Quantum Inf Process (2017) 16:38
DOI 10.1007/s11128-016-1470-1

Violation of Bell inequalities from S_4 symmetry: the three orbits case

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Received: 26 May 2016 / Accepted: 8 November 2016 / Published online: 27 December 2016

Abstract The recently proposed (Güney and Hillery in Phys Rev A 90:062121, 2014; Phys Rev A 91:052110, 2015) group theoretical approach to the problem of violating the Bell inequalities is applied to S_4 group. The Bell inequalities based on the choice of three orbits in the representation space corresponding to standard representation of S_4 are derived and their violation is described. The corresponding nonlocal games are analyzed.

Keywords Bell inequalities · Nonlocal games · Group theoretical methods

1 Introduction

The famous Bell inequalities [1] provide the necessary conditions for any theory to be a local realistic one. Their importance stems from the observation that they can be violated in quantum theory. As a result the Bell inequalities can be used for test of entanglement and as a basis for protocols in quantum cryptography [2].

Bell inequalities have been studied intensively by numerous authors. Their various forms have been derived [3–9] characterized by the number of parties, measurement settings and outcomes for each measurement (for a review, see [10,11]).

Recently, there appeared interesting papers [12,13] where the group theoretical methods have been proposed as a tool for analyzing the quantum mechanical violation

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of Bell inequalities. Examples of Bell inequalities based on representations of some finite groups were presented there. Further example has been considered in Ref. [14]. It is based on S_4 symmetry and its standard irreducible representation. The resulting Bell inequality is obtained by selecting two generic orbits determined by the geometry of tetrahedron. In the present paper we provide further examples of Bell inequalities related to the symmetry of tetrahedron. They result from the particular choices of three generic orbits.

The paper is organized as follows. In Sect. 2 we discuss the Bell inequalities from the point of view of the existence of joint probability distribution and describe the group theoretical approach to the problem of their violation proposed by Güney and Hillery. This approach is then applied in Sect. 3 to the symmetric group S_4. Three examples of quantum mechanical violation of Bell inequalities are presented. They are based on the specific choice of orbits in the standard representation of S_4. In each of three cases we consider the set of states arising from the choice of three orbits. The results are interpreted in Sect. 4 in the framework of game theory. Section 5 is devoted to some conclusions. Some technical details are relegated to Appendix.

2 Bell inequalities

Quantum mechanical violation of Bell inequalities is closely related to the existence of noncommuting observables. In two elegant papers [15,16], Fine provided a particulary transparent interpretation of Bell inequalities (see also [17,18]). Assume that we have a number of random variables possessing joint probability distribution. Bell inequalities concern the joint probability distributions of some subsets of the initial set of random variables. They result from the assumption that these distributions can be obtained as marginals from the original joint probability distribution. What is even more important is that the Bell inequalities form also the sufficient conditions for the existence of joint distribution returning other probabilities as marginals. In fact, the latter condition provides a set of linear equations for the joint distribution which possess the whole family of solutions. We are interested in solutions belonging to the interval $\langle 0, 1\rangle$. The possibility of selecting such solutions relies on the validity of Bell inequalities.

Fine's theorem explains the origin of quantum machanical violation of Bell inequalities. Due to the uncertainty principle the joint probability can be constructed only for the set of mutually commuting observables. Therefore, no inequality of Bell type could be derived for joint probabilities of commuting observables if these probabilities emerged as marginals from joint distribution for larger set of, in general, noncommuting observables.

Let us illustrate the above discussion by a simple example. Let $\hat{A}$ be some observable with the spectral decomposition

$$\hat{A} = \sum_i a_i \hat{\Pi}_i \tag{1}$$

where $\hat{\Pi}_i$ are the projectors on the relevant eigenspaces (we shall assume our space of states is finite-dimensional). Consider any state $\hat{\rho}$ and let [19]

$$C(\zeta) = \mathrm{Tr}\left(e^{i\zeta\hat{A}}\hat{\rho}\right) = \sum_i e^{i\zeta a_i}\mathrm{Tr}\left(\hat{\Pi}_i\hat{\rho}\right) \equiv \sum_i e^{i\zeta a_i} p_i \tag{2}$$

be the generating function for the moments of $\hat{A}$:

$$\left\langle \hat{A}^n \right\rangle = \mathrm{Tr}\left(\hat{A}^n\hat{\rho}\right) = \sum_i a_i^n p_i = \left(-i\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n C(\zeta)\Big|_{\zeta=0}. \tag{3}$$

The probability distribution is obtained by Fourier transform

$$p(a) = \frac{1}{2\pi}\int \mathrm{d}\zeta\, e^{-i\zeta a} C(\zeta) = \sum_i p_i \delta(a - a_i). \tag{4}$$

Assume that now we have two observables,

$$\hat{A}_1 \equiv \sum_i a_{1i}\hat{\Pi}_{1i}, \quad \hat{A}_2 \equiv \sum_k a_{2k}\hat{\Pi}_{2k}. \tag{5}$$

The generating function for the moments $\left\langle A_1^{n_1} A_2^{n_2}\right\rangle$ reads

$$\begin{aligned} C(\zeta_1,\zeta_2) &= \mathrm{Tr}\left(e^{i\zeta_1\hat{A}_1}e^{i\zeta_2\hat{A}_2}\hat{\rho}\right) = \sum_{i,k} e^{i\zeta_1 a_{1i}}e^{i\zeta_2 a_{2k}}\mathrm{Tr}\left(\hat{\Pi}_{1i}\hat{\Pi}_{2k}\hat{\rho}\right) \\ &\equiv \sum_{i,k} e^{i\zeta_1 a_{1i}}e^{i\zeta_2 a_{2k}} p_{ik}. \end{aligned} \tag{6}$$

We are tempted to define the joint probability as

$$\begin{aligned} p(a_1,a_2) &\equiv \frac{1}{4\pi^2}\int \mathrm{d}\zeta_1\mathrm{d}\zeta_2 e^{-i(\zeta_1 a_1 + \zeta_2 a_2)} C(\zeta_1,\zeta_2) \\ &= \sum_{i,k} p_{ik}\delta(a_1 - a_1 i)\,\delta(a_2 - a_{2k}). \end{aligned} \tag{7}$$

Due to

$$\sum_k p_{ik} = \sum_k \mathrm{Tr}\left(\hat{\Pi}_{1i}\hat{\Pi}_{2k}\hat{\rho}\right) = \mathrm{Tr}\left(\hat{\Pi}_{1i}\left(\sum_k \hat{\Pi}_{2k}\right)\hat{\rho}\right) = \mathrm{Tr}\left(\hat{\Pi}_{1i}\hat{\rho}\right) = p_{1i} \tag{8}$$

single probability densities can be obtained as marginals

$$p_1(a_1) = \int \mathrm{d}a_2\, p(a_1,a_2). \tag{9}$$

To have the genuine probability distribution we must assume $p_{ik} \geq 0$. Then the last expression (7) provides a finite positive measure on $\mathbb{R}^2$. Therefore, by Bochner theorem $C(\zeta_1,\zeta_2)$ is positive definite function [20]. In particular

$$C(\zeta_1, \zeta_2) = \overline{C(-\zeta_1, -\zeta_2)} \tag{10}$$

or

$$\mathrm{Tr}\left(e^{i\zeta_1 \hat{A}_1} e^{i\zeta_2 \hat{A}_2} \hat{\rho}\right) = \mathrm{Tr}\left(e^{i\zeta_2 \hat{A}_2} e^{i\zeta_1 \hat{A}_1} \hat{\rho}\right). \tag{11}$$

Assuming that (11) holds for all states $\hat{\rho}$ we find

$$e^{i\zeta_1 \hat{A}_1} e^{i\zeta_2 \hat{A}_2} = e^{i\zeta_2 \hat{A}_2} e^{i\zeta_1 \hat{A}_1} \tag{12}$$

or $\left[\hat{A}_1, \hat{A}_2\right] = 0$. We see that the joint probability can be defined only for commuting variables.

Taking into account Fine's results one concludes that the general scheme for deriving the Bell inequalities is quite simple. The relevant combination of probabilities is written in terms of marginals of the joint probability distribution, assumed to exist, arriving at the expression $\sum_\alpha c(\alpha)\, p(\alpha)$, where $c(\alpha)$ are integers equal to the number of times $p(\alpha)$ appears in the sum. Due to $0 \leq p(\alpha) \leq 1$, $\sum\limits_\alpha p(\alpha) = 1$ one obtains

$$\min_\alpha c(\alpha) \leq \sum_\alpha c(\alpha)\, p(\alpha) \leq \max_\alpha c(\alpha). \tag{13}$$

In order to get the standard form of Bell inequalities one should express $p(\alpha)$ in terms of relevant correlation functions.

In order to establish the violation of Bell inequalities in quantum mechanics one has to construct the particular examples. In two papers mentioned above [12,13] Güney and Hillery proposed to use the group theoretical methods. Consider some finite group G and its irreducible representation D. The space carrying the representation D becomes the space of states of one party. One selects an orbit $\{D(g)|\varphi\rangle\}_{g\in G}$ in such a way that it decomposes into disjoint sets of orthonormal bases. These bases define the spectral decompositions of observables entering the example. The space of states of the second party carries the second representation in the product $D \otimes D$; the corresponding orbit reads $\{D(g)|\psi\rangle\}_{g\in G}$ and defines the observables of second party.
Let us construct the operator [12,13]

$$X(\varphi, \psi) \equiv \sum_{g\in G} \left(D(g)|\varphi\rangle \otimes D(g)|\psi\rangle\right)\left(\langle\varphi| D^+(g) \otimes \langle\psi| D^+(g)\right). \tag{14}$$

Defining

$$\begin{aligned} &|g, \varphi\rangle \equiv D(g)|\varphi\rangle, \quad |g, \psi\rangle \equiv D(g)|\psi\rangle \\ &|g, \varphi, \psi\rangle \equiv |g, \varphi\rangle \otimes |g, \psi\rangle \end{aligned} \tag{15}$$

one finds for arbitrary bipartite state $|\chi\rangle$

$$\langle\chi| X |\chi\rangle = \sum_{g\in G} |\langle g, \varphi, \psi|\chi\rangle|^2. \tag{16}$$

The right-hand side of Eq. (16) represents the sum of probabilities of particular outcomes of measurement performed on observables defined by the orbits $\{|g, \varphi\rangle\}$ and $\{|g, \psi\rangle\}$. Its maximal value corresponds to maximal eigenvalue of X. In this way we obtain a kind of Cirel'son bound [21] for the class of states under consideration.

On the other hand one easily derives the Bell inequality involving the sum of probabilities on the right-hand side of Eq. (16). To this end one assumes the existence of joint probability distribution for all observables defined by both orbits (note that the ones belonging to one orbit in general do not commute) and uses the inequalities (13).

It remains to find the maximal eigenvalue of X. To this end assume that in the decomposition of $D \otimes D$ into irreducible pieces,

$$D \otimes D = \bigoplus_s D^{(s)} \tag{17}$$

each $D^{(s)}$ appears only once. Then, by Schur's lemma, $X(\varphi, \psi)$ is diagonal and reduces to a multiple of unity on each irreducible component. Using the orthogonality relations it is easy to see that the relevant eigenvalues of $X(\varphi, \psi)$ are [13]

$$\frac{|G|}{d_s} \parallel (|\varphi\rangle \otimes |\psi\rangle)_s \parallel^2 \tag{18}$$

where $|G|$ is the order of G, d_s is the dimension of $D^{(s)}$ and $(|\varphi\rangle \otimes |\psi\rangle)_s$ is the projection of $|\varphi\rangle \otimes |\psi\rangle$ on the carrier space of $D^{(s)}$.

In general, in order to violate the Bell inequality it is necessary to consider a number of orbits. To this end one considers the orbits generated by N pairs of vectors $(|\varphi_n\rangle, |\psi_n\rangle)$ and the corresponding operators $X(\varphi_n, \psi_n)$. They mutually commute so the eigenvalues of

$$X = \sum_{n=1}^{N} X(\varphi_n, \psi_n) \tag{19}$$

are the sums of eigenvalues of all $X(\varphi_n, \psi_n)$. In this way one can maximize the sum of probabilities

$$\sum_{n=1}^{N} \sum_{g \in G} |\langle g, \varphi_n, \psi_n | \chi\rangle|^2 \tag{20}$$

and proceed as above.

3 The S_4 group: three orbits

S_4 is the group of order 24. It has 5 conjugacy classes. There exist five irreducible representations of S_4: trivial representation, the alternating representation, the homomorphic two-dimensional one and two three-dimensional representations, D and $\widetilde{D}$ [22]. All representations can be made real orthogonal.

Consider three-dimensional representation D. The matrices representing the transpositions generate D. They are written out explicitly in Ref. [14] and will be not reported here. S_4 is the symmetry of tetrahedron as can be easily seen using the explicit form of the representation D [14]. In fact, the vectors $\vec{c}_1 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}\right)$, $\vec{c}_2 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}\right)$, $\vec{c}_3 = \left(-\frac{1}{3}, \frac{\sqrt{8}}{3}, 0\right)$ and $\vec{c}_4 = (1, 0, 0)$ are the vertices of regular tetrahedron and form an (degenerate) orbit of S_4.

The generic orbit consists of 24 states. According to the discussion presented in Sect. 2 we look for the orbit consisting of eight triples of orthonormal vectors. Due to the fact that the action of S_4 (in the representation D under consideration) reduces to the symmetries of tetrahedron (rotations around symmetry axes and reflections in symmetry planes) this is a simple problem of three-dimensional Euclidean geometry. The elements of the orbit are denoted as $\left|x_\alpha^i\right\rangle$, $i = 1, \ldots, 8$, $\alpha = 0, 1, 2$. They obey $\left\langle x_\alpha^i | x_\beta^i \right\rangle = \delta_{\alpha\beta}$, $i = 1, \ldots, 8$ (no summation over i). Consequently, we are dealing with eight observables a_i for Alice and eight observables b_i for Bob,

$$a_i = \sum_{\alpha=0}^{2} \alpha \left|x_\alpha^i\right\rangle\left\langle x_\alpha^i\right|, \quad b_i = \sum_{\beta=0}^{2} \beta \left|x_\beta^i\right\rangle\left\langle x_\beta^i\right|. \tag{21}$$

The explicit form of the vectors $\left|x_\alpha^i\right\rangle$, obtained by considering the elementary geometry of rotations and reflections of tetrahedron, is given in Appendix.

According to the discussion presented in Sect. 2 we should also know the details of the decomposition of the product representation $D \otimes D$ into irreducible pieces. As explained in Ref. [14] (see also [22]) it reads

$$D \otimes D = D \oplus \widetilde{D} \oplus D_2 \oplus D_0. \tag{22}$$

The Clebsh–Gordan coefficients, i.e., the elements of the matrix C relating the product basis to the one in which the decomposition (22) is explicit were computed in [14]. They read

$$C = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}. \tag{23}$$

Table 1 Maximal quantum mechanical values of the sum of probabilities for three examples of three orbits case

No.	First orbit	Second orbit	Third orbit	$\lambda_{\max}(X)$
I	$\lvert\varphi_1\rangle = \lvert x_0^1\rangle$	$\lvert\varphi_2\rangle = \lvert x_0^1\rangle$	$\lvert\varphi_3\rangle = \lvert x_0^1\rangle$	16.09
	$\lvert\psi_1\rangle = \lvert x_1^4\rangle$	$\lvert\psi_2\rangle = \lvert x_0^7\rangle$	$\lvert\psi_3\rangle = \lvert x_1^5\rangle$	
	$\lambda_{\max} \approx 7.40$	$\lambda_{\max} \approx 4.57$	$\lambda_{\max} \approx 4.12$	
II	$\lvert\varphi_1\rangle = \lvert x_0^1\rangle$	$\lvert\varphi_2\rangle = \lvert x_0^1\rangle$	$\lvert\varphi_3\rangle = \lvert x_0^1\rangle$	18.51
	$\lvert\psi_1\rangle = \lvert x_2^3\rangle$	$\lvert\psi_2\rangle = \lvert x_1^6\rangle$	$\lvert\psi_3\rangle = \lvert x_0^1\rangle$	
	$\lambda_{\max} \approx 5.12$	$\lambda_{\max} \approx 5.30$	$\lambda_{\max} \approx 8.00$	
III	$\lvert\varphi_1\rangle = \lvert x_0^1\rangle$	$\lvert\varphi_2\rangle = \lvert x_0^1\rangle$	$\lvert\varphi_3\rangle = \lvert x_0^1\rangle$	17.38
	$\lvert\psi_1\rangle = \lvert x_2^5\rangle$	$\lvert\psi_2\rangle = \lvert x_1^4\rangle$	$\lvert\psi_3\rangle = \lvert x_1^8\rangle$	
	$\lambda_{\max} \approx 3.35$	$\lambda_{\max} \approx 7.40$	$\lambda_{\max} \approx 6.63$	

In order to construct the examples of Bell inequalities violation one has to select the relevant operator X given by Eqs. (14) and (19), i.e., the number N, and N pairs of vectors (φ_n, ψ_n), $n = 1, \ldots, N$. In all our examples $N = 3$ and, moreover, the states describing both parties belong to the same orbit; however, in each case the orbit of the second party ("Bob") is shifted with respect to the one of the first parties ("Alice"). The latter implies that all $|\varphi_n\rangle, |\psi_n\rangle$ are of the form $\left|x_\alpha^i\right\rangle$ for some α and i. Equations (18) and (23) allow us to compute all eigenvalues of arbitrary operator X (φ, ψ). In all examples given below the largest eigenvalue $\lambda_{\max}$ corresponds to the scalar component in the decomposition (22). The results of three examples we considered are summarized in Table 1.

The corresponding sums of probabilities appearing on the right-hand side of Eq. (16) are written out explicitly in Appendix 2. Having computed the (maximal) quantum mechanical values of the relevant sums of probabilities one can study the corresponding Bell inequalities. To this end we compute the coefficients $c(\alpha)$ entering the inequalities (13). There are 16 observables: 8 for Alice and 8 for Bob. Therefore, the assumed joint probability is defined for 3^{16} configurations. We used computer to check, for three examples above (cf. Table 1), how many times any given configuration appears in 72 terms of classical counterpart of the right-hand side of Eq. (16). The result are summarized in Appendix 2. It follows that the relevant sums of probabilities have the upper bounds 16, 18 and 16 for Examples I, II and III, respectively. This implies that in all three examples the Bell inequalities are violated.

4 Interpretation in terms of game theory

As it has been described in Refs. [12,13] the Bell inequalities can be discussed in terms of a nonlocal game. To this end we assume there are two players, Alice and Bob and an arbitrator who sends Alice a value s and Bob a value t, $s = 1, 2, \ldots, 8$,

Table 2 Winning configurations for nonlocal game defined by three orbits from Example I

s, t	Alice, Bob	s, t	Alice, Bob
14	01, 10, 22	51	01, 10, 22
15	01, 10, 22	52	01, 10, 22
17	00, 12, 21	56	02, 10, 21
24	02, 11, 20	65	01, 12, 20
25	01, 10, 22	67	02, 10, 21
28	02, 11, 20	68	00, 11, 22
34	00, 11, 22	71	00, 12, 21
37	00, 11, 22	73	00, 11, 22
38	02, 10, 21	76	01, 12, 20
41	01, 10, 22	82	02, 11, 20
42	02, 11, 20	83	01, 12, 20
43	00, 11, 22	86	00, 11, 22

$t = 1, 2 \ldots, 8$; assume that all of 64 possible values of (s, t) are equally likely. After receiving the numbers s and t from an arbitrator both Alice nad Bob transmit back the numbers a and b, respectively, where $a = 0, 1, 2$, $b = 0, 1, 2$. They win iff the configuration $(a_s = a, b_t = b)$ appears in the sum of probabilities corresponding to the right-hand side of Eq. (16).

Let us consider for definiteness Example I. Using (36) we get the set of winning values given in Table 2.

Following Ref. [13] we can show that the maximal classical probability of winning the game is determined by Bell inequality. In fact, let $f_A(s)$ and $f_B(t)$ be the strategies of Alice and Bob, respectively; the function $f_{A,B}$ take their values in the set {0, 1, 2}. Let $F(a, b; s, t)$ be the characteristic function for the set of winning strategies. Then the winning probability for the given strategies f_A, f_B is

$$\frac{1}{64} \sum_{a,b=0}^{2} \sum_{s,t=1}^{8} F(a, b; s, t)\, \delta_{a, f_A(s)} \delta_{b, f_B(t)}. \tag{24}$$

Now, the sum entering the left-hand side of Bell inequality can be written as

$$\sum_{a,b=0}^{2} \sum_{s,t=1}^{8} F(a, b; s, t)\, p(a_s = a, b_t = b) \tag{25}$$

which is bounded, in Example I, by 16 provided $p(a_s = a, b_t = b)$ can be derived from a joint probability distribution. However, defining

$$p(a_1, \ldots, a_8, b_1, \ldots, b_8) \equiv \prod_{k=1}^{8} \delta_{a_k, f_A(k)} \delta_{b_k, f_B(k)} \tag{26}$$

we find that $p\,(a_s, b_t)$ are derived as marginals from the above joint probability. Therefore, the success probability for any classical strategy $(f_A(s), f_B(t))$ cannot exceed $\frac{16}{64} = 0, 25$.

Note that the optimal strategy saturating this limit always exists. To see this let $\alpha = \left(\underline{a}_1, \ldots, \underline{a}_8, \underline{b}_1, \ldots, \underline{b}_8\right)$ be one of the configurations for which $c(\alpha)$ attains its maximal value. Then the Bell inequality is saturated for the joint distribution probability $p(\alpha) = 1$, $p(\alpha') = 0$ for $\alpha' \neq \alpha$. Such distribution can be written in form (24) with $f_A(s) = \underline{a}_s$, $f_B(t) = \underline{b}_t$.

In the quantum strategy Alice and Bob share the state corresponding to the maximal eigenvalue of $\sum_{n=1}^{3} X\,(\varphi_n, \psi_n)$. If they receive the numbers s, t from an arbitrator, they measure a_s (Alice) and b_t (Bob), respectively, and send the result to the arbitrator. The probability of winning in Example I is then $\frac{16{,}09}{64} \simeq 0, 2514$ which exceeds (although only slightly) the classical bound. Other examples can be treated similarly.

5 Conclusions

Nonlocality which manifests itself in violation of Bell inequalities is an inherent property of quantum theory. On the other hand most physical systems exhibit some symmetries described by the relevant groups of transformations. It is, therefore, interesting to study in more detail the relationship between nonlocality and symmetries.

The space of states of any quantum system carries some unitary representation of the relevant symmetry group. The states are classified according to their transformation properties under the action of symmetries. One can pose the general question concerning the structure of "invariant" Bell inequalities and degree of their violation; in other words, to study the counterpart of Cirel'son bound in the symmetric context. Some steps in this direction were made in [12,13] and slightly more sophisticated example was presented in [14]. In the present paper we provide further example which, basically, differ from the previous ones in the number of orbits taken into account. Generally speaking, the degree of violation of Bell inequality depends on the symmetry group under consideration, an unitary representation entering the game and the choice of orbits. As far as the very group and its representations are concerned, the scheme proposed in [12,13] is rather simple and transparent. Actually, it can be easily generalized from finite to compact groups; as it is easily seen the only important point is the existence of finite invariant measure on the group. What concerns the question of the proper choice of orbits the situation is less clear. In the existing examples choosing one orbit is not sufficient to violate a Bell inequality. Two orbits suffice but, as it follows from the results of the present paper, three orbits give stronger violation. The choice of orbits may be critical because they contain, in general, nonorthogonal states leading to noncommuting projectors, i.e., incompatible measurements characteristic for quantum theory. The problem deserves further study. We believe some light may be shed on it by using the method of induced representations.

The advantage of our construction is that it can formulated in terms of elementary Euclidean geometry. As it has been already noticed in Ref. [14], it can be extended to all Platonic solids and, presumably, to S_n group acting as a symmetry group of the simplest regular $n-1$-dimensional polyhedron.

We do not have much to say about the experimental prospects. Up to now the experiments concerning the violation of Bell inequalities were performed with the use of photons, atoms (which are less likely to suffer from detection loophole) and on hybrid atom–photon systems (for a review see [11]). The relevant symmetries are rotations and reflections. The states can be classified by angular momentum and parity. In order to reduce rotations to finite subgroup the rotation group must be spontaneously broken to some finite subgroup by crystallization. Therefore, the experiments concerning violation of Bell inequalities based on finite symmetries should likely concern solid state phenomena.

Acknowledgements Katarzyna Bolonek-Lasoń would like to acknowledge Prof. Piotr Kosiński for helpful discussions and suggestions. Research of Katarzyna Bolonek-Lasoń was supported by the NCN Grant No. DEC-2012/05/D/ST2/00754.

Appendix 1

Below we give the explicit form of the vectors $\left|x_\alpha^i\right\rangle$. In order to construct them it is sufficient to find a vector which (a) belongs to the generic orbit under the action of symmetries of tetrahedron; (b) transforms into two orthogonal vectors under the action of two such symmetries. It is easy to check that, for example,

$$\left|x_0^1\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} \tag{27}$$

obeys these conditions. In this way we obtain

$$\begin{aligned}
&a_1: \quad \left(x_0^1, x_1^1, x_2^1\right) \\
&\left|x_0^1\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}, \quad \left|x_1^1\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{1}{2}\left(1 - \frac{\sqrt{3}}{3}\right) \\ \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right) \end{bmatrix}, \quad \left|x_2^1\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right) \\ -\frac{1}{2}\left(1 - \frac{\sqrt{3}}{3}\right) \end{bmatrix} \qquad (28) \\
&a_2: \quad \left(x_0^2, x_1^2, x_2^2\right) \\
&\left|x_0^2\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-3\sqrt{2} - \sqrt{3} - \sqrt{6}\right) \\ \frac{1}{18}\left(-3 + 5\sqrt{3} - 2\sqrt{6}\right) \\ \frac{1}{6}\left(-1 - 2\sqrt{2} + \sqrt{3}\right) \end{bmatrix}, \left|x_1^2\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-\sqrt{3} + 2\sqrt{6}\right) \\ \frac{1}{18}\left(9 - \sqrt{3} - 2\sqrt{6}\right) \\ -\frac{1}{6}\left(1 + 2\sqrt{2} + \sqrt{3}\right) \end{bmatrix},
\end{aligned}$$

$$\left|x_2^2\right\rangle = \begin{bmatrix} \frac{1}{9}\left(3\sqrt{2}-\sqrt{3}-\sqrt{6}\right) \\ -\frac{1}{9}\left(3+2\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{3}\left(1-\sqrt{2}\right) \end{bmatrix}, \tag{29}$$

$$a_3: \quad \left(x_0^3, x_1^3, x_2^3\right)$$

$$\left|x_0^3\right\rangle = \begin{bmatrix} \frac{1}{9}\left(3\sqrt{2}-\sqrt{3}-\sqrt{6}\right) \\ \frac{1}{18}\left(3+5\sqrt{3}-2\sqrt{6}\right) \\ \frac{1}{6}\left(1+2\sqrt{2}+\sqrt{3}\right) \end{bmatrix}, \left|x_1^3\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-\sqrt{3}+2\sqrt{6}\right) \\ -\frac{1}{18}\left(9+\sqrt{3}+2\sqrt{6}\right) \\ \frac{1}{6}\left(1+2\sqrt{2}-\sqrt{3}\right) \end{bmatrix},$$

$$\left|x_2^3\right\rangle = \begin{bmatrix} -\frac{1}{9}\left(3\sqrt{2}+\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{9}\left(3-2\sqrt{3}-\sqrt{6}\right) \\ \frac{1}{3}\left(-1+\sqrt{2}\right) \end{bmatrix}, \tag{30}$$

$$a_4: \quad \left(x_0^4, x_1^4, x_2^4\right)$$

$$\left|x_0^4\right\rangle = \begin{bmatrix} \frac{1}{9}\left(3\sqrt{2}-\sqrt{3}-\sqrt{6}\right) \\ \frac{1}{18}\left(3-\sqrt{3}+4\sqrt{6}\right) \\ \frac{1}{6}\left(3+\sqrt{3}\right) \end{bmatrix}, \left|x_1^4\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-\sqrt{3}+2\sqrt{6}\right) \\ \frac{1}{9}\left(\sqrt{3}+2\sqrt{6}\right) \\ -\frac{\sqrt{3}}{3} \end{bmatrix},$$

$$\left|x_2^4\right\rangle = \begin{bmatrix} -\frac{1}{9}\left(3\sqrt{2}+\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{18}\left(-3-\sqrt{3}+4\sqrt{6}\right) \\ \frac{1}{6}\left(-3+\sqrt{3}\right) \end{bmatrix}, \tag{31}$$

$$a_5: \quad \left(x_0^5, x_1^5, x_2^5\right)$$

$$\left|x_0^5\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-\sqrt{3}+2\sqrt{6}\right) \\ \frac{1}{18}\left(9-\sqrt{3}-2\sqrt{6}\right) \\ \frac{1}{6}\left(1+2\sqrt{2}+\sqrt{3}\right) \end{bmatrix}, \left|x_1^5\right\rangle = \begin{bmatrix} -\frac{1}{9}\left(3\sqrt{2}+\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{18}\left(-3+5\sqrt{3}-2\sqrt{6}\right) \\ \frac{1}{6}\left(1+2\sqrt{2}-\sqrt{3}\right) \end{bmatrix},$$

$$\left|x_2^5\right\rangle = \begin{bmatrix} \frac{1}{9}\left(3\sqrt{2}-\sqrt{3}-\sqrt{6}\right) \\ -\frac{1}{9}\left(3+2\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{3}\left(-1+\sqrt{2}\right) \end{bmatrix}, \tag{32}$$

$$a_6: \quad \left(x_0^6, x_1^6, x_2^6\right)$$

$$\left|x_0^6\right\rangle = \begin{bmatrix} -\frac{1}{9}\left(3\sqrt{2}+\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{18}\left(-3-\sqrt{3}+4\sqrt{6}\right) \\ \frac{1}{6}\left(3-\sqrt{3}\right) \end{bmatrix}, \ \left|x_1^6\right\rangle = \begin{bmatrix} \frac{1}{9}\left(3\sqrt{2}-\sqrt{3}-\sqrt{6}\right) \\ \frac{1}{18}\left(3-\sqrt{3}+4\sqrt{6}\right) \\ -\frac{1}{6}\left(3+\sqrt{3}\right) \end{bmatrix},$$

$$\left|x_2^6\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-\sqrt{3}+2\sqrt{6}\right) \\ \frac{1}{9}\left(\sqrt{3}+2\sqrt{6}\right) \\ \frac{\sqrt{3}}{3} \end{bmatrix}, \tag{33}$$

$$a_7 : \quad \left(x_0^7, x_1^7, x_2^7\right)$$

$$\left|x_0^7\right\rangle = \begin{bmatrix} \frac{1}{9}\left(3\sqrt{2}-\sqrt{3}-\sqrt{6}\right) \\ \frac{1}{18}\left(3+5\sqrt{3}-2\sqrt{6}\right) \\ -\frac{1}{6}\left(1+2\sqrt{2}+\sqrt{3}\right) \end{bmatrix}, \ \left|x_1^7\right\rangle = \begin{bmatrix} \frac{1}{9}\left(-\sqrt{3}+2\sqrt{6}\right) \\ -\frac{1}{18}\left(9+\sqrt{3}+2\sqrt{6}\right) \\ \frac{1}{6}\left(-1-2\sqrt{2}+\sqrt{3}\right) \end{bmatrix},$$

$$\left|x_2^7\right\rangle = \begin{bmatrix} -\frac{1}{9}\left(3\sqrt{2}+\sqrt{3}+\sqrt{6}\right) \\ \frac{1}{9}\left(3-2\sqrt{3}-\sqrt{6}\right) \\ \frac{1}{3}\left(1-\sqrt{2}\right) \end{bmatrix}, \tag{34}$$

$$a_8 : \quad \left(x_0^8, x_1^8, x_2^8\right)$$

$$\left|x_0^8\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right) \\ \frac{1}{2}\left(1-\frac{\sqrt{3}}{3}\right) \end{bmatrix}, \ \left|x_1^8\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{1}{2}\left(1-\frac{\sqrt{3}}{3}\right) \\ -\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right) \end{bmatrix},$$

$$\left|x_2^8\right\rangle = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}. \tag{35}$$

Appendix 2

To make the results slightly more transparent we write out explicitly the sum of probabilities appearing on the right-hand side of Eq. (16). They read:

Example I

$$\begin{aligned} S_1 \equiv\ & P\left(a_1=0, b_4=1\right) + P\left(a_1=1, b_5=0\right) + P\left(a_1=2, b_7=1\right) \\ & + P\left(a_2=0, b_4=2\right) + P\left(a_2=1, b_8=1\right) + P\left(a_2=2, b_5=2\right) \\ & + P\left(a_3=0, b_4=0\right) + P\left(a_3=1, b_8=0\right) + P\left(a_3=2, b_7=2\right) \\ & + P\left(a_4=0, b_3=0\right) + P\left(a_4=1, b_1=0\right) + P\left(a_4=2, b_2=0\right) \end{aligned}$$

$$
\begin{aligned}
&+ P\,(a_5 = 0, b_1 = 1) + P\,(a_5 = 1, b_6 = 0) + P\,(a_5 = 2, b_2 = 2)\\
&+ P\,(a_6 = 0, b_5 = 1) + P\,(a_6 = 1, b_7 = 0) + P\,(a_6 = 2, b_8 = 2)\\
&+ P\,(a_7 = 0, b_6 = 1) + P\,(a_7 = 1, b_1 = 2) + P\,(a_7 = 2, b_3 = 2)\\
&+ P\,(a_8 = 0, b_3 = 1) + P\,(a_8 = 1, b_2 = 1) + P\,(a_8 = 2, b_6 = 2)\\
&+ P\,(a_1 = 0, b_7 = 0) + P\,(a_1 = 1, b_4 = 0) + P\,(a_1 = 2, b_5 = 2)\\
&+ P\,(a_2 = 0, b_5 = 1) + P\,(a_2 = 1, b_4 = 1) + P\,(a_2 = 2, b_8 = 0)\\
&+ P\,(a_3 = 0, b_8 = 2) + P\,(a_3 = 1, b_7 = 1) + P\,(a_3 = 2, b_4 = 2)\\
&+ P\,(a_4 = 0, b_1 = 1) + P\,(a_4 = 1, b_2 = 1) + P\,(a_4 = 2, b_3 = 2)\\
&+ P\,(a_5 = 0, b_6 = 2) + P\,(a_5 = 1, b_2 = 0) + P\,(a_5 = 2, b_1 = 2)\\
&+ P\,(a_6 = 0, b_7 = 2) + P\,(a_6 = 1, b_8 = 1) + P\,(a_6 = 2, b_5 = 0)\\
&+ P\,(a_7 = 0, b_1 = 0) + P\,(a_7 = 1, b_3 = 1) + P\,(a_7 = 2, b_6 = 0)\\
&+ P\,(a_8 = 0, b_2 = 2) + P\,(a_8 = 1, b_6 = 1) + P\,(a_8 = 2, b_3 = 0)\\
&+ P\,(a_1 = 0, b_5 = 1) + P\,(a_1 = 1, b_7 = 2) + P\,(a_1 = 2, b_4 = 2)\\
&+ P\,(a_2 = 0, b_8 = 2) + P\,(a_2 = 1, b_5 = 0) + P\,(a_2 = 2, b_4 = 0)\\
&+ P\,(a_3 = 0, b_7 = 0) + P\,(a_3 = 1, b_4 = 1) + P\,(a_3 = 2, b_8 = 1)\\
&+ P\,(a_4 = 0, b_2 = 2) + P\,(a_4 = 1, b_3 = 1) + P\,(a_4 = 2, b_1 = 2)\\
&+ P\,(a_5 = 0, b_2 = 1) + P\,(a_5 = 1, b_1 = 0) + P\,(a_5 = 2, b_6 = 1)\\
&+ P\,(a_6 = 0, b_8 = 0) + P\,(a_6 = 1, b_5 = 2) + P\,(a_6 = 2, b_7 = 1)\\
&+ P\,(a_7 = 0, b_3 = 0) + P\,(a_7 = 1, b_6 = 2) + P\,(a_7 = 2, b_1 = 1)\\
&+ P\,(a_8 = 0, b_6 = 0) + P\,(a_8 = 1, b_3 = 2) + P\,(a_8 = 2, b_2 = 0) \qquad (36)
\end{aligned}
$$

Example II

$$
\begin{aligned}
S_2 \equiv\; & P\,(a_1 = 0, b_3 = 2) + P\,(a_1 = 1, b_2 = 0) + P\,(a_1 = 2, b_6 = 0)\\
&+ P\,(a_2 = 0, b_1 = 1) + P\,(a_2 = 1, b_3 = 0) + P\,(a_2 = 2, b_6 = 2)\\
&+ P\,(a_3 = 0, b_2 = 1) + P\,(a_3 = 1, b_6 = 1) + P\,(a_3 = 2, b_1 = 0)\\
&+ P\,(a_4 = 0, b_7 = 1) + P\,(a_4 = 1, b_5 = 2) + P\,(a_4 = 2, b_8 = 0)\\
&+ P\,(a_5 = 0, b_7 = 0) + P\,(a_5 = 1, b_8 = 1) + P\,(a_5 = 2, b_4 = 1)\\
&+ P\,(a_6 = 0, b_1 = 2) + P\,(a_6 = 1, b_3 = 1) + P\,(a_6 = 2, b_2 = 2)\\
&+ P\,(a_7 = 0, b_5 = 0) + P\,(a_7 = 1, b_4 = 0) + P\,(a_7 = 2, b_8 = 2)\\
&+ P\,(a_8 = 0, b_4 = 2) + P\,(a_8 = 1, b_5 = 1) + P\,(a_8 = 2, b_7 = 2)\\
&+ P\,(a_1 = 0, b_6 = 1) + P\,(a_1 = 1, b_3 = 0) + P\,(a_1 = 2, b_2 = 2)\\
&+ P\,(a_2 = 0, b_6 = 0) + P\,(a_2 = 1, b_1 = 0) + P\,(a_2 = 2, b_3 = 1)\\
&+ P\,(a_3 = 0, b_6 = 2) + P\,(a_3 = 1, b_1 = 2) + P\,(a_3 = 2, b_2 = 0)\\
&+ P\,(a_4 = 0, b_5 = 0) + P\,(a_4 = 1, b_8 = 1) + P\,(a_4 = 2, b_7 = 2)\\
&+ P\,(a_5 = 0, b_8 = 2) + P\,(a_5 = 1, b_4 = 2) + P\,(a_5 = 2, b_7 = 1)\\
&+ P\,(a_6 = 0, b_3 = 2) + P\,(a_6 = 1, b_2 = 1) + P\,(a_6 = 2, b_1 = 1)
\end{aligned}
$$

$$
\begin{aligned}
&+P(a_7=0,b_4=1)+P(a_7=1,b_8=0)+P(a_7=2,b_5=1)\\
&+P(a_8=0,b_5=2)+P(a_8=1,b_7=0)+P(a_8=2,b_4=0)\\
&+P(a_1=0,b_1=0)+P(a_1=1,b_1=1)+P(a_1=2,b_1=2)\\
&+P(a_2=0,b_2=0)+P(a_2=1,b_2=1)+P(a_2=2,b_2=2)\\
&+P(a_3=0,b_3=0)+P(a_3=1,b_3=1)+P(a_3=2,b_3=2)\\
&+P(a_4=0,b_4=0)+P(a_4=1,b_4=1)+P(a_4=2,b_4=2)\\
&+P(a_5=0,b_5=0)+P(a_5=1,b_5=1)+P(a_5=2,b_5=2)\\
&+P(a_6=0,b_6=0)+P(a_6=1,b_6=1)+P(a_6=2,b_6=2)\\
&+P(a_7=0,b_7=0)+P(a_7=1,b_7=1)+P(a_7=2,b_7=2)\\
&+P(a_8=0,b_8=0)+P(a_8=1,b_8=1)+P(a_8=2,b_8=2) \qquad (37)
\end{aligned}
$$

Example III

$$
\begin{aligned}
S_3\equiv\ &P(a_1=0,b_5=2)+P(a_1=1,b_7=0)+P(a_1=2,b_4=0)\\
&+P(a_2=0,b_8=0)+P(a_2=1,b_5=1)+P(a_2=2,b_4=1)\\
&+P(a_3=0,b_7=1)+P(a_3=1,b_4=2)+P(a_3=2,b_8=2)\\
&+P(a_4=0,b_2=1)+P(a_4=1,b_3=2)+P(a_4=2,b_1=1)\\
&+P(a_5=0,b_2=0)+P(a_5=1,b_1=2)+P(a_5=2,b_6=2)\\
&+P(a_6=0,b_8=1)+P(a_6=1,b_5=0)+P(a_6=2,b_7=2)\\
&+P(a_7=0,b_3=1)+P(a_7=1,b_6=0)+P(a_7=2,b_1=0)\\
&+P(a_8=0,b_6=1)+P(a_8=1,b_3=0)+P(a_8=2,b_2=2)\\
&+P(a_1=0,b_4=1)+P(a_1=1,b_5=0)+P(a_1=2,b_7=1)\\
&+P(a_2=0,b_4=2)+P(a_2=1,b_8=1)+P(a_2=2,b_5=2)\\
&+P(a_3=0,b_4=0)+P(a_3=1,b_8=0)+P(a_3=2,b_7=2)\\
&+P(a_4=0,b_3=0)+P(a_4=1,b_1=0)+P(a_4=2,b_2=0)\\
&+P(a_5=0,b_1=1)+P(a_5=1,b_6=0)+P(a_5=2,b_2=2)\\
&+P(a_6=0,b_5=1)+P(a_6=1,b_7=0)+P(a_6=2,b_8=2)\\
&+P(a_7=0,b_6=1)+P(a_7=1,b_1=2)+P(a_7=2,b_3=2)\\
&+P(a_8=0,b_3=1)+P(a_8=1,b_2=1)+P(a_8=2,b_6=2)\\
&+P(a_1=0,b_8=1)+P(a_1=1,b_8=2)+P(a_1=2,b_8=0)\\
&+P(a_2=0,b_7=2)+P(a_2=1,b_7=0)+P(a_2=2,b_7=1)\\
&+P(a_3=0,b_5=0)+P(a_3=1,b_5=2)+P(a_3=2,b_5=1)\\
&+P(a_4=0,b_6=2)+P(a_4=1,b_6=1)+P(a_4=2,b_6=0)\\
&+P(a_5=0,b_3=0)+P(a_5=1,b_3=2)+P(a_5=2,b_3=1)\\
&+P(a_6=0,b_4=2)+P(a_6=1,b_4=1)+P(a_6=2,b_4=0)\\
&+P(a_7=0,b_2=1)+P(a_7=1,b_2=2)+P(a_7=2,b_2=0)\\
&+P(a_8=0,b_1=2)+P(a_8=1,b_1=0)+P(a_8=2,b_1=1) \qquad (38)
\end{aligned}
$$

Table 3 Coefficients $c(\alpha)$ entering Eq. (13)

Example I		Example II		Example III	
$c(\alpha)$	No. of configurations	$c(\alpha)$	No. of configurations	$c(\alpha)$	No. of configurations
1	12 960	1	9 720	1	18 360
2	159 408	2	126 576	2	115 596
3	645 408	3	510 480	3	474 696
4	1 729 188	4	1 514 862	4	1 445 778
5	3 479 760	5	3 182 904	5	3 286 224
6	5 424 408	6	5 374 584	6	5 510 160
7	6 896 016	7	7 139 664	7	7 178 976
8	7 261 569	8	7 822 791	8	7 670 547
9	6 410 016	9	6 903 648	9	6 795 936
10	4 866 480	10	5 058 216	10	5 012 208
11	3 176 496	11	3 006 000	11	3 087 504
12	1 758 348	12	1 506 186	12	1 567 458
13	808 704	13	613 800	13	638 280
14	311 040	14	208 008	14	196 812
15	90 720	15	55 584	15	41 400
16	15 876	16	11 673	16	4 761
17	0	17	1 656	17	0
18	0	18	144	18	0
19	0	19	0	19	0
20	0	20	0	20	0

Therefore, the corresponding Bell inequalities take the form

$$S_1 \leq 16 \tag{39}$$
$$S_2 \leq 18 \tag{40}$$
$$S_3 \leq 16. \tag{41}$$

They were obtained by assuming the existence of joint distribution of random variables $a_1, \ldots, a_8, b_1, \ldots, b_8$ and computing the coefficients $c(\alpha)$ defined in Sect. 2. More precisely, for each example we write all probabilities entering the sums S_1, S_2, S_3 as the marginals of joint probability distribution. As a result we obtain the expressions of the form

$$S = \sum_{\alpha} c(\alpha) p(\alpha) \tag{42}$$

where α runs over all 3^{16} configurations of the variables $a_1, \ldots, a_8, b_1, \ldots, b_8$. The results of numerical computations are summarized in Table 3.

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