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The problem of spinning-up an axially symmetric spacecraft subjected to an external torque constant in magnitude and parallel to the symmetry axis is considered. The existing exact analytic solution for an axially symmetric body is applied for the first time to this problem. The proposed solution is valid for any initial conditions of attitude and angular velocity and for any length of time and rotation amplitude. Furthermore, the proposed solution can be numerically evaluated up to any desired level of accuracy. Numerical experiments and comparison with an existing approximated solution and with the integration of the equations of motion are reported in the paper. Finally, a new approximated solution obtained from the exact one is introduced in this paper.

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1. Introduction

The problem of spinning-up a spacecraft is of high importance for many space missions. In particular, as many axially symmetric spacecraft are spin-stabilized a spinning-up maneuver is needed in order to bring them from an initial condition typically close to rest to the nominal spinning condition. The spacecraft may need spin-down and spin-up maneuvering capabilities also to perform reorientation maneuvers.

The spin-up problem of spacecraft has been previously studied. For the general case of a spin-up maneuver of a spherically symmetric or an axially symmetric spacecraft subjected to constant torques, several approximated analytic solutions were introduced. Longuski [1] proposes a formulation for angular velocity and Euler angles for both symmetric and near symmetric spacecraft, valid for small angular velocity and rotational angles. Another approximated formulation for the angular velocity and Euler angles of a spinning spacecraft, valid for small angles only, is given by Wie [3]. Ayoubi and Longuski [4,5] introduce approximated solutions for the inertial transverse velocity, inertial displacement and axial velocity of a spinning-up axially symmetric spacecraft, under the assumptions of small Euler angles and linear behavior of the spin rate. Longuski [6] considers the case of a spinning spacecraft with constant spinning rate and subjected to transverse body-fixed torques. The author proposes an approximated solution for the attitude, rotational and translational motions of the spacecraft while assuming small angular excursions of the spin axis with respect to an inertially fixed direction. Other authors analyze the motion of a spinning spacecraft in particular situations. Ayoubi and Longuski [7,8] study the asymptotic behavior of the motion of spinning spacecraft subjected to constant torques in all the three axis of the body frame, introducing
new methods for damping the spinning axis of the spacecraft. The problem of controlling the spinning axis of a spacecraft was also studied by other authors \[14,16,15\].

The present paper applies for the first time to the best knowledge of the authors an exact analytic solution for both the kinematics and the dynamics of the spinning-up maneuver of an axially symmetric spacecraft. The exact solution for the evolution in time of the direction cosine matrix for an axially symmetric rigid body subjected to an external torque constant in magnitude and parallel to the symmetry axis which was introduced by Romano \[17,18\] is applied. In particular, this exact solution is generalized in the present paper in order to be applicable to the important case of zero initial angular velocity component along the symmetry axis.

Furthermore, the present paper introduces a new approximated solution in terms of direction cosine matrix obtained from the exact solution and a new approximated solution in terms of Euler angles as an extension of the existing solution due to Wie \[3\].

The newly proposed exact and approximated solutions are compared by numerical experiments with the numerical integration of the equations of motion and with the existing approximated solution by Wie \[3\]. Several comparison metrics are used including computational time, precision, rotational amplitude and initial angular velocity.

The paper is organized as follows: Section 2 introduces the equations of motion for the spin-up maneuver of a spacecraft together with the existing approximated solution due to Wie \[3\] and the existing exact analytic solution due to Romano \[18\]. Section 3 introduces the generalization of Romano’s solution and its application to the spin-up problem. The same section introduces also the new approximated solution in terms of the direction cosine matrix and the new approximated solution in terms of Euler angles. Section 4 reports the numerical experiments. Finally, Section 5 concludes the paper.

2. Equations of motion for the spin-up maneuver and existing solutions

This section introduces the spin-up problem for a spacecraft consisting of an axially symmetric rigid body and subjected to a constant external torque along the symmetry axis. This system constitutes a model for an axially symmetric spacecraft actuated by jet thrusters.

The section introduces first the equations of motion, then the existing exact analytic solution for the dynamics problem and the existing solutions for the kinematics problem due to Wie \[3\] and Romano \[18\].

The dynamics of the rotational motion of a rigid body is governed by Euler’s equation \[3\]

\[
\dot{\mathbf{I}} = \mathbf{M}
\]

which in vectorial form can be expressed as

\[
\dot{\mathbf{I}} = \mathbf{M}
\]

where

\[
\mathbf{M} = M_3 \mathbf{e}_z
\]

having the third axis parallel to the axis of symmetry;

2. the external torque is constant in magnitude and directed along the axis of symmetry, i.e.

\[
\mathbf{M} = M_3 \mathbf{e}_z
\]

Therefore, it results in

\[
I_1 = I_2 = I \neq I_3
\]

and Euler’s equation can be written in scalar form along the reference frame \(B\) as

\[
I_1 \dot{p} - (I_1 - I_3)qr = 0
\]

\[
I_2 \dot{q} + (I_2 - I_3)pr = 0
\]

\[
I_3 \dot{r} = M_3
\]

The kinematics of the rotational motion of a rigid body in terms of direction cosine matrix is governed by Darboux’ equation \[3\]

\[
\dot{\mathbf{R}}_{NB} = \mathbf{R}_{NB} \Omega(\omega)
\]

where

\[
\Omega(\omega) = \begin{bmatrix} 0 & -q & p \\ q & 0 & -r \\ -p & r & 0 \end{bmatrix}
\]

The kinematics of the rotational motion of a rigid body in terms of Euler angles (sequence 1–2–3) is governed by the following differential equations \[3\]:

\[
\dot{\theta}_1 = \frac{p \cos \theta_3 - q \sin \theta_3}{\cos \theta_2}
\]

\[
\dot{\theta}_2 = p \sin \theta_3 + q \cos \theta_3
\]
\[ \dot{\theta}_3 = (-p \cos \theta_3 + q \sin \theta_3) \tan \theta_2 + r. \]  
\[ (11) \]

2.1. Dynamic problem: exact solution

The exact analytic solution for the dynamic problem is existing and well known and it is given by [18]

\[ p(t) = p_0 \cos [f(t)] + q_0 \sin [f(t)] \]  
\[ (12) \]
\[ q(t) = -p_0 \sin [f(t)] + q_0 \cos [f(t)] \]  
\[ (13) \]
\[ r(t) = r_0 + \frac{I}{I_3}Ut \]  
\[ (14) \]

with

\[ U = \frac{M_3}{I} \]  
\[ (15) \]

and

\[ f(t) = A \left( r_0 t + \frac{U}{2}t^2 \right). \]  
\[ (16) \]

In Eq. (16), the constant A is defined as

\[ A = \frac{I - I_3}{I_3} \]  
\[ (17) \]

and

\[ r_0 = \frac{r_0 I_3}{I}. \]  
\[ (18) \]

2.2. Kinematic problem: existing approximated solution in terms of Euler Angles

The approximated solution for the kinematic problem in terms of Euler angles as proposed by Wie [3] is outlined in this section. The following simplifying assumptions are made to obtain the approximated solution:

1. The initial angular velocity along the symmetry axis is zero, i.e. \( r_0 = 0 \).
2. The initial value of the Euler angles is zero (the inertial frame is taken as coincident with the initial body frame).
3. The Euler angles \( \theta_1 \) and \( \theta_2 \) as well as their time derivatives are small at any time during the motion.

With these assumptions Eqs. (12)-(14) become

\[ p(t) = p_0 \cos \left[ \frac{U}{2}t^2 \right] + q_0 \sin \left[ \frac{U}{2}t^2 \right] \]  
\[ (19) \]
\[ q(t) = -p_0 \sin \left[ \frac{U}{2}t^2 \right] + q_0 \cos \left[ \frac{U}{2}t^2 \right] \]  
\[ (20) \]
\[ r(t) = \frac{I}{I_3}Ut, \]  
\[ (21) \]

and (9)-(11) become

\[ \dot{\theta}_1 = p \cos \theta_3 - q \sin \theta_3 \]  
\[ (22) \]
\[ \dot{\theta}_2 = p \sin \theta_3 + q \cos \theta_3 \]  
\[ (23) \]
\[ \dot{\theta}_3 = r. \]  
\[ (24) \]

The solution of Eq. (24), taking into account Eq. (21), is

\[ \dot{\theta}_3(t) = \frac{I}{2I_3} Ut^2. \]  
\[ (25) \]

By substituting Eqs. (19) and (20) into Eqs. (22) and (23), after some simplifications, it results in

\[ \dot{\theta}_1 = p_0 \cos \left[ \frac{U^2}{2} t \right] - q_0 \sin \left[ \frac{U^2}{2} t \right] \]  
\[ (26) \]
\[ \dot{\theta}_2 = q_0 \cos \left[ \frac{U^2}{2} t \right] + p_0 \sin \left[ \frac{U^2}{2} t \right]. \]  
\[ (27) \]

The solution of this differential equations is

\[ \theta_1(t) = p_0 \sqrt{\frac{1}{2U}} \int_0^t \cos \theta \frac{d \theta}{\sqrt{\theta}} - q_0 \sqrt{\frac{1}{2U}} \int_0^t \sin \theta \frac{d \theta}{\sqrt{\theta}} \]  
\[ (28) \]
\[ \theta_2(t) = q_0 \sqrt{\frac{1}{2U}} \int_0^t \sin \theta \frac{d \theta}{\sqrt{\theta}} + p_0 \sqrt{\frac{1}{2U}} \int_0^t \cos \theta \frac{d \theta}{\sqrt{\theta}} \]  
\[ (29) \]
where

\[ \theta^* = \frac{U^2}{2} t. \]  
\[ (30) \]

The integrals in Eqs. (28) and (29) are obtained from the Fresnel integrals [19]

\[ S_f(s) = \int_0^s \sin \left( \frac{U^2}{2} t \right) \frac{d t}{t} \]  
\[ (31) \]
\[ C_f(s) = \int_0^s \cos \left( \frac{U^2}{2} t \right) \frac{d t}{t} \]  
\[ (32) \]
with the substitution of variable \( \theta = \frac{U^2}{2} t \).

Furthermore, the time limit of (28) and (29) provides the steady-state values of the precession and nutation angles [3]:

\[ \theta_1(\infty) = p_0 \sqrt{\frac{1}{2U}} \frac{\pi}{2} - q_0 \sqrt{\frac{1}{2U}} \frac{\pi}{2} \]  
\[ (33) \]
\[ \theta_2(\infty) = q_0 \sqrt{\frac{1}{2U}} \frac{\pi}{2} + p_0 \sqrt{\frac{1}{2U}} \frac{\pi}{2}. \]  
\[ (34) \]

2.3. Kinematic problem: existing exact analytic solution in terms of direction cosine matrix

The exact analytic solution in terms of direction cosine matrix for the kinematic problem at hand is proposed by Romano [18]. This solution is outlined below.

The absolute angular momentum of an axially symmetric rigid body can be written as (Reduction Theorem in Ref. [20])

\[ h = I \omega + (I_3 - I)(\omega \cdot \varepsilon) \varepsilon. \]  
\[ (35) \]

The angular velocity of the spacecraft, at any instant of time, can be expressed by

\[ \omega = \omega_h + \omega_e. \]  
\[ (36) \]
whereas obtained from Eq. (35):
\[
\omega_h = \frac{h}{I}
\] (37)
and
\[
\omega_e = A(\omega_h \cdot e)e.
\] (38)
Finally, from Eqs. (37) and (1) it results in
\[
\dot{h} = I\dot{\omega}_h = M.
\] (39)

Therefore, the evolution in time of the angular momentum of an axially symmetric spacecraft subjected to the torque \(M\) is equivalent to the one of a “virtual” spherical body whose inertia is the transversal inertia of the considered body subjected to the same torque \(M\). Accordingly, the angular velocity \(\omega\) of the axially symmetric body can be calculated from the angular velocity \(\omega_h\) of the “virtual” spherical body (Eq. (36) and Fig. 2). Moreover, the orientation of the body frame \(B\) with respect to the inertial frame \(N\) can be obtained as the composition of the orientation of the axially symmetric body with respect to the “virtual” spherical body and the orientation of the spherical body with respect to the inertial frame. Therefore, it results in
\[
R_{BN}(t) = R_{BS}(t)R_{SN}(t).
\] (40)

**Proof of Eq. (40).** Eq. (36) can be written as
\[
\omega = \omega_e + R_{BS}\omega_h.
\] (41)
By applying the skew-symmetric matrix operator to both sides of Eq. (41) it yields
\[
\dot{\Omega}(\omega) = \dot{\Omega}(\omega_e) + \dot{\Omega}(R_{BS}\omega_h).
\] (42)
By considering the property [18]
\[
\dot{\Omega}(R_{BS}\omega_h) = R_{BS}\dot{\Omega}(\omega_h)R_{BS}
\] (43)

\[\text{Fig. 2. Conceptual sketch of the spin-up maneuver for an axially symmetric spacecraft represented by its inertial ellipsoid. The gray area represents the “virtual” sphere. The applied torque } M \text{ has direction of the vector } e \text{ and magnitude } U.\]

together with Eq. (7), Eq. (42) becomes
\[
\dot{R}_{BN} = \dot{R}_{BS}R_{SN} + R_{BS}\dot{R}_{BS}.
\] (44)
which represents the time derivative of Eq. (40).

Without losing generality it is assumed that the frames \(B, S\) and \(N\) have superimposed axes at the initial time \(t=0\) s. Therefore, \(R_{BN}(0), R_{BS}(0)\) and \(R_{SN}(0)\) are identity matrices. Notably, since \(\omega_e\) is parallel to \(e\) by construction, \(R_{BS}\) expresses an elementary rotation about the axis \(e\) at any time \(t\).

**Orientation of \(B\) with respect to \(S\):** By assuming the torque as in Eq. (2) and by taking into account Eqs. (41) and (15), the solution of Eq. (39) is
\[
q_k(t) = p_0
\] (45)
\[
q_k(t) = q_0
\] (46)
\[
r_k(t) = r_k^0 + Ut,
\] (47)
and it yields
\[
p_e(t) = 0
\] (48)
\[
q_e(t) = 0
\] (49)
\[
r_e(t) = A(r_k^0 + Ut),
\] (50)
Therefore, from Eqs. (48)–(50) and (7) it finally yields
\[
R_{BS}(t) = \begin{bmatrix}
\cos[f(t)] & \sin[f(t)] & 0 \\
-\sin[f(t)] & \cos[f(t)] & 0 \\
0 & 0 & 1
\end{bmatrix},
\] (51)
where \(f(t)\) is defined in Eq. (16).

**Orientation of \(S\) with respect to \(N\):** The matrix \(R_{SN}(t)\) in Eq. (40) can be expressed as follows, as stated in Corollary 2 of Ref. [18] (with \(K = S\) and \(R_{BK}\) equal to the identity matrix because the torque is parallel to the third axis of \(S\): \(R_{NS} = [r_{kj}], \quad k, j = 1, 2, 3\) (52)

\[
r_{k1} = \frac{i(w_k - \bar{w}_k)}{1 + |w_k|^2}, \quad k = 1, 2, 3
\] (53)
\[
r_{k2} = \frac{w_k + \bar{w}_k}{1 + |w_k|^2}, \quad k = 1, 2, 3
\] (54)
and
\[
r_{k3} = \frac{1 - |\bar{w}_k|^2}{1 + |w_k|^2}, \quad k = 1, 2, 3.
\] (55)
The kinematic differential equation
\[
\dot{R}_{NS} = R_{NS}\dot{\Omega}(\omega_h)
\] (56)
in terms of stereographic complex rotation variables is in Riccati form [18]:
\[
w_k = \frac{p_k - iq_k}{2}w_k^2 - ir_k w_k + \frac{p_k + iq_k}{2}, \quad k = 1, 2, 3.
\] (57)
In case of \(p_0\) and \(q_0\) not both zero and \(r_k \neq 0\), Eq. (57) admits the following exact analytic solution [18]:
\[
w_k(t, c_k) = \left(1 + \frac{h}{2}\sqrt{\frac{1}{U}}\right) [\exp(2z) + G(z, c_k)]
\] (58)
\[
G(z, c_k) = \frac{2_1 F_1 \left( \frac{3 - \nu}{2}, \frac{5}{2}; z^2 \right) (\nu - 1) z^2 + 6_1 F_1 \left( 1 - \frac{1 - \nu}{2}; \frac{3}{2}; z^2 \right) c_k v z - 3_1 F_1 \left( \frac{1 - \nu}{2}, \frac{3}{2}; z^2 \right)}{1_1 F_1 \left( - \frac{\nu}{2}, \frac{1}{2}; z^2 \right) c_k + 1_1 F_1 \left( 1 - \frac{\nu}{2}, \frac{3}{2}; z^2 \right) z},
\]

(59)

\[
z = \frac{(1 + i)(r_0 + Ut)}{2\sqrt{U}},
\]

(60)

and

\[
\nu = - \frac{- i(p_0^2 + q_0^2)}{4U}.
\]

(61)

The confluent hypergeometric function \( _1 F_1 \) in Eq. (59) is defined as

\[
_1 F_1 (\alpha, \beta; \gamma) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{\beta_k} \frac{\gamma^k}{k!} |\gamma| < \infty, \quad \beta \neq 0, -1, -2, \ldots.
\]

(62)

The Pochhammer symbol is defined as

\[
(\star)_k = 1, \quad (\bullet)_k = (k+1)(k+2) \ldots (k+k+1), \quad k = 1, 2, \ldots
\]

(63)

Finally, the constants \( c_k \) are defined as [18]

\[
c_1 = \frac{(1+i)}{6\sqrt{U}} \left[ \frac{1_1 F_1 \left( \frac{1 - \nu}{2}, \frac{3}{2}; \frac{2}{2}; \frac{2}{2U} \right)}{1_1 F_1 \left( 1 - \frac{\nu}{2}, \frac{3}{2}; \frac{3}{2}; \frac{2}{2U} \right)} \left( \frac{1}{2} \right)^{1+i} \right]
\]

(64)

\[
c_2 = \frac{(1-i)}{6\sqrt{U}} \left[ \frac{1_1 F_1 \left( \frac{1 - \nu}{2}, \frac{3}{2}; \frac{2}{2}; \frac{2}{2U} \right) \left( \frac{1}{2} \right)^{1+i} \left( \frac{1}{2} \right)^{1+i} \right]
\]

(65)

\[
c_3 = \frac{(1+i)}{6r_0^2 \sqrt{U}} \left[ \frac{1_1 F_1 \left( \frac{1 - \nu}{2}, \frac{3}{2}; \frac{2}{2}; \frac{2}{2U} \right)}{1_1 F_1 \left( 1 - \frac{\nu}{2}, \frac{3}{2}; \frac{3}{2}; \frac{2}{2U} \right)} \left( \frac{1}{2} \right)^{1+i} \right]
\]

(66)

with

\[
g = 1_1 F_1 \left( \frac{3 - \nu}{2}, \frac{5}{2}; \frac{2}{2U} \right) (\nu - 1)r_0^2.
\]

(67)

In the particular case of \( p_0 \) and \( q_0 \) both zero, Eq. (57) admits the analytic solutions

\[
w_1(t) = - \sin \left( \frac{r_0 + Ut}{\sqrt{2}} \right) t - i \cos \left( \frac{r_0 + Ut}{\sqrt{2}} \right) t
\]

(68)

\[
w_2(t) = \cos \left( \frac{r_0 + Ut}{\sqrt{2}} \right) t - i \sin \left( \frac{r_0 + Ut}{\sqrt{2}} \right) t
\]

(69)

\[
w_3(t) = 0.
\]

(70)

### 3. New analytic solutions for the spin-up problem

This section introduces new analytic solutions for the considered problem. This section first describes the generalization of Romano’s exact analytic solution valid for any initial condition and its application to the spin-up problem. Then, the section introduces a new approximated solution in terms of direction cosine matrix and a new approximated solution in terms of Euler angles, as an extension of Wie’s solution.

#### 3.1. New generalized exact analytic solution in terms of direction cosine matrix

In practical application, an axially symmetric spacecraft is usually subjected to a nutation motion, while the component of the angular velocity along the axis of symmetry is close to zero. In case of \( r_0 \neq 0 \) rad/s the exact analytic solution of the spin-up motion is the one reported in Section 2.3. However, that solution is not applicable as it is when \( r_0 = 0 \) rad/s because the constant \( c_3 \) of Eq. (66) cannot be evaluated as the term \( r_0 \) is at denominator. Therefore, the third line \( b_3 \) of the matrix \( R_{NS}(t) \) cannot be evaluated.

This limitation of the method is given below for the first time removed by using the properties of direction cosine matrices.

The matrix \( R_{NS}(t) \) can be written, in general form, as

\[
R_{NS}(t) = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}.
\]

(71)

Since \( R_{NS}(t) \) is orthogonal (\( R_{NS} = R_{NS}^T \)) and \( \det[R_{NS}] = 1 \), it is possible to define the system

\[
\begin{align*}
r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} &= 0 \\
r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} &= 0 \\
r_{11}r_{21}r_{32} + r_{12}r_{22}r_{33} + r_{13}r_{23}r_{31} - r_{11}r_{23}r_{32} - r_{12}r_{21}r_{33} &= 1
\end{align*}
\]

where \( r_{31}, r_{32} \) and \( r_{33} \) are the unknown terms. The other terms \( r_{11}, r_{12}, r_{13}, r_{21}, r_{22} \) and \( r_{23} \) are defined in Eqs. (53) and (55). The solution of the system is given by

\[
r_{31} = \frac{r_{11}r_{23} + r_{13}r_{21}}{ar_{23}^2 - 2r_{11}r_{13}r_{21}r_{23} - 2dr_{12}r_{22} + br_{12}^2 + cr_{11}^2}
\]

(72)

\[
r_{32} = \frac{r_{13}r_{21} - r_{11}r_{23}}{ar_{23}^2 - 2r_{11}r_{13}r_{21}r_{23} - 2dr_{12}r_{22} + br_{12}^2 + cr_{11}^2}
\]

(73)

\[
r_{33} = \frac{-r_{12}r_{21} + r_{11}r_{23}}{ar_{23}^2 - 2r_{11}r_{13}r_{21}r_{23} - 2dr_{12}r_{22} + br_{12}^2 + cr_{11}^2}
\]

(74)

where

\[
a = r_{21}^2 + r_{22}^2,
\]

(75)

\[
b = r_{21}^2 + r_{23}^2,
\]

(76)

\[
c = r_{22}^2 + r_{23}^2.
\]

(77)
the coefficients $c_1$ and $c_2$ can be written as

$$c_1 = \frac{(1 - i)(-I_3 i(p_0 + q_0) r_0 + 2IU)}{2I(p_0 - iq_0)\sqrt{U}}$$

$$c_2 = \frac{(1 + i)(I_3 (p_0 - iq_0) r_0 + 2IU)}{2I(p_0 - iq_0)\sqrt{U}}$$

with the additional assumption that both $p_0$ and $q_0$ are not zero. The hypergeometric functions in Eq. (59) are first expanded in Taylor series around the point $r_0 = 0$ and the terms of order higher than two are neglected. Then the resulting functions are further expanded in Taylor series around the point $(p_0^2 + q_0^2) = 0$ and the terms of order higher than one are neglected. Thus, it yields

$$iF_1 \left( \frac{3 - \nu}{2}, z^2 \right) \approx -\frac{4k_4 e^{\nu z^2/2} + iI(p_0^2 + q_0^2)F_1 \left( \frac{3}{2}, 3 \frac{1}{2}, \frac{1}{2} \right)}{8\nu^2 U^4}$$

with

$$k_3 = 2I^2 U + 3r_0(-2 + i t^2 U)$$

$$k_4 = -2I^2 U + 3r_0(-2 - i t^2 U)$$

$$k_5 = -70\nu t^2 U(12I^2 r_0^2 - 3I_3 r_0(2 + I_3 r_0)U - t^2(2 + I_3 r_0(2 + I_3 r_0)U))$$

$$k_6 = i2t^2 U^2 (i + t^2 U) + iI_3 r_0 U(-3 + 2t^2 U)$$

$$k_7 = 35I^2 (p_0^2 + q_0^2) t^2 U^2 F_1 \left( \frac{2}{2}, 5 \frac{1}{2}, \frac{1}{2} \right)$$

$$k_8 = -2I^2 U + 3r_0(-2I t U + I_3 r_0(-i + t^2 U))$$

and

$$\text{Erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} \, dt.$$
where

\[ F(t) = -1 F_1 \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) t^2 U + 1 F_1 \left( \frac{3}{2} \frac{3}{2} \frac{3}{2} \right) t^2 U \].

(95)

3.3. New approximated solution in terms of Euler angles

A new approximated solution in terms of Euler angles for the kinematic problem is here introduced as an extension of Wie’s solution. Unlike the original solution by Wie reported in Section 2.2 of the present paper, this new approximated solution is valid also when \( \theta_1(0) \neq 0, \ \theta_2(0) \neq 0, \ \theta_3(0) \neq 0 \)

and

\( r_0 \neq 0, \)

(97)

while still it is assumed that the Euler angles \( \theta_1 \) and \( \theta_2 \) as well as their time derivatives are small at any time during the motion.

With these assumptions, the solution of Eq. (24), taking into account Eq. (14), is

\[ \theta_2(t) = \frac{1}{2} t^2 + r_0 t + \theta_3(0). \]

(98)

By substituting Eqs. (12), (13) and (98) into Eqs. (22) and (23), after some simplification, it results in

\[ \dot{\theta}_1 = p_0^* \cos \left( \frac{r_0^* + Ut}{2U} \right) + q_0^* \sin \left( \frac{r_0^* + Ut}{2U} \right) \]

(99)

\[ \dot{\theta}_2 = p_0^* \cos \left( \frac{r_0^* + Ut}{2U} \right) - q_0^* \sin \left( \frac{r_0^* + Ut}{2U} \right) \]

(100)

where

\[ p_0^* = p_0 \cos \left( \frac{r_0^*}{2U} - \frac{\theta_1(0)}{2U} \right) + q_0 \sin \left( \frac{r_0^*}{2U} - \frac{\theta_1(0)}{2U} \right) \]

(101)

\[ q_0^* = p_0 \sin \left( \frac{r_0^*}{2U} - \frac{\theta_1(0)}{2U} \right) - q_0 \cos \left( \frac{r_0^*}{2U} - \frac{\theta_1(0)}{2U} \right). \]

(102)

The solution of this differential equations, taking into account the initial conditions, is

\[ \theta_1(t) = p_0^* \left[ \frac{1}{2U} \int_0^t \cos \frac{\theta}{\sqrt{\theta}} \sqrt{\theta} \right] - q_0^* \left[ \frac{1}{2U} \int_0^t \sin \frac{\theta}{\sqrt{\theta}} \sqrt{\theta} \right] + \theta_1(0) \]

(103)

\[ \theta_2(t) = \frac{1}{2U} \left[ \int_0^t \frac{\sin \theta}{\sqrt{\theta}} \sqrt{\theta} \right] - q_0 \left[ \frac{1}{2U} \int_0^t \cos \frac{\theta}{\sqrt{\theta}} \sqrt{\theta} \right] + \theta_2(0) \]

(104)

where

\[ \theta^* = \left( \frac{r_0^* + Ut}{2U} \right) \]

(105)

and

\[ \theta = \frac{r_0^*}{2U} \]

(106)

The integrals in Eqs. (104) and (105) are obtained after substitution of variable defined in Eqs. (105) and (106). The time limit of Eqs. (104) and (105) provides the steady-state precession and nutation angles:

\[ \theta_1(\infty) = p_0^* \left[ \frac{1}{2U} \int_0^t \cos \frac{\theta}{\sqrt{\theta}} \sqrt{\theta} \right] - q_0^* \left[ \frac{1}{2U} \int_0^t \sin \frac{\theta}{\sqrt{\theta}} \sqrt{\theta} \right] + \theta_1(0) \]

(107)

\[ \theta_2(\infty) = \frac{1}{2U} \left[ \int_0^t \frac{\sin \theta}{\sqrt{\theta}} \sqrt{\theta} \right] - q_0 \left[ \frac{1}{2U} \int_0^t \cos \frac{\theta}{\sqrt{\theta}} \sqrt{\theta} \right] + \theta_2(0). \]

(108)

4. Numerical experiments

The numerical results of comparisons among the methods introduced above for the analysis of a spin-up maneuver of a spacecraft are presented in this section.

The spin-up maneuvers of two sample spacecraft have been considered for the numerical experiments. While the spin-up maneuver of the sample spacecraft #1 is driven by a torque about the axis of minimum inertia, the spin-up maneuver of the sample spacecraft #2 is driven by a torque about the axis of maximum inertia. Table 1 lists the numerical values for the sample spacecraft #1. The sample spacecraft #2 and the related numerical experiments are introduced in Section 4.2. The values of \( p_0 \) are set to different levels from \( 10^{-4} \) rad/s to 1 rad/s in order to analyze the problem in the case of small angles and
angular velocity components as well as in the case of large angles and angular velocity components.

The same maneuver has been simulated with the generalized exact analytic solution, the existing approximated solution due to Wie, the new approximated solution from the exact one and the numerical integration of the equations of motion. All of the numerical evaluations of analytic expressions have been conducted by using a precision of $10^{-15}$. The numerical evaluation of the analytic solutions for both the attitude and the angular velocity has been repeated at every 0.1 s. Finally, the kinematic differential equations (Eqs. (9)–(11)) have been numerically propagated with an explicit 8th order Runge–Kutta method with different accuracies ($10^{-15}$, $10^{-10}$ and $10^{-5}$) as specified for different presented cases.

Table 1
Sample spacecraft #1: inertia and torque values ($I_1$, $I_3$ and $M_3$ correspond to the ones in Ref. [3, p. 385]) and initial conditions used for the related numerical experiments.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit of measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$</td>
<td>4223</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>768</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>10</td>
<td>N m</td>
</tr>
<tr>
<td>$\theta_1(0)$</td>
<td>0</td>
<td>rad</td>
</tr>
<tr>
<td>$\theta_2(0)$</td>
<td>0</td>
<td>rad</td>
</tr>
<tr>
<td>$\theta_3(0)$</td>
<td>0</td>
<td>rad</td>
</tr>
<tr>
<td>$p_0$</td>
<td>Various cases (from $10^{-4}$ to 1)</td>
<td>rad/s</td>
</tr>
<tr>
<td>$q_0$</td>
<td>0</td>
<td>rad/s</td>
</tr>
<tr>
<td>$r_0$</td>
<td>0</td>
<td>rad/s</td>
</tr>
<tr>
<td>Duration</td>
<td>Various (from 300 to 2000)</td>
<td>s</td>
</tr>
</tbody>
</table>

Fig. 3. Sample spacecraft #1: time history of the Euler angles in the case of $p_0 = 10^{-2}$ rad/s.

Fig. 4. Sample spacecraft #1: time history of the Euler angles in the case of $p_0 = 10^{-1}$ rad/s.
Since the exact analytic solution provides the orientation of the rigid body in terms of direction cosine matrix the Euler angles (of the sequence 1–2–3) have been computed with the following equations:

\[ \theta_1 = \arctan \left( \frac{R_{BN}[3,2]}{R_{BN}[3,3]} \right) \]  

\[ \theta_2 = \arcsin(R_{BN}[3,1]) \]  

\[ \theta_3 = \arctan \left( \frac{R_{BN}[2,1]}{R_{BN}[1,1]} \right) \]

in order to compare the results with the other methods.

### 4.1. Numerical experiments with the generalized exact analytic solution

In this subsection the comparative results obtained with the generalized exact analytic solution, the existing approximated solution and the numerical integration of the kinematic equations are discussed. First, the spin-up maneuver of sample spacecraft #1 (see Table 1) is discussed.

The time histories of the Euler angles of the considered maneuvers have been simulated with the three solutions and then compared. In the cases of the exact or approximated analytic solutions the simulation entailed a numerical evaluation of the angles repeated every 0.1 s. For this analysis the differential kinematic equations (Eqs. (9)–(11)) were integrated with high accuracy \((10^{-15})\).

**Fig. 3** shows the time history of the Euler angles in the case of \(p_0 = 10^{-1} \text{ rad/s}\). At any instant of time, the results from the generalized exact analytic solution and the approximated solution are very close to the results obtained from the numerical integration and the corresponding curves are basically superimposed.

**Fig. 4** shows the time history of the Euler angles in the case of \(p_0 = 10^{-1} \text{ rad/s}\). Notably in this case while the angles computed with the exact analytic solution still match the angles obtained with the numerical integration, the approximated solution fails to be accurate after a time of about 35 s. The reason for this behavior is that the values of the angles \(\theta_1\) and \(\theta_2\) are not small anymore and therefore the assumptions on which the approximated solution is based are no more valid.

**Fig. 5** illustrates the time history of the quaternions, computed from the Euler angles. Clearly, as in the case of the Euler angles, only the history obtained with the exact analytic solution closely matches the numerical integration.

**Figs. 6 and 7** report the time history of the Euler angles and quaternions respectively in the case of \(p_0 = 1 \text{ rad/s}\). Similar conclusions as in the case of **Figs. 4 and 5** hold.

An interesting feature of the exact analytic solution, as it can be observed from **Fig. 3, 4 and 6**, is that for values of initial angular velocity up to \(10^{-1} \text{ rad/s}\), the angles become periodic after a certain time. However, if \(p_0\) is higher than \(10^{-1} \text{ rad/s}\), the angles are divergent and no periodicities occur at least for the considered duration of the maneuver (300 s). For the case of **Fig. 3** where the approximated solution gives an accurate solution for the
angles define now a “pseudo-clothoid” curve with a “rise-up transition time” of approximately 150 s (see also Fig. 4).

Fig. 10 shows the curve, obtained with the exact analytic solution, in the case of \( p_0 = 1 \text{ rad/s} \). The angles do not describe a clothoid, since they diverge (Fig. 6).

In order to further evaluate the behavior of the solution in case of \( p_0 = 1 \text{ rad/s} \), a simulation has been done for the same maneuver analyzed above but with the duration of 21,000 s. The results obtained confirm that \( \theta_1 \) and \( \theta_3 \) diverge. Notably however the angle \( \theta_2 \) reaches an asymptote of value 1.553 rad evaluated with the exact analytic solution.

To further compare the results of the exact analytic solution, the numerical integration and the approximated solution, the two cumulative error parameters are defined as follows:

\[
E^{\text{NI}}_j = \sum_{t=0}^{T_f} \sqrt{(E\theta_j(t) - \text{NI}\theta_j(t))^2}, \quad j = 1, 2, 3
\]

\[
E^{\text{SB}}_j = \sum_{t=0}^{T_f} \sqrt{(E\theta_j(t) - \text{SB}\theta_j(t))^2}, \quad j = 1, 2, 3,
\]

where the final time is set to 500 s for these comparisons. The angles are evaluated with a time step of 0.1 s. In this analysis, the kinematic equations have been integrated with different levels of accuracies \( 10^{-15}, 10^{-10} \) and \( 10^{-5} \), while the numerical evaluation of the exact and approximated solutions has been set to \( 10^{-15} \) for all of the cases.

Tables 2–4 list the cumulative errors, for several values of \( p_0 \). Notably the accuracy of the numerical integration affects the cumulative errors \( E^{\text{NI}}_j \). In particular, the numerical integration closely matches the exact analytic solution only when conducted with large accuracy, for every value of \( p_0 \). Furthermore, as it can be observed from the cumulative errors \( E^{\text{SB}}_j(\theta_1) \) and expected, the approximated solution produces results close to the exact analytic solution only for small initial angular velocity.

4.2. Further numerical experiments with the generalized exact analytic solution

In order to study a maneuver for a spacecraft with maximum moment of inertia about the symmetry axis, sample spacecraft #2 (see Table 5) has been considered and all of the numerical experiments of Section 4.1 have been repeated for this case. The values of \( q_0 \) are set to different levels from \( 10^{-4} \text{ rad/s} \) to 1 rad/s in order to analyze the problem in the case of small angles and angular velocity components as well as in the case of large angles and angular velocity components.

Figs. 11–13 show the plots of \( \theta_2 \) versus \( \theta_1 \) evaluated with the exact analytic solution in the case of several values of \( q_0 \). In the case of \( q_0 = 10^{-2} \text{ rad/s} \), the angles define the clothoid, whose parametric expression is given by Eq. (112). In the case of larger \( q_0 \), the angles describe a "pseudo-clothoid" curve since the angles do not diverge.

Notably, in the case of sample spacecraft #2, \( \theta_1 \) and \( \theta_2 \) always converge contrarily to what happens to sample

![Fig. 6. Sample spacecraft #1: time history of the Euler angles in the case of \( p_0 = 1 \text{ rad/s} \).

By plotting the angle \( \theta_2 \) versus \( \theta_1 \) in the case of small initial angular velocity and attitude angles, an Euler curve or a clothoid is obtained, whose parametric expression is given by the Fresnel integrals as follows (see also Fig. 8) [19]:

\[
\int_0^t \cos \left( \frac{z^2}{2} \right) dz = \int_0^t \sin \left( \frac{z^2}{2} \right) dz.
\]  

The center of the spiral in Fig. 8a resides at asymptotic value of the angles \((\theta_1(\infty), \theta_2(\infty))\). By adding the time as third dimension, the curve in Fig. 8b is obtained. These curves can be obtained by using either the generalized exact analytic solution, the numerical integration or the approximated solution.

Fig. 9 shows the plots of \( \theta_2 \) versus \( \theta_1 \) evaluated with the exact analytic solution in the case of \( p_0 = 10^{-1} \text{ rad/s} \). The
spacecraft #1 (see Figs. 9 and 10). This behavior is as expected due to the fact that the spinning motion for the sample spacecraft #2 is about the axis of maximum moment of inertia, while it is about the axis of minimum inertia for the sample spacecraft #1.

Tables 6–8 report the cumulative errors, for several values of $q_0$. Similar to the case of sample spacecraft #1 the numerical integration closely matches the exact analytic solution only when conducted with large accuracy, for every $q_0$. Moreover, the approximated solution matches the exact analytic solution for small $q_0$ only.

4.3. Evaluation of the required computational time

For the numerical experiments reported in this section, both the evaluation of the generalized exact analytic solution and the numerical integration of the equations of motion (order 8 Runge–Kutta) have been implemented in Mathematica 9 [21] on a computer with an Intel Core i7 3.4 GHz CPU and 8.0 GB of RAM. In particular the spin-up maneuver of the sample spacecraft #1 described in Table 1 ($p_0 = 10^{-2}$ rad/s) was simulated with several levels of precision and accuracy, with a maneuver duration between...
The plots were obtained by using the exact analytic solution.

Fig. 9. Sample spacecraft #1: two-dimension plot (on the left) and three-dimension plot (on the right) of the curve ($\theta_1$, $\theta_2$) (pseudo-clothoid) in the case of $p_0 = 10^{-1}$ rad/s (same case of Fig. 4). The plots were obtained by using the exact analytic solution.

Table 2
Sample spacecraft #1: cumulative errors for the angle $\theta_1$. The cumulative error with respect to the numerical integration is given for several accuracies.

<table>
<thead>
<tr>
<th>$p_0$ (rad/s)</th>
<th>$E^\text{NI}_{\theta_1}$ (rad)</th>
<th>$E^\text{NIT}_{\theta_1}$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>$1.2 \times 10^{-7}$</td>
<td>$1.8 \times 10^{-6}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$1.3 \times 10^{-6}$</td>
<td>$4.8 \times 10^{-6}$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$1.3 \times 10^{-5}$</td>
<td>$2.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>$4.2 \times 10^{-4}$</td>
<td>$4.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$10^0$</td>
<td>$2.7 \times 10^{-4}$</td>
<td>$2.8 \times 10^{-4}$</td>
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</table>

Table 3
Sample spacecraft #1: cumulative errors for the angle $\theta_2$. The cumulative error with respect to the numerical integration is given for several accuracies.

<table>
<thead>
<tr>
<th>$p_0$ (rad/s)</th>
<th>$E^\text{NI}_{\theta_2}$ (rad)</th>
<th>$E^\text{NIT}_{\theta_2}$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>$1.3 \times 10^{-7}$</td>
<td>$1.8 \times 10^{-6}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$1.4 \times 10^{-6}$</td>
<td>$4.7 \times 10^{-6}$</td>
</tr>
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<td>$1.6 \times 10^{-5}$</td>
<td>$2.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>$2.3 \times 10^{-4}$</td>
<td>$2.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>$10^0$</td>
<td>$8.1 \times 10^{-5}$</td>
<td>$8.9 \times 10^{-5}$</td>
</tr>
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</table>

Table 4
Sample spacecraft #1: cumulative errors for the angle $\theta_1$. The cumulative error with respect to the numerical integration is given for several accuracies.

<table>
<thead>
<tr>
<th>$p_0$ (rad/s)</th>
<th>$E^\text{NI}_{\theta_1}$ (rad)</th>
<th>$E^\text{NIT}_{\theta_1}$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>$6.2 \times 10^{-9}$</td>
<td>$1.2 \times 10^{-8}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
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<tr>
<td>$10^{-1}$</td>
<td>$4.1 \times 10^{-6}$</td>
<td>$3.9 \times 10^{-5}$</td>
</tr>
<tr>
<td>$10^0$</td>
<td>$1.0 \times 10^{-4}$</td>
<td>$1.1 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 5
Sample spacecraft #2: inertia and torque values ($I$, $I_3$ and $M_3$ correspond to the ones in Ref. [4]) and initial conditions used for the related numerical experiments.

<table>
<thead>
<tr>
<th>Parameter</th>
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<th>Unit of measure</th>
</tr>
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<tbody>
<tr>
<td>$I$</td>
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</tr>
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</tr>
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<td>rad</td>
</tr>
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<td>$q(0)$</td>
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<td>rad/s</td>
</tr>
<tr>
<td>$q_0$</td>
<td>Various cases (from $10^{-4}$ to 1)</td>
<td>rad/s</td>
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<tr>
<td>$T_0$</td>
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<td>rad/s</td>
</tr>
<tr>
<td>Duration</td>
<td>Various (from 300 to 2000)</td>
<td>s</td>
</tr>
</tbody>
</table>
50 s and 2000 s. The time required for the CPU to compute the time history of the Euler angle was determined with the \textit{Timing} function \cite{21} available in Mathematica.

Fig. 14 represents the required computational time for the two solutions, with different accuracies. The generalized analytic solution needs an average time of 1 ms to compute the angles at a certain instant of time of the motion and the precision does not influence the required CPU time. On the contrary, the computational time needed for the numerical integration depends on the time of the motion, since the solver integrates the equations from the initial time (0 s) up to the selected instant of time. Unlike the exact solution, the required CPU time depends on the precision of the solver. Notably a higher computational time with a peak of 6 ms has been observed when the motion is evaluated in between 100 s and 200 s with the

Fig. 15 reports a detailed view of the data in Fig. 14 for duration of the maneuver between 1 s and 60 s. Notably the numerical evaluation of the exact analytic solution is always faster than the numerical integration.

4.4. Experiments with the new approximated solution in terms of direction cosine matrix

The maneuver of the sample spacecraft #1 listed in Table 1 has been simulated with the new approximated solution in terms of direction cosine matrix and the

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig11.png}
\caption{Sample spacecraft #2: two-dimension plot (on the left) and three-dimension plot (on the right) of the curve $(\theta_1, \theta_2)$ (clothoid) in the case of $q_0 = 10^{-2}$ rad/s. The plots were obtained by using the exact analytic solution.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig12.png}
\caption{Sample spacecraft #2: two-dimension plot (on the left) and three-dimension plot (on the right) of the curve $(\theta_1, \theta_2)$ (clothoid) in the case of $q_0 = 10^{-1}$ rad/s. The plots were obtained by using the exact analytic solution.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig13.png}
\caption{Sample spacecraft #2: two-dimension plot (on the left) and three-dimension plot (on the right) of the curve $(\theta_1, \theta_2)$ (clothoid) in the case of $q_0 = 1$ rad/s. The plots were obtained by using the exact analytic solution.}
\end{figure}
The spin-up maneuver of the sample spacecraft #1 (Table 1) has been simulated with the new approximated solution in terms of Euler angles, the existing approximated solution introduced by Wie and the numerical integration of the kinematic equations (accuracy of $10^{-15}$). The value of $p_0$ was set to $10^{-4}$ rad/s and $r_0$ was changed from $10^{-4}$ rad/s to $10^{-2}$ rad/s in order to maintain $\theta_1$ and $\theta_2$ small.

Results demonstrate that the cumulative error between the proposed approximated solution and the numerical integration is always lower than the cumulative error between the existing approximated solution and the integration, for each angle. In particular, the main improvement concerns the third angle $\theta_3$ since in the proposed solution the term $r_0$ is not neglected and proportional to the time (see Eqs. (98) and (25)).

5. Conclusions

This paper proposes an exact analytic solution for both the kinematics and the dynamics of the spinning-up
maneuver of an axially symmetric spacecraft subjected to a constant torque along the axis of symmetry. This solution is obtained by applying the existing exact analytic solution for the evolution in time of the direction cosine matrix for an axially symmetric rigid body subjected to an external torque constant in magnitude to the considered problem. The solution is generalized in order to be applicable to any initial angular velocity, including the important case of zero angular velocity component along the symmetry axis. Numerical experiments and comparison with the existing approximated solution introduced by Wie and the numerical integration of the equations of motion show that the proposed solution is capable to compute results comparable with high accuracy integration for any initial angular velocity, length of time and rotation amplitude. Moreover, the evaluation of the maneuver with the exact analytic solution is computationally more efficient than the numerical integration of the equations of motion. Some unexpected peaks in computational time with the exact analytic solution were observed due to the evaluation of the hypergeometric functions. However, the small computational time required by the proposed exact analytic solution opens scenarios for real time applications.

A new approximated solution in terms of direction cosine matrix is obtained from the exact one. Numerical experiments demonstrate that the solution is valid for small rotation amplitude only.

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References