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Equilibrium Force Allocations and Attack Probabilities

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Abstract

We model the struggle between terrorist and conventional forces as a *Colonel Blotto* game, replacing Powers and Shen's (2006) mathematical expression for the probability of target destruction by a more rigorously derived approximation from a diffusion-based Lanchester analysis. We then use the resulting equilibrium solutions for force allocations and attack probabilities to make inferences about terrorist attackers and government defenders that are roughly consistent with empirical findings. Our analysis reveals that the loss function of a government/society plays a central role in determining the types of targets likely to be attacked by terrorists in "peacetime" and "wartime", leading to a much more frequent selection of "trophy" targets in peacetime.

Keywords – Terrorism risk, force allocations, attack probabilities, game theory, Lanchester equations, power-law distributions.

1. Introduction

To study the problem of terrorism risk, we model the struggle between terrorist and conventional forces as a *Colonel Blotto* game. This approach arises from the confluence of three distinct research streams: (1) the game-theoretic analyses of terrorism provided by Major (2002) and Powers and Shen (2006); (2) the introduction of diffusion processes into Lanchester-like combat analyses, first proposed by Perla and Lehoczky (1977), and more recently developed by Powers (2008) and Gudmundsson et al. (2008); and (3) the empirical analysis of terrorist-destroyed-target distributions conducted by Johnson et al. (2005). Given that many of the relevant mathematical theorems are published elsewhere, we confine the present study primarily to the implications of those results, and provide all new derivations in a technical appendix.

Most significantly, we replace Powers and Shen's (2006) mathematical expression for the conditional probability of destruction of a target, given that that target is selected for attack by terrorists, by a more rigorously derived approximation from a diffusion-based Lanchester analysis. We then use the resulting equilibrium solutions for force allocations and attack probabilities to make inferences about terrorist attackers and government defenders that are roughly consistent with the empirical findings of Johnson et al. (2005). In addition to providing explicit forms for the force-allocation and attack-probability strategies, our analysis reveals that the loss function of a government (*qua* society) plays a central role in determining the actions of attackers. Distinguishing between the risk attitudes of "peacetime" and "wartime" governments, we find that there is a much more frequent selection of "trophy" targets in peacetime.

2. Prior Work

2.1. The *Colonel Blotto* Game

Given a finite set of potential targets, let W denote the combined monetary/human-life value[1] of a particular target, which is assumed to be directly proportional to that target's (three-dimensional) physical volume, V; that is, $W \propto V$. Next, let A and D denote the sizes of the forces allocated to the target by the terrorist attackers and government defenders, respectively, where the attackers' (but not the defenders') *total* forces are assumed to be fixed *a priori*.

In the *Colonel Blotto* game, the attackers and the defenders must allocate their total forces across the various targets without knowing their opponents' strategies. In the simplest version of the game, the player that assigns the higher level of force to a given target prevails at that target; in a more sophisticated version, a player's probability of prevailing would be an increasing function of the player's force allocation (for a fixed allocation made by the player's opponent). For our purposes, we will say that the attackers prevail at a given target if they succeed in destroying the target, and that the defenders prevail by preserving the target, while explicitly acknowledging that any target that is attacked is *partially* damaged. A player's payoff from the game is then the expected value of that player's total gain or loss from the outcomes at the various targets.

Powers and Shen (2006) proposed that the attackers' conditional probability of destroying a particular target, given that that target is selected for attack, be written as

$$p = \exp\left(-\frac{A^s D^s}{V^s}\right)\left(\frac{A^c}{A^c + D^c}\right), \qquad (1)$$

[1] The use of a hybrid monetary *and* human-life value scale is a quantitative simplification that bears further study. For the present, one could think of W as consisting of two components, one for monetary worth and one for human lives, and simply assume that the two components always increase or decrease in direct proportion to each other.

where the first factor on the right-hand side of equation (1) represents the probability that the attackers avoid detection prior to their attack (derived from a simple search model), and the second factor represents the probability that the attackers are then successful in destroying the target (derived from a classical gambler's ruin model). In the above expression, the constants $s > 1$ and $0 < c < 1$ are scale parameters. Powers and Shen (2006) also assumed that the attackers' gain associated with damage to, and/or destruction of, a target of physical volume V is given by $Gain_A(V) \propto V^{\lambda}$, for some positive constant λ, and that the game is zero-sum (so that the defenders' corresponding loss is given by $Loss_D(V) \propto V^{\lambda}$). They then used equation (1) to prove three theorems.

The first theorem addresses the case in which terrorists attack all of the targets simultaneously, and shows that there exists a Cournot-Nash equilibrium in which both the attackers and defenders allocate their forces to each target in direct proportion to the square root of the target's volume. The second theorem addresses the case in which terrorists attack only one target, selected at random, and shows that there exists a Cournot-Nash equilibrium in which both sides again allocate their forces in direct proportion to the square roots of a target's volume. Finally, the third theorem reveals that if the probability with which the attackers select a target at random (in the setting of the second theorem) is treated as a strategic decision of the attackers, then no Cournot-Nash equilibrium with pure-strategy force allocations can exist.

Given that the setting of the second theorem appears more relevant to today's War on Terror, we will work with that model in the present study. However, because of potential weaknesses with the expression in equation (1) (e.g., the right-hand side approaches zero as A

tends to infinity)[2], we will replace that probability with a more rigorously derived expression based upon a stochastic Lanchester model tailored specifically to terrorism combat.

2.2. The Lanchester Paradigm

The most widely studied mathematical model of military combat is that proposed by Lanchester (1916), which may be described by a system of differential equations of the form

$$dA = -k_1 A^{\alpha_1} D^{\delta_1} dt \tag{2}$$

$$dD = -k_2 A^{\alpha_2} D^{\delta_2} dt, \tag{3}$$

where: $A = A(t)$ and $D = D(t)$ denote, respectively, the sizes of the attackers' and defenders' forces at time $t \geq 0$; k_1, k_2 are positive constants denoting, respectively, the defenders' and attackers' effective destruction rates; and α_1, α_2 and δ_1, δ_2 are real-valued constants reflecting the fundamental nature of the combat under study. In his original formulation, Lanchester (1916) considered two cases – one for "ancient" warfare, in which $\alpha_1 = 1,\ \delta_1 = 1,\ \alpha_2 = 1,\ \delta_2 = 1$, and one for "modern" warfare, in which $\alpha_1 = 0,\ \delta_1 = 1,\ \alpha_2 = 1,\ \delta_2 = 0$.

Gudmundsson et al. (2008) considered a special case of (2) and (3) designed specifically for terrorism combat:

$$dA = -\frac{k_1'}{V^q} ADdt \tag{4}$$

$$dD = -k_2 Adt, \tag{5}$$

where: V (as before) denotes the physical volume of the target under attack; q denotes a positive power-transformation constant used to recognize the appropriate domain of combat (e.g., $q = 1/3$ if a building can be attacked through only its ground-level perimeter, $q = 2/3$ if a

[2] This means that as the terrorists' forces increase in magnitude, the disadvantage of size in terms of avoiding detection eventually outweighs the benefit of size in combat. While this implication may be realistic in certain scenarios, it is easily challenged. For example, the September 11 attacks suggest a small role for detection in even the boldest of attacks when the target is inadequately defended.

building can be attacked anywhere along its surface, as by a fuel-filled airplane, and $q = 1$ if a bomb can be planted anywhere within a building); and $k_1' = k_1 V^q$. Rewriting A and D in terms of a single variable, $U(A,D)$, Gudmundsson et al. (2008) replaced the system (4), (5) with the stochastic differential equation

$$dU = Udt + \sigma U^{\gamma/2} dZ,$$

where: dZ is a standard Brownian motion; σ is the associated infinitesimal standard deviation; and $\gamma \in [0,2]$. They then identified the attackers' probability of victory with the probability of first-passage to the state $D = 0 \Leftrightarrow U = 1$, and derived the following approximation for the attackers' conditional probability of destroying a particular target, given that that target is selected for attack:

$$p = \begin{cases} 1 - \dfrac{k_1' D^2}{2k_2 V^q A} \quad \text{for } A > \dfrac{k_1' D^2}{2k_2 V^q} \\ 0 \quad \text{for } A \leq \dfrac{k_1' D^2}{2k_2 V^q} \end{cases}. \qquad (6)$$

3. Analytical Results

Substituting equation (6) for equation (1) in a modified version[3] of Powers and Shen's (2006) second theorem, we obtain the following result. (The proof is provided in the appendix.)

Theorem 1: There exists a Cournot-Nash equilibrium in which the attackers' and defenders' force allocations to a particular target are given by $A \propto W^a$ and $D \propto W^d$, respectively, and the attackers' probability of selecting the target is given by $\pi \propto W^r$, for constants a, d, and r such that the probability of target destruction (p) is 0.

[3] In addition to the stated revision of the target-destruction probability, our analysis differs from Powers and Shen's (2006) in that (1) the defenders' total forces are not fixed *a priori*, and (2) every target that is attacked is assumed to be partially damaged.

Although Theorem 1 states that no target can be destroyed in equilibrium, it is important to recall our assumption that any target attacked will be partially damaged. Obviously, the occurrence of another September 11-like event, in which major targets are destroyed completely, would cast serious doubt on the validity of the above result.[4] However, any attack with only partial damage would be consistent with it.

From the first-order conditions of the optimization problem underlying Theorem 1 (shown in the proof), we know that the result is subject to the constraints

$$2d + r + \lambda = q + 2a \tag{7}$$

and

$$d = (q + a)/2. \tag{8}$$

Assuming that q and λ are known, this leaves three unknown constants – d, a, and r – but only two equations – (7) and (8) – to specify them.

Fortunately, there is one additional piece of information that we have not yet used – the fact that in a real-world *multi-period* setting, the attackers are able to move first (e.g., with the September 11 strikes), and thus are able to select the equilibrium constant r (which the defenders then are forced to follow in all subsequent plays of the game). Given the privilege of selecting this constant, the attackers will do so in a way that maximizes the *expected value* (or average value) of their gain – that is, the weighted sum of $Gain_A$ over all of the targets, where each target is weighted by its corresponding probability of partial damage (i.e., its probability of being attacked, since there is no chance of target destruction under Theorem 1).

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[4] In fact, for Theorem 1 to be consistent with reality, one must view the September 11 attacks as a formal initiation of hostilities, only after which the *Colonel Blotto* game actually began.

Assuming, as suggested by the empirical work of Kaizoji and Kaizoji (2008), that the distribution of available target values follows a continuous power law with positive constant t,[5] it is not difficult to derive the following result (proved in the appendix).

Theorem 2: For the Cournot-Nash equilibrium described in Theorem 1, the attackers can maximize their expected gain by choosing

$$r = t' - 1 - \lambda, \tag{9}$$

for any constant t' that is greater than or equal to t.

Substituting equation (9) into equations (7) and (8) then yields

$$a = t' - 1 \tag{10}$$

and

$$d = \left(q + t' - 1\right)/2. \tag{11}$$

4. Discussion

In light of equations (9) through (11), our Cournot-Nash-equilibrium result may be restated as follows.

Corollary 1: There exists a Cournot-Nash equilibrium in which the attackers' and defenders' pure-strategy force allocations to a particular target are given by $A \propto W^{t'-1}$ and $D \propto W^{(q+t'-1)/2}$, respectively, and the attackers' probability of selecting the target is given by $\pi \propto W^{t'-1-\lambda}$.

To interpret this result, we must know something about the positive constants q, t', and λ. For simplicity, we will assume that $q = 1$ – that is, that the domain of combat includes the entire three-dimensional volumes of the targets. Furthermore, consistent with Kaizoji and

[5] Formally, this means that the probability density function of available target values is given by $f(W) \sim W^{-t}$.

Kaizoji (2008), we will assume that $t = 2.35$ for industrially developed nations,[6] and further that $t' = t = 2.35$. (In less-developed nations, one would expect the value of t' to be somewhat larger, since the distribution of property values would tend to have a thinner tail.) This leaves the gain/loss-function constant, λ, for further investigation.

Consider then the government defenders' loss function, $Loss_D(W) \propto W^{\lambda}$. Figure 1 shows that this function is: (1) concave downward for values of λ between 0 and 1; (2) linear for $\lambda = 1$; and (3) concave upward for values of λ greater than 1. At first blush, it seems reasonable that the loss function should be concave downward, since a government would tend to experience decreasing marginal losses as the monetary/human-life values of the terrorists' targets increase. For example, one might argue that for the U.S. government's September 11 losses to be doubled, the terrorists would have to destroy a target of more than twice the monetary/human-life value of the September 11 targets.

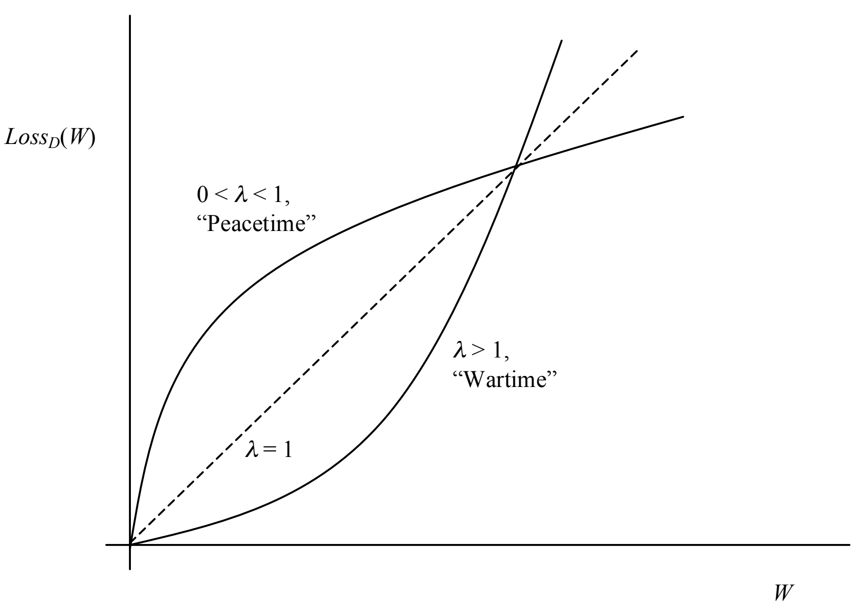

Figure 1. Government's Loss Function for Various Values of λ

[6] Kaizoji and Kaizoji (2008) provided annual estimates of t for Japanese land values during the period 1981-2002. Those estimates vary from a low of about 2.0 to a high of about 2.7. We have selected the approximate sample mean (2.35) of the estimates for our analysis.

Note, however, that a concave-downward function corresponds to an assumption of *risk proneness* on the part of the government defenders (i.e., given the choice between any random lottery and a fixed amount equal to the lottery's expected value, they would prefer the lottery itself). Thus, such an assumption appears somewhat inconsistent with the generally observed *risk-averse* nature of governments (e.g., their seemingly cautious behavior in responding to life-threatening crises).

To place this issue in some perspective, one might distinguish between two distinctly different societies, one – like the U.S. – that has enjoyed a long period of domestic peacetime, and another – like Israel – that has experienced an extensive period of terrorist activity. While it is true that *all* governments, in moments of crisis, manifest risk-averse tendencies, such behavior is not necessarily characteristic of more mundane periods. More precisely, in a nation used to peace, it is quite likely that – apart from specific moments of crisis – both the populace and government tend to "take chances" by preferring an abundance of personal liberty and relaxed government security. In a nation used to war, however, a more restrictive, security-conscious view would tend to prevail even in the best of times.

For these reasons, it makes sense to model the U.S. and other "peacetime" governments as risk-prone decision makers (with concave-downward loss functions; i.e., $0 < \lambda < 1$), while modeling the Israeli and other "wartime" governments as risk-averse decision makers (with concave-upward loss functions; i.e., $\lambda > 1$). Hypothetically, we will select $\lambda = 0.5$ for peacetime nations and $\lambda = 1.5$ for wartime nations.[7]

[7] Note that, under our zero-sum assumption, the selection of the defenders' loss function immediately implies the form of the attackers' gain function. This is quite reasonable if the attackers' utility (gratification) arises directly from the defenders' disutility (frustration).

We now return to the above Cournot-Nash-equilibrium result and suggest that, in practice, one should expect to find results similar to the following (in industrially developed nations):

$$A \propto W^{t'-1} = W^{1.35}, \tag{12}$$

$$D \propto W^{(q+t'-1)/2} = W^{1.175}, \tag{13}$$

$$\pi \propto W^{t'-1-\lambda} = \begin{cases} W^{0.85} \text{ for peacetime nations} \\ W^{-0.15} \text{ for wartime nations} \end{cases}. \tag{14}$$

To compare these implications with the empirical distributions of *destroyed* (rather than simply *available*) target values estimated by Johnson et al. (2005),[8] one first must multiply π (for wartime nations) by the probability density function associated with a power-law constant of t. This yields

$$g(W) \propto \pi f(W) \sim W^{t'-1-\lambda-t} = W^{-2.5}. \tag{15}$$

Interestingly, the constant in the power-law distribution implied by approximation (15), $\tau = 2.5$, happens to be identical to the constant estimated by Johnson et al. (2005) for less-developed wartime nations. However, our figure is substantially higher than Johnson et al.'s (2005) estimate for industrially developed wartime nations ($\tau = 1.71$) (although the latter estimate could be obtained quite readily by changing the assumption of $t' = 2.35$ to the equally permissible $t' = 3.14$).

Given the highly subjective procedure for selecting the various model constants (q, t', and λ), one should not read too much into either of these comparisons. Rather, we simply would observe that our results appear to be in the same ballpark as Johnson et al.'s (2005), which

[8] Johnson et al. (2005) argued that the distribution of destroyed target values follows a continuous power law with positive constant τ (i.e., the probability density function is given by $g(W) \sim W^{-\tau}$). They further estimated the values of τ for both industrially developed nations and less-developed nations.

affords some support for the following *qualitative* observations from approximations (12) through (14):

- In both peacetime and wartime, government defenders tend to allocate forces in slightly lower proportion to high-value targets than do terrorist attackers.
- In peacetime, terrorist attackers tend to give substantial weight to high-value targets; however, such targets actually are avoided in wartime.

5. Conclusions

In the present study, we have modeled the struggle between terrorist and conventional forces as a *Colonel Blotto* game. We first replaced Powers and Shen's (2006) mathematical expression for the conditional probability of destruction of a particular target, given that that target is selected for attack by terrorists, by a more rigorously derived approximation from a diffusion-based Lanchester analysis, and then used the resulting equilibrium solutions for force allocations and attack probabilities to make inferences about terrorist attackers and government defenders. A brief analysis showed that these solutions are roughly consistent with the empirical findings of Johnson et al. (2005).

Our analysis revealed that the loss function of a government plays a central role in determining the types of targets likely to be attacked by terrorists in "peacetime" and "wartime", respectively. Specifically, we found that terrorists tend to select high-value ("trophy") targets much more frequently in peacetime than in wartime.

Investigating how a government's loss function depends on its society's perception of conflict-related risk is crucial to a thorough understanding of the behavior of both governments and terrorists. We believe this is a promising area for further research.

References

Gudmundsson, B. I., Powers, M. R., and Zhang, L. (2008), "The mathematics of terrorism risk: Lanchester SDEs and the probability of target destruction," working paper, Center for Insurance and Risk Management, Tsinghua University.

Johnson, N. F., Spagat, M., Restrepo, J. A., Becerra, O., Bohórquez, J. C., Suárez, N., Restrepo, E. M., and Zarama, R. (2005), "Universal patterns underlying ongoing wars and terrorism," *arXiv:physics/0506213v1*, available at http://xxx.lanl.gov/abs/physics/0506213.

Kaizoji, T. and Kaizoji, M. (2008), "A mechanism leading from bubbles to crashes: the case of Japan's land market," *arXiv:cond-mat/0312404v2*, available at http://arxiv.org/abs/cond-mat/0312404v2.

Lanchester, F. W. (1916), "Aircraft in warfare: the dawn of the fourth arm – the principle of concentration," *Engineering*, Vol. 98, pp. 422-423.

Major, J. A. (2002), "Advanced techniques for modeling terrorism risk," *Journal of Risk Finance*, Vol. 4, No. 1, pp. 15-24.

Perla, P. P. and Lehoczky, J. P. (1977), "A new approach to the analysis of stochastic Lanchester processes – time evolution," technical report, Carnegie-Mellon University Department of Statistics.

Powers, M. R. (2008), "Lanchester resurgent? the mathematics of terrorism risk," *Journal of Risk Finance*, Vol. 9, No. 3.

Powers, M. R. and Shen, Z. (2006), "Colonel Blotto in the war on terror: implications for event frequency," paper presented at the American Risk and Insurance Association Annual Meeting, Washington, DC.

Appendix

Proof of Theorem 1:

Let $i = 1,2,\ldots,n$ be the index for the various targets, and let

$$E[Gain_A] = \sum_{i=1}^{n} \pi_i(\varepsilon_A + p_i k_A)V_i^{\lambda} = \sum_{i=1}^{n} \pi_i(\varepsilon_A + p_i k_A)\upsilon^{\lambda} W_i^{\lambda} \text{ and}$$

$$E[Loss_D] = \sum_{i=1}^{n} \pi_i(\varepsilon_D + p_i k_D)V_i^{\lambda} = \sum_{i=1}^{n} \pi_i(\varepsilon_D + p_i k_D)\upsilon^{\lambda} W_i^{\lambda},$$

where: ε_A, ε_D are positive constants reflecting the amount of partial damage sustained by any target that is attacked; k_A, k_D are positive constants reflecting the additional damage sustained by a target that is destroyed; υ is a positive constant such that $\upsilon W_i = V_i$; and $p_i = 1 - \dfrac{k_1' D_i^2}{2k_2 V_i^q A_i} = 1 - \dfrac{k_1' D_i^2}{2k_2 \upsilon^q W_i^q A_i}$. To solve the joint optimization problem

$$\underset{A_1,\ldots,A_n}{Max}\ E[Gain_A] \quad s.t. \sum_{i=1}^{n} A_i = A^* \text{ and}$$

$$\underset{D_1,\ldots,D_n}{Min}\ E[Loss_D] \quad s.t. \sum_{i=1}^{n} D_i < \infty,$$

where A^* is a positive constant denoting the attackers' total forces (fixed *a priori*), we seek solutions satisfying

$$grad(E[Gain_A]) - \mu_A grad\left(\sum_{i=1}^{n} A_i\right) = 0 \text{ and}$$

$$\frac{\partial E[Loss_D]}{\partial D_i} = 0 \text{ for } i = 1,2,\ldots,n,$$

where μ_A is a Lagrange multiplier. In other words, we wish to solve the system of first-order equations

$$\frac{\partial p_i}{\partial A_i}\pi_i k_A \upsilon^{\lambda} W_i^{\lambda} = \mu_A \text{ and} \tag{A1}$$

$$\frac{\partial p_i}{\partial D_i}\pi_i k_D \upsilon^{\lambda} W_i^{\lambda} = 0 \tag{A2}$$

for $i = 1,2,\ldots,n$, subject to the second-order conditions

$$\frac{\partial^2 E\left[Gain_A\right]}{\partial A_i^2} = \frac{\partial^2 p_i}{\partial A_i^2}\pi_i k_A \upsilon^{\lambda} W_i^{\lambda} < 0 \text{ and} \tag{A3}$$

$$\frac{\partial^2 E\left[Loss_D\right]}{\partial D_i^2} = \frac{\partial^2 p_i}{\partial D_i^2}\pi_i k_D \upsilon^{\lambda} W_i^{\lambda} > 0 \tag{A4}$$

for $i = 1,2,\ldots,n$.

Now let $A_i = \alpha W_i^a$ and $D_i = \delta W_i^d$ denote the equilibrium-allocation solutions, where

$$\alpha = A^* \Big/ \sum_{j=1}^{n} W_j^a,$$

and let $\pi_i = \rho W_i^r$ denote the attackers' probability of selecting target i, where

$$\rho = 1 \Big/ \sum_{j=1}^{n} W_j^r.$$

Since

$$\frac{\partial p_i}{\partial A_i} = \frac{k_1' D_i^2}{2k_2 \upsilon^q W_i^q A_i^2},$$

it follows from equation (A1) that

$$\frac{k_1' D_i^2}{2k_2 \upsilon^q W_i^q A_i^2}\rho W_i^r k_A \upsilon^{\lambda} W_i^{\lambda} = \frac{k_1' \delta^2 W_i^{2d}}{2k_2 \upsilon^q W_i^q \alpha^2 W_i^{2a}}\rho W_i^r k_A \upsilon^{\lambda} W_i^{\lambda} = \mu_A. \tag{A5}$$

Then, since inequality (A3) always holds, we can collect the exponents of W_i in equation (A5) to conclude

$$2d + r + \lambda = q + 2a.$$

Turning to the defenders' allocations, we note that

$$\frac{\partial p_i}{\partial D_i} = -\frac{k_1' D_i}{k_2 \upsilon^q W_i^q A_i}$$

is negative for all $D_i > 0$, and so there is no internal solution to equation (A2) (and indeed, inequality (A4) also fails). Consequently, the values of D_i that minimize $E\left[Loss_D\right]$ must lie at the boundary provided by equation (6); that is,

$$D_i = \sqrt{\frac{2k_2 \upsilon^q W_i^q A_i}{k_1'}} = \sqrt{\frac{2k_2 \upsilon^q \alpha}{k_1'}} W_i^{(q+a)/2},$$

for which $p_i = 0$. This implies

$$\delta = \sqrt{\frac{2k_2 \upsilon^q \alpha}{k_1'}} \text{ and}$$

$$d = (q + a)/2.$$

Proof of Theorem 2:

From the proof of Theorem 1, we know that

$$E\left[Gain_A\right] = \sum_{i=1}^{n} \pi_i (\varepsilon_A + p_i k_A) \upsilon^\lambda W_i^\lambda = \sum_{i=1}^{n} \rho W_i^r \varepsilon_A \upsilon^\lambda W_i^\lambda = \left(\varepsilon_A \upsilon^\lambda \Big/ \sum_{j=1}^{n} W_j^r\right) \sum_{i=1}^{n} W_i^{r+\lambda}.$$

Given that the distribution of W_i is continuous with probability density function

$$f(W) \sim W^{-t},$$

it follows that

$$f(W) \propto (W + c)^{-t}$$

for some positive constant c, and

$$E\left[Gain_A\right] = \left(\varepsilon_A \upsilon^\lambda / nE\left[W_i^r\right]\right) nE\left[W_i^{r+\lambda}\right] = \varepsilon_A \upsilon^\lambda \left(E\left[W_i^{r+\lambda}\right] / E\left[W_i^r\right]\right)$$

$$= \varepsilon_A \upsilon^\lambda \frac{\int_0^\infty W^{r+\lambda}(W+c)^{-t}\,dW}{\int_0^\infty W^r (W+c)^{-t}\,dW},$$

which is finite for $r < t-1-\lambda$ and diverges to positive infinity for $r \geq t-1-\lambda$. (Actually, the above ratio of integrals possesses the indeterminate form ∞/∞ for $r \geq t-1$, but one can interpret this as divergence to positive infinity by viewing $\frac{\int_0^\infty W^{r+\lambda}(W+c)^{-t}\,dW}{\int_0^\infty W^r (W+c)^{-t}\,dW}$ as

$\lim_{z\to\infty} \frac{\int_0^z W^{r+\lambda}(W+c)^{-t}\,dW}{\int_0^z W^r (W+c)^{-t}\,dW}$ and applying l'Hôpital's rule.)