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A class of two-point sixth-order multiple-zero finders of modified double-Newton type and their dynamics

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ABSTRACT

Under the assumption of the known multiplicity of zeros of nonlinear equations, a class of two-point sextic-order multiple-zero finders and their dynamics are investigated in this paper by means of extensive analysis of modified double-Newton type of methods. With the introduction of a bivariate weight function dependent on function-to-function and derivative-to-derivative ratios, higher-order convergence is obtained. Additional investigation is carried out for extraneous fixed points of the iterative maps associated with the proposed methods along with a comparison with typically selected cases. Through a variety of test equations, numerical experiments strongly support the theory developed in this paper. In addition, relevant dynamics of the proposed methods is successfully explored for various polynomials with a number of illustrative basins of attraction.

1. Introduction

Root-finding of nonlinear equations of the form $f(x) = 0$ has been one of the most frequently occurring problems in scientific work. In rare cases, it is possible to solve the governing equations exactly. In most cases, however, only approximate solutions may resolve the real problems handling such as weather forecast, accurate positioning of satellite systems in the desired orbit, measurement of earthquake magnitudes and other high-level engineering technologies. Among simple-zero finders, the most widely accepted classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

solves $f(x) = 0$ without difficulty, provided that a good initial guess x_0 is chosen near the zero α. Under the assumption that the multiplicity m is known a priori, it is of considerable interest to design efficient methods for locating repeated zeros of $f(x)$. For the zero α with a given multiplicity of $m \geq 1$, modified Newton's method [36,37] in the following form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

is frequently used by many researchers. It is known that numerical scheme (1.2) is a second-order one-point optimal [25] method on the basis of Kung–Traub's conjecture [25] that any multipoint method [35] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. Other higher-order multiple-zero finders can be found in papers [15,17–20,26,27,40,45].

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Given a known multiplicity of $m > 1$, we propose in this paper a family of new two-point sixth-order multiple-zero finders of modified double-Newton type by adding the second step to (1.2) of the form:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - Q_f(x_n)\cdot\dfrac{f(y_n)}{f'(y_n)}, \end{cases} \tag{1.3}$$

where the desired form of the weight function Q_f using only two-point functional information at x_n and y_n will be extensively studied for maximal order of convergence in Section 2.

This paper is divided into six sections. Investigated in Section 2 is methodology and convergence analysis for newly proposed multiple-zero finders. A main theorem is established to state convergence order of six as well as to derive asymptotic error constants and error equations by use of a family of bivariate weight functions Q_f dependent on two principal roots of function-to-function and derivative-to-derivative ratios. In Section 3, special forms of weight functions are considered based on polynomials and rational functions with labeled case numbers. Section 4 discusses the extraneous fixed points and related dynamics behind the basins of attraction. Tabulated in Section 5 are computational results for a variety of numerical examples. Table 6 compares the magnitudes of $e_n = x_n - \alpha$ among those of typically selected cases of the proposed methods. Dynamical properties of the proposed methods along with their illustrative basins of attraction are displayed with detailed analyses and comments. Overall conclusion as well as possible future work is briefly discussed at the end of the final section.

2. Methodology and convergence analysis

Let a function $f:\mathbb{C}\to\mathbb{C}$ have a repeated zero α with integer multiplicity $m > 1$ and be analytic [1] in a small neighborhood of α. Then, given an initial guess x_0 sufficiently close to α, new iterative methods proposed in (1.3) to find an approximate zero α of multiplicity m will take the specific form of:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - Q_f(u,s)\cdot\dfrac{f(y_n)}{f'(y_n)}, \end{cases} \tag{2.1}$$

where

$$u = \left[\frac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{2.2}$$

$$s = \left[\frac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}, \tag{2.3}$$

and where $Q_f:\mathbb{C}^2\to\mathbb{C}$ is holomorphic [21,39] in a neighborhood of $(0, 0)$. Since u and s are respectively a one-to-m and a one-to-$(m$-1) multiple-valued functions, we consider their principal analytic branches [1]. Hence, it is convenient to treat u as a principal root given by $u = \exp[\frac{1}{m}\mathrm{Log}(\frac{f(y_n)}{f(x_n)})]$, with $\mathrm{Log}(\frac{f(y_n)}{f(x_n)}) = \mathrm{Log}\left|\frac{f(y_n)}{f(x_n)}\right| + i\,\mathrm{Arg}(\frac{f(y_n)}{f(x_n)})$ for $-\pi < \mathrm{Arg}(\frac{f(y_n)}{f(x_n)}) \le \pi$; this convention of $\mathrm{Arg}(z)$ for $z\in\mathbb{C}$ agrees with that of $\mathrm{Log}[z]$ command of Mathematica [44] to be employed later in numerical experiments of Section 5. By means of further inspection of u, we find that $u = \left|\frac{f(y_n)}{f(x_n)}\right|^{\frac{1}{m}}\cdot\exp[\frac{i}{m}\,\mathrm{Arg}(\frac{f(y_n)}{f(x_n)})] = O(e_n)$. Similarly we treat $s = \left|\frac{f'(y_n)}{f'(x_n)}\right|^{\frac{1}{m-1}}\cdot\exp[\frac{i}{m-1}\,\mathrm{Arg}(\frac{f'(y_n)}{f'(x_n)})] = O(e_n)$. In addition, we find that $O(\frac{f(y_n)}{f'(y_n)}) = O(e_n^2)$.

Definition 2.1 (Error equation, asymptotic error constant, order of convergence). Let $\{x_0, x_1, \ldots, x_n, \ldots\}$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p\in\mathbb{R}$ and $b\in\mathbb{R}-\{0\}$ such that the following error equation holds

$$e_{n+1} = b{e_n}^p + O(e_n^{p+1}), \tag{2.4}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [42].

In this paper, we investigate the maximal convergence order of proposed methods (2.1). We here establish a main theorem describing the convergence analysis regarding proposed methods (2.1) and find out how to construct the weight function Q_f for sextic-order convergence. Hence, it suffices to consider the weight function Q_f with $O(Q_f(u,s)) = O(e_n^4)$ due to the fact that $O(\frac{f(y_n)}{f'(y_n)}) = O(e_n^2)$.

Applying the Taylor's series expansion of f about α, we get the following relations:

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}{e_n}^m[1 + \theta_2 e_n + \theta_3 e_n^2 + \theta_4 e_n^3 + \theta_5 e_n^4 + \theta_6 e_n^5 + \theta_7 e_n^6 + O(e_n^7)], \tag{2.5}$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}e_n{}^{m-1}\Bigg[1 + \frac{m+1}{m}\theta_2 e_n + \frac{m+2}{m}\theta_3 e_n^2 + \frac{m+3}{m}\theta_4 e_n^3 + \frac{m+4}{m}\theta_5 e_n^4 + \frac{m+5}{m}\theta_6 e_n^5 + \frac{m+6}{m}\theta_7 e_n^6 + O(e_n^7)\Bigg], \tag{2.6}$$

where $\theta_k = \frac{m!}{(m-1+k)!}\frac{f^{(m-1+k)}(\alpha)}{f^{(m)}(\alpha)}$ for $k \in \mathbb{N} - \{1\}$. For convenience, we denote e_n by e without subscript n whenever required to do so. Dividing (2.5) by (2.6), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{e}{m} - \frac{\theta_2 e^2}{m^2} + \frac{Y_3 e^3}{m^3} + \frac{Y_4 e^4}{m^4} + \frac{Y_5 e^5}{m^5} + \frac{Y_6 e^6}{m^6} + O(e^7), \tag{2.7}$$

where $Y_3 = (1+m)\theta_2^2 - 2m\theta_3, Y_4 = -(1+m)^2\theta_2^3 + m(4+3m)\theta_2\theta_3 - 3m^2\theta_4$, $Y_5 = (1+m)^3\theta_2^4 - 2m(1+m)(3+2m)\theta_2^2\theta_3 + 2m^2(3+2m)\theta_2\theta_4 + 2m^2((2+m)\theta_3^2 - 2m\theta_5)$ and $Y_6 = -(1+m)^4\theta_2^5 + m(1+m)^2(8+5m)\theta_2^3\theta_3 - m^2(1+m)(9+5m)\theta_2^2\theta_4 + m^2\theta_2(-(2+m)(6+5m)\theta_3^2 + m(8+5m)\theta_5) + m^3((12+5m)\theta_3\theta_4 - 5m\theta_6)$.

Thus, from relation (2.7), we obtain

$$y_n = \alpha + \frac{\theta_2 e^2}{m} - \frac{Y_3 e^3}{m^2} - \frac{Y_4 e^4}{m^3} - \frac{Y_5 e^5}{m^4} - \frac{Y_6 e^6}{m^5} + O(e^7). \tag{2.8}$$

$$\begin{aligned} f(y_n) = {} & \frac{f^{(m)}(\alpha)}{m!}\Big(\frac{\theta_2}{m}\Big)^m e^{2m}\{1 - \frac{Y_3}{\theta_2}e + \frac{(m-1)Y_3^2 - 2Y_4\theta_2 + 2\theta_2^4}{2m\theta_2^2}e^2 \\ & - \frac{(m-1)(m-2)Y_3^3 + 6Y_5\theta_2^2 + 6Y_3\theta_2(Y_4 - mY_4 + (m+1)\theta_2^3)}{6m^2\theta_2^3}e^3 \\ & + \frac{1}{24m^3\theta_2^4}[(m-1)(m-2)(m-3)(m-3)Y_3^4 + 24(m-1)Y_3Y_5\theta_2^2 \\ & + 12Y_3^2\theta_2(-(m-1)(m-2)Y_4 + m(m+1)\theta_2^3) \\ & + 12\theta_2^2((m-1)Y_4^2 - 2Y_6\theta_2 - 2(m+1)Y_4\theta_2^3 + 2m\theta_2^4\theta_3)]e^4 + O(e^5)\}. \end{aligned} \tag{2.9}$$

Expanding $f'(y_n)$ about the α leads us to the following relation:

$$\begin{aligned} f'(y_n) = {} & \frac{f^{(m)}(\alpha)}{(m-1)!}\Big(\frac{\theta_2}{m}\Big)^{m-1} e^{2m-2}\{1 + \frac{(1-m)Y_3}{m\theta_2}e + \frac{(m-1)(m-2)Y_3^2 + 2\theta_2((1-m)Y4 + (1+m)\theta_2^3)}{2m^2\theta_2^2}e^2 \\ & - \frac{(m-1)(m-2)(m-3)Y_3^3 - 6\theta_2((m-1)(m-2)Y_3Y_4 - (m-1)Y_5\theta_2 - m(m+1)Y_3\theta_2^3)}{6m^3\theta_2^3}e^3 \\ & + \frac{1}{24m^4\theta_2^4}[(m-1)(m-2)(m-3)(m-4)Y_3^4 + 24(m-1)(m-2)Y_3Y_5\theta_2^2 \\ & + 12(m-1)Y_3^2\theta_2(-(m-2)(m-3)Y_4 + m(m+1)\theta_2^3) \\ & + 12\theta_2^2((m-1)(m-2)Y_4^2 - 2(m-1)Y_6\theta_2 - 2m(m+1)Y_4\theta_2^3 + 2m(m+2)\theta_2^4\theta_3)e^4 + O(e^5)\}. \end{aligned} \tag{2.10}$$

By Taylor's expansion or multinomial expansion, we get expressions u in (2.2) and s in (2.3) as follows:

$$u = \frac{\theta_2}{m}e - \frac{Y_3 + \theta_2^2}{m^2}e^2 + \frac{-2Y_4 + \theta_2(2Y_3 + (m+3)\theta_2^2 - 2m\theta_3)}{2m^3}e^3 + \frac{W_4}{6m^4}e^4 + \frac{W_5}{24m^5}e^5 + O(e^6), \tag{2.11}$$

where $W_4 = (2m^2+3m+7)\theta_2^4 + 3\theta_2^2((m+5)Y_3 - 2m(m+1)\theta_3) + 6(Y_5 - mY_3\theta_3) + \theta_2(-6Y_4 + 6m^2\theta_4)$, and $W_5 = (6m^3 + 11m^2 + 6m + 25)\theta_2^5 + 4\theta_2^3((2m^2+3m+13)Y_3 - 3m(m+1)(2m+1)\theta_3) + 24(-Y_6 + mY_4\theta_3) + 24m^2Y_3\theta_4 + \theta_2^2(-12(m+5)Y_4 + 24m^2(m+1)\theta_4) + \theta_2[12(2Y_3^2 + 2Y_5 - 2m(m+1)Y_3\theta_3 + m^2(m+1)\theta_3^2) - 24m^3\theta_5]$.

$$\begin{aligned} s = {} & \frac{\theta_2}{m}e - \frac{(m-1)Y_3 + (m+1)\theta_2^2}{(m-1)m^2}e^2 + \left(\frac{\theta_2[2(m^2-1)Y_3 + (m+1)(m^2+3m-2)\theta_2^2 - 2m(m^2+m-2)\theta_3]}{2(m-1)^2m^3} - \frac{Y_4}{m^3}\right)e^3 \\ & + \frac{R_4}{6(m-1)^3m^4}e^4 + \frac{R_5}{24(m-1)^4m^5}e^5 + O(e^6), \end{aligned} \tag{2.12}$$

where $R_4 = -6(m-1)^3Y_5 - (m+1)^2(2m^3+m^2+5m-6)\theta_2^4 + 6(m-1)^2m(m+2)Y_3\theta_3 + 3(m-1)(m+1)[(4-5m-m^2)Y_3 + 2m^2(m+2)\theta_3]\theta_2^2 - 6(m-1)^2[m^2(m+3)\theta_4 - (m+1)Y_4]\theta_2$, and $R_5 = -24(m-1)^4Y_6 + (m+1)^2(2m-1)(-24+22m-m^2+4m^3+3m^4)\theta_2^5 + 24(m-1)^3m(m+2)Y_4\theta_3 + \theta_2^3[4(m+1)^2(m-1)(-12+11m+m^2+2m^3)Y_3 - 12m(m-1)(m+2)(-4+3m+3m^3+2m^4)\theta_3] + 24(m-1)^3m^2(m+3)Y_3\theta_4 + \theta_2^2[-12(m-1)^2(m+1)(-4+5m+m^2)Y_4 + 24(m-1)^2m^3(m+1)(m+3)\theta_4] + \theta_2[24(m-1))^3(m+1)(Y_3^2+Y_5) - 24(m-1)^2m^2(m+1)(m+2)Y_3\theta_3 + 12m^3((m-1)(m+2))^2\theta_3^2 - 24(m-1)^3m^3(m+4)\theta_5]$.

With the use of u and s in (2.11) and (2.12), expanding Taylor series of $Q_f(u,s)$ about $(0,0)$ up to fourth-order terms we find:

$$\begin{aligned} Q_f(u,s) &= Q_{00} + Q_{10}u + Q_{20}u^2 + Q_{30}u^3 + Q_{40}u^4 + Q_{01}s + Q_{02}s^2 + Q_{03}s^3 + Q_{04}s^4 \\ &\quad + u(Q_{11}s + Q_{12}s^2 + Q_{13}s^3) + u^2(Q_{21}s + Q_{22}s^2) + Q_{31}u^3 s + O(e^5). \end{aligned} \tag{2.13}$$

Hence by substituting (2.5)–(2.13) into the proposed method (2.1), we obtain the error equation as

$$x_{n+1} - \alpha = y_{n+1} - \alpha - Q_f(u,s) \cdot \frac{f(y_n)}{f'(y_n)} = L_2e^2 + L_3e^3 + L_4e^4 + L_5e^5 + L_6e^6 + O(e^7), \tag{2.14}$$

where $L_2 = \frac{(m-Q_{00})\theta_2}{m^2}$ and the coefficients $L_i (3 \le i \le 6)$ generally depend on m, the parameters $Q_{jk}(j,k = 0,1\ldots)$ and $\theta_i(i = 1,2,\ldots)$. Solving $L_2 = 0$ independently of θ_i for Q_{00}, we get

$$Q_{00} = m. \tag{2.15}$$

Substituting $Q_{00} = m$ into $L_3 = 0$ and simplifying, we obtain $\frac{(Q_{01}+Q_{10})}{m^3}\theta_2^2 = 0$, from which

$$Q_{01} = -Q_{10} \tag{2.16}$$

follows independently of θ_2. Substituting $Q_{00} = m, Q_{01} = -Q_{10}$ into $L_4 = 0$ and simplifying yields:

$$-\frac{m^2 + \delta - 2Q_{10} - m(1+\delta)}{(m-1)m^4}\theta_2^3 = 0, \tag{2.17}$$

from which

$$Q_{10} = \frac{1}{2}(m-1)(m-\delta) \tag{2.18}$$

is found independently of θ_2, with $\delta = Q_{02} + Q_{11} + Q_{20}$.

Substituting $Q_{00} = m, Q_{01} = -Q_{10}, Q_{10} = \frac{1}{2}(m-1)(m-\delta)$ into $L_5 = 0$ and simplifying yields:

$$L_5 = \frac{m(3m^2 + 6m - 5) + 8(Q_{02} - Q_{20}) + \delta(m+1)^2 - 4\rho(m-1)}{4(m-1)m^5}\theta_2^4 - \frac{(3m+\delta)}{2m^4}\theta_2^2\theta_3 = 0, \tag{2.19}$$

where

$$\rho = Q_{03} + Q_{12} + Q_{21} + Q_{30}. \tag{2.20}$$

We first let $L_5 = L_{51}\theta_2^4 + L_{52}\theta_2^2\theta_3$. To make $L_5 = 0$ independently of θ_2 and θ_3, we solve $L_{51} = 0$ and $L_{52} = 0$ simultaneously for Q_{11} and Q_{02}. As a result, we get:

$$Q_{11} = -\frac{\rho(m-1)}{2} - 2(2m + Q_{20}), \quad Q_{02} = \frac{\rho(m-1)}{2} + m + Q_{20}. \tag{2.21}$$

Thus we compactly find $Q_{01} = -2(m-1)m$ and $Q_{10} = 2(m-1)m$. Substituting $Q_{00}, Q_{01}, Q_{10}, Q_{11}, Q_{02}$ into L_6, we obtain

$$L_6 = \theta_2^2\left(\phi_1\theta_2^3 + \phi_2\theta_2\theta_3 + \frac{1}{m^3}\theta_4\right), \tag{2.22}$$

where

$$\phi_1 = \frac{1}{m^6}\left[\frac{-8(Q_{12} + 2Q_{21} + 3Q_{30}) + \rho(m^2 + 2m + 9)}{4(m-1)} + \frac{\tau_0 - 12Q_{20}}{3(m-1)^2} - \tau_1\right] \tag{2.23}$$

and

$$\phi_2 = -\frac{1}{m^4}\left[\frac{m^2 + 5m - 4}{m-1} + \frac{\rho}{2m}\right] \tag{2.24}$$

with $\tau_0 = m^5 + 7m^4 + 2m^3 - 17m^2 - m$ and $\tau_1 = Q_{04} + Q_{13} + Q_{22} + Q_{31} + Q_{40}$.

The consequence of the analysis carried out thus far immediately leads us to the following theorem.

Theorem 2.1. *Let $m \in \mathbb{N} - \{1\}$ be given. Let $f : \mathbb{C} \to \mathbb{C}$ have a zero α of multiplicity m and be analytic in a small neighborhood of α. Let $\theta_j = \frac{m!}{(m-1+j)!} \cdot \frac{f^{(m-1+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j \in \mathbb{N} - \{1\}$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $Q_f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in a neighborhood of $(0,0)$. Let $Q_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial u^j \partial s^j}Q_f(u,s)|_{(u=0,s=0)}$ for $1 \le i,j \le 4$. Suppose that $Q_{00} = m$, $Q_{01} = -2m(m-1)$, $Q_{02} = m + Q_{20} + \frac{(m-1)}{2}(Q_{03} + Q_{12} + Q_{21} + Q_{30})$, $Q_{10} = -Q_{01}$ and $Q_{11} = -(3m + Q_{02} + Q_{20})$ hold. Then iterative methods (2.1) are of sextic-order and possess the following error equation:*

$$e_{n+1} = \theta_2^2\left(\phi_1\theta_2^3 + \phi_2\theta_2\theta_3 + \frac{1}{m^3}\theta_4\right)e_n^6 + O(e_n^7), \tag{2.25}$$

where $\phi_i (1 \le i \le 2)$ are given in (2.23) and (2.24), respectively.

3. Special cases of weight functions

Using relations (2.2), (2.3), (2.13), (2.15), (2.16), (2.18) and (2.21), the Taylor-polynomial form of $Q_f(u, s)$ is easily given by

$$Q_f(u,s) = m + 2m(m-1)u + Q_{20}u^2 + Q_{30}u^3 + Q_{40}u^4 - 2m(m-1)s + s^2\left[m + Q_{20} + \frac{(m-1)}{2}\rho\right] + Q_{03}s^3 + Q_{04}s^4 + u\left[-\left\{2(2m+Q_{20}) + \frac{(m-1)}{2}\rho\right\}s + Q_{12}s^2 + Q_{13}s^3\right] + u^2(Q_{21}s + Q_{22}s^2) + u^3Q_{31}s, \quad (3.1)$$

where ρ is introduced in (2.20). Special cases of $Q_f(u, s)$ are considered here. In each case, relevant coefficients are determined based on relations (2.15), (2.16), (2.18) and (2.21).

Although a variety of forms of weight functions $Q_f(u, s)$ are available in view of bivariate Taylor-polynomial forms shown by (3.1), we will limit ourselves to several cases of weight functions comprising low-order polynomials or simple rational functions.

Case 1: Second-order bivariate polynomial weight functions: $Q_{12} = Q_{21} = Q_{30} = Q_{03} = Q_{04} = Q_{22} = Q_{13} = Q_{31} = Q_{40} = 0, \rho = Q_{03} + Q_{12} + Q_{21} + Q_{30} = 0$.

$$Q_f(u,s) = m + 2m(m-1)(u-s) + Q_{20}u^2 - 2su(2m + Q_{20}) + s^2(m + Q_{20}). \quad (3.2)$$

Case 1A: When $Q_{20} = 0$

$$Q_f(u,s) = m[1 + 2(m-1)(u-s) - 4us + s^2]. \quad (3.3)$$

Case 1B: When $Q_{20} = -m$

$$Q_f(u,s) = m[1 + 2(m-1)(u-s) - u^2 - 2us]. \quad (3.4)$$

Case 1C: When $Q_{20} = -2m$

$$Q_f(u,s) = m[1 + 2(m-1)(u-s) - 2u^2 - s^2]. \quad (3.5)$$

Case 2: Second-order bivariate rational weight functions

$$Q_f(u,s) = \frac{b_0 + b_1u + b_2s + b_3us}{1 + a_1u + a_2s + a_3us}, \quad (3.6)$$

where $b_0 = m, b_2 = \frac{b_1+(2-b_1)m}{m-1}, a_1 = 2 + \frac{b_1}{m} - 2m, a_2 = \frac{b_1+(4-b_1)m-4m^2+2m^3}{m(m-1)}$ and $a_3 = \frac{b_3}{m} + 3$ with b_1 and b_3 as free parameters.

Case 2A: When $b_3 = b_2 = 0$

$$Q_f(u,s) = \frac{m + b_1u}{1 + a_1u + a_2s + a_3su}, \quad (3.7)$$

with $a_1 = -\frac{2m(m-2)}{m-1}, b_1 = \frac{2m}{m-1}, a_2 = 2(m-1), a_3 = 3$.

Case 2B: When $b_1 = b_3 = 0$

$$Q_f(u,s) = \frac{m + b_2s}{1 + a_1u + a_2s + a_3su}, \quad (3.8)$$

with $a_1 = 2 - 2m, b_2 = \frac{2m}{m-1}, a_2 = \frac{2(2-2m+m^2)}{m-1}, a_3 = 3$.

Case 2C: When $a_3 = b_1 = 0$

$$Q_f(u,s) = \frac{m + b_2s + b_3su}{1 + a_1u + a_2s}, \quad (3.9)$$

with $a_1 = 2(1-m), a_2 = \frac{2(2-2m+m^2)}{m-1}, b_2 = \frac{2m}{m-1}, b_3 = -3m$.

Case 3: Weight function as a sum of two univariate functions

$$Q_f(u,s) = G_f(u) + K_f(s), \quad (3.10)$$

where $G_f, K_f : \mathbb{C} \to \mathbb{C}$ are analytic in a neighborhood of the origin and satisfy the following relations.

$$G_0 = -K_0 + m, G_1 = 2m(m-1), G_2 = -K_2 - 3m, G_3 = -K_3 + \frac{4(K_2+m)}{m-1}, K_1 = -2m(m-1),$$

with $G_i = \frac{G_f^{(i)}(0)}{i!}, K_j = \frac{K_f^{(j)}(0)}{j!}$ for $0 \le i, j \le 4$.

Case 3A: (Sum of two second-order univariate polynomials)

$$Q_f(u,s) = m\big[1 + 2(m-1)(u-s) - 2u^2 - s^2\big], \tag{3.11}$$

which is identical with **Case 1C**.

Case 3B: (Sum of two first-order univariate rational functions)

$$Q_f(u,s) = \frac{d_0 + d_1 u}{1+cu} + \frac{r_0 + r_1 s}{1+qs}, \tag{3.12}$$

with $c = \frac{7}{4(m-1)}, d_1 = \frac{7d_0}{4(m-1)} + 2m(m-1), r_0 = m - d_0,\ r_1 = -\frac{d_0 + m(8m^2-16m+7)}{4(m-1)}, q = \frac{1}{4(m-1)}$ and d_0 as a free parameter.

Case 3C:

$$Q_f(u,s) = \frac{m + d_1 u}{1+cu} + \frac{r_1 s}{1+qs}, \tag{3.13}$$

with $c = \frac{7}{4(m-1)}, d_1 = \frac{m(8m^2-16m+15)}{4(m-1)}, r_1 = -2m(m-1)$ and $q = \frac{1}{4(m-1)}$.

Case 3D:

$$Q_f(u,s) = \frac{d_1 u}{1+cu} + \frac{m + r_1 s}{1+qs}, \tag{3.14}$$

with $c = \frac{7}{4(m-1)}, d_1 = 2m(m-1), r_1 = -\frac{m(8m^2-16m+7)}{4(m-1)}$ and $q = \frac{1}{4(m-1)}$.

We can find that **Case 3D** yields the same $Q_f(u,s)$ as that of **Case 3C** via direct computation of the given coefficients.

Case 4: Weight function as a product of two univariate functions

$$Q_f(u,s) = G_f(u) \times K_f(s), \tag{3.15}$$

where $G_f, K_f : \mathbb{C} \to \mathbb{C}$ are analytic in a neighborhood of the origin and satisfy the following relations.

$$\begin{aligned} G_0 &= \frac{m}{K_0}, \\ G_1 &= -\frac{2m(m-1)}{K_0}, \\ G_2 &= -\frac{m[-K_2 + K_0(4m^2 - 8m + 1)]}{K_0^2}, \\ G_3 &= -\frac{m[K_3(1-m) - 4K_2 m(m-2) + 2K_0(4m^4 - 16m^3 + 17m^2 - 2m - 1)]}{(m-1)K_0^2}, \\ K_1 &= -2(m-1)K_0, \end{aligned}$$

with $G_i = \frac{G_f^{(i)}(0)}{i!}, K_j = \frac{K_f^{(j)}(0)}{j!}$ for $0 \le i, j \le 4$.

Case 4A: (Product of two univariate rational functions)

$$Q_f(u,s) = \frac{m + a_1 u + a_2 u^2}{1 + b_1 u + b_2 u^2} \times \frac{d_0 + d_1 s}{1 + c_1 s}, \tag{3.16}$$

with $d_0 = 1, d_1 = c_1 - 2(m-1), a_1 = 2m \cdot \frac{b_2 - 2c_1 - 2 + c_1^2 - 2m(b_2 - 4 + c_1 + c_1^2) + (b_2 - 4 + m^2 c_1(6 + c_1)) - 2c_1 m^3}{(m-1)(2(m-1)c_1 - 4m^2 + 8m - 1)}$,
$a_2 = m \cdot \frac{2(m-1)c_1(2+b_2) + 3b_2 - 4m^2 + 8m - 5}{2(m-1)c_1 - 4m^2 + 8m - 1}$ and $b_1 = 2 \cdot \frac{(m-1)^2(b_2 + c_1^2) - 4c_1 m(m-1)(m-2) + 4m^4 - 16m^3 + 17m^2 - 2m - 1}{(m-1)(2(m-1)c_1 - 4m^2 + 8m - 1)}$.

Case 4B: (Product of two second-order univariate polynomials)

$$Q_f(u,s) = \frac{m + a_1 u}{1 + b_1 u} \times \frac{1 + d_1 s}{1 + c_1 s}, \tag{3.17}$$

where $a_1 = \frac{m(11-8m+4m^2)}{4(m-1)}$, $b_1 = \frac{(3+8m-4m^2)}{4(m-1)}$, $c_1 = \frac{(5-8m+4m^2)}{4(m-1)}$ and $d_1 = -\frac{(3-8m+4m^2)}{4(m-1)}$.

Case 4C:

$$Q_f(u,s) = \frac{m + a_1 u}{1 + b_1 u + b_2 u^2} \times \frac{1}{1 + c_1 s}, \tag{3.18}$$

where $a_1 = \frac{2m(4m^4-16m^3+31m^2-30m+13)}{(m-1)(4m^2-8m+7)}$, $b_1 = \frac{4(2m^2-4m+3)}{(m-1)(4m^2-8m+7)}$, $b_2 = -\frac{4m^2-8m+3}{4m^2-8m+7}$ and $c_1 = 2(m-1)$.

Case 4D:

$$Q_f(u,s) = \frac{m}{1 + b_1 u + b_2 u^2} \times \frac{1 + d_1 s}{1 + c_1 s}, \tag{3.19}$$

where $b_1 = -2(m-1)$, $b_2 = -1 \pm 2\lambda$, $c_1 = \frac{-2\pm\lambda}{m-1}$ and $d_1 = \frac{-2(2-2m+m^2)\pm\lambda}{m-1}$, with $\lambda = \sqrt{(m-1)^2+2}$.

In the next section, we will discuss the extraneous fixed points [24,43] of Q_f and relevant dynamics associated with their basins of attraction. The dynamics behind basins of attraction was initiated by Stewart [41] and followed by works of Amat et al. [2–5], Scott et al. [38], Chun et al. [10], Chun and Neta [11], Chicharro et al. [8], Cordero et al. [16], Neta et al. [30], [33], Argyros and Magreñan [7], Magreñan [29], Magreñan et al. [28], Andreu et al. [6] and Chun et al. [12]. The only papers comparing basins of attraction for methods to obtain multiple roots are due to Neta et al. [31], Neta and Chun [32,34], and Chun and Neta [13,14].

4. Extraneous fixed points

In general, multipoint iterative methods [22,23] finding a zero α of a nonlinear equation $f(x) = 0$ can be written as

$$x_{n+1} = R_f(x_n), n = 0, 1, \dots, \tag{4.1}$$

where a fixed point ξ of R_f is α. The iteration function R_f, however, might possess other fixed points $\xi \neq \alpha$. Such fixed points are called the *extraneous fixed points* of the iteration function R_f. Extraneous fixed points may form attractive, indifferent or repulsive cycles as well as other periodic orbits to display chaotic dynamics of the basin of attraction under investigation.

Investigation of such dynamics clearly motivates our current analysis, which enables us to write the proposed method (2.1) in the following form:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{4.2}$$

where $H_f(x_n) = m + \frac{u^m}{s^{m-1}} \cdot Q_f(u,s)$ can be regarded as a weight function of the classical Newton's method. It is obvious that α is a fixed point of R_f. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

We limit ourselves to paying a special attention to several cases **1C, 2A, 3C, 4C** in order to explore further properties of extraneous fixed points and relevant dynamics associated with their basins of attraction. By closely following the works of Chun et al. [9,13,34] and Neta et al. [30,33,34], we construct $H_f(x_n) = m + \frac{u^m}{s^{m-1}} \cdot Q_f(u,s)$ in (4.2). We then apply a polynomial $f(z) = (z^2-1)^m$ to $H_f(x_n)$ and construct a weight function $H(z)$, with a change of a variable $\zeta = z^2$, in the form of

$$H(z) = \mathcal{A} \cdot \frac{F(\zeta)}{D(\zeta)}, \tag{4.3}$$

where $\mathcal{A}$ is a constant which may be dependent on m but independent of the extraneous fixed points of H; $F(\zeta)$ and $D(\zeta)$ have no common factors; $F(\zeta)$ may indeed contain the extraneous fixed points H. Thus the extraneous fixed points ξ of H can be found from zeros ζ of $F(\zeta)$ via relation $\xi = \zeta^{\frac{1}{2}}$. Note that F is a finite sum of rational powers in ζ. It must be emphasized that any general algebraic ways of zero-finding of $F(\zeta)$ seem to be infeasible. By a suitable change of variables for the terms with rational powers, $F(\zeta)$ can be transformed into a multivariate polynomial, which can be solved with known polynomial root-finding methods. Constant $\mathcal{A}$ and functions $F(\zeta)$, $D(\zeta)$ of $H(z)$ are explicitly displayed for cases **1C, 2A, 3C, 4C** in Table 1.

It is our main interest to investigate the complex dynamics of the iterative map R_p of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H(z_n), \tag{4.4}$$

in connection with the basins of attraction for a variety of polynomials $p(z_n)$ and a weight function $H(z_n)$. Indeed, $R_p(z)$ represents the classical Newton's method with weight function $H(z)$ and may possess its fixed points as zeros of $p(z)$ or extraneous fixed points associated with $H(z)$. As a result, basins of attraction for the fixed points or the extraneous fixed points as well as their attracting periodic orbits may make an impact on the complicated and chaotic complex dynamics whose visual description for various polynomials will be shown in the latter part of Section 5.

At this point, we now wish to describe the dynamical behavior of (4.4) for selected cases for values of $m \in \{2, 3, 4, 5\}$. Table 2 lists corresponding extraneous fixed points ξ of H for values of $2 \le m \le 5$. All the fixed points ξ of H in each case show their stability in Table 3.

In the latter part of Section 5, we will explore complex dynamics associated with the basins of attraction for iterative maps (4.4) when applied to various polynomials.

Table 1
$\mathcal{A}$, $F(\zeta)$ and $D(\zeta)$ for typical cases.

Case	$\mathcal{A}$	$F(\zeta)$	$D(\zeta)$
1C	m	$2^{\frac{m+1}{m-1}}\zeta^{\frac{2}{m-1}}t_1-(\zeta-1)^2(\zeta+1)^{\frac{1}{m-1}}t_2$	$2^{\frac{5m-3}{m-1}}\zeta^{\frac{2m}{m-1}}(\zeta+1)$
2A	m	$(\zeta+1)\mu_1+\zeta^{\frac{m}{m-1}}\mu_2$	$(\zeta+1)[\mu_1+\zeta^{\frac{m}{m-1}}\mu_3]$
3C	1	$\zeta^{\frac{m}{m-1}}\rho_1-(\zeta-1)(\zeta+1)^{\frac{1}{m-1}}\rho_2$	$[2^{\frac{4m-3}{m-1}}(m-1)\zeta^{\frac{m}{m-1}}+(\zeta-1)(\zeta+1)^{\frac{1}{m-1}}]\rho_3$
4C	m	$(m-1)(\zeta-1)(\zeta+1)^{\frac{1}{m-1}}\sigma_1+\zeta^{\frac{m}{m-1}}\sigma_2$	$[2^{\frac{m}{m-1}}\zeta^{\frac{m}{m-1}}+(m-1)(\zeta-1)(\zeta+1)^{\frac{1}{m-1}}]\sigma_1$

where $t_1=1+(4m-7)\zeta+(19-8m)\zeta^2+(19+4m)\zeta^3$, $t_2=2^{\frac{3m-2}{m-1}}(m-1)\zeta^{\frac{m}{m-1}}+(\zeta-1)(\zeta+1)^{\frac{1}{m-1}}$, $\mu_1=(m-1)(\zeta-1)(\zeta+1)^{\frac{1}{m-1}}[-3+(8m-5)\zeta]$, $\mu_2=2^{\frac{2m-1}{m-1}}[1-4m+2m^2+2(-2+m)\zeta+(-5+10m-2m^2)\zeta^2]$, $\mu_3=2^{\frac{3m-2}{m-1}}[m(m-2)-(2-4m+m^2)\zeta]$, $\rho_1=2^{\frac{4m-3}{m-1}}(m-1)[1-16m+8m^2-2(23-24m+8m^2)\zeta+(-19+32m+8m^2)\zeta^2]$, $\rho_2=48m^2-96m+55-2(64m^3-144m^2+104m-31)\zeta+(128m^3-336m^2+240m-53)\zeta^2$, $\rho_3=2(\zeta+1)(-7+(16m-9)\zeta)$, $\sigma_1=(\zeta+1)(\nu_0+\zeta\nu_1+\nu_2\zeta^2)$, $\sigma_2=2^{\frac{m}{m-1}}[\nu_0+\zeta\nu_3+\zeta^2\nu_4+\zeta^3\nu_5]$, $\nu_0=-(m-1)(2m-1)(2m-3)$, $\nu_1=2(4m^3-28m^2+43m-27)$, $\nu_2=-61+165m-148m^2+60m^3$, $\nu_3=1-45m+80m^2-60m^3+16m^4$, $\nu_4=-163+371m-356m^2+164m^3-32m^4$, $\nu_5=16m^4+28m^3-120m^2+165m-65$.

Table 2
Extraneous fixed points $\xi=\zeta^{1/2}$ for selected cases with $2\le m\le 5$.

Case	m	ξ	no. of ξ
	2	$\pm 0.516521i$, $\pm 0.392347\pm 0.594746i$, $\pm 0.500255\pm 0.141729i$	10
1C	3	$\pm 0.373652\pm 0.698108i$, $\pm 0.449253\pm 0.157328i$	8
	4	$\pm 0.372206\pm 0.726675i$, $\pm 0.431928\pm 0.163164i$	8
	5	$\pm 0.371518\pm 0.739565i$, $\pm 0.423776\pm 0.165934i$	8
	2	$\pm 0.293287\pm 0.699989i$, $\pm 0.596503\pm 0.107471i$	8
2A	3	$\pm 0.252854\pm 0.782644i$, $\pm 0.545882\pm 0.144661i$	8
	4	$\pm 0.237208\pm 0.812658i$, $\pm 0.524521\pm 0.158536i$	8
	5	$\pm 0.228688\pm 0.828309i$, $\pm 0.512699\pm 0.165833i$	8
	2	$\pm 0.345662\pm 0.609862i$, $\pm 0.552447\pm 0.0751945i$	8
3C	3	$\pm 0.352025\pm 0.681211i$, $\pm 0.488042\pm 0.117962i$	8
	4	$\pm 0.356666\pm 0.709113i$, $\pm 0.462851\pm 0.134584i$	8
	5	$\pm 0.359362\pm 0.723901i$, $\pm 0.449299\pm 0.143406i$	8
	2	$\pm 0.286835\pm 0.655947i$, $\pm 0.240302i$, ± 0.620034, ± 0.650152	10
4C	3	± 0.736042, $\pm 0.248379\pm 0.666306i$, ± 0.552873	8
	4	± 0.787799, $\pm 0.240897\pm 0.652507i$, ± 0.531382	8
	5	± 0.823542, $\pm 0.240365\pm 0.64212i$, ± 0.519868	8

Table 3
Absolute values of multipliers of the extraneous fixed points for selected cases with $2\le m\le 5$.

Case	m	$\lvert R_p{}'(\xi)\rvert$	no. of ξ
	2	1.0000000000000007, 1.0000000000000002, 1.0000000000000002, 1.0000000000000036, 1.0000000000000036	10
1C	3	1.0000000000000189, 1.0000000000000189, 1.0000000000000049, 1.0000000000000049	8
	4	1.000000000000824, 1.000000000000824, 1.000000000000039, 1.000000000000039	8
	5	1.0000000000592706, 1.0000000000592706, 0.9999999999996363, 0.9999999999996363	8
	2	0.9999999999999998, 0.9999999999999998, 0.9999999999999936, 0.9999999999999936	8
2A	3	1.000000000000002, 1.000000000000002, 0.9999999999999697, 0.9999999999999697	8
	4	1.0000000000000044, 1.0000000000000044, 1.000000000000013, 1.000000000000013	8
	5	0.9999999999999274, 0.9999999999999274, 0.9999999999999896, 0.9999999999999896	8
	2	1.0000000000000004, 1.0000000000000004, 0.9999999999999969, 0.9999999999999969	8
3C	3	1.0000000000000038, 1.0000000000000038, 0.9999999999999628, 0.9999999999999628	8
	4	0.9999999999999313, 0.9999999999999313, 1.0000000000000049, 1.0000000000000049	8
	5	0.9999999999999976, 0.9999999999999976, 0.9999999999998596, 0.9999999999998596	8
	2	1.0, 1.0, 1.0000000000001026, 0.9999999999999983, 0.9999999999999909	10
4C	3	1.0000000000003915, 1.0000000000000002, 1.0000000000000002, 1.0000000000000133	8
	4	0.9999999999384795, 0.9999999999999797, 0.9999999999999797, 1.000000000000486	8
	5	0.9999999973848279, 0.9999999999999746, 0.9999999999999746, 0.9999999999941207	8

5. Numerical experiments and complex dynamics

We first begin this section with computational aspects of proposed methods (2.1) for a variety of test functions in comparison with typically selected cases. Later on in the latter part of this section, we will explore the complex dynamics behind the basins of attraction of iterative maps (4.4) for iterative maps of the selected cases.

A variety of numerical experiments have been carried out with Mathematica programming to confirm the developed theory. Throughout these experiments, we have maintained 100 digits of minimum number of precision, via Mathematica command

Table 4
Additional test functions $f_i(x)$, zeros α and initial guesses x_0.

i	$f_i(x)$	α	m	x_0
1	$x^2[x^3-\log(1+x^2)]^2$	0	6	-0.01
2	$[x^2-\sin x-3]^2$	≈ 1.97932014655621	2	2.0
3	$[2x-\pi+\cos x\log(x^2+1)]^4$	$\frac{\pi}{2}$	4	1.53
4	$[2x^3+e^{-x^2}+\sin x-2[^7$	$\approx 0.916952932621001$	7	0.73
5	$[x-\sqrt{3}x^3\cos\frac{\pi x}{6}+\frac{1}{x^2+1}-\frac{11}{5}+4\sqrt{3}](x-2)^4$	2	5	1.87
6	$\exp\left[\frac{(x^3+1)^2}{x^5+7\cos(x^3+1)}\right]-1$	$\frac{1+i\sqrt{3}}{2}$	2	$0.52+0.85i,\ i=\sqrt{-1}$
7	$[x\log x-\sqrt{x}+x^2]^4$	1	4	1.05

Here $\log z(z\in\mathbb{C})$represents a principal analytic branch with $-\pi\le Im(\log z)<\pi$.

$\$MinPrecision=100$, to achieve the specified accuracy. In case that α is not exact, it is replaced by a more accurate value which has more number of significant digits than the assigned $\$MinPrecision=100$.

Definition 5.1 (Computational convergence order)**.** Assume that theoretical asymptotic error constant $\eta=\lim_{n\to\infty}\frac{|e_n|}{|e_{n-1}|^p}$ and convergence order $p\ge 1$ are known. Define $p_n=\frac{\log|e_n/\eta|}{\log|e_{n-1}|}$ as the computational convergence order. Note that $\lim_{n\to\infty}p_n=p$.

Remark 5.1. Note that p_n requires knowledge at two points x_n, x_{n-1}, while the usual COC(computational order of convergence) $\frac{\log(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)}{\log(|x_{n-1}-x_{n-2}|/|x_{n-2}-x_{n-3}|)}$ does require knowledge at four points $x_n, x_{n-1}, x_{n-2}, x_{n-3}$. Hence p_n can be handled with a less number of working precision digits than the usual COC whose number of working precision digits is at least p times as large as that of p_n.

Computed values of x_n are accurate up to $\$MinPrecision$ significant digits. If α has the same accuracy of $\$MinPrecision$ as that of x_n, then $e_n=x_n-\alpha$ would be nearly zero and hence computing $|e_{n+1}|/e_n^p|$ would unfavorably break down. To clearly observe the convergence behavior, we desire α to have more significant digits that are Φ digits higher than $\$MinPrecision$. To supply such α, a set of following Mathematica commands are used:

$$sol = FindRoot[f(x),\{x,x_0\},PrecisionGoal\to\Phi+\$MinPrecision,$$
$$WorkingPrecision\to 2*\$MinPrecision];$$
$$\alpha = sol[[1,2]]$$

In this experiment, we assign $\Phi=16$. As a result, the numbers of significant digits of x_n and α are found to be 100 and 116, respectively. Nonetheless, the limited paper space allows us to list both of them only up to 15 significant digits. We set the error bound ϵ to $\frac{1}{2}\times 10^{-80}$ satisfying $|x_n-\alpha|<\epsilon$.

Iterative methods (2.1) with cases **1C, 2A, 3C, 4C** were respectively identified by **W1C, W2A, W3C, W4C**, being W-prefixed. Methods **W1C, W2A, W3C, W4C** have been successfully applied to the test functions F_1-F_4 below:

$$\begin{cases}\mathbf{W1C}: F_1(x)=[\cos(\frac{\pi x}{2})+x^2-\pi]^5, m=5, \alpha\approx -2.03472489627913\\ \mathbf{W2A}: F_2(x)=[\cos(x^2-1)-x\log(x^2-\pi)+1]^2(x^2-1-\pi), m=3, \alpha=\sqrt{1+\pi},\\ \mathbf{W3C}: F_3(x)=[\sin^{-1}(x-1)+e^{x^2}-3]^3, m=3, \alpha\approx 1.04148187058433,\\ \mathbf{W4C}: F_4(x)=(9-2x-2x^4+\cos 2x)(5-x-x^4-\sin^2 x), m=2, \alpha\approx 1.29173329244360,\\ \text{where } \log z(z\in\mathbb{C}) \text{ represents a principal analytic branch such that } -\pi<Im(\log z)\le\pi\end{cases}$$

As seen in Table 5, they clearly confirmed sextic-order convergence. The values of computational asymptotic error constant agree up to 7 significant digits with η. It appears that the computational convergence order well approaches 6.

Table 4 shows additional test functions to further confirm the convergence behavior of proposed scheme (2.1).

In Table 6, we compare numerical errors $|x_n-\alpha|$ of proposed methods **W1C, W2A, W3C, W4C**. The least errors within the prescribed error bound are highlighted in bold face. Within two iterations, in view of strict comparison, Method **W1C** shows slightly better convergence for $f_1, f_2, f_3, f_5, f_6, f_7$, while method **W2A** for f_4. By inspecting the asymptotic error constant $\eta(\theta_i,m,Q_f)=\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p}$ when p is known, we find that the local convergence is dependent on the function $f(x)$, an initial value x_0, the multiplicity m, the zero α itself and the weight function Q_f. Accordingly, for a given set of test functions, one method is hardly expected to always show better performance than the others.

This paper proposes sextic-order multiple-zero finders with a bivariate weight function dependent on two principal roots of $\left[\frac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}$ and $\left[\frac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}$. We find it very important to properly select initial values influencing the convergence behavior of iterative methods. To ensure the convergence of iterative map (4.4) viewed as Newton's method with a weight function $H(z)$, it requires good initial values close to zero α. It is, however, a difficult task to determine how close the initial values are to zero α, since initial values are generally dependent upon computational precision, error bound and the given function $f(x)$ under consideration. One effective way of selecting stable initial values is to directly use visual basins of attraction. Since the area of

Table 5
Convergence for test functions $F_1(x) - F_4(x)$ with methods **W1C, W2A, W3C, W4C.**

MT	F	n	x_n	$\|F(x_n)\|$	$\|x_n-\alpha\|$	$\|e_n/e_{n-1}^6\|$	η	p_n
		0	−2.1	0.00174325	0.0652751			
		1	−2.03472492017726	9.654×10^{-36}	4.913×10^{-8}	0.3089431095	0.4282207521	6.11963
W1C	F_1	2	−2.03472489627913	4.001×10^{-228}	5.348×10^{-43}	0.4282207000		6.00000
		3	−2.03472489627913	0.0×10^{-497}	0.0×10^{-99}			
		0	2.0	0.0140817	0.0350903			
		1	2.03509028144049	3.312×10^{-20}	2.389×10^{-8}	26.31721953	38.01718897	6.10980
W2A	F_2	2	2.03509033057253	4.271×10^{-125}	7.977×10^{-47}	38.01716758		6.00000
		3	2.03509033057253	0.0×10^{-299}	0.0×10^{-99}			
		0	1.084	0.0335181	0.0425181			
		1	1.04148199694198	7.415×10^{-19}	1.263×10^{-7}	21.38733354	26.44207159	6.06718
W3C	F_3	2	1.04148187058433	4.582×10^{-118}	1.076×10^{-40}	26.44205449		6.00000
		3	1.04148187058433	0.0×10^{-298}	0.0×10^{-99}			
		0	1.35	0.388805	0.0582667			
		1	1.29173359504765	9.435×10^{-12}	3.026×10^{-7}	7.733068545	12.59469262	6.17159
W4C	F_4	2	1.29173329244360	9.635×10^{-75}	9.670×10^{-39}	12.59465793		6.00000
		3	1.29173329244360	0.0×10^{-198}	0.0×10^{-99}			

MT = method.

Table 6
Comparison of $|x_n-\alpha|$ for selected multiple-zero finders.

$f, x_0; m$	$\|x_n-\alpha\|$	**W1C**	**W2A**	**W3C**	**W4C**
f_1, −0.01; 6	$\|x_1-\alpha\|$	1.24e-13*	1.29e-13	1.33e-13	4.61e-13
	$\|x_2-\alpha\|$	**4.93e-79**	6.81e-79	8.35e-79	5.06e-75
f_2, 2.0; 2	$\|x_1-\alpha\|$	5.75e-12	6.71e-12	7.01e-12	9.29e-12
	$\|x_2-\alpha\|$	**2.90e-69**	8.68e-69	1.16e-68	8.52e-68
f_3, 1.53; 4	$\|x_1-\alpha\|$	6.49e-8	6.59e-8	7.48e-8	1.91e-7
	$\|x_2-\alpha\|$	**1.51e-42**	1.78e-42	4.15e-42	3.93e-39
f_4, 0.73; 7	$\|x_1-\alpha\|$	3.12e-11	2.94e-11	3.47e-11	2.01e-10
	$\|x_2-\alpha\|$	2.43e-62	**1.63e-62**	5.13e-62	1.30e-56
f_5, 1.87; 5	$\|x_1-\alpha\|$	2.29e-7	2.37e-7	2.41e-7	4.52e-7
	$\|x_2-\alpha\|$	**3.35e-41**	4.39e-41	4.87e-41	5.45e-39
f_6, 0.52	$\|x_1-\alpha\|$	3.41e-10	1.02e-9	9.30e-10	1.20e-9
$+0.85i$; 2	$\|x_2-\alpha\|$	**2.26e-57**	4.61e-54	2.37e-54	1.45e-53
f_7, 1.05; 4	$\|x_1-\alpha\|$	1.56e-8	1.64e-8	1.78e-8	4.42e-8
	$\|x_2-\alpha\|$	**2.15e-47**	3.12e-47	5.46e-47	3.75e-44

$1.24e-13$ denotes 1.24×10^{-13}.

Table 7
Average number of iterations per point for each example (1–6).

Example	GKN1C	GKN2A	GKN3C	GKN4C
1 $m=2$	18.0428	9.4718	10.7086	8.1742
2 $m=3$	40.0000	15.1676	26.5189	7.7825
3 $m=3$	-	17.4269	28.5655	9.9977
4 $m=4$	-	20.5243	-	8.8558
5 $m=5$	-	22.7542	-	13.1151
6 $m=5$	-	24.9346	-	14.9314
Average	-	18.3799	-	10.4761

convergence can be seen on the basins of attraction, it would be reasonable to say that larger area of convergence indicates a better method. Clearly a quantitative analysis is necessary for measuring the size of area of convergence.

To this end, we provide Table 7 featuring a statistical data describing the average number of iterations per point. In the following 6 examples, we take a 6 by 6 square centered at the origin and containing all the zeros of the given functions. We assume that all zeros are of the same multiplicity m. We then take 360,000 equally spaced points in the square as initial points for the iterative methods. We color the point based on the root it converged to. This way we can find out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial point.

We now are ready to discuss the complex dynamics of iterative map (4.4) applied to various polynomials. To continue our discussion, let us first identify four members of iterative map (4.4) associated with Cases **1C, 2A, 3C, 4C** by **GKN1C**, **GKN2A**, **GKN3C**, **GKN4C**, respectively.

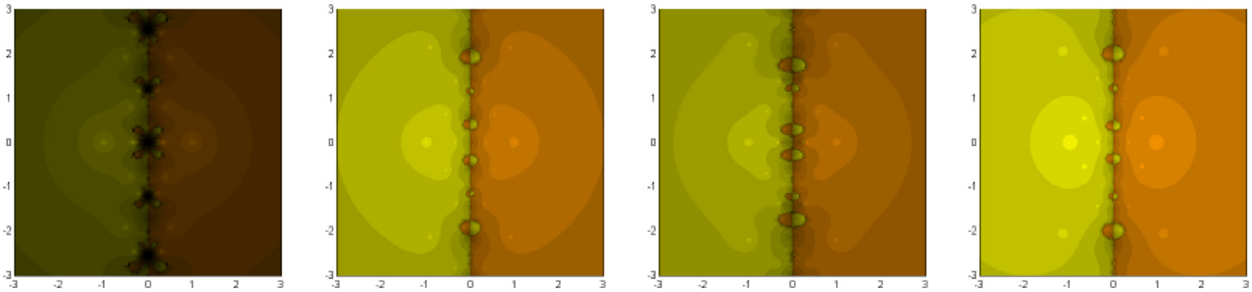

Fig. 1. The leftmost for **1C**, second for **2A**, third for **3C** and the rightmost for **4C** for the roots of the polynomial $(z^2 - 1)^2$.

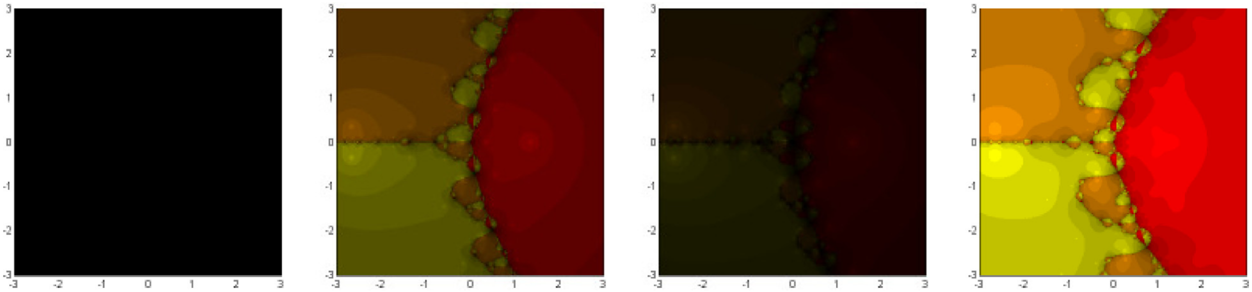

Fig. 2. The leftmost for **1C**, second for **2A**, third for **3C** and the rightmost for **4C** for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Table 8
CPU time (in seconds) required for each example (1–6) using a Dell Multiplex-990.

Example	GKN1C	GKN2A	GKN3C	GKN4C
1 $m = 2$	1935.63	1086.62	1230.25	987.27
2 $m = 3$	13229.31	5423.05	9185.28	2914.50
3 $m = 3$	-	5537.33	9083.11	3275.10
4 $m = 4$	-	8979.37	-	3520.85
5 $m = 5$	-	8766.24	-	5183.44
6 $m = 5$	-	10636.10	-	6586.66
Average	-	6738.12	-	3744.64

Table 9
Number of points requiring 40 iterations for each example (1–6).

Example	GKN1C	GKN2A	GKN3C	GKN4C
1 $m = 2$	1523	601	601	601
2 $m = 3$	361201	9	921	2
3 $m = 3$	-	4159	24154	1128
4 $m = 4$	-	4	-	0
5 $m = 5$	-	73	-	7
6 $m = 5$	-	7065	-	817
Average	-	1985.17	-	425.83

Example 1. As a first example, we have taken a quadratic polynomial raised to the power of 2 with all real roots:

$$p_1(z) = (z^2 - 1)^2. \tag{5.1}$$

Clearly the roots are $\pm$ 1 with multiplicity 2. Basins of attraction for iterative maps **GKN1C**, **GKN2A**, **GKN3C**, and **GKN4C** are illustrated in Fig. 1. Each basin is painted in a different color. At a root its color is white, while getting darker for more iterations required for convergence within the iteration limit. At black points, we recognize that the corresponding iterative maps did not converge within the iteration limit of 40 currently prescribed in this experiment. Based on the displayed results in Fig. 1, we find that it is clear that **GKN1C** was the worst and **GKN2A** and **GKN4C** are better.

If we look at the first row of Table 7, we find that these are the cases with the lowest number of iterations per point. **GKN4C** took less CPU time (see Table 8) than all the other methods. In terms of the number of points requiring 40 iterations (see Table 9)

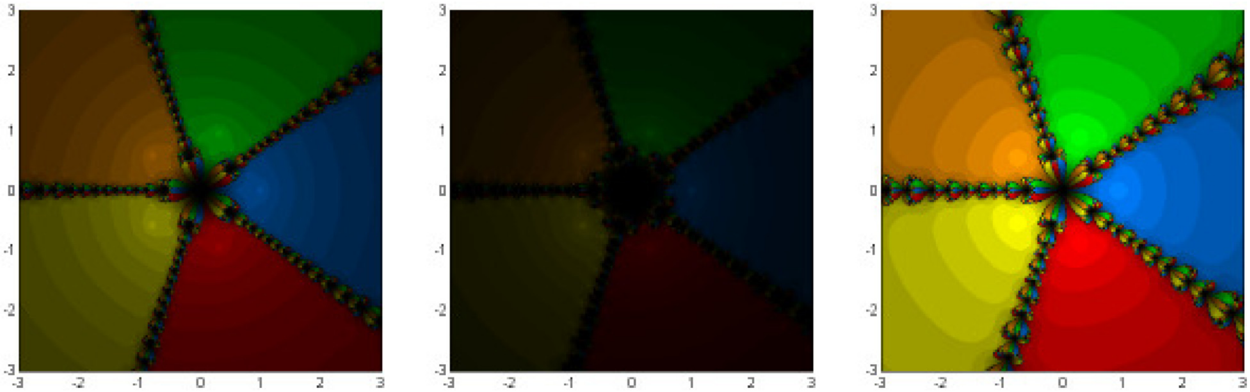

Fig. 3. The leftmost for **2A**, second for **3C** and the rightmost for **4C** for the roots of the polynomial $(z^5 - 1)^3$.

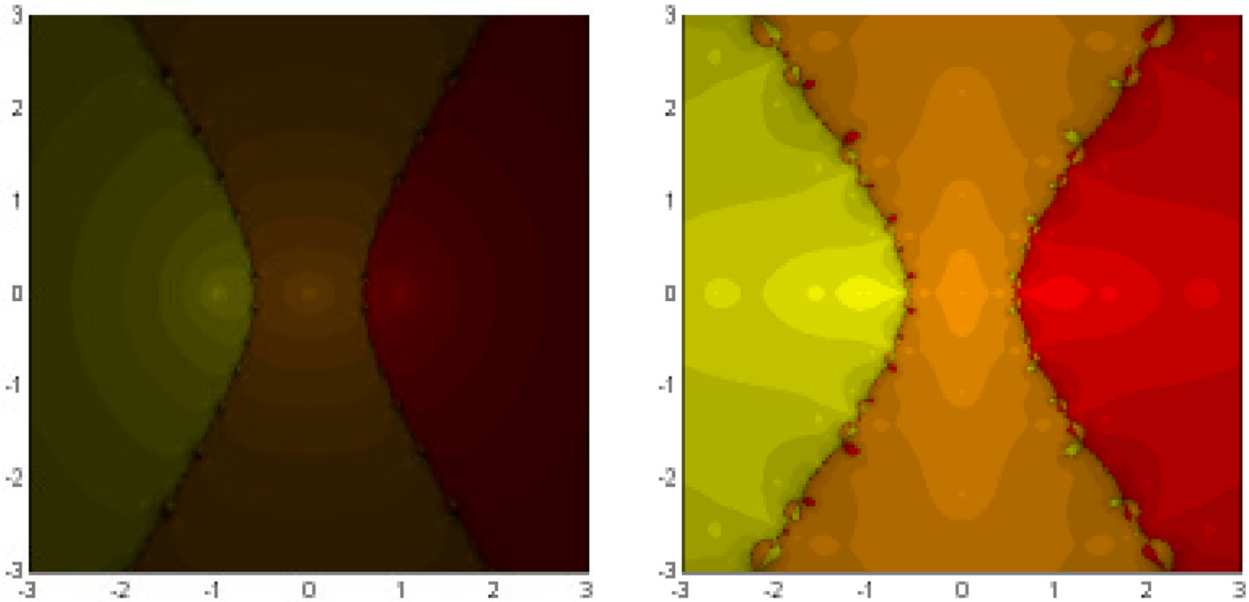

Fig. 4. The left for **2A** and the right for **4C** for the roots of the polynomial $(z^3 - z)^4$.

we find that **GKN1C** is the worst. **GKN2A**, **GKN3C** and **GKN4C** have the same number of points requiring 40 iterations and those are the best cases.

Example 2. In our second example, we have taken a cubic polynomial raised to the power of 3:

$$p_2(z) = (z^3 + 4z^2 - 10)^3. \tag{5.2}$$

Basins of attraction for **GKN1C**, **GKN2A**, **GKN3C**, and **GKN4C** are illustrated in Fig. 2. The worst are **GKN1C** and **GKN3C**. The best is **GKN4C**. This is also seen in Table 7. In terms of CPU time (see Table 8) again **GKN4C** is the fastest. The slowest are **GKN1C** and **GKN3C**. The lowest number of points requiring 40 iterations (see Table 8) is for **GKN4C** followed by **GKN2A**. In the following examples we will not show **GKN1C** because of its poor performance.

Example 3. As a third example, we have taken a quintic polynomial raised to the power of 3:

$$p_3(z) = (z^5 - 1)^3. \tag{5.3}$$

Basins of attraction for **GKN2A**, **GKN3C**, and **GKN4C** are illustrated in Fig. 3. **GKN3C** is the worst as in the previous example and we should exclude it from the other runs. The best is **GKN4C**. If we examine the average number of iterations per point (see Table 7), we arrive at the same conclusion with **GKN4C** requiring about 10 iterations per point. On the other hand, the CPU time for **4C** was the smallest (3275 s) followed by **GKN2A** (5537 s) .

Example 4. As a fourth example, we have taken a different cubic polynomial raised to the power of 4:

$$p_4(z) = (z^3 - z)^4. \tag{5.4}$$

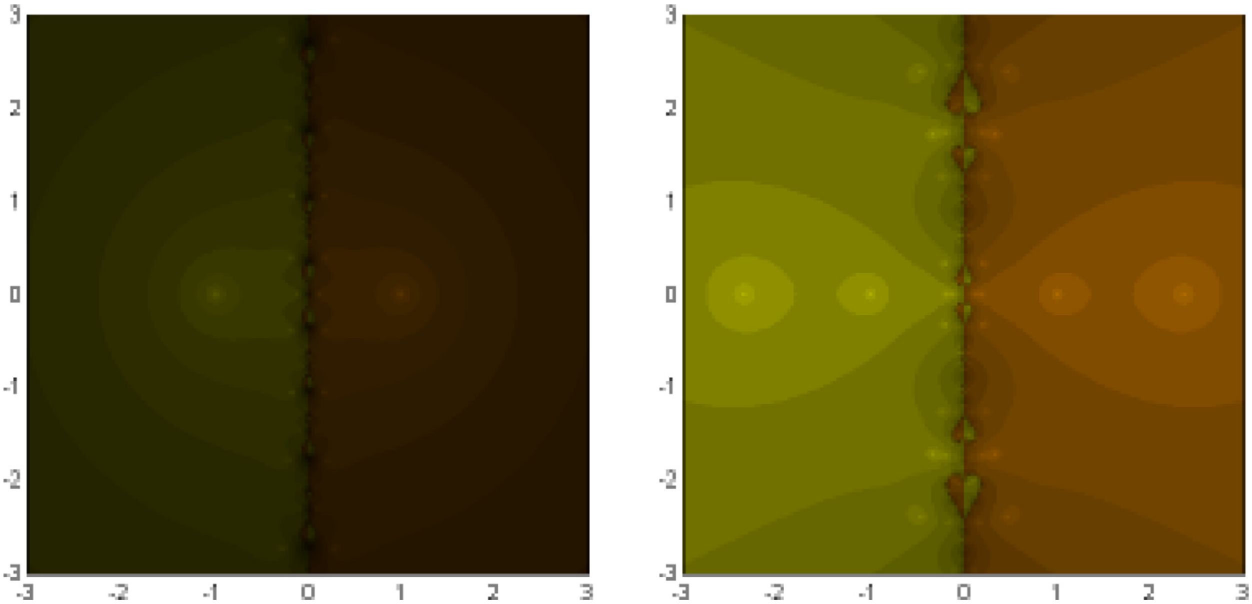

Fig. 5. The left for **2A** and the right for **4C** for the roots of the polynomial $(z^2 - 1)^5$.

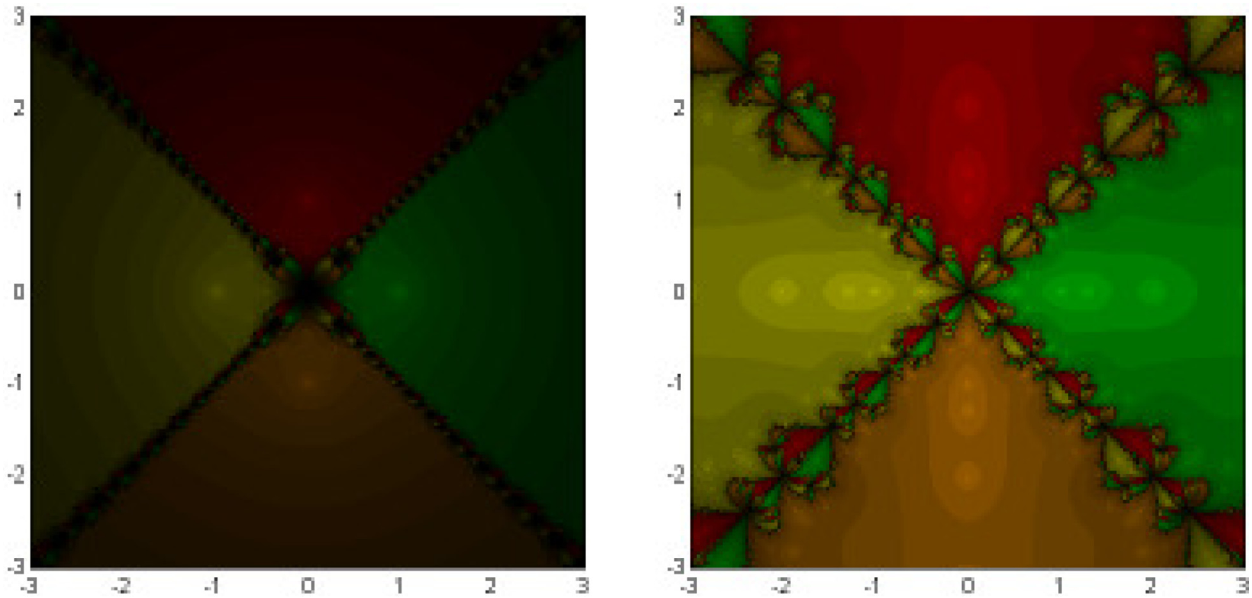

Fig. 6. The left for **2A** and the right for **4C** for the roots of the polynomial $(z^4 - 1)^5$.

Now all the roots are real. Basins of attraction for **GKN2A**, and **GKN4C** are illustrated in Fig. 4. Again **GKN4C** is best. Quantitatively, we find that **GKN4C** requires an average of 8.8 iterations per point (see Table 7). **GKN4C** is the fastest as before (see Table 8) and have **no** black points, as can be seen in Table 9.

Example 5. As a fifth example, we have taken a quadratic polynomial raised to the power of 5:

$$p_5(z) = (z^2 - 1)^5. \tag{5.5}$$

Basins of attraction for **GKN2A** and **GKN4C** are illustrated in Fig. 5. The conclusions are the same as in the previous example.

Example 6. As a last example, we have taken a quartic polynomial raised to the power of 5:

$$p_6(z) = (z^4 - 1)^5. \tag{5.6}$$

Basins of attraction for **GKN2A** and **GKN4C** are illustrated in Fig. 6. It is clear that we have the same conclusions. The best is **GKN4C**.

In summary, we find that **GKN4C** is faster, requires less iterations per point on average, and have much fewer points requiring 40 iterations (see the last row of Table 9).

We have shown a technique of improving convergence order of our family of proposed methods (2.1) with the introduction of a bivariate weight function. One such technique is to express the weight function in terms of two functional ratios, one of which is a function-to-function ratio and the other of which is a derivative-to-derivative ratio. To determine what type of initial values of the proposed iterative methods chosen near the zero α must be given for their ensured convergence, we should carefully investigate the dynamics behind the basins of attraction for extraneous fixed points of the corresponding iterative maps applied to a well-known polynomial $p(z) = (z^2 - 1)^m$. In our future work developing a family of new higher-order multiple-zero finders, it would be essential to improve our current approach with the use of principal analytic branches of two functional ratios in selecting free parameters of the weight functions that enhance relevant basins of attraction for a wide class of polynomials.

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