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The Nicolas and Robin inequalities with sums of two squares

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Abstract In 1984, G. Robin proved that the Riemann hypothesis is true if and only if the *Robin inequality* $\sigma(n) < e^{\gamma} n \log\log n$ holds for every integer $n > 5040$, where $\sigma(n)$ is the sum of divisors function, and γ is the Euler–Mascheroni constant. We exhibit a broad class of subsets $\mathcal{S}$ of the natural numbers such that the Robin inequality holds for all but finitely many $n \in \mathcal{S}$. As a special case, we determine the finitely many numbers of the form $n = a^2 + b^2$ that do not satisfy the Robin inequality. In fact, we prove our assertions with the *Nicolas inequality* $n/\varphi(n) < e^{\gamma} \log\log n$; since $\sigma(n)/n < n/\varphi(n)$ for $n > 1$ our results for the Robin inequality follow at once.

Keywords Nicolas inequality · Robin inequality · Sums of two squares

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1 Introduction

Let $\varphi(n)$ denote the *Euler function*. In 1903 Landau (see [4, pp. 217–219]) showed that

$$\overline{\lim_{n\to\infty}} \frac{n}{\varphi(n)\log\log n} = e^{\gamma}, \tag{1}$$

where γ is the *Euler–Mascheroni constant*. Eighty years later, in a highly interesting work, Nicolas [5] proved that the inequality

$$\frac{n}{\varphi(n)} > e^{\gamma}\log\log n$$

holds for infinitely many natural numbers n. Moreover, if N_k denotes the product of the first k primes, he proved that

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma}\log\log N_k$$

holds for every $k \geqslant 1$ on the *Riemann hypothesis* (RH). Assuming RH is false, he also showed there are both infinitely many k for which this inequality holds and infinitely many k for which it does not hold. To acknowledge the many contributions of Nicolas to this subject, we denote by $\mathcal{N}$ the set of numbers $n \in \mathbb{N}$ that satisfy the *Nicolas inequality*:

$$\frac{n}{\varphi(n)} < e^{\gamma}\log\log n. \tag{2}$$

The principal aim of this paper is to exhibit a broad class of infinite subsets $\mathcal{S} \subset \mathbb{N}$ such that this inequality holds for all but finitely many $n \in \mathcal{S}$. This class includes a set that contains all natural numbers which can be expressed as a sum of two squares.

Let $\sigma(n)$ be the *sum of divisors function*. The analogue of (1) for this function was obtained by Gronwall [2], who proved that

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n\log\log n} = e^{\gamma}.$$

Robin [7] showed that if RH is true, then the *Robin inequality*:

$$\frac{\sigma(n)}{n} < e^{\gamma}\log\log n \tag{3}$$

holds for every integer $n > 5040$, whereas if RH is false, then this inequality fails for infinitely many n. We denote by $\mathcal{R}$ the set of numbers $n \in \mathbb{N}$ that satisfy (3). In view of the elementary inequality

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \quad (n > 1),$$

it is clear that $\mathcal{N} \subset \mathcal{R}$. Thus, for the class of subsets $\mathcal{S} \subset \mathbb{N}$ considered in the present paper, the Robin inequality holds for all but finitely many $n \in \mathcal{S}$.

Our work was originally inspired by a recent paper of Choie et al. [1], which establishes the inclusion in $\mathcal{R}$ of various infinite subsets of the natural numbers $\mathbb{N}$. In particular, in [1] it is shown that $\mathcal{R}$ contains every square-free number $n > 30$, every odd integer $n > 9$, every powerful number $n > 36$, and every integer $n > 1$ not divisible by the fifth power of some prime. As a consequence it follows that the RH holds iff the Robin inequality holds for all natural numbers n divisible by the fifth power of some prime. Note that this criterion does not have the restriction $n \geqslant 5041$. Another "5041-free" criterion was given earlier by Lagarias [3], who showed that RH is true iff

$$\sigma(n) \leqslant H_n + e^{H_n} \log H_n,$$

where

$$H_n = \sum_{j \leqslant n} \frac{1}{j} \quad (n \geqslant 1).$$

To state our results more precisely, let $\mathbb{P}$ denote the set of prime numbers, and for any subset $\mathcal{A} \subset \mathbb{P}$, put

$$\pi_{\mathcal{A}}(x) = \#\{p \leqslant x : p \in \mathcal{A}\}.$$

Let $\mathcal{P}$ be an arbitrary (fixed) subset of $\mathbb{P}$ such that

$$\overline{\delta} = \varlimsup_{x\to\infty} \frac{\pi_{\mathcal{P}}(x)}{\pi(x)} < 1 \quad \text{and} \quad \underline{\delta} = \varliminf_{x\to\infty} \frac{\pi_{\mathcal{P}}(x)}{\pi(x)} > 0, \tag{4}$$

where $\pi(x) = \#\{p \leqslant x\}$ as usual. Let $\mathcal{Q}$ denote the complementary set of primes (i.e., $\mathcal{Q} = \mathbb{P} \backslash \mathcal{P}$), and note that

$$\varlimsup_{x\to\infty} \frac{\pi_{\mathcal{Q}}(x)}{\pi(x)} = 1 - \underline{\delta} < 1 \quad \text{and} \quad \varliminf_{x\to\infty} \frac{\pi_{\mathcal{Q}}(x)}{\pi(x)} = 1 - \overline{\delta} > 0. \tag{5}$$

In this paper, we work with the set $\mathcal{S} = \mathcal{S}(\mathcal{P})$ defined by

$$\mathcal{S} = \left\{ n \in \mathbb{N} \ : \ \text{if } p \in \mathcal{Q} \text{ and } p \mid n, \ \text{then } p^2 \mid n \right\}. \tag{6}$$

Our main result is the following:

Theorem 1 *The set $\mathcal{N}$ contains all but finitely many of the numbers in $\mathcal{S}$.*

Corollary 1 *Of the numbers n which do not satisfy the Nicolas inequality, all but finitely many are divisible by a prime $q \in \mathcal{Q}$ such that $q^2 \nmid n$.*

In particular, for any fixed $a, m \in \mathbb{N}$ with $\gcd(a, m) = 1$, one can put

$$\mathcal{P} = \{p \in \mathbb{P} \ : \ p \not\equiv a \pmod{m}\}$$

and apply Corollary 1 to deduce the following:

Corollary 2 *Of the numbers n which do not satisfy the Nicolas inequality, all but finitely many are divisible by a prime $q \equiv a \pmod{m}$ such that $q^2 \nmid n$.*

In Sect. 3 we examine more closely the special case

$$\mathcal{P} = \{p \in \mathbb{P} : p \equiv 1 \pmod{4}\} \cup \{2\}.$$

Note that the corresponding set $\mathcal{S}$ contains all natural numbers of the form $n = a^2 + b^2$ (since, by a theorem of Fermat, every prime $q \equiv 3 \pmod{4}$ appears with even multiplicity in the prime factorization of n if and only if n can be written as a sum of two squares). Using effective bounds from [6] on the number of primes in arithmetic progressions modulo 4, we are able to determine the set $\mathcal{S} \setminus \mathcal{N}$ completely, leading to:

Theorem 2 *The set $\mathcal{S} \setminus \mathcal{N}$ contains precisely* 347 *natural numbers. In particular, there are precisely* 246 *numbers which can be expressed as a sum of two squares and such that the Nicolas inequality* (2) *does not hold, the largest of which is the number* 525095813442228128l0.

As an application, we obtain the unconditional result that

$$\{1, 2, 4, 5, 8, 9, 10, 16, 18, 20, 36, 72, 180, 360, 720\}$$

is a complete list of those natural numbers which can be expressed as a sum of two squares and such that the Robin inequality (3) does not hold; this result is consistent with the truth of the Riemann Hypothesis.

Results like those of Theorem 2 can be established for certain quadratic forms other than $a^2 + b^2$. For example, using similar techniques one finds that there are precisely 261 numbers that can be expressed in the form $n = a^2 + 3b^2$ and for which the Nicolas inequality (2) does not hold, the largest of which is the number 3979993613l188090700.

Throughout the paper, any implied constants in the symbols $O, \ll, \gg$ and $\asymp$ depend (at most) on the set $\mathcal{P}$ and are absolute otherwise. We recall that for positive functions f, g the notations $f = O(g)$, $f \ll g$ and $g \gg f$ are all equivalent to the assertion that $f \leqslant cg$ for some constant $c > 0$, and the notation $f \asymp g$ means that $f \ll g$ and $g \ll f$.

2 Proof of Theorem 1

For every natural number n we put

$$F(n) = \frac{n}{\varphi(n)} = \prod_{p \mid n} \frac{p}{p-1}.$$

Note that

$$F(n) = F(\kappa(n)) \quad \text{and} \quad \omega(n) = \omega(\kappa(n)), \tag{7}$$

where $\omega(n)$ is the number of distinct prime divisors of n, and $\kappa(n)$ is the square-free kernel of n:

$$\kappa(n) = \prod_{p \mid n} p.$$

Let

$$\mathcal{N}^\circ = \mathbb{N} \setminus \mathcal{N} = \left\{ n \in \mathbb{N} \ : \ F(n) \geqslant e^\gamma \log\log n \right\},$$

and for every integer $k \geqslant 0$, let

$$\mathcal{V}_k = \{n \in \mathbb{N} \ : \ \omega(n) \geqslant k\} \quad \text{and} \quad \mathcal{W}_k = \mathcal{S} \cap \mathcal{N}^\circ \cap \mathcal{V}_k.$$

Since $\mathcal{V}_0 = \mathbb{N}$, Theorem 1 is the assertion that $\mathcal{W}_0 = \mathcal{S} \cap \mathcal{N}^\circ$ is a finite set. In view of the next lemma, it suffices to show that $\mathcal{W}_k = \varnothing$ for some k.

Lemma 1 *For every $k \geqslant 0$, $\mathcal{W}_0 \backslash \mathcal{W}_k$ is a finite set.*

Since $\omega(n) < k$ and $F(n) \geqslant e^\gamma \log\log n$ for all $n \in \mathcal{W}_0 \backslash \mathcal{W}_k$, Lemma 1 is an immediate consequence of the following:

Lemma 2 *For every constant $K > 0$, there are at most finitely many natural numbers n such that $\omega(n) \leqslant K$ and $F(n) \geqslant e^\gamma \log\log n$.*

Proof If $\overline{p}_1, \overline{p}_2, \ldots$ is the sequence of consecutive prime numbers, then for any such number n we have

$$\prod_{j \leqslant K} \frac{\overline{p}_j}{\overline{p}_j - 1} \geqslant \prod_{p \mid n} \frac{p}{p-1} = F(n) \geqslant e^\gamma \log\log n;$$

this shows that n is bounded by a constant which depends only on K. □

For every natural number n, let

$$s(n)=\Big(\prod_{\substack{p\mid n\\ p\in\mathcal{P}}}p\Big)\Big(\prod_{\substack{q\mid n\\ q\in\mathcal{Q}}}q^2\Big),$$

and put

$$\mathcal{Y}=\{n\in\mathbb{N}:n=s(n)\}.$$

Note that $\mathcal{Y}\subset\mathcal{S}$. The following statements are elementary:

($\mathscr{C}_1$) if $n=pm$ with $p\in\mathcal{P}$ and $p\nmid m$, then $n\in\mathcal{Y}$ if and only if $m\in\mathcal{Y}$;
($\mathscr{C}_2$) if $n=q^2m$ with $q\in\mathcal{Q}$ and $q\nmid m$, then $n\in\mathcal{Y}$ if and only if $m\in\mathcal{Y}$;
($\mathscr{C}_3$) $s(n)\in\mathcal{S}$ for all n;
($\mathscr{C}_4$) $\kappa(s(n))=\kappa(n)$ for all n;
($\mathscr{C}_5$) $s(n)\mid n$ for all $n\in\mathcal{S}$; in particular, $s(n)\leqslant n$.

Lemma 3 *If $\mathcal{W}_k\neq\varnothing$ and m_k is the least integer in $\mathcal{W}_k$, then $m_k\in\mathcal{Y}$.*

Proof Clearly, $s(m_k)\in\mathcal{S}$ by ($\mathscr{C}_3$). Combining ($\mathscr{C}_4$) with (7) one sees that

$$F(s(n))=F(n)\quad\text{and}\quad\omega(s(n))=\omega(n)\qquad(n\in\mathbb{N}).$$

Then, using ($\mathscr{C}_5$) it follows that

$$F(s(m_k))=F(m_k)\geqslant e^\gamma\log\log m_k\geqslant e^\gamma\log\log s(m_k),$$

which shows that $s(m_k)\in\mathcal{N}^\circ$. Finally, $s(m_k)\in\mathcal{V}_k$ since

$$\omega(s(m_k))=\omega(m_k)\geqslant k.$$

Thus, we have shown that $s(m_k)\in\mathcal{S}\cap\mathcal{N}^\circ\cap\mathcal{V}_k=\mathcal{W}_k$. Since m_k is the *least* integer in $\mathcal{W}_k$, the equality $m_k=s(m_k)$ follows from ($\mathscr{C}_5$), hence $m_k\in\mathcal{Y}$. □

Next, for every integer $k\geqslant 0$ let

$$\mathcal{Z}_k=\{n\in\mathbb{N}:\Omega(n)=k\}\quad\text{and}\quad\mathcal{T}_k=\mathcal{N}^\circ\cap\mathcal{Y}\cap\mathcal{Z}_k.$$

Here, $\Omega(n)$ is the number of prime divisors of n, counted with multiplicity. Using Lemma 3 one sees that if $\mathcal{W}_\ell\neq\varnothing$ and m_ℓ is the least integer in $\mathcal{W}_\ell$, then $m_\ell\in\mathcal{T}_k$ for some $k\geqslant\ell$; in particular,

$$\bigcup_{k\geqslant\ell}\mathcal{T}_k=\varnothing\quad\Longrightarrow\quad\mathcal{W}_\ell=\varnothing.$$

As we mentioned earlier, in order to prove Theorem 1 it suffices to show that $\mathcal{W}_\ell = \varnothing$ for some ℓ, hence it is enough to show that $\mathcal{T}_k \neq \varnothing$ for at most finitely many integers $k \geqslant 0$.

When $\mathcal{T}_k \neq \varnothing$ we shall use the following notation. Let n_k denote the least integer in $\mathcal{T}_k$. Let $\widehat{p}_k$ be the largest prime $p \in \mathcal{P}$ that divides n_k, and put $\widehat{p}_k = 1$ if no such prime exists. Similarly, let $\widehat{q}_k$ be the largest prime $q \in \mathcal{Q}$ that divides n_k, and set $\widehat{q}_k = 1$ if no such prime exists. Finally, let

$$P_k^+ = \max\{\widehat{p}_k, \widehat{q}_k\} \quad \text{and} \quad P_k^- = \min\{\widehat{p}_k, \widehat{q}_k\}. \tag{8}$$

Note that P_k^+ is the largest prime factor of n_k.

Lemma 4 *Suppose* $\mathcal{T}_k \neq \varnothing$:

(i) *if* $p \in \mathcal{P}$ *with* $p < \widehat{p}_k$, *then* $p \mid n_k$;
(ii) *if* $q \in \mathcal{Q}$ *with* $q < \widehat{q}_k$, *then* $q \mid n_k$.

Proof Suppose on the contrary that $p \in \mathcal{P}$ with $p < \widehat{p}_k$ and $p \nmid n_k$. Since $n_k = s(n_k)$ we can write $n_k = \widehat{p}_k m$ with $\widehat{p}_k \nmid m$. Put $n^* = pm$. Since $n_k \in \mathcal{N}^\circ$, $F(p) > F(\widehat{p}_k)$, and $n^* < n_k$, it follows that

$$F(n^*) = F(p)\,F(m) > F(\widehat{p}_k)\,F(m) = F(n_k) \geqslant e^\gamma \log\log n_k > e^\gamma \log\log n^*,$$

where we have used the fact that F is multiplicative; this shows that $n^* \in \mathcal{N}^\circ$. As $n_k \in \mathcal{Y}$, $(\mathscr{C}_1)$ implies that $n^* \in \mathcal{Y}$. Finally, since Ω is (completely) additive, we see that

$$\Omega(n^*) = \Omega(m) + 1 = \Omega(n_k) = k,$$

which shows that $n^* \in \mathcal{Z}_k$, and thus $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$ (the least number in $\mathcal{T}_k$), and this contradiction completes our proof of (i). Using $(\mathscr{C}_2)$, the proof of (ii) is similar; we omit the details. □

Lemma 5 *Suppose that* $\mathcal{T}_k \neq \varnothing$ *and* $\widehat{p}_k < \widehat{q}_k$. *Then there is at most one prime* $p \in \mathcal{P}$ *such that* $\widehat{p}_k < p < \widehat{q}_k$.

Proof Suppose on the contrary that there are two primes $p_1, p_2 \in \mathcal{P}$ such that $\widehat{p}_k < p_1 < p_2 < \widehat{q}_k$. Since $n_k = s(n_k)$ we can write $n_k = \widehat{q}_k^{\,2} m$, and it is clear that $\gcd(m, p_1 p_2 \widehat{q}_k) = 1$. Put $n^* = p_1 p_2 m$. Since $n_k \in \mathcal{N}^\circ$, $F(p_1 p_2) > F(\widehat{q}_k^{\,2})$, and $n^* < n_k$, we have

$$F(n^*) = F(p_1 p_2)\,F(m) > F(\widehat{q}_k^{\,2})\,F(m) = F(n_k) \geqslant e^\gamma \log\log n_k > e^\gamma \log\log n^*,$$

which shows that $n^* \in \mathcal{N}^\circ$. As $n_k \in \mathcal{Y}$, $(\mathscr{C}_1)$ implies that $n^* \in \mathcal{Y}$. Finally, since

$$\Omega(n^*) = \Omega(m) + 2 = \Omega(n_k) = k,$$

we see that $n^* \in \mathcal{Z}_k$, and thus $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$, and this contradiction implies the result. □

Lemma 6 *Suppose that $\mathcal{T}_k \neq \varnothing$ and $\widehat{p}_k > \widehat{q}_k$. Let p be the largest prime in $\mathcal{P}$ that is less than $\widehat{p}_k$, and let q be the smallest prime in $\mathcal{Q}$ that is greater than $\widehat{q}_k$. Then $q > p/2$.*

Proof Suppose on the contrary that $q \leqslant p/2$. Since $n_k = s(n_k)$ and $p \mid n_k$ (by Lemma 4) but $q \nmid n_k$ (since $q > \widehat{q}_k$), we can write $n_k = p\widehat{p}_k m$, where $\gcd(m, p\widehat{p}_k q) = 1$. Put $n^* = q^2 m$. As in the proofs of Lemmas 4 and 5, we see that $n^* \in \mathcal{Y} \cap \mathcal{Z}_k$. Since $p < \widehat{p}_k$ and $q \leqslant p/2$, we have

$$F(p\widehat{p}_k) = \frac{p\widehat{p}_k}{(p-1)(\widehat{p}_k - 1)} < \frac{p^2}{(p-1)^2} < \frac{q}{q-1} = F(q^2);$$

therefore,

$$F(n^*) = F(q^2)\,F(m) > F(p\widehat{p}_k)\,F(m) = F(n_k) \geqslant e^\gamma \log\log n_k > e^\gamma \log\log n^*,$$

which shows that $n^* \in \mathcal{N}^\circ$. Thus, $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$, and this contradiction implies the result. □

As mentioned above, in order to prove Theorem 1 it suffices to show that $\mathcal{T}_k \neq \varnothing$ for at most finitely many integers $k \geqslant 0$. Arguing by contradiction, we shall assume that the set

$$\mathcal{K} = \{k \geqslant 0 \ : \ \mathcal{T}_k \neq \varnothing\}$$

has infinitely many elements.

Since $\Omega(n_k) = k$, we see that $n_k \to \infty$ as $k \to \infty$ with $k \in \mathcal{K}$; using Lemma 2 it follows that $\omega(n_k) \to \infty$ as well, and therefore $P_k^+ \to \infty$.

We claim that

$$\widehat{p}_k \asymp \widehat{q}_k \quad (k \in \mathcal{K}), \tag{9}$$

which by (8) is equivalent to

$$P_k^+ \asymp P_k^- \quad (k \in \mathcal{K}). \tag{10}$$

To see this, we express $\mathcal{K}$ as a disjoint union $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ [resp. $\mathcal{B}$] is the set of numbers $k \in \mathcal{K}$ for which $\widehat{p}_k < \widehat{q}_k$ [resp. $\widehat{p}_k > \widehat{q}_k$]. To prove (9) it suffices to show:

($\mathscr{D}_1$) $\widehat{p}_k \gg \widehat{q}_k$ for all $k \in \mathcal{A}$;
($\mathscr{D}_2$) $\widehat{p}_k \ll \widehat{q}_k$ for all $k \in \mathcal{B}$.

We use the following result, which is an easy consequence of the prime number theorem:

Lemma 7 *Let $c_\mathcal{P} = \overline{\delta} / \underline{\delta}$ and $c_\mathcal{Q} = \big(1 - \underline{\delta}\big) / \big(1 - \overline{\delta}\big)$. For every $\varepsilon > 0$ there is a number $x_0(\varepsilon)$ such that for all $x > x_0(\varepsilon)$:*

(i) *if p is the smallest prime in $\mathcal{P}$ greater than x, then $p \leqslant (c_{\mathcal{P}} + \varepsilon)\, x$;*
(ii) *if q is the smallest prime in $\mathcal{Q}$ greater than x, then $q \leqslant \left(c_{\mathcal{Q}} + \varepsilon\right) x$;*
(iii) *if p is the largest prime in $\mathcal{P}$ less than x, then $p \geqslant \left(c_{\mathcal{P}}^{-1} - \varepsilon\right) x$;*
(iv) *if q is the largest prime in $\mathcal{Q}$ less than x, then $q \geqslant \left(c_{\mathcal{Q}}^{-1} - \varepsilon\right) x$.*

To prove $(\mathscr{D}_1)$ we can assume that $\mathcal{A}$ is an infinite set. Let $k \in \mathcal{A}$, so that $\widehat{p}_k < \widehat{q}_k$. Since $\widehat{q}_k = P_k^+ \to \infty$ as $k \to \infty$ with $k \in \mathcal{A}$, the assertion $(\mathscr{D}_1)$ then follows from Lemmas 5 and 7.

To prove $(\mathscr{D}_2)$ we can assume that $\mathcal{B}$ is an infinite set. Let $k \in \mathcal{B}$, so that $\widehat{p}_k > \widehat{q}_k$. Let p, q be defined as in Lemma 6. Since $\widehat{p}_k = P_k^+ \to \infty$ as $k \to \infty$ with $k \in \mathcal{B}$, on combining Lemmas 6 and 7 it follows that

$$\widehat{p}_k \ll p \ll q \ll \widehat{q}_k,$$

which proves $(\mathscr{D}_2)$ and completes our Proof of (9).

Next, for every $n \in \mathbb{N}$ let

$$\omega_{\mathcal{P}}(n) = \#\{p \in \mathcal{P} : p \mid n\} \quad \text{and} \quad \omega_{\mathcal{Q}}(n) = \#\{q \in \mathcal{Q} : q \mid n\}.$$

We claim that

$$\omega_{\mathcal{P}}(n_k) \asymp \omega_{\mathcal{Q}}(n_k) \qquad (k \in \mathcal{K}). \tag{11}$$

Indeed, by Lemma 4 it follows that $\omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k)$ and $\omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{Q}}(\widehat{q}_k)$. Therefore, using the prime number theorem together with (4), (5) and (9) we have

$$\omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) \asymp \frac{\widehat{p}_k}{\log \widehat{p}_k} \asymp \frac{\widehat{q}_k}{\log \widehat{q}_k} \asymp \pi_{\mathcal{Q}}(\widehat{q}_k) = \omega_{\mathcal{Q}}(n_k),$$

which proves (11).

Finally, we need the following relation:

$$\log \kappa(n_k) \asymp \omega(n_k) \log \omega(n_k) \qquad (k \in \mathcal{K}). \tag{12}$$

To prove this, observe that Definition (8) and Lemma 4 together imply

$$\prod_{p \leqslant P_k^-} p \mid \kappa(n_k) \quad \text{and} \quad \kappa(n_k) \mid \prod_{p \leqslant P_k^+} p.$$

Consequently,

$$\sum_{p \leqslant P_k^-} \log p \leqslant \log \kappa(n_k) \leqslant \sum_{p \leqslant P_k^+} \log p,$$

and also

$$\pi(P_k^-) \leqslant \omega(n_k) \leqslant \pi(P_k^+).$$

By the prime number theorem, for either choice of the sign $\pm$ we have

$$\sum_{p\leqslant P_k^{\pm}} \log p \sim P_k^{\pm} \quad \text{and} \quad \pi(P_k^{\pm}) \sim \frac{P_k^{\pm}}{\log P_k^{\pm}} \qquad (k \to \infty,\ k \in \mathcal{K}),$$

therefore in view of (10) we see that

$$\log \kappa(n_k) \asymp P_k^+ \quad \text{and} \quad \omega(n_k) \asymp \frac{P_k^+}{\log P_k^+},$$

and (12) follows immediately.

Now we come to the heart of the argument. To complete the proof of Theorem 1, we seek a contradiction to our assumption that $\mathcal{K}$ is an infinite set. For this, it is enough to prove both of the following statements with a suitably chosen real number $\varepsilon > 0$:

($\mathscr{E}_1$) the inequality $n_k \leqslant \kappa(n_k)^{1+\varepsilon}$ holds for at most finitely many $k \in \mathcal{K}$;
($\mathscr{E}_2$) the inequality $n_k > \kappa(n_k)^{1+\varepsilon}$ holds for at most finitely many $k \in \mathcal{K}$.

In view of (11) and (12), there is a constant $C > 1$ such that the inequalities

$$\omega_{\mathcal{P}}(n_k) \leqslant (C-1)\, \omega_{\mathcal{Q}}(n_k) \tag{13}$$

and

$$\log \kappa(n_k) \leqslant C\, \omega(n_k) \log \omega(n_k) \tag{14}$$

both hold if k is sufficiently large. Let C be fixed, and put $\varepsilon = C^{-3}$.

To prove ($\mathscr{E}_1$), we suppose on the contrary that $n_k \leqslant \kappa(n_k)^{1+\varepsilon}$ holds for infinitely many $k \in \mathcal{K}$. Let k be large, and put

$$r = \omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) \quad \text{and} \quad s = \omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{Q}}(\widehat{q}_k)$$

By what we have already seen it is clear that $\min\{r, s\} \to \infty$ as $k \to \infty$ with $k \in \mathcal{K}$, thus by (13) we have

$$r \leqslant (C-1)s \tag{15}$$

if k is large enough. By Lemma 4 and the fact that $n_k \in \mathcal{Y}$, it follows that

$$n_k = \Bigg(\prod_{\substack{p\leqslant \widehat{p}_k\\ p\in\mathcal{P}}} p\Bigg)\Bigg(\prod_{\substack{q\leqslant \widehat{q}_k\\ q\in\mathcal{Q}}} q^2\Bigg) \quad \text{and} \quad \kappa(n_k) = \Bigg(\prod_{\substack{p\leqslant \widehat{p}_k\\ p\in\mathcal{P}}} p\Bigg)\Bigg(\prod_{\substack{q\leqslant \widehat{q}_k\\ q\in\mathcal{Q}}} q\Bigg).$$

Hence, our assumption that $n_k \leqslant \kappa(n_k)^{1+\varepsilon}$ implies that

$$\kappa(n_k) \geqslant \left(\frac{n_k}{\kappa(n_k)}\right)^{1/\varepsilon} = \Bigg(\prod_{\substack{q \leqslant \widehat{q}_k \\ q \in \mathcal{Q}}} q\Bigg)^{1/\varepsilon}. \tag{16}$$

If $\overline{p}_1, \overline{p}_2, \ldots$ is the sequence of consecutive prime numbers, then by the prime number theorem (and recalling our choice of ε) we derive that

$$\log \kappa(n_k) \geqslant C^3 \sum_{\substack{q \leqslant \widehat{q}_k \\ q \in \mathcal{Q}}} \log q \geqslant C^3 \sum_{p \leqslant \overline{p}_s} \log p \sim C^3 \overline{p}_s \sim C^3 s \log s$$

as $k \to \infty$ with $k \in \mathcal{K}$. On the other hand, using (14), (15) and the fact that $\omega(n_k) = r + s$, it follows that

$$\log \kappa(n_k) \leqslant C(r+s)\log(r+s) \leqslant C^2 s \log(Cs) \sim C^2 s \log s.$$

Since $C^3 > C^2$, these two inequalities for $\log \kappa(n_k)$ lead to a contradiction once k is sufficiently large, and this completes the proof of $(\mathscr{E}_1)$.

To prove $(\mathscr{E}_2)$ we use some ideas from Choie et al. [1]. Suppose that $n_k > \kappa(n_k)^{1+\varepsilon}$, and put $t = \omega(n_k)$. We claim that either

$$\sum_{p \leqslant \overline{p}_t} \log p < (1+\varepsilon)^{-1/2}\, \overline{p}_t, \tag{17}$$

or

$$\overline{p}_t \leqslant \exp\left(2/\log(1+\varepsilon)\right). \tag{18}$$

Assuming the claim, it is easy to see that $\omega(n_k)$ is bounded above by a constant K that depends only on ε. By Lemma 2, n_k can take only finitely many distinct values, which implies $(\mathscr{E}_2)$.

To prove the claim, assume that (17) fails:

$$\log(\overline{p}_1 \cdots \overline{p}_t) = \sum_{p \leqslant \overline{p}_t} \log p \geqslant (1+\varepsilon)^{-1/2}\, \overline{p}_t.$$

Thanks to Rosser and Schoenfeld [8] it is known that

$$\prod_{p \leqslant x} \frac{p}{p-1} \leqslant e^{\gamma}\left(\log x + \frac{1}{\log x}\right) \qquad (x > 1).$$

Therefore, taking $x = \overline{p}_t$ and noting that $\kappa(n_k) \geqslant \overline{p}_1 \cdots \overline{p}_t$, we derive that

$$\begin{aligned} e^\gamma\left(\log \overline{p}_t + \frac{1}{\log \overline{p}_t}\right) &\geqslant \prod_{j=1}^{t} \frac{\overline{p}_j}{\overline{p}_j - 1} \geqslant \frac{n_k}{\varphi(n_k)} \geqslant e^\gamma \log\log n_k \\ &> e^\gamma \log((1+\varepsilon)\log\kappa(n_k)) \\ &\geqslant e^\gamma \log\big((1+\varepsilon)\log(\overline{p}_1 \cdots \overline{p}_t)\big) \\ &\geqslant e^\gamma \log\left((1+\varepsilon)^{1/2}\,\overline{p}_t\right) = e^\gamma\left(\log \overline{p}_t + 0.5\log(1+\varepsilon)\right); \end{aligned}$$

that is,

$$\frac{1}{\log \overline{p}_t} \geqslant 0.5\log(1+\varepsilon),$$

which is equivalent to (18). This proves the claim and completes our proof of Theorem 1.

3 Proof of Theorem 2

We continue to use the notation of the previous section, but we focus on the special case that

$$\begin{aligned} \mathcal{P} &= \{p \in \mathbb{P} \ : \ p \equiv 1 \pmod 4\} \cup \{2\}, \\ \mathcal{Q} &= \{q \in \mathbb{P} \ : \ q \equiv 3 \pmod 4\}. \end{aligned}$$

Note that the corresponding set $\mathcal{S}$ contains every natural number that can be expressed as a sum of two squares. As before, we write

$$\mathcal{T}_k = \left\{n \in \mathbb{N} \ : \ F(n) \geqslant e^\gamma \log\log n, \ n = s(n), \ \text{and} \ \Omega(n) = k\right\}$$

and put

$$\mathcal{K} = \{k \geqslant 0 \ : \ \mathcal{T}_k \neq \varnothing\}.$$

Lemma 8 *If* $k \in \mathcal{K}$, *then* $P_k^- < 50000$.

Proof For every real number $x \geqslant 10$, let

- $g_{\mathcal{P}}(x) =$ the smallest prime in $\mathcal{P}$ greater than x;
- $g_{\mathcal{Q}}(x) =$ the smallest prime in $\mathcal{Q}$ greater than x;
- $\ell_{\mathcal{P}}(x) =$ the largest prime in $\mathcal{P}$ less than x;
- $\ell_{\mathcal{Q}}(x) =$ the largest prime in $\mathcal{Q}$ less than x.

Also, put

$$\vartheta_{\mathcal{P}}(x)=\sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}}\log p \quad\text{and}\quad \vartheta_{\mathcal{Q}}(x)=\sum_{\substack{q\leqslant x\\ q\in\mathcal{Q}}}\log q.$$

Using the explicit bounds of Theorems 1 and 2 of Ramaré and Rumely [6], we see that the inequalities

$$0.49\,x<\vartheta_{\mathcal{P}}(x)<0.51\,x \quad\text{and}\quad 0.49\,x<\vartheta_{\mathcal{Q}}(x)<0.51\,x \tag{19}$$

hold for all $x\geqslant 45000$ (note that $\vartheta_{\mathcal{P}}(x)=\log 2+\theta(x;4,1)$ and $\vartheta_{\mathcal{Q}}(x)=\theta(x;4,3)$ in the notation of [6]). Consequently, for any $x\geqslant 50000$ we have

$$\tfrac{49}{51}\,x<\ell_{\mathcal{P}}(x)<x<g_{\mathcal{P}}(x)<\tfrac{51}{49}\,x$$

and

$$\tfrac{49}{51}\,x<\ell_{\mathcal{Q}}(x)<x<g_{\mathcal{Q}}(x)<\tfrac{51}{49}\,x.$$

Now suppose that $P_k^-\geqslant 50000$. Using Lemma 5 and the preceding bounds we have

$$\widehat{q}_k<g_{\mathcal{P}}(g_{\mathcal{P}}(\widehat{p}_k))<\left(\tfrac{51}{49}\right)^2\widehat{p}_k.$$

On the other hand, by Lemma 6 we have

$$\tfrac{51}{49}\,\widehat{q}_k>g_{\mathcal{Q}}(\widehat{q}_k)>\tfrac{1}{2}\,\ell_{\mathcal{P}}(\widehat{p}_k)>\tfrac{49}{102}\,\widehat{p}_k.$$

Hence, it follows that

$$0.92\,\widehat{q}_k<\widehat{p}_k<2.2\,\widehat{q}_k. \tag{20}$$

By Lemma 4 it is clear that

$$\log\kappa(n_k)=\sum_{\substack{p\leqslant\widehat{p}_k\\ p\in\mathcal{P}}}\log p+\sum_{\substack{q\leqslant\widehat{q}_k\\ q\in\mathcal{Q}}}\log q=\vartheta_{\mathcal{P}}(\widehat{p}_k)+\vartheta_{\mathcal{Q}}(\widehat{q}_k).$$

On the other hand, arguing as in the proof of Theorem 1, it follows from (16) that

$$\log\kappa(n_k)\geqslant\varepsilon^{-1}\vartheta_{\mathcal{Q}}(\widehat{q}_k)$$

if $\varepsilon>0$ is fixed and $n_k\leqslant\kappa(n_k)^{1+\varepsilon}$. Combining the two preceding results with (19), we see that

$$0.51\,(\widehat{p}_k+\widehat{q}_k)\geqslant\vartheta_{\mathcal{P}}(\widehat{p}_k)+\vartheta_{\mathcal{Q}}(\widehat{q}_k)\geqslant\varepsilon^{-1}\vartheta_{\mathcal{Q}}(\widehat{q}_k)\geqslant 0.49\,\varepsilon^{-1}\widehat{q}_k$$

since $P_k^- \geqslant 50000$; taking into account (20), we further have

$$0.51\,(1+2.2)\,\widehat{q}_k \geqslant 0.51\,(\widehat{p}_k+\widehat{q}_k) \geqslant 0.49\,\varepsilon^{-1}\widehat{q}_k,$$

which implies that $\varepsilon \geqslant 0.3002$. Thus, for the smaller value $\varepsilon = 0.3$, we see that the condition $n_k \leqslant \kappa(n_k)^{1.3}$ implies $P_k^- < 50000$.

On the other hand, if $n_k > \kappa(n_k)^{1.3}$, we put $t = \omega(n_k)$ as in the proof of Theorem 1. Since $\varepsilon = 0.3$, we derive from (17) and (18) that either

$$\vartheta(\overline{p}_t) = \sum_{p \leqslant \overline{p}_t} \log p < (1.3)^{-1/2}\,\overline{p}_t < 0.88\,\overline{p}_t, \tag{21}$$

or

$$\overline{p}_t \leqslant \exp(2/\log 1.3) < 2045.$$

Using again Theorems 1 and 2 of Ramaré and Rumely [6] (see also [8]), it is easy to see that the inequality (21) implies $\overline{p}_t < 300$, hence the inequality $\overline{p}_t < 2045$ holds in both cases. It follows that $t < 310$, and therefore,

$$\min\{\pi_{\mathcal{P}}(\widehat{p}_k), \pi_{\mathcal{Q}}(\widehat{q}_k)\} = \min\{\omega_{\mathcal{P}}(n_k), \omega_{\mathcal{Q}}(n_k)\} \leqslant \omega(n_k) = t < 310,$$

which implies that $P_k^- < 5000$. This completes the proof. □

Corollary 3 *If $k \in \mathcal{K}$, then $k < 10000$.*

Proof For any $k \in \mathcal{K}$ we have

$$k = \Omega(n_k) = \omega_{\mathcal{P}}(n_k) + 2\,\omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) + 2\,\pi_{\mathcal{Q}}(\widehat{q}_k).$$

If $P_k^- = \widehat{p}_k$ (i.e., $\widehat{p}_k < \widehat{q}_k$), then by Lemmas 5 and 8 it follows that

$$\begin{aligned} k &\leqslant \max_{p<50000} \left\{\pi_{\mathcal{P}}(p) + 2\,\pi_{\mathcal{Q}}\left(g_{\mathcal{P}}(g_{\mathcal{P}}(p))\right)\right\} \\ &\leqslant \pi_{\mathcal{P}}(50000) + 2\,\pi_{\mathcal{Q}}\left(g_{\mathcal{P}}(g_{\mathcal{P}}(50000))\right) = 7718. \end{aligned}$$

If $P_k^- = \widehat{q}_k$ (i.e., $\widehat{q}_k < \widehat{p}_k$), then by Lemmas 6 and 8 it follows that

$$\begin{aligned} k &\leqslant \max_{q<50000} \max_{\substack{p\in\mathbb{P} \\ \ell_{\mathcal{P}}(p)<2g_{\mathcal{Q}}(q)}} \left\{\pi_{\mathcal{P}}(p) + 2\,\pi_{\mathcal{Q}}(q)\right\} \\ &= \max_{q<50000} \max_{\substack{p\in\mathbb{P} \\ \ell_{\mathcal{P}}(p)<2g_{\mathcal{Q}}(q)}} \left\{1 + \pi_{\mathcal{P}}(\ell_{\mathcal{P}}(p)) + 2\,\pi_{\mathcal{Q}}(q)\right\} \\ &\leqslant \max_{q<50000} \left\{1 + \pi_{\mathcal{P}}(2\,g_{\mathcal{Q}}(q)) + 2\,\pi_{\mathcal{Q}}(q)\right\} \\ &\leqslant 1 + \pi_{\mathcal{P}}(2\,g_{\mathcal{Q}}(50000)) + 2\,\pi_{\mathcal{Q}}(50000) = 9951. \end{aligned}$$

The result follows. □

Now let $\overline{\overline{p}}_1, \overline{\overline{p}}_2, \ldots$ be the sequence of consecutive primes in $\mathcal{P}$, and let $\overline{\overline{q}}_1, \overline{\overline{q}}_2, \ldots$ be the consecutive primes in $\mathcal{Q}$. For any integers $r, s \geqslant 0$, let

$$N_{r,s} = \Big(\prod_{i=1}^{r} \overline{\overline{p}}_i\Big)\Big(\prod_{j=1}^{s} \overline{\overline{q}}_j^{\,2}\Big).$$

It is easy to see that $N_{r,s} \in \mathcal{Y}$ for all $r, s \geqslant 0$, and for every $k \in \mathcal{K}$ one has

$$n_k = N_{r,s}, \qquad \widehat{p}_k = \overline{\overline{p}}_r, \qquad \widehat{q}_k = \overline{\overline{q}}_s \quad \text{and} \quad k = r + 2s,$$

where $r = \omega_{\mathcal{P}}(n_k)$ and $s = \omega_{\mathcal{Q}}(n_k)$. By a straightforward computation, one verifies the following:

Lemma 9 *If $r, s \geqslant 0$, then $N_{r,s} \in \mathcal{N}^\circ$ if and only if the pair (r, s) lies in the set*

$$\begin{aligned}\mathcal{X} = \{&(0,0), (1,0), (0,1), (2,0), (1,1), (2,1), (1,2), (3,1), (2,2), (4,1),\\ &(3,2), (2,3), (4,2), (3,3), (5,2), (4,3), (3,4), (5,3), (4,4), (6,3),\\ &(5,4), (4,5), (7,3), (6,4), (5,5), (7,4), (6,5), (7,5), (8,5)\}\,.\end{aligned}$$

We remark that, in view of Corollary 3, it suffices to check the condition $N_{r,s} \in \mathcal{N}^\circ$ only for those pairs (r, s) with $r + 2s < 10000$.

Corollary 4 *If $k \in \mathcal{K}$, then $k \leqslant 18$.*

Corollary 5 *If $n \in \mathcal{S} \cap \mathcal{N}^\circ$, $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, then $(r, s) \in \mathcal{X}$. In particular, $\omega(n) \leqslant 13$.*

Proof Since

$$F(N_{r,s}) = \Big(\prod_{i=1}^{r} \frac{\overline{\overline{p}}_i}{\overline{\overline{p}}_i - 1}\Big)\Big(\prod_{j=1}^{s} \frac{\overline{\overline{q}}_j}{\overline{\overline{q}}_j - 1}\Big) \geqslant \Big(\prod_{\substack{p \mid n\\ p\in\mathcal{P}}} \frac{p}{p-1}\Big)\Big(\prod_{\substack{q \mid n\\ q\in\mathcal{Q}}} \frac{q}{q-1}\Big) = F(n)$$

and

$$n \geqslant s(n) = \Big(\prod_{\substack{p \mid n\\ p\in\mathcal{P}}} p\Big)\Big(\prod_{\substack{q \mid n\\ q\in\mathcal{Q}}} q^2\Big) \geqslant \Big(\prod_{i=1}^{r} \overline{\overline{p}}_i\Big)\Big(\prod_{j=1}^{s} \overline{\overline{q}} q_j^2\Big) = N_{r,s},$$

we have

$$F(N_{r,s}) \geqslant F(n) \geqslant e^\gamma \log\log n \geqslant e^\gamma \log\log N_{r,s},$$

which shows that $N_{r,s} \in \mathcal{N}^\circ$. □

We now turn to a description of our method for generating the elements of $\mathcal{S}\setminus\mathcal{N} = \mathcal{S}\cap\mathcal{N}^\circ$. For any given $n\in\mathcal{S}\cap\mathcal{N}^\circ$ with $r=\omega_{\mathcal{P}}(n)$ and $s=\omega_{\mathcal{Q}}(n)$, we can write

$$s(n) = p_1\cdots p_r\, q_1^2\cdots q_s^2,$$

where $p_1<\cdots<p_r$ are primes in $\mathcal{P}$ and $q_1<\cdots<q_s$ are primes in $\mathcal{Q}$. For fixed $i=1,\ldots,r$, let γ_i be the largest non-negative integer such that the number

$$\Big(\prod_{\ell=1}^{i-1}\overline{\overline{p}}_\ell\Big)\Big(\prod_{\ell=i}^{r}\overline{\overline{p}}_{\ell+\gamma_i}\Big)\Big(\prod_{j=1}^{s}\overline{\overline{q}}_j^{\,2}\Big)$$

lies in $\mathcal{N}^\circ$, which exist by Lemma 2. Using an argument similar to that in the proof of Lemma 4, one can deduce that

$$\overline{\overline{p}}_i \leqslant p_i \leqslant \overline{\overline{p}}_{i+\gamma_i}\qquad (i=1,\ldots,r). \tag{22}$$

Similarly, for fixed $j=1,\ldots,s$, let δ_j be the largest non-negative integer such that the number

$$\Big(\prod_{i=1}^{r}\overline{\overline{p}}_i\Big)\Big(\prod_{\ell=1}^{j-1}\overline{\overline{q}}_\ell^{\,2}\Big)\Big(\prod_{\ell=j}^{s}\overline{\overline{q}}_{\ell+\delta_j}^{\,2}\Big)$$

lies in $\mathcal{N}^\circ$. Then,

$$\overline{\overline{q}}_j \leqslant q_j \leqslant \overline{\overline{q}}_{j+\gamma_j}\quad (j=1,\ldots,s). \tag{23}$$

Therefore, for fixed $(r,s)\in\mathcal{X}$, if $n\in\mathcal{S}\cap\mathcal{N}^\circ$ with $r=\omega_{\mathcal{P}}(n)$ and $s=\omega_{\mathcal{Q}}(n)$, then the number $s(n)$ must lie in the finite set $\mathcal{A}_{r,s}$ of integers of the form

$$m = p_1\cdots p_r\, q_1^2\cdots q_s^2, \tag{24}$$

where $p_1<\cdots<p_r$ are primes in $\mathcal{P}$, $q_1<\cdots<q_s$ are primes in $\mathcal{Q}$, the primes p_i and q_j satisfy the bounds (22) and (23), and $m\in\mathcal{N}^\circ$. The set $\mathcal{A}_{r,s}$ can be explicitly determined by a numerical computation, and we obtain a finite list of "admissible" values for the quantity $s(n)$.

To determine explicitly all of the numbers $n\in\mathcal{S}\cap\mathcal{N}^\circ$ with $r=\omega_{\mathcal{P}}(n)$ and $s=\omega_{\mathcal{Q}}(n)$, for every $m\in\mathcal{A}_{r,s}$ we need to find all such numbers for which $s(n)=m$. To do this, factor m as in (24). For fixed $i=1,\ldots,r$, let α_i be the largest integer such that the number $mp_i^{\alpha_i-1}$ lies in $\mathcal{N}^\circ$. Similarly, for fixed $j=1,\ldots,s$, let β_j be the largest integer such that the number $mq_j^{\beta_j-1}$ lies in $\mathcal{N}^\circ$. Put

$$M = m\cdot p_1^{\alpha_1-1}\cdots p_r^{\alpha_r-1}q_1^{\beta_1-1}\cdots q_s^{\beta_s-1}.$$

Then, it is easy to see that $m \mid n$ and $n \mid M$ for any $n \in \mathcal{S} \cap \mathcal{N}^\circ$ such that $s(n) = m$. Hence, n can take only finitely many values which can be determined explicitly for each $m \in \mathcal{A}_{r,s}$.

For example, taking $r = s = 2$ we find that

{4410, 8820, 10890, 13230, 17640, 21780, 22050, 26460, 30870, 35280, 39690, 44100, 52920, 61740, 66150, 70560, 79380, 88200, 92610, 105840, 110250}

is a complete list of the numbers $n \in \mathcal{S} \setminus \mathcal{N}$ with $\omega_{\mathcal{P}}(n) = \omega_{\mathcal{Q}}(n) = 2$. Examining the lists generated as (r, s) varies over the pairs in $\mathcal{X}$, we are lead to the statement of Theorem 2.

4 Evaluation of $\overline{\lim}_{n\in\mathcal{S}} \frac{n}{\varphi(n)\log\log n}$ and $\overline{\lim}_{n\in\mathcal{S}} \frac{\sigma(n)}{n\log\log n}$

We conclude the paper by giving two propositions and two corollaries that yield the analogue of the work of Landau [4] and Gronwall [2] for any set $\mathcal{S}$ of the form (6) and for the set of natural numbers equal to a sum of two squares. In fact, Corollary 6 shows that Theorem 1 is nontrivial in the sense that $F(n)/\log\log n$ cannot be bounded away from e^γ by any positive constant for all large $n \in \mathcal{S}$. We will use the notation $f(n) = o(g(n))$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 0$.

Proposition 1 *Let $\{a_n\}$ be an infinite sequence of positive integers such that if we write $a_n = \prod_p p^{v(p,n)}$ we have:*

(i) $\kappa(a_n) = \prod_{p\leqslant n} p$ *(i.e., $v(p,n) = 0 \iff p > n$);*
(ii) $a_n = \exp(n^{1+o(1)})$*;*
(iii) $\lim_{n\to\infty} v(p,n) = \infty$ *for each p.*

Then,

$$\lim_{n\to\infty} \frac{\sigma(a_n)}{a_n \log\log a_n} = e^\gamma.$$

Proof For all $n \geqslant 1$, let

$$b_n = \prod_{p\leqslant n} p \quad \text{and} \quad c_n = \frac{\sigma(a_n)}{a_n}\frac{\varphi(b_n)}{b_n},$$

and observe that (i) implies

$$c_n = \left(\prod_{p\leqslant n} \frac{p^{v(p,n)+1} - 1}{p^{v(p,n)}(p-1)}\right)\left(\prod_{p\leqslant n} \frac{p-1}{p}\right) = \prod_{p\leqslant n}\left(1 - \frac{1}{p^{v(p,n)+1}}\right).$$

Since $v(p,n)+1 \geqslant 2$ for every prime $p \leqslant n$, we have for any $m \leqslant n$:

$$1 \geqslant c_n > \prod_{p \leqslant m}\left(1-\frac{1}{p^{v(p,n)+1}}\right)\prod_{p>m}\left(1-\frac{1}{p^2}\right).$$

Using (iii) we have for every fixed integer m:

$$1 \geqslant \overline{\lim_{n\to\infty}}\, c_n \geqslant \underline{\lim_{n\to\infty}}\, c_n \geqslant \prod_{p>m}\left(1-\frac{1}{p^2}\right).$$

The product on the right tends to one as $m \to \infty$, hence $\lim_{n\to\infty} c_n = 1$; therefore,

$$\lim_{n\to\infty}\frac{\sigma(a_n)}{a_n \log n} = \lim_{n\to\infty}\frac{b_n}{\varphi(b_n)\log n}.$$

Our assumption (ii) implies that $\log\log a_n = (1+o(1))\log n$, and using Mertens' theorem (see, e.g., [8]) we have

$$\frac{\varphi(b_n)}{b_n} = \prod_{p\leqslant n}\left(1-\frac{1}{p}\right) = (1+o(1))\,\frac{e^{-\gamma}}{\log n},$$

and the result follows. □

Using similar ideas (and an easier argument) one can obtain the following analogue of Proposition 1 for the Euler totient function:

Proposition 2 *Let* $\{a_n\}$ *be an infinite sequence of positive integers such that*:

(i) $\kappa(a_n) = \prod_{p\leqslant n} p$;
(ii) $a_n = \exp(n^{1+o(1)})$.

Then,

$$\lim_{n\to\infty}\frac{a_n}{\varphi(a_n)\log\log a_n} = e^{\gamma}.$$

Corollary 6 *For any set* $\mathcal{S}$ *defined by* (6), *we have*

$$\overline{\lim_{n\in\mathcal{S}}}\,\frac{\sigma(n)}{n\log\log n} = \overline{\lim_{n\in\mathcal{S}}}\,\frac{n}{\varphi(n)\log\log n} = e^{\gamma}.$$

Proof Since

$$\overline{\lim_{n\to\infty}}\,\frac{\sigma(n)}{n\log\log n} = \overline{\lim_{n\to\infty}}\,\frac{n}{\varphi(n)\log\log n} = e^{\gamma}$$

by [2] and [4], respectively, it suffices to show that there is a sequence $\{a_n\}$ in $\mathcal{S}$ such that

$$\lim_{n\to\infty} \frac{\sigma(a_n)}{a_n \log\log a_n} = \lim_{n\to\infty} \frac{a_n}{\varphi(a_n)\log\log a_n} = e^\gamma.$$

Let $a_1 = 1$, and for every integer $n \geqslant 2$, let

$$b_n = \prod_{p\leqslant n} p, \qquad d_n = \left\lfloor n^{(\log n)^{-1/2}} \right\rfloor \quad \text{and} \quad a_n = b_n^{d_n}.$$

It is easy to see that $d_n \geqslant 2$ for $n \geqslant 2$, $d_n = n^{o(1)}$, and d_n tends to infinity with n. Clearly, $a_n \in \mathcal{S}$ for all $n \geqslant 1$, and by the Prime Number Theorem in the form $\sum_{p\leqslant x} \log p = x(1+o(1))$ as $x \to \infty$ we see that

$$\log a_n = d_n \log b_n = n^{o(1)} \sum_{p\leqslant n} \log p = n^{1+o(1)} \quad (n\to\infty).$$

The sequence $\{a_n\}$ therefore satisfies the hypotheses of Propositions 1 and 2, and the result follows. □

Corollary 7 *We have*

$$\overline{\lim_{n=a^2+b^2}} \frac{\sigma(n)}{n\log\log n} = \overline{\lim_{n=a^2+b^2}} \frac{n}{\varphi(n)\log\log n} = e^\gamma.$$

Proof Defining a_n for all $n \geqslant 1$ as in the proof of Corollary 6, it is easy to see that the sequence $\{a_n^2\}$ satisfies the hypotheses of Propositions 1 and 2; it follows that

$$\overline{\lim_{n=a^2}} \frac{\sigma(n)}{n\log\log n} = \overline{\lim_{n=a^2}} \frac{n}{\varphi(n)\log\log n} = e^\gamma,$$

and this implies the stated result. □

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