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QUASITORIC MANIFOLDS WHICH ARE NOT TORIC ORIGAMI

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ABSTRACT. We construct 6-dimensional quasitoric manifolds which are not toric origami manifolds.

INTRODUCTION

Origami manifolds appeared in differential geometry recently as a generalization of symplectic manifolds [3]. Toric origami manifolds are in turn generalizations of symplectic toric manifolds. Toric origami manifolds are a special class of $2n$-dimensional compact manifolds with an effective action of a half-dimensional compact torus T^n. In this paper we consider the following question. How large is this class? Which manifolds with half-dimensional torus actions are toric origami manifolds? In particular, which quasitoric manifolds admit toric origami structures? In [7] Masuda and Park proved

Theorem 1. *Any simply connected compact smooth 4-manifold M with an effective smooth action of T^2 is equivariantly diffeomorphic to a toric origami manifold.*

In particular, any 4-dimensional quasitoric manifold is toric origami. The same question about higher dimensions was open. Here we prove the negative result.

Theorem 2. *There exist 6-dimensional quasitoric manifolds, which are not equivariantly homeomorphic to any toric origami manifold.*

We will describe an obstruction for a quasitoric 6-manifold to be toric origami and present a large series of examples, where such an obstruction appears. In spite of topological nature of the task, the proof is purely discrete geometrical: it relies on metric and coloring properties of planar graphs.

1. TOPOLOGICAL PRELIMINARIES

1.1. **Quasitoric manifolds.** The subject of this subsection originally appeared in the seminal work of Davis and Januszkiewicz [4]. The modern exposition and technical details can be found in [2, Ch.7].

Let T^n be a compact n-dimensional torus. The standard representation of T^n is a representation $T^n \curvearrowright \mathbb{C}^n$ by coordinate-wise rotations. The action of T^n on a manifold M^{2n} is called locally standard, if M has an atlas of standard charts, each isomorphic to a subset of the standard representation. In the following M is supposed to be compact.

Since the orbit space $\mathbb{C}^n/T^n$ of the standard representation is a nonnegative cone $\mathbb{R}^n_{\geqslant} = \{x \in \mathbb{R}^n \mid x_i \geqslant 0\}$, the orbit space of any locally standard action has the structure of a compact manifold with corners. Let $\mathcal{F}(Q)$ denote the set of facets of Q (i.e. faces of codimension 1). For each facet F of Q consider a stabilizer subgroup $\lambda(F) \subset T^n$, which preserves points over the interior of F. This subgroup is 1-dimensional and connected, thus it has the form $\{(t^{\lambda_1}, \dots, t^{\lambda_n}) \mid t \in T^1\} \subset T^n$, for

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some primitive integral vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, defined uniquely up to a common sign. Thus, a primitive integral vector (up to sign) $\Lambda(F) \in \mathbb{Z}^n/\pm$ is associated with any facet F of Q. This map $\Lambda\colon \mathcal{F}(Q) \to \mathbb{Z}^n/\pm$ is called a *characteristic function* (or a characteristic map). It satisfies the following so called $(*)$-condition:

$$(*) \qquad \begin{array}{c}\text{If facets } F_1, \dots, F_s \text{ intersect, then the set of vectors}\\ \Lambda(F_1), \dots, \Lambda(F_s) \text{ is a part some basis of } \mathbb{Z}^n.\end{array}$$

Here we actually take not a class $\Lambda(F_i) \in \mathbb{Z}^n/\pm$, but one of its two particular representatives in $\mathbb{Z}^n$. Obviously, the condition does not depend on the choice of sign, thus $(*)$ is well defined. The same convention appears further in the text without special mention.

Definition 1.1. *A manifold M^{2n} with a locally standard action of T^n is called quasitoric, if the orbit space M/T^n is homeomorphic to a simple polytope as a manifold with corners.*

Recall that a convex polytope P of dimension n is called simple if any of its vertices lies in exactly n facets. In other words, a simple polytope is a polytope which is at the same time a manifold with corners. Considering manifolds with corners, simple polytopes are the simplest geometrical examples one can imagine. This makes the definition of quasitoric manifold very natural.

Let P be a simple polytope and Λ be a characteristic function, i.e. any map $\Lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n/\pm$ satisfying $(*)$-condition. The pair (P, Λ) is called a *characteristic pair.* According to [4], there is a one-to-one correspondence

$$\{\text{quasitoric manifolds}\} \leftrightsquigarrow \{\text{characteristic pairs}\}$$

up to equivariant homeomorphism on the left-hand side and combinatorial equivalence on the right-hand side. The quasitoric manifold associated with a characteristic pair (P, Λ) will be denoted $M_{(P,\Lambda)}$. Let η denote the projection to the orbit space $\eta\colon M_{(P,\Lambda)} \to P$. Each facet $F \in \mathcal{F}(P)$ determines a characteristic submanifold $N_F \overset{\text{def}}{=} \eta^{-1}(F) \subset M_{(P,\Lambda)}$ of dimension $2n-2$. On its own, the manifold N_F is again a quasitoric manifold with the orbit space F.

1.2. **Toric origami manifolds.** In the following subsections we recall the definitions and properties of toric origami manifolds and origami templates. More detailed exposition of this theory can be found in [3], [7] or [6].

A *folded symplectic form* on a $2n$-dimensional smooth manifold M is a closed 2-form ω whose top power ω^n vanishes transversally on a subset Z and whose restriction to points in Z has maximal rank. Then Z is a codimension-one submanifold of M called the *fold.* The pair (M, ω) is called a *folded symplectic manifold.* If Z is empty, ω is a genuine symplectic form and (M, ω) is a genuine symplectic manifold according to classical definition.

Since the restriction of ω to Z has maximal rank, it has a one-dimensional kernel at each point of Z. This determines a line field on Z called the *null foliation.* If the null foliation is the vertical bundle of some principal S^1-fibration $Z \to Y$ over a compact base Y, then the folded symplectic form ω is called an *origami form* and the pair (M, ω) is called an *origami manifold.*

The action of a torus T (of any dimension) on an origami manifold (M, ω) is called Hamiltonian if it admits a moment map $\mu\colon M \to \mathfrak{t}^*$ to the dual Lie algebra of the torus, which satisfies the conditions: (1) μ is equivariant with respect to the given action of T on M and the trivial action of T on $\mathfrak{t}^*$; (2) μ collects Hamiltonian functions, that is, $d\langle\mu, V\rangle = \omega(V^{\#}, \cdot)$, where $\langle\mu, V\rangle$ is the function on M, taking the value $\langle\mu(x), V\rangle$ at a point $x \in M$, $V^{\#}$ is a vector flow on M, generated by $V \in \mathfrak{t}$, and $\omega(V^{\#}, \cdot)$ is its dual 1-form.

Definition 1.2. *A toric origami manifold (M, ω, T, μ), abbreviated as M, is a compact connected origami manifold (M, ω) equipped with an effective Hamiltonian action of a torus T with $\dim T = \frac{1}{2}\dim M$ and with a choice of a corresponding moment map μ.*

1.3. **Symplectic toric manifolds.** When the fold Z is empty, a toric origami manifold is a *symplectic toric manifold.* In this case the image $\mu(M)$ of the moment map is a Delzant polytope in $\mathfrak{t}^*$, and the map $\mu\colon M \to \mu(M)$ itself can be identified with the map to the orbit space $\eta\colon M \to M/T^n$. A classical theorem of Delzant [5] says that symplectic toric manifolds are classified by the images of their moment maps in $\mathfrak{t}^* \cong \mathbb{R}^n$. In other words, there is a one-to-one correspondence

$$\{\text{symplectic toric manifolds}\} \leftrightsquigarrow \{\text{Delzant polytopes}\}$$

up to equivariant symplectomorphism on the left-hand side, and affine equivalence on the right-hand side. Let us recall the notion of Delzant polytope.

Definition 1.3. *A simple convex polytope $P \subset \mathbb{R}^n$ is called Delzant, if its normal fan is smooth (with respect to a fixed lattice $\mathbb{Z}^n \subset \mathbb{R}^n$).*

Construction 1.4 (Topological model of symplectic toric manifold)**.** Let P be a Delzant polytope in $\mathbb{R}^n$. For a facet $F \in \mathcal{F}(P)$ consider its outward primitive normal vector $\tilde{\nu}(F) \in \mathbb{Z}^n$. Consider the corresponding vector modulo sign: $\nu(F) \in \mathbb{Z}^n/\pm$. By the definition of Delzant polytope, $\nu\colon \mathcal{F}(P) \to \mathbb{Z}^n/\pm$ satisfies $(*)$, thus provides an example of a characteristic function. The quasitoric manifold

$$M_P \stackrel{\text{def}}{=} M_{(P,\nu)}.$$

is the symplectic toric manifold corresponding to P (up to equivariant homeomorphism).

1.4. **Origami templates.** To generalize Delzant correspondence to toric origami manifolds we need a notion of an origami template, which we review next.

Let $\mathcal{D}_n$ denote the set of all (full-dimensional) Delzant polytopes in $\mathbb{R}^n$ (w.r.t. a fixed lattice) and $\mathcal{F}_n$ the set of all their facets.

Definition 1.5. *An origami template is a triple (Γ, Ψ_V, Ψ_E), where*

- *Γ is a connected finite graph (loops and multiple edges are allowed) with the vertex set V and edge set E;*
- *$\Psi_V\colon V \to \mathcal{D}_n$;*
- *$\Psi_E\colon E \to \mathcal{F}_n$;*

subject to the following conditions:

1. *If $e \in E$ is an edge of Γ with endpoints $v_1, v_2 \in V$, then $\Psi_E(e)$ is a facet of both polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$, and these polytopes coincide near $\Psi_E(e)$ (this means there exists an open neighborhood U of $\Psi_E(e)$ in $\mathbb{R}^n$ such that $U \cap \Psi_V(v_1) = U \cap \Psi_V(v_2)$).*
2. *If $e_1, e_2 \in E$ are two edges of Γ adjacent to $v \in V$, then $\Psi_E(e_1)$ and $\Psi_E(e_2)$ are disjoint facets of $\Psi_V(v)$.*

The facets of the form $\Psi_E(e)$ for $e \in E$ are called the fold facets of the origami template.

For convenience in the following we call the vertices of graph Γ the nodes. One can simply view an origami template as a collection of (possibly overlapping) Delzant polytopes $\{\Psi_V(v) \mid v \in V\}$ in the same ambient space, with some gluing data, encoded by a template graph Γ (see Fig. 1).

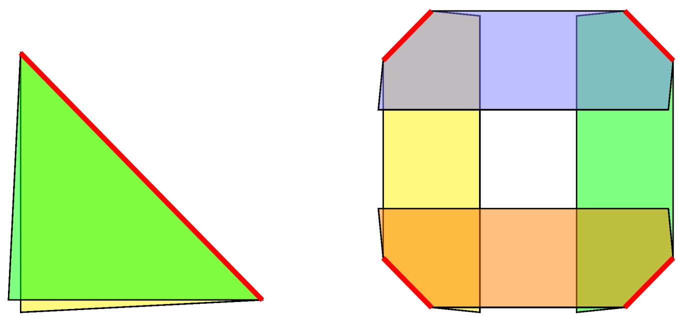

FIGURE 1. Examples of origami templates in dim = 2. Fold facets are shown in red.

Theorem 3 ([3]). *Assigning the moment data of a toric origami manifold induces a one-to-one correspondence*

$$\{\textit{toric origami manifolds}\} \leftrightsquigarrow \{\textit{origami templates}\}$$

up to equivariant origami symplectomorphism on the left-hand side, and affine equivalence on the right-hand side.

Construction 1.6 (Topological construction of toric origami manifold). Consider an origami template $O = (\Gamma, \Psi_V, \Psi_E)$, $\Gamma = (V, E)$. For each node $v \in V$ the Delzant polytope $\Psi_V(v) \in \mathcal{P}_n$ gives a symplectic toric manifold $M_{\Psi_V(v)}$, see construction 1.4. Now do the following procedure:

1 Take a disjoint union of all manifolds $M_{\Psi_V(v)}$ for $v \in V$;
2 For each edge $e \in E$ with distinct endpoints v_1 and v_2 take an equivariant connected sum of $M_{\Psi_V(v_1)}$ and $M_{\Psi_V(v_2)}$ along the characteristic submanifold $N_{\Psi_E(e)}$ (which is embedded in both manifolds);
3 For each loop $e \in E$ based at $v \in V$ take a real blow up of normal bundle to the submanifold $N_{\Psi_E(e)}$ inside $M_{\Psi_V(v)}$.

Step 2 makes sense because of pt.1 of Definition 1.5. Indeed, the polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$ agree near $\Psi_E(e)$, thus $M_{\Psi_V(v_1)}$ and $M_{\Psi_V(v_2)}$ have equivariantly homeomorphic neighborhoods around $N_{\Psi_E(e)}$, so the connected sum is well defined. Pt. 2 of Definition 1.5 ensures that surgeries do not touch each other, so all the connected sums and blow ups can be taken simultaneously.

Denote the resulting manifold of this construction by $M_O = M_{(\Gamma,\Psi_V,\Psi_E)}$. This is exactly the toric origami manifold associated with O via Theorem 3.

Example 1.7. Let us construct a toric origami manifold X, corresponding to the origami template, made of two triangles (Fig. 1, left). The symplectic toric 4-manifold corresponding to a triangle is known to be the complex projective plane $\mathbb{C}P^2$. The characteristic submanifold corresponding to the fold facet is a projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Thus, X is a connected sum of two copies of $\mathbb{C}P^2$ along the line $\mathbb{C}P^1$, which lies in both. It is easily seen that this manifold is S^4.

An origami template $O = (\Gamma, \Psi_V, \Psi_E)$ (and the corresponding manifold M_O) is called coörientable if Γ has no loops (i.e. edges based at one point). If M_O is coörientable, then the action of T^n on M_O is locally standard [6, lemma 5.1]. The converse is also true. In the following we consider only coörientable templates and toric origami manifolds.

Construction 1.8 (Orbit space of toric origami manifold). The orbit space $Q = M_{(\Gamma,\Psi_V,\Psi_E)}/T^n$ of a (coörientable) toric origami manifold is a manifold with corners. It can be described as a topological space obtained by gluing polytopes $\Psi_V(v)$ along fold facets. More precisely,

$$Q = \bigsqcup_{v \in V} (v, \Psi_V(v)) \big/ \sim, \tag{1.1}$$

where $(u, x) \sim (v, y)$ if there exists an edge e with endpoints u and v, and $x = y \in \Psi_E(e)$. Facets of Q are given by non-fold facets of polytopes $\Psi_V(v)$ identified in the same way. To make this precise, let us call non-fold facets $F_1 \in \mathcal{F}(\Psi_V(v_1))$ and $F_2 \in \mathcal{F}(\Psi_V(v_2))$ elementary neighboring w.r.t. to the edge $e \in E$ (with endpoints v_1 and v_2) if $F_1 \cap \Psi_E(e) = F_2 \cap \Psi_E(e)$. The relation of elementary neighborliness generates an equivalence relation $\leftrightarrow$ on the set of all non-fold facets of all polytopes $\Psi_V(v)$. Define the facet $[F]$ of the orbit space Q as a union of facets in one equivalence class:

$$[F] \stackrel{\text{def}}{=} \bigsqcup_{\substack{v \in V, G \in \mathcal{F}(\Psi_V(v)),\\ G \text{ is not fold}, G \leftrightarrow F}} (v, G) \big/ \sim, \qquad [F] \in \mathcal{F}(Q), \tag{1.2}$$

where $\sim$ is the same as in (1.1).

Let us define a primitive normal vector to the facet $[F]$ of Q by $\nu([F]) \stackrel{\text{def}}{=} \nu(F) \in \mathbb{Z}^n/\pm$. It is well defined since $\nu(F) = \nu(G)$ for $F \leftrightarrow G$.

Note that the relation of elementary neighborliness determines a connected subgraph $\Gamma_{[F]}$ of Γ. All facets $G \leftrightarrow F$ are Delzant and lie in the same hyperplane $H_{[F]}$. Thus we obtain an induced origami template

$$O_{[F]} = (\Gamma_{[F]}, \Psi_V|_{\Gamma_{[F]}} \cap H_{[F]}, \Psi_E|_{\Gamma_{[F]}} \cap H_{[F]}) \tag{1.3}$$

of dimension $n-1$. In particular, if $\eta\colon M_O \to Q$ denotes the projection to the orbit space, then the characteristic submanifold $\eta^{-1}([F])$ is the toric origami manifold of dimension $2n-2$ generated by the origami template $O_{[F]}$.

Extending the origami analogy, we can think of the orbit space Q as "unfolding" the origami template and then smoothening the angles adjacent to the former fold facets (remember that we have to identify neighboring faces!).

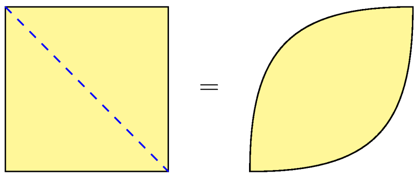

FIGURE 2. The orbit space of a manifold S^4, corresponding to the origami template shown on Fig. 1, left.

It is easy to see that the orbit space $Q = M_{(\Gamma, \Psi_V, \Psi_E)}/T^n$ has the same homotopy type as the graph Γ, thus Q is either contractible (when Γ is a tree) or homotopy equivalent to a wedge of circles. This observation shows that whenever the template graph Γ has cycles, the corresponding toric origami manifold cannot be quasitoric. As an example, the origami template shown on Fig. 1, at the right corresponds to the origami manifold which is not quasitoric. Since we want to find a quasitoric manifold which is not toric origami, we need to consider only the cases when the orbit space is contractible. Thus in the following Γ is supposed to be a tree.

2. Weighted simplicial spheres

In the previous section we have seen that quasitoric manifolds are encoded by the orbit spaces (which are simple polytopes) and characteristic functions (which are colorings of facets by elements of $\mathbb{Z}^n/\pm$). It will be easier, however, to work with the dual objects, which we call weighted simplicial spheres.

Recall that a *simplicial poset* is a finite partially ordered set S such that: (1) There is a unique minimal element $\varnothing \in S$, (2) For each $I \in S$ the interval subset $[\varnothing, I] \stackrel{\text{def}}{=} \{J \in S \mid J \leqslant I\}$ is isomorphic to the poset of faces of $(k-1)$-dimensional simplex (i.e. Boolean lattice of rank k). In this case the element I is said to have rank k and dimension $k-1$. The elements of S are called simplices and elements of rank 1 are called vertices. The set of vertices of S is denoted $\operatorname{Vert}(S)$.

A simplicial poset is called pure, if all maximal simplices have the same dimension. A simplicial poset S is called a simplicial complex, if for any subset of vertices $\sigma \subseteq \operatorname{Vert}(S)$, there exists at most one simplex whose vertex set is σ.

Construction 2.1. It is convenient to visualize simplicial posets using their geometrical realizations. Assign the geometrical simplex Δ_I of dimension $\operatorname{rank}(I) - 1$ to each $I \in S$ and attach them together according to the order relation in S. More formally, the geometric realization of S is the topological space

$$|S| \stackrel{\text{def}}{=} \bigsqcup_{I \in S} (I, \Delta_I)/\sim,$$

where $(I_1, x_1) \sim (I_2, x_2)$ if $I_1 < I_2$ and $x_1 = x_2 \in \Delta_{I_1} \subset \Delta_{I_2}$. See details in [2].

A simplicial poset S is called a triangulated sphere if $|S|$ is homeomorphic to a sphere. S is called a PL-sphere if the barycentric subdivision S' (which is a simplicial complex) is PL-homeomorphic to the boundary of a simplex. In dimension 2, which is the only important case for us, these two notions are equivalent. In the sequel we call either of them *simplicial spheres.*

Construction 2.2. Let us define a connected sum of two simplicial spheres along their vertices. At first we should exclude certain degenerate situations.

For every $I < J$ in S there is a complementary simplex $J \setminus I \in S$. In other words, $J \setminus I$ is the face of J complementary to the face I. Define a *link* of a simplex $I \in S$ as a partially ordered set $\operatorname{link}_S I = \{J \setminus I \mid J \in S, J \geqslant I\}$ with the order relation induced from S. Define an *open star* of a simplex $I \in S$ as a subset $\operatorname{star}^\circ_S I \stackrel{\text{def}}{=} \{J \in S \mid J \geqslant I\}$. There is a natural surjective map of sets $D_I\colon \operatorname{star}^\circ_S I \to \operatorname{link}_S I$ sending J to $J \setminus I$. We call a simplex I *admissible* if D_I is injective.

Note that in a simplicial complex every simplex is admissible. One can view admissibility as the property of being "locally a simplicial complex".

Let us define the connected sum of two simplicial posets S_1 and S_2 along admissible vertices. Let $i_1 \in S_1$ and $i_2 \in S_2$ be admissible vertices, and suppose there exists an isomorphism of posets $\xi\colon \operatorname{link}_{S_1} i_1 \to \operatorname{link}_{S_2} i_2$ (thus an isomorphism of open stars, by admissibility). Consider a poset

$$S_1 \,{}_{i_1}\#_{i_2}\, S_2 \stackrel{\text{def}}{=} (S_1 \setminus \operatorname{star}^\circ_{S_1} i_1) \sqcup (S_2 \setminus \operatorname{star}^\circ_{S_2} i_2)/\sim, \tag{2.1}$$

where $I_1 \in \operatorname{link}_{S_1} i_1 \subset S_1$ is identified with $I_2 \in \operatorname{link}_{S_2} i_2 \subset S_2$ whenever $I_2 = \xi(I_1)$. The order relation on $S_1 \,{}_{i_1}\#_{i_2}\, S_2$ is induced from S_1 and S_2 in a natural way. The poset $S_1 \,{}_{i_1}\#_{i_2}\, S_2$ is simplicial. If S_1, S_2 are simplicial spheres, then so is $S_1 \,{}_{i_1}\#_{i_2}\, S_2$ (this property may break for non-admissible vertices).

Definition 2.3. *Let S be a pure simplicial poset of dimension $n-1$. A map $\Lambda\colon \operatorname{Vert}(S) \to \mathbb{Z}^n/\pm$ is called a weighting if, for every simplex $I \in S$ with vertices $i_1, \ldots, i_n$, the vectors $\Lambda(i_1), \ldots, \Lambda(i_n)$ span $\mathbb{Z}^n$. The pair (S, Λ) is called a weighted simplicial poset.*

Definition 2.4. *Let (S_1, Λ_1) and (S_2, Λ_2) be weighted simplicial posets. Let i_1, i_2 be admissible vertices of S_1, S_2 such that there exists an isomorphism $\xi\colon \operatorname{link}_{S_1} i_1 \to$*

$\mathrm{link}_{S_2} i_2$ *preserving weights:* $(\Lambda_2 \circ \xi)|_{\mathrm{link}_{S_1} i_1} = \Lambda_1|_{\mathrm{link}_{S_1} i_1}$. *Then* Λ_1, Λ_2 *induce the weight* Λ *on the connected sum* $S_1\, {}_{i_1}\#_{i_2}\, S_2$. *The weighted simplicial poset* $(S_1\, {}_{i_1}\#_{i_2}\, S_2, \Lambda)$ *is called a weighted connected sum of* (S_1, Λ_1) *and* (S_2, Λ_2).

Construction 2.5. Let (P, Λ) be a characteristic pair (see section 1). Let $K_P = \partial P^*$ be the dual simplicial sphere to a simple polytope P. Since there is a natural correspondence $\mathrm{Vert}(K_P) = \mathcal{F}(P)$ we get the weighting $\Lambda\colon \mathrm{Vert}(K_P) \to \mathbb{Z}^n/\pm$. This defines a weighted sphere (K_P, Λ). In particular, any Delzant polytope P defines a weighted sphere (K_P, ν), where $\nu(F)$ is the normal vector to $F \in \mathcal{F}(P) = \mathrm{Vert}(K_P)$ modulo sign (construction 1.4).

Construction 2.6. Let $O = (\Gamma, \Psi_V, \Psi_E)$ be an origami template and M_O be the corresponding toric origami manifold. Suppose that Γ is a tree. The orbit space $Q = M_O/T^n$ is homeomorphic to an n-dimensional disc. The face structure of Q defines a poset S_Q, whose elements are faces of Q ordered by reversed inclusion (it is easy to show that such poset is simplicial). In particular, $\mathrm{Vert}(S_Q) = \mathcal{F}(Q)$. Normal vectors to facets of Q (construction 1.8) determine the characteristic function $\nu\colon \mathcal{F}(Q) \to \mathbb{Z}^n/\pm$, $\nu([F]) = \nu(F)$. Thus there is a weighted simplicial poset (S_Q, ν) associated with a toric origami manifold M_O.

Construction 2.7. If Γ is a tree, then the simplicial poset S_Q is the connected sum of simplicial spheres $K_{\Psi_V(v)}$ along vertices, corresponding to fold facets:

$$S_Q \cong \mathop{\#}_{\Gamma} K_{\Psi_V(v)}. \tag{2.2}$$

Let us introduce a notation to make this precise. Let e be an edge of Γ, and v be its endpoint. Let $i_{v,e}$ be the vertex of $K_{\Psi_V(v)}$ corresponding to the facet $\Psi_E(e) \subset \Psi_V(v)$. Then (2.2) denotes the connected sum of all simplicial spheres $K_{\Psi_V(v)}$ along vertices $i_{v,e}, i_{u,e}$ for all edges $e = \{v, u\}$ of graph Γ. This simultaneous connected sum is well defined. Indeed, if $e_1 \neq e_2 \in E$ are two edges emanating from $v \in V$, then vertices i_{v,e_1} and i_{v,e_2} are not adjacent in $K_{\Psi_V(v)}$ by pt.2 of Definition 1.5. Therefore, open stars $\mathrm{star}^{\circ}_{K_{\Psi_V(v)}} i_{v,e_1}$ and $\mathrm{star}^{\circ}_{K_{\Psi_V(v)}} i_{v,e_2}$, which we remove in (2.1), do not intersect. Also note that all vertices $i_{v,e}$ are admissible, since the spheres $K_{\Psi_V(v)}$ are simplicial complexes.

Each sphere $K_{\Psi_V(v)}$ comes equipped with a weighting $\nu_v\colon \mathrm{Vert}(K_{\Psi_V(v)}) \to \mathbb{Z}^n/\pm$, since $\Psi_V(v)$ is Delzant. By pt.1 of Definition 1.5 these weightings agree on identified links. Therefore we have an isomorphism of weighted spheres

$$(S_Q, \nu) \cong \mathop{\#}_{\Gamma}(K_{\Psi_V(v)}, \nu_v). \tag{2.3}$$

3. Proof of Theorem 2

Suppose that a quasitoric manifold $M_{(P,\Lambda)}$ is equivariantly homeomorphic to the origami manifold $M_{(\Gamma,\Psi_V,\Psi_E)}$, Γ is a tree. First, the orbit spaces should be isomorphic as manifolds with corners: $P \cong Q = M_O/T^n$. Second, $M_{(P,\Lambda)} \overset{T}{\cong} M_O$ implies that stabilizers of the torus actions coincide for the corresponding faces of orbit spaces. Thus characteristic functions on P and Q taking values in $\mathbb{Z}^n/\pm$ are the same. Hence, the weighted simplicial spheres (K_P, Λ) and $(S_Q, \nu) \cong \#_{\Gamma}(K_{\Psi_V(v)}, \nu)$ are isomorphic. So far to prove Theorem 2 it is sufficient to prove

Proposition 3.1. *There exists a simple 3-dimensional polytope* P *and a characteristic function* $\Lambda\colon \mathcal{F}(P) \to \mathbb{Z}^3/\pm$ *such that the dual weighted sphere* (K_P, Λ) *cannot be represented as a connected sum, along a tree, of weighted spheres dual to Delzant polytopes.*

We proceed by steps. At first notice that any simplicial 2-sphere, which is a simplicial complex, is dual to some simple 3-polytope by Steinitz's theorem (see e.g. [10]). Thus it is sufficient to prove the statement for weighted simplicial complexes-spheres of dimension 2.

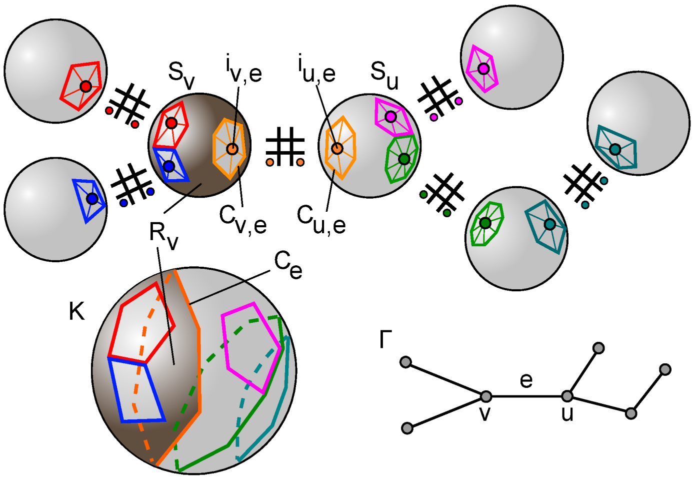

FIGURE 3. Connected sum of spheres along a tree

Construction 3.2. We introduce some notation in addition to that of construction 2.7, see Fig. 3. As before, let $\Gamma = (V, E)$ be a tree. Suppose that a simplicial $(n-1)$-sphere S_v is associated with each node $v \in V$, and for each edge $e \in E$ with an endpoint $v \in V$ there is an admissible vertex $i_{v,e} \in S_v$ subject to the following conditions: (1) $\operatorname{link}_{S_v} i_{v,e}$ is isomorphic to $\operatorname{link}_{S_u} i_{u,e}$ for any edge e with endpoints v, u; (2) Vertices i_{v,e_1}, i_{v,e_2} are different and not adjacent in S_v for any two edges $e_1 \neq e_2$ emanating from v. Then we can form a connected sum along Γ as in construction 2.7: $K = \#_\Gamma S_v$. For each $v \in V$ consider the simplicial subposet

$$R_v = S_v \Big\backslash \bigsqcup_{e \in E, v \in e} \operatorname{star}^\circ_{S_v} i_{v,e}. \tag{3.1}$$

This subposet will be called a region. Denote $\operatorname{link}_{S_v} i_{v,e}$ by $C_{v,e}$. By construction, $C_{v,e}$ is attached to $C_{u,e}$ if $e = \{v, u\}$. The resulting $(n-2)$-dimensional simplicial subposet of K is denoted by C_e. Since $i_{v,e}$ is admissible, the subposet $C_e \cong C_{v,e} = \operatorname{link}_{S_v} i_{v,e}$ is a simplicial $(n-2)$-sphere.

We get a collection of $(n-2)$-dimensional cycles $C_e, e \in E$, dividing the $(n-1)$-sphere K into regions $R_v, v \in V$. If $e = \{v, u\}$, then R_v and R_u share a common border C_e. Note that cycles C_e are mutually ordered, meaning that each C_e lies at one side of any other cycle. Though the cycles may have common points (as schematically shown on Fig. 3) and even coincide (in this case the region between them coincides with both of them).

On the other hand, any collection of mutually ordered $(n-2)$-dimensional spherical cycles in K defines the representation of K as a connected sum of smaller simplicial spheres. A representation $K = \#_\Gamma S_v$ will be called a *slicing*.

Define the *width* of a slicing Θ to be the maximal number of vertices in its regions:

$$\operatorname{wid}(\Theta) \stackrel{\text{def}}{=} \max\{|\operatorname{Vert}(R_v)| \mid v \in V\}. \tag{3.2}$$

Define the *fatness* of a sphere K as the minimal width of all its possible slicings:

$$\operatorname{ft}(K) \stackrel{\text{def}}{=} \min\{\operatorname{wid}(\Theta) \mid \Theta \text{ is a slicing of } K\}. \tag{3.3}$$

The essential step in the proof of Proposition 3.1 is the following.

Lemma 3.3. *Let K be an $(n-1)$-dimensional simplicial sphere and* $\Lambda\colon \operatorname{Vert}(K) \to \mathbb{Z}^n/\pm$ *a weighting. Let r denote the number of different values of this weighting,* $r = |\Lambda(\operatorname{Vert}(K))|$. *Suppose that* $\operatorname{ft}(K) > 2r$. *Then (K, Λ) cannot be represented as a connected sum, along a tree, of simplicial spheres dual to Delzant polytopes.*

Proof. Assume the converse. Then $(K,\Lambda) \cong \#_\Gamma(K_{\Psi_V(v)}, \nu_v)$, where $\Psi_V(v)$ are Delzant polytopes. If we forget the weights, this defines a slicing Θ of K. The width of every slicing of K is greater than $2r$ by the definition of fatness. In particular, $\operatorname{wid}(\Theta) > 2r$. Thus there exists a node v of Γ such that $|\operatorname{Vert}(R_v)| > 2r$.

The region R_v is a subcomplex of $K_{\Psi_V(v)}$. The restriction of Λ to the subset $\operatorname{Vert}(R_v)$ coincides with the restriction of $\nu\colon \operatorname{Vert}(K_{\Psi_V(v)}) \to \mathbb{Z}^n/\pm$ to $\operatorname{Vert}(R_v)$. Recall, that $\tilde{\nu}(F) \in \mathbb{Z}^n$ is the outward normal vector to the facet $F \in \mathcal{F}(\Psi_V(v)) = \operatorname{Vert}(K_{\Psi_V(v)})$, and $\nu(F) \in \mathbb{Z}^n/\pm$ is its class modulo sign. The outward normal vectors to facets of a convex polytope are mutually distinct, thus $|\tilde{\nu}(\operatorname{Vert} R_v)| = |\operatorname{Vert}(R_v)|$ and, therefore, $|\nu(\operatorname{Vert}(R_v))| \geqslant |\operatorname{Vert}(R_v)|/2$. Thus $|\Lambda(\operatorname{Vert}(R_v))| = |\nu(\operatorname{Vert}(R_v))| > r$, — the contradiction, since r is the total number of values of Λ. ☐

So far we may find counterexamples to origami realizability among polytopes, which are $\mathbb{Z}^n$-colored with a small number of colors, but whose dual simplicial spheres have large fatness. Of course such examples do not exist for $n = 2$ — this would contradict Theorem 1. The existence of 2-spheres satisfying conditions of Lemma 3.3 is thus our next and primary goal. At first, we prove that any 2-sphere admits a characteristic function with few values; then construct 2-spheres of arbitrarily large fatness.

Lemma 3.4. *Any simplicial 2-sphere K admits a weighting* $\Lambda\colon \operatorname{Vert}(K) \to \mathbb{Z}^3/\pm$ *such that* $|\Lambda(\operatorname{Vert}(K))| \leqslant 4$.

Proof. Four color theorem states that there exists a proper vertex-coloring: $\operatorname{Vert}(K) \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Now replace colors by integral vectors $\alpha_1 \mapsto (1,0,0)$, $\alpha_2 \mapsto (0,1,0)$, $\alpha_3 \mapsto (0,0,1)$, $\alpha_4 \mapsto (1,1,1)$. This gives the required characteristic function. ☐

Proposition 3.5. *For any $N > 0$ there exists a simplicial 2-sphere K such that* $\operatorname{ft}(K) > N$.

Proof.

Construction 3.6. Let K be a simplicial 2-complex. Define a (piecewise Riemannian) metric g and measure μ on $|K|$ in such a way that each triangle $|I| \subset |K|$ becomes an equilateral Euclidian triangle with the standard metric and edge length 1. Thus the area of each triangle is $\sqrt{3}/4$.

Let $L(\gamma)$ denote the length of a piecewise smooth curve γ in $|K|$. If $C \subset K$ is a closed 1-dimensional cycle (simplicial subcomplex), then, obviously,

$$L(|C|) = |\operatorname{Vert}(C)|. \tag{3.4}$$

A cycle C divides K into two subcomplexes K_+ and K_-, each homeomorphic to a closed 2-disc (we suppose $C \subset K_+, K_-$). Let us estimate the number of vertices in K_- in terms of its area (K_+ is similar). Let $\mathcal{V}_-, \mathcal{E}_-, \mathcal{T}_-$ denote the number of vertices, edges and triangles in K_-. By the definition of measure, $\mathcal{T}_- = \frac{4}{\sqrt{3}}\mu(|K_-|)$. We have $\mathcal{V}_- - \mathcal{E}_- + \mathcal{T}_- = 1$ (Euler characteristic of K_-) and $\mathcal{E}_- < 3\mathcal{T}_-$ (by counting pairs $e \subset t$, where e is an edge and t is a triangle). Therefore,

$$\mathcal{V}_- \leqslant \frac{8}{\sqrt{3}}\mu(|K_-|). \tag{3.5}$$

Let $\mathbb{S}_R$ be a 2-dimensional round sphere of radius R, with the standard metric g_s and measure μ_s. A piecewise smooth closed curve $\gamma \subset \mathbb{S}_R$ without self-intersections divides $\mathbb{S}_R$ into two regions A_+, A_-. The isoperimetric inequality on a sphere (see e.g. [8, Ch.4]) has the form $R^2 L_s(\gamma)^2 \geqslant \mu_s(A_+)\mu_s(A_-)$, where $L_s(\gamma)$ is the length of γ. Since $\mu_s(\mathbb{S}_R) = 4\pi R^2$ we may assume that $\mu_s(A_+) \geqslant 2\pi R^2$ (otherwise consider A_- instead), thus

$$\mu_s(A_-) \leqslant \frac{L_s(\gamma)^2}{2\pi}. \tag{3.6}$$

Let K be a 2-dimensional simplicial sphere and R, c_1, c_2, c_3, c_4 be positive real numbers. Suppose there exists a bijective piecewise smooth map $f\colon |K| \to \mathbb{S}_R$ such that

$$c_1 L(\gamma) \leqslant L_s(f(\gamma)) \leqslant c_2 L(\gamma), \tag{3.7}$$

$$c_3 \mu(\Omega) \leqslant \mu_s(f(\Omega)) \leqslant c_4 \mu(\Omega), \tag{3.8}$$

for each piecewise smooth curve $\gamma \subset |K|$ and measurable set $\Omega \subset |K|$. Numbers c_1, c_2, c_3, c_4 will be called Lipschitz constants of the map f.

Lemma 3.7. *In the above setting, suppose the cycle $C \subset K$ contains at most N vertices. Then either K_+ or K_- contains at most $\frac{4N^2c_2^2}{\sqrt{3}\pi c_3}$ vertices.*

Proof. Among two regions $f(|K_-|), f(|K_+|) \subset \mathbb{S}_R$ let $f(|K_-|)$ be the one with the smaller area. Combine (3.4), (3.5), (3.6), (3.7), and (3.8):

$$V_- \leqslant \frac{8}{\sqrt{3}}\mu(|K_-|) \leqslant \frac{8\mu_s(f(|K_-|))}{\sqrt{3}c_3} \leqslant \frac{8L_s(f(|C|))^2}{2\sqrt{3}\pi c_3} \leqslant \frac{4N^2c_2^2}{\sqrt{3}\pi c_3}. \tag{3.9}$$

□

Lemma 3.8. *If Θ is a slicing $K = \#_\Gamma S_v$ and* $\operatorname{wid}(\Theta) \leqslant N$, *then* $\deg v \leqslant 2(N-2)$ *for any node v of Γ.*

Proof. Denote the degree of v by d. By construction, the region R_v is obtained from a sphere S_v by removing d open stars which correspond to the edges of Γ emanating from v. The complex R_v itself can be considered as a plane graph. Denote the numbers of its vertices, edges and faces by $\mathcal{V}, \mathcal{E}, \mathcal{R}$ respectively. By the definition of the width, we have $\mathcal{V} \leqslant N$. We also have $\mathcal{V} - \mathcal{E} + \mathcal{R} = 2$, and $2\mathcal{E} \geqslant 3\mathcal{R}$ (each region has at least 3 edges). Thus, $\mathcal{V} \geqslant 2 + \frac{1}{2}\mathcal{R}$. Notice that each removed open star represents a face of graph R_v, therefore, $d \leqslant \mathcal{R} \leqslant 2(\mathcal{V}-2) \leqslant 2(N-2)$. □

Lemma 3.9. *Let K be a 2-dimensional simplicial sphere endowed with the map f to the round sphere, satisfying Lipschitz bounds* (3.7) *and* (3.8). *For a natural number N set $A = \frac{4N^2c_2^2}{\sqrt{3}\pi c_3}$ and $B = 2(N-2)$. If $|\operatorname{Vert}(K)| > \max(AB + N, 2A)$, then $\operatorname{ft}(K) > N$.*

Proof. Assume the contrary: $\operatorname{ft}(K) \leqslant N$. Then there is a slicing $K = \#_\Gamma S_v$ in which every region R_v has at most N vertices. Consequently, any cycle $C_e, e \in E$ has at most N vertices. By Lemma 3.7, the cycle C_e divides K into two parts, one of which has $\leqslant A$ vertices. Since $|\operatorname{Vert}(K)| > 2A$, the other part has $> A$ vertices. Assign a direction to each edge e of Γ in such a way that e points from the larger component of $K \setminus C_e$ to the smaller, where the "size" means the number of vertices.

Γ is a tree, therefore there exists a source u, i.e. a node from which all adjacent edges emanate. Let d denote the degree of u and $\Gamma_1, \ldots, \Gamma_d$ the connected components of $\Gamma \setminus u$. By Lemma 3.8 we have $d \leqslant B$. By the construction of the directions of edges, $|\operatorname{Vert}(\bigsqcup_{\Gamma_i} R_v)| \leqslant A$ for each Γ_i. Thus $|\operatorname{Vert}(K)| < |\operatorname{Vert}(R_u)| + \sum_{i=1}^{d} |\operatorname{Vert}(\bigsqcup_{\Gamma_i} R_v)| \leqslant N + AB$ — the contradiction. □

Lemma 3.10. *For any $N > 0$ there exists a 2-dimensional simplicial sphere K such that:*

(1) *There exists a piecewise smooth map $f\colon |K| \to \mathbb{S}_R$ satisfying Lipschitz bounds* (3.7) *and* (3.8) *for some constants $c_1, c_2, c_3, c_4, R > 0$*
(2) $|\operatorname{Vert}(K)| > \max(AB + N, 2A)$, *where A and B are defined in Lemma 3.9.*

Proof. Start with the boundary of a regular tetrahedron with edge length 1: $L = \partial\Delta^3$. The projection from the center of L to the circumsphere $f\colon L \to \mathbb{S}_R$ is obviously Lipschitz for some constants $c_1, c_2, c_3, c_4 > 0$. Now subdivide each triangle of $|L|$ into q^2 smaller regular triangles as shown on Fig. 4.

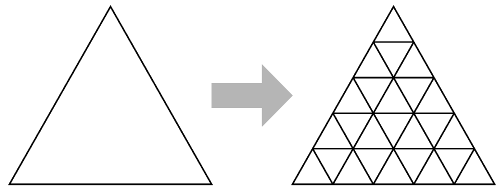

FIGURE 4. Subdivision of a regular triangle.

This results in a simplicial complex $L_{(q)}$. As a space with metric and measure $|L_{(q)}|$ is homothetic to $|L|$ with a linear scaling factor q. Thus there exists a map $f_{(q)}\colon |L_{(q)}| \to \mathbb{S}_{qR}$ with the same Lipschitz constants as f. The number of vertices $|\operatorname{Vert}(L_{(q)})|$ can be made arbitrarily large. □

Lemmas 3.10 and 3.9 conclude the proof of Proposition 3.5. □

Remark 3.11. Actually, in the proof of Lemma 3.10 we could have started with any simplicial sphere L, take any piecewise smooth map $f\colon |L| \to \mathbb{S}_R$, find Lipschitz constants $c_2, c_3 > 0$ (they exist by the standard calculus arguments), and then apply the same subdivision procedure. We used the boundary of a regular simplex, because in this case Lipschitz map is constructed easily and admits an explicit computation.

We give a concrete example of a quasitoric manifold which is not toric origami, by performing this computation. The calculations themselves are elementary thus omitted. It is sufficient to construct a simplicial sphere for $N = 8$. For a projection map from the boundary of a regular tetrahedron to the circumscribed sphere we have Lipschitz constants $c_2 = 3$, $c_3 = \frac{1}{3}$. Thus $\max(AB + N, 2A) \approx 15251.14$. Subdivide each triangle in the boundary of a regular tetrahedron in q^2 small triangles where $q \geqslant 88$. This gives a simplicial sphere K with at least 15490 vertices and the same Lipschitz constants as $\partial\Delta^3$. Thus $\operatorname{ft}(K) > 8$. Now take the dual simple polytope P of K, consider any proper facet-coloring in four colors and assign a characteristic function Λ, as described in Lemma 3.4. This gives a characteristic pair (P, Λ), whose corresponding quasitoric manifold is not toric origami.

Of course, all our estimations are very rough, and, probably, there are better ways to construct fat spheres. For sure, there exist 2-spheres of fatness 9 with less than 15490 vertices.

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