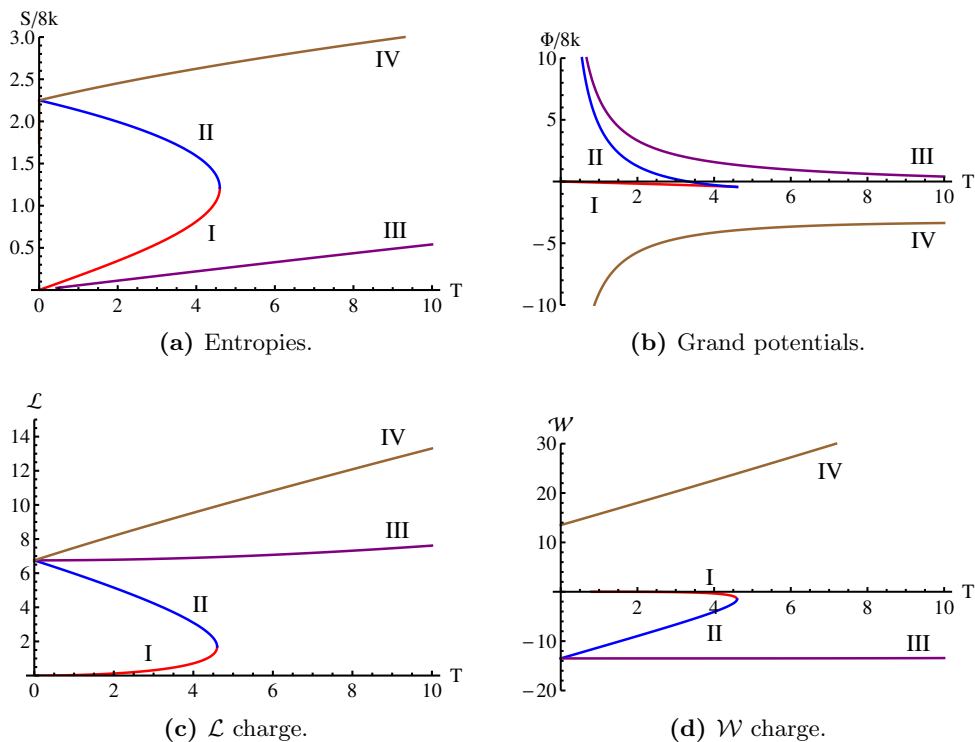**(c)** $\mathcal{L}$ charge.

(d) $\mathcal{W}$ charge.

Figure 3. Entropies, Grand potentials and extensive variables for the four branches as a function of the temperature, at fixed α. I,II and III branches have been plotted for $\alpha = 3$, while branch IV has $\alpha = -3$.

using (4.32) as a function of α and T only. The first holonomy condition (4.13) is linear in $\mathcal{W}$ and can be easily solved;

$$\mathcal{W} = -\frac{12\alpha^2\mathcal{L} + 12\mathcal{L}^2 - 3T^2}{36\alpha}. \tag{4.42}$$

Plugging this into the second holonomy condition (4.14), twe obtain the following quartic equation for $\mathcal{L}$.

$$256\mathcal{L}^4 - 576\alpha^2\mathcal{L}^3 + (432\alpha^4 - 96\,T^2\mathcal{L}^2 + (36\alpha^2T^2 - 108\,\alpha^6)\mathcal{L} + 27\alpha^4T^2 + 9T^4 = 0. \tag{4.43}$$

This implies the existence of four branches. Even though the number of branches is different from the ones found in last section, one can see that appropriately gluing together these branches, one obtains the solutions we studied in section 4.5. For positive temperature, the only branches with positive entropy can be found in figure 3. Note that branch IV has been plotted for a negative value of α because its entropy is negative otherwise.

One can check that I and II branches map back to the $n = 0$ branch from the previous section, while III and IV are related to the $n = 5$ branch. Figure 3b shows the grand potential (4.31) as a function of the temperature for fixed chemical potential. In the case of negative α, the only sensible branch is IV, and it dominates the thermodynamics. However, for a positive value of α, branch I ($n = 0$) takes over.

We should note that the phase diagrams displayed in section 4.5 and 4.6 look very similar to the ones obtained for the asymptotically AdS higher spin black holes discussed in [24–27]. This is no surprise since the holonomy equations are identical. The Lifshitz black hole differs from the AdS higher spin black hole however in the identification of temperature and chemical potential as well as the charges. Hence the physical interpretation of the quantities and physical constraints (such as positive temperature) are different.

It is interesting to study the high temperature limit of these solutions. Branch I cannot reach high temperatures at fixed α. However, in the high α limit, the temperature can be arbitrarily high at the point of maximum entropy. This point is defined by a concrete value of θ, so the high α limit can only be reached by taking $\mathcal{L}$ to infinity, as can be seen by looking at equations (4.39). In that case the temperature grows like $\mathcal{L}$ while the entropy grows like $\sqrt{\mathcal{L}}$. This implies

$$S_{n=0} \sim \sqrt{T}. \tag{4.44}$$

The same can be checked for branch IV. In the limit of high temperature, one finds that

$$\mathcal{L} = \frac{\sqrt{3}T}{4} + 3^{3/4}\sqrt{\frac{T}{32}}|\alpha| + \mathcal{O}(1), \tag{4.45}$$

$$S = -\frac{3^{1/4}\mathrm{sgn}(\alpha)}{\sqrt{8}}\sqrt{T} + \mathcal{O}(1). \tag{4.46}$$

Hence for negative α we obtain again

$$S_{n=5} \sim \sqrt{T}. \tag{4.47}$$

This temperature scaling (4.44) and (4.47) is expected for a theory dual to a quantum field theory with $z = 2$ anisotropic Lifshitz scaling symmetry in two dimensions [62].

4.7 Local stability in the grand canonical ensemble

Local thermodynamical stability is associated with the subadditivity of the entropy, as discussed in [58, 59] this condition is equivalent to demanding that the Hessian matrices of $-S$ and $-\beta\Phi$ are positive definite.

$$H_{mn} = \frac{\partial^2(-S)}{\partial x_m \partial x_n}, \qquad W_{mn} = \frac{\partial^2(-\beta\Phi)}{\partial y_m \partial y_n} \tag{4.48}$$

Which Matrix one has to consider depends on whether one describes the thermodynamic state of the system in terms of extensive parameters x_i or intensive parameters y_i respectively.

In the case of our Lifshitz black hole solution, the extensive parameters can be regarded as the charges $\mathcal{L}$ and $\mathcal{W}$, while the intensive parameters can be regarded as β and $\beta\alpha$. Evaluation of the eigenvalues of the Hessian H_{nm} for the $n = 5$ branch shows that this condition can't be satisfied for any value of $\mathcal{L}$ and $\mathcal{W}$, so the $n = 5$ branch is locally unstable. Demanding positive definiteness of the Hessian for the $n = 0$ branch requires that $\theta \in \left(-3\cos^{-1}\left(\frac{3^{1/4}}{\sqrt{2}}\right), 0\right)$. This is exactly the regime of θ covered by the curve representing branch I in figure 3.

One can further check that this result is consistent with the description in terms of potentials. Computation of the eigenvalues of W_{nm} for the four branches studied in section 4.6, indeed shows that branch I is locally stable, while II is not.

4.8 Metric and black hole gauge

We now investigate the question whether a gauge exists in which the metric of the Lifshitz black hole solutions displays a regular horizon. In fact, we demonstrate that for some branches one can maintain radial gauge and choose some of the residual gauge such that g_{tt} contains a double zero and g_{xx} is regular.

We begin again with the ansatz (3.9), (3.10) and the flatness conditions (4.4), where again the barred sector is determined by the non-rotating condition $\bar{a}_x = -a_x^T$ and $\bar{a}_t = a_t^T$. We also regard equations (4.5) as a reparametrization of $w_{t,2}, l_{t,1}, l_{x,-1}$, and $w_{x,-2}$ as functions of the charges and potentials $\mathcal{L}, \mathcal{W}, \mu_1, \mu_2$, and the residual gauge parameter $w_{x,0}$. Next, we solve for the value of $w_{x,0}$ for which the corresponding metric derived from (2.5) has a double zero in g_{tt} at some value of $\rho = \rho_h$, the location of the corresponding horizon. To do this, first we note that the metric component g_{tt} can be written as

$$g_{tt} = -(e^{2\rho}p_1 - e^{-2\rho}p_2)^2 - (e^{\rho}p_3 - e^{-\rho}p_4)^2 \tag{4.49}$$

where p_i are ρ-independent coefficients given by

$$p_1 = \mu_2 \tag{4.50}$$

$$p_3 = \mu_1 - \mu_2 \frac{w_{x,0}}{3} \tag{4.51}$$

$$p_2 = -\left(\frac{w_{x,0}}{3}\right)^3 \frac{\mu_1}{4} + \mathcal{W}\mu_1 + \mathcal{L}^2\mu_2 + \left(\frac{w_{x,0}}{3}\right)^4 \frac{\mu_2}{16} + \frac{w_{x,0}}{3}2\mathcal{W}\mu_2 + \frac{1}{2}\mathcal{L}\frac{w_{x,0}}{3}\left(2\mu_1 + \frac{w_{x,0}}{3}\mu_2\right) \tag{4.52}$$

$$p_4 = \mathcal{L}\mu_1 - \left(\frac{w_{x,0}}{3}\right)^2 \frac{3}{4}\mu_1 + \frac{w_{x,0}}{3}\mathcal{L}\mu_2 + \left(\frac{w_{x,0}}{3}\right)^3 \frac{\mu_2}{4} + 2\mathcal{W}\mu_2 \tag{4.53}$$

It is clear that g_{tt} is zero if and only if each term in parentheses on the right hand side of (4.49) is zero for the same value of ρ_h which implies that $p_2/p_1 = (p_4/p_3)^2$. Using the expressions above for $p_1, \ldots, p_4$, this constraint is equivalent to the following cubic equation for $w_{x,0}$:

$$w_{x,0}^3 - 36\mathcal{L}w_{x,0} - 108\mathcal{W} = 0. \tag{4.54}$$

The three solutions are given by

$$w_{x,0} = 4\sqrt{3\mathcal{L}}\cos\left(\frac{\cos^{-1}(\sin\theta)}{3} + m\frac{2\pi}{3}\right), \tag{4.55}$$

with $m = 0, 1, 2$. However the only solution with a positive and real horizon $\rho_h = \sqrt[4]{p_2/p_1} = \sqrt{p_4/p_3}$ is the one with $m = 2$, which can be simplified to

$$w_{x,0} = -4\sqrt{3\mathcal{L}}\sin\left(\frac{\theta}{3}\right). \tag{4.56}$$

The horizon is then located at

$$\rho_h = \sqrt{\mathcal{L}\left(2\cos\frac{2\theta}{3} - 1\right)}. \tag{4.57}$$

It seems that we did not need to impose the holonomy conditions in order to find this black hole gauge. However, we still need to check that the metric and the spin three field in this gauge are smooth around the cycle $t \sim t + 2\pi i$. this implies the following conditions

$$1 = \left.\sqrt{\frac{g_{tt}}{-2g_{\rho\rho}}}\right|_{\rho_h}, \qquad 1 = \left.\sqrt{\frac{\psi_{xtt}}{-2\psi_{x\rho\rho}}}\right|_{\rho_h}. \tag{4.58}$$

Direct substitution of the charges and sources for the six branches found in previous sections shows that only the $n = 0, 4$ cases satisfy these identities. This can mean that this gauge is appropriate for those two solutions, while the other branches require giving up the radial gauge chosen in equation (2.7). As we have argued in section 4.5, the $n = 3$ branch does not seem to be physically sensible. For this reason we will focus our attention in branch $n = 0$. The values of the spin fields at the horizon in this branch obey the following relations

$$g_{tt}|_{\rho_h} = 0, \qquad g'_{tt}|_{\rho_h} = 0, \qquad g_{xx}|_{\rho_h} = 4\mathcal{L}, \qquad \psi_{xxx}|_{\rho_h} = 2\mathcal{W}. \tag{4.59}$$

So we can recast our expresion for the entropy as

$$S = \frac{4k}{\pi} A \cos\left[\frac{1}{3}\sin^{-1}\left(\frac{3^{\frac{3}{2}}\psi_3}{A^3}\right)\right] \tag{4.60}$$

where

$$A = 2\pi\sqrt{g_{xx}|_{\rho_h}}, \qquad \psi_3 = \psi_{xxx}|_{\rho_h}, \tag{4.61}$$

which is very similar to the entropy formula found for asymptotically AdS higher spin black holes [34]. It would be interesting to investigate whether the local thermodynamic instability of the $n = 5$ branch discussed in section 4.7 and the absence of a regular horizon are related. However, it is an open and interesting question, if for the $n = 5$ branch there is a more general radial gauge choice (along the lines of [20]) which has a regular horizon.

5 Generalizations

In this section we will present some observations on possible generalizations of our $\text{SL}(3, \mathbb{R})$ results obtained in the previous sections.

5.1 Rotating solutions

In the present paper we have limited ourselves to non-rotating solutions, for which the connections A and $\bar{A}$ are related by equation (4.1). Since the two Chern-Simons connections $A, \bar{A}$ are independent, it is clear that constructing a solution with angular momentum entails lifting the condition (4.1). This also means that there will be two holonomy conditions

for the A and the $\bar{A}$ connection. Recall that in the $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ black hole first discussed in [23] a rotating higher spin black hole is obtained by choosing modular parameter to be complex $\tau = \Omega + i\beta$, where Ω is the potential dual to the angular momentum. For the Lifshitz black holes this cannot work quite the same way and we present some observations here. Note that in the holographic dictionary or the stress energy complex of a Lifshitz theory (5.1) the angular momentum (i.e. the momentum along the x direction if we take x to be compact) is identified with $\mathcal{L} - \bar{\mathcal{L}}$, whose conjugate potential is $\mu_1 - \bar{\mu}_1$ and the energy is identified with $\mathcal{W} + \bar{\mathcal{W}}$, whose conjugate potential is $\mu_2 + \bar{\mu}_2$. Hence it is likely that a rotating solution can be constructed by choosing a connections with $\mu_1 \neq \bar{\mu}_1$ and keeping the indentification of the temperature β the same as in the non-rotating case. The expressions for the metric and higher spin fields are much more complicated. This implies also that the analysis of the black hole gauge done section 4.8 becomes more involved, and we leave these questions for future work. We also note that, to our knowledge, no rotating Lifshitz black hole solutions have been constructed using the standard supergravity actions. Hence constructing such solutions in higher spin gravity might be interesting.

5.2 Lifshitz vacuum for $hs(\lambda)$

In this section we discuss some steps in generalizing the construction of Lifshitz black holes from $\mathrm{SL}(3,\mathbb{R})$ to $hs(\lambda)$, note that this generalization will also include the case of $\mathrm{SL}(N,\mathbb{R})$ by choosing $\lambda = N$, where the infinite-dimensional Lie algebra reduces to $\mathrm{SL}(N,\mathbb{R})$. Our conventions for $hs(\lambda)$ are summarized in appendix A.2.

A Lifshitz vacuum in the $hs(\lambda)$ theory can be easily constructed as follows

$$
\begin{aligned}
a_t &= \frac{1}{\sqrt{\mathrm{tr}(V_{s-1}^s V_{-(s-1)}^s)}} V_{s-1}^s, & a_x &= \frac{1}{\sqrt{\mathrm{tr}(V_1^2 V_{-1}^2)}} V_1^2 \\
\bar{a}_t &= \frac{1}{\sqrt{\mathrm{tr}(V_{s-1}^s V_{-(s-1)}^s)}} V_{-(s-1)}^s, & \bar{a}_x &= -\frac{1}{\sqrt{\mathrm{tr}(V_1^2 V_{-1}^2)}} V_{-1}^2.
\end{aligned} \tag{5.1}
$$

Note that since

$$
[V_1^2, V_{s-1}^s]_\star = 0, \qquad [V_{-1}^2, V_{-(s-1)}^s]_\star = 0, \tag{5.2}
$$

this satisfies the flatness condition for a connection in the radial gauge. The gauge connections A_μ and the metric are obtained from (5.1) by adapting the formulae (2.5) and using $b = \exp(\rho V_0^2)$ It follows that the metric is of the form.

$$
ds^2 = -e^{2(s-1)\rho} dt^2 + e^{2\rho} dx^2 + d\rho^2. \tag{5.3}
$$

Hence we can realize an asymptotically Lifshitz metric in the $hs(\lambda)$ theory for any $z = 2, 3, 4, \cdots$, by setting $s = z + 1$. Note that some higher spin fields will be non-vanishing for this $hs(\lambda)$ Lifshitz vacuum. By setting $\lambda = N$, the infinite-dimensional $hs(\lambda)$ gauge algebra truncates to a finite-dimensional $\mathrm{SL}(N,\mathbb{R})$, and the connections give Lifshitz vacua with $z = N - 1, N - 2, \cdots, 2$. Note that the generators V_0^2, V_1^2 and V_2^3 form a Lifshitz sub algebra. The generalization of the evolution equations (3.15) and (3.16) to the case of $hs(\lambda)$ is an interesting open problem.

5.3   An $hs(\lambda)$ Lifshitz black hole

Here we limit ourself to the BH for $z = 2$, which is related to the $hs(\lambda)$ black hole with a chemical potential for the spin three charge, which is most extensively studied in the literature. The connection is given by

$$a_x = V_1^2 + \tilde{\mathcal{L}}V_{-1}^2 + \tilde{\mathcal{W}}V_{-2}^3 + \tilde{\mathcal{U}}V_{-3}^4 + \cdots \qquad (5.4)$$

$$a_t = \tilde{\mu}_1 a_x + \tilde{\mu}_2 (a_x \star a_x)\mid_{\text{traceless}} . \qquad (5.5)$$

Here, $\tilde{\mathcal{L}}, \tilde{\mathcal{W}}, \tilde{\mathcal{U}}$, etc are associated with charges of spin $2, 3, 4, \cdots$. We have tilded all quantities to distinguish them from the quantities appearing in the higher spin black hole reviewed in the appendix A.2.

By construction the connection (5.4) satisfied the flatness condition. To define a regular black hole in a higher spin Chern-Simons theory one has to impose a holonomy condition on the gauge connection around the euclidean time circle. The holonomy condition which we choose is again that the holonomy is equal to the BTZ holonomy for the $hs(\lambda)$ black hole defined in appendix A.2. One might object that in the case of the Lifshitz BH this condition seems less well motivated since there is no analog of a BTZ black hole for an asymptotically Lifshitz spacetime, however a better way to think about this is that the BTZ holonomy simply states that the holonomy of the BH is in the center of $hs(\lambda)$ (see [30] for a discussion on how the center of $hs(\lambda)$ is defined).

If we compare the holonomy associated with a_t defined in (5.4) and the higher spin black hole holonomy (A.9) one recognizes that they are the same upon the following identifications

$$\tilde{\mu}_1 = 2\pi\tau, \qquad \tilde{\mu}_2 = -2\pi\alpha. \qquad (5.6)$$

Furthermore the charges can also be identified

$$\tilde{\mathcal{L}} = -\frac{2\pi}{k}\mathcal{L}, \qquad \tilde{\mathcal{W}} = -\frac{\pi}{2k}\mathcal{W}, \quad \cdots \qquad (5.7)$$

Since there is a one-to-one map of parameters one might ask how this can be different than the $hs(\lambda)$ [55]. The answer lies in the fact that while (this was true for the SL(3, $\mathbb{R}$) case too) the holonomy conditions have the same functional form, the interpretations of $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are different. The inverse temperature β and the chemical potential $\tilde{\alpha}$ can be related to $\tilde{\mu}_1$ and $\tilde{\mu}_2$ following the the SL(3, $\mathbb{R}$) Lifshitz black hole example

$$\tilde{\mu}_1 = \beta\tilde{\alpha}, \qquad \tilde{\mu}_2 = \beta. \qquad (5.8)$$

This means that the most natural regime for the Lifshitz black hole , i.e. $\tilde{\beta}$ finite and $\tilde{\alpha}$ small, is not the same regime as the one which allows the perturbative solution of the holonomy conditions first obtained in [55]. Indeed if we take the limit $\tilde{\alpha} \to 0$, this is equivalent for the higher spin black hole to taking the limit $\tau \to 0$ and keeping α finite, i.e. taking an infinite temperature limit and finite chemical potential.

6   Discussion

In this paper we have discussed the construction of holographic spacetimes dual to field theory with Lifshitz $z = 2$ scaling symmetry . In addition we have constructed black hole solutions in these theories. One interesting feature of these theories is that the connections, holonomy conditions and thermodynamic relations are all very similar to the higher spin black holes first constructed in [23]. This can be traced back to the fact that the Lifshitz black hole connections and the higher spin black hole connections are related by replacing t, x by $\bar{z}, z$ respectively. Note however that the interpretation of the parameters is quite different. First, the holographic identification of the stress energy complex of the QFT with Lifshitz symmetry and the role of the fields $\mathcal{L}$ and $\mathcal{W}$ are quite different for the Lifshitz theory compared to the W_3 CFT. Second, for the Lifshitz black hole solutions the identification of the temperature and higher spin chemical potential is in some sense reversed compared to the higher spin black hole, this leads to a different interpretation of the thermodynamics. The solution of the holonomy conditions has different branches, which we can interpret as different thermodynamic phases. We have shown that only one branch (branch I of section 4.6) has 1. positive entropy and 2. positive temperature, 3. is locally thermodynamically stable and 4. enjoys a radial gauge with a regular horizon. All other branches do not satisfy one or more of these conditions and are therefore physically not satisfying.

We have briefly discussed generalizations of the black hole solutions found in this paper. It would be interesting to study Lifshitz black hole solutions in $hs(\lambda)$ further, since there exists a concrete proposal for a dual CFT and the Lifshitz theories could be interpreted as deformations of the CFT. Furthermore since it is possible to couple scalar matter consistently there are independent probes of the geometry of the black hole. To make progress one has to solve the holonomy conditions either exactly or maybe less ambitiously determine wether it is possible to solve the holonomy conditions perturbatively for small $\tilde{\alpha}$ and finite temperature. We plan to return to these interesting questions in the future.

A   Conventions

In this appendix we present some details on the conventions and explicit representations of the Lie algebras used in the main body of the paper.

A.1 Explicit $SL(3, \mathbb{R})$ representation

The $SL(2, \mathbb{R})$ generators of the principal embedding are given by the following matrices

$$L_{-1} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

and the spin 3 generators, on which we omit the superscript $^{(3)}$ for notational simplicity, are as follows:

$$W_{-2} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{A.2}$$

$$W_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}. \tag{A.3}$$

If we define $(T_1, T_2, \ldots, T_8) = (L_1, L_0, L_{-1}, W_2, \ldots W_{-2})$, then traces of all pairs of generators are given by

$$\mathrm{tr}(T_i T_j) = \begin{pmatrix} & & -4 & 0 & \cdots & & 0 \\ & 2 & & \vdots & \ddots & & \vdots \\ -4 & & & 0 & \cdots & & 0 \\ \hline 0 & \cdots & 0 & & & & 4 \\ & & & & -1 & & \\ \vdots & \ddots & \vdots & & & \frac{2}{3} & \\ & & & & -1 & & \\ 0 & \cdots & 0 & 4 & & & \end{pmatrix} \tag{A.4}$$

A.2 $hs(\lambda)$ conventions and black hole

Here we follow the conventions of [28] and [30]. The main formulas we use are, the lone star products

$$V_m^s \star V_n^t = \frac{1}{2} \sum_{u=1,2,\cdots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda) V_{m+n}^{s+t-u}. \tag{A.5}$$

The star product is used to define the commutator between Lie algebra generator and is denotes by $[\cdot, \cdot]_\star$. For the elements of the Lie-algebra V_m^s one has $|m| < s$ (the generators are zero otherwise). The elements $V_{-1,0,1}^2$ form a SL(2,R) sub algebra and V_m^s form spin s representation

$$[V_m^2, V_n^t]_* = \Big(m(t-1) - n\Big) V_{m+n}^t. \tag{A.6}$$

The algebra has a unit element denoted by X_0^1, the trace is defines by

$$Tr(X) = X|_{V_0^1}. \tag{A.7}$$

A $hs(\lambda)$ black hole with a chemical potential for the spin 3 charge (this can be generalized to arbitrary spin s) has the following connections

$$a_z = V_1^2 - \frac{2\pi}{k}\mathcal{L}V_{-1}^2 - \frac{\pi}{2k}\mathcal{W}V_{-2}^3 + \mathcal{U}V_{-3}^4 + \cdots,$$
$$a_{\bar{z}} = -\frac{\alpha}{\bar{\tau}}(a_z \star a_z)\mid_{\text{traceless}}. \tag{A.8}$$

The holonomy around the time circle is given by $H = e^\omega$ with

$$\omega = 2\pi\big(\tau a_z + \bar{\tau}a_{\bar{z}}\big). \tag{A.9}$$

The holonomy condition for the black hole is that the holonomy is the same as the holonomy of the BTZ black hole

$$H = H_{BTZ}. \tag{A.10}$$

where ω_{BTZ} is given by

$$\omega_{BTZ} = 2\pi\tau V_1^2 + \frac{\pi}{\tau}V_{-1}^2. \tag{A.11}$$

This condition is equivalent to the following conditions on the powers of ω (see eq. 2.17 of [30]).

$$Tr(\omega^n) = \frac{1}{\lambda}\lim_{t\to 0}\left(\partial_t^n \frac{\sin \pi\lambda t}{\sin \pi t}\right). \tag{A.12}$$

These conditions have been solved perturbatively in the chemical potential α and one gets the charges $\mathcal{L}, \mathcal{W}, \mathcal{U}, \cdots$ as a power series in α (and depending on τ), such that as $\alpha \to 0$ one gets back the BTZ black hole.

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