Multiple Bifurcations in a Reaction-Diffusion Problem

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Abstract—We study multiple bifurcations in a system of reaction-diffusion equations defined on the unit square. First, we investigate linear stability of the system at the uniform steady-state solution. Then we discuss necessary conditions for mode interactions. In particular, we show that a corank-four bifurcation point of the system can be easily generated by the mode interactions of two double-bifurcation points. We also study the possibilities of preserving the multiplicity of bifurcation points in the discrete system. A continuation-BCG algorithm proposed by Chien et al. is exploited to trace the solution branches. Finally, sample numerical results are reported.

Keywords—Reaction-diffusion equations, Double bifurcation, Continuation methods.

1. INTRODUCTION

In this paper, we will be concerned with the numerical solutions of a system of two-dimensional reaction-diffusion equations

\begin{align*}
\frac{\partial u}{\partial t} & = d_1 \Delta u + f(u, v, \lambda), \\
\frac{\partial v}{\partial t} & = d_2 \Delta v + g(u, v, \lambda),
\end{align*}

for all \( t \geq 0 \), subject to the homogeneous Dirichlet boundary conditions

\begin{align*}
u(x, y, t) &= u_0, \\
v(x, y, t) &= v_0, \\
(x, y) &\in \partial \Omega.
\end{align*}

Here the unknowns \( u, v \) are state variables which represent concentrations of some intermediate chemicals in the reaction, \( d_1 \) and \( d_2 \) are diffusion rates, while \( \lambda \) is one of the control parameters in the system, e.g., initial or final products, catalysts, temperature, etc. Furthermore, \( (u_0, v_0) \) is a uniform steady-state solution which is independent of the variables \( t, x, y \), and satisfies

\[ f(u_0, v_0, \lambda) = g(u_0, v_0, \lambda) = 0. \]

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A typical example of (1.1) is the well-known Brusselator equations (cf. [1–3]) where
\[
\begin{align*}
f(u, v, \lambda) &= -(\lambda + 1)u + u^2v + \alpha, \\
g(u, v, \lambda) &= \lambda u - u^2v,
\end{align*}
\] (1.3)
with boundary conditions
\[
u(x, y, t) = \alpha, \quad v(x, y, t) = \frac{\lambda}{\alpha}, \quad (x, y) \in \partial\Omega.
\] (1.4)

The interaction of diffusion and reaction, measured by the diffusion term \( \Delta \) with rates \( d_1, d_2 \) and the reaction terms \( f(u, v, \lambda), g(u, v, \lambda) \), can yield rich bifurcation phenomena. During the past decades, some researchers have used various mathematical methods to analyze multiple bifurcation behavior of certain systems of reaction-diffusion equations such as (1.1), see, e.g., [3,4]. On the other hand, numerical methods and computer simulations are also applied to tackle this problem. We refer to [5–7] for details. Recently, Chien et al. [8] have investigated multiple bifurcations of the one-dimensional counterpart of (1.1) by using continuation methods [9,10]. The unsymmetric Lanczos method was used to solve the linear systems of equations associated with the discrete problem. Bifurcation points along the trivial solution curve were also treated numerically.

In this paper, we extend our study to the two-dimensional problems, namely, (1.1). It is obvious that the bifurcation scenario of (1.1) is more complicated than the one-dimensional counterpart. Actually, a corank-four bifurcation point of (1.1) defined on the unit square can be easily generated by the mode interactions [4,11,12] of two double-bifurcation points. It would be interesting to study how many nontrivial solution curves bifurcate from a corank-four bifurcation point. We show that for a bifurcation problem with multiple parameters, a multiple bifurcation of corank greater than or equal to two can persist after discretizations if one chooses the parameters properly. By doing this, one may reduce imperfection of the bifurcations.

This paper is organized as follows. In Section 2, we study linear stability of system (1.1) at the uniform steady-state solution. Then we discuss necessary conditions for mode interaction. In particular, we show that a corank-four bifurcation point of (1.1) defined on a unit square can be easily generated by the mode interactions of two double-bifurcation points. In Section 3, we study the central difference analogue of (1.1) in the context of continuation methods. In particular, we study the possibilities of preserving the multiplicity of bifurcation points in the discrete problems. A continuation-BCG algorithm described in [13] is discussed in Section 4 to trace the bifurcating solution branches. Finally, our numerical results are reported in Section 5, where the Brusselator is used as our test problem.

2. LINEAR STABILITY ANALYSIS

2.1. General System

To simplify the discussions and the numerical approximations, we shift the homogeneous states \((u_0, v_0)\) to \((0, 0)\) by the transformation \((u, v) = (u_0 + \bar{u}, v_0 + \bar{v})\). Furthermore, we incorporate explicitly the length \(\ell\) into the equations by the transformation \(x = \ell\tilde{x}\), which changes the domain \(\Omega = [0, \ell] \times [0, 1]\) to the unit square \(\tilde{\Omega} = [0, 1] \times [0, 1]\), and equations (1.1) into
\[
\begin{align*}
\frac{\partial \bar{u}}{\partial \ell} &= d_1 \left( \frac{1}{\ell^2} \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) + f(u_0 + \bar{u}, v_0 + \bar{v}, \lambda), \\
\frac{\partial \bar{v}}{\partial \ell} &= d_2 \left( \frac{1}{\ell^2} \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right) + g(u_0 + \bar{u}, v_0 + \bar{v}, \lambda),
\end{align*}
\] (2.1)
in \(\tilde{\Omega} = [0, 1] \times [0, 1]\).

For simplicity, we denote \(\tilde{\Omega}, \tilde{x}, f(u_0 + \bar{u}, v_0 + \bar{v}, \lambda), g(u_0 + \bar{u}, v_0 + \bar{v}, \lambda),\) and \(\bar{u}, \bar{v}\) again by \(\Omega, x, f(u, v, \lambda), g(u, v, \lambda)\) and \(u, v\), respectively, when no confusion arises.
Let
\[ C^2_0(\Omega) := \{ u \in C^2(\Omega); u|_{\partial\Omega} = 0 \} \]
and
\[ X := (C^2_0(\Omega))^2, \quad Y := (C(\Omega))^2. \]

We rewrite (2.1) as an operator equation
\[
\frac{\partial u}{\partial t} = \Phi(u, \lambda),
\]
where \( u := (u, v) \), and the mapping \( \Phi : X \times \mathbb{R} \to Y \) is defined by
\[
\Phi(u, \lambda) := \begin{pmatrix}
\frac{d_1 \partial^2 u}{\ell^2 \partial x^2} + \frac{d_1 \partial^2 u}{\partial y^2} & 0 \\
0 & \frac{d_2 \partial^2 v}{\ell^2 \partial x^2} + \frac{d_2 \partial^2 v}{\partial y^2}
\end{pmatrix} + \begin{pmatrix}
f_0(u, v, \lambda) \\
g_0(u, v, \lambda)
\end{pmatrix}. \]

Differentiating \( \Phi \) with respect to \( u \) at the homogeneous equilibrium \( u_0 = (u_0, v_0) = (0, 0) \), we obtain the linearization \( L \) of \( \Phi \),
\[
L := D_u \Phi(u_0, \lambda) = \begin{pmatrix}
\frac{d_1 \partial^2 u}{\ell^2 \partial x^2} + \frac{d_1 \partial^2 u}{\partial y^2} & 0 \\
0 & \frac{d_2 \partial^2 v}{\ell^2 \partial x^2} + \frac{d_2 \partial^2 v}{\partial y^2}
\end{pmatrix} + \begin{pmatrix}
f_0'(0, 0, \lambda) \\
g_0'(0, 0, \lambda)
\end{pmatrix}.
\]

Thereafter, stabilities of \( u_0 \) can be analyzed via solutions of the variational problem
\[
\frac{\partial u}{\partial t} = Lu. \tag{2.4}
\]
To examine the spectrum of \( L \), we observe that the direct sum
\[
X = \bigoplus_{m,n=1}^\infty \oplus X_{m,n}, \quad X_{m,n} := \left\{ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \sin m\pi x \cdot \sin n\pi y; \ c_1, c_2 \in \mathbb{R} \right\}, \quad m, n \in \mathbb{N} \tag{2.5}
\]
holds, and the fact that \( L \) maps \( X_{m,n} \) into itself. More precisely,
\[
L \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \sin m\pi x \cdot \sin n\pi y = \begin{pmatrix}
-d_1 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + f_0^u \\
-s_0^u
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \sin m\pi x \cdot \sin n\pi y.
\]

Here and in the sequel, \( f_0^u, f_0^v, \ldots \) represent the partial derivatives of \( f \) with respect to \( u \) and \( v \) at \((0, 0, \lambda)\), respectively. The restriction of \( L \) in the subspace \( X_{m,n} \) is a \( 2 \times 2 \) matrix
\[
M_{m,n}(\lambda, \ell) := L|_{X_{m,n}} = \begin{pmatrix}
-d_1 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + f_0^u \\
-s_0^u
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad m, n = 1, 2, \ldots \tag{2.6}
\]
The eigenvalues of $L$ consist of those of $M_{m,n} \in \mathbb{R}^{2\times 2}$, $m, n = 1, 2, \ldots$. On the other hand, the eigenvalues $\nu_1, \nu_2$ of a matrix $M \in \mathbb{R}^{2\times 2}$ have the properties
\[
\det(M) = \nu_1 \nu_2, \quad \text{trace } (M) = \nu_1 + \nu_2. \tag{2.7}
\]
In particular, if $\det(M) > 0$ and $\text{trace } (M) = 0$, then $\nu_1$ and $\nu_2$ are pure imaginary.

Since
\[
\det(M_{m,n}(\lambda, \ell)) = \left( -d_1 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + f_u^0 \right) \left( -d_2 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + g_x^0 \right) - f_v^0 g_y^0, \tag{2.8}
\]
\[
\text{trace } (M_{m,n}(\lambda, \ell)) = \left( -d_1 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + f_u^0 \right) + \left( -d_2 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + g_x^0 \right), \tag{2.9}
\]
a stationary bifurcation occurs on $u = (u, v) \equiv 0$ at $\lambda = \lambda_0$ if and only if there exist $m_0, n_0 \in \mathbb{N}$ such that
\[
\det(M_{m_0,n_0}(\lambda_0, \ell)) = 0. \tag{2.10}
\]
Similarly, a Hopf bifurcation takes place at $\lambda = \lambda^*$ if and only if for some $m_0, n_0 \in \mathbb{N}$, we have
\[
\text{trace } (M_{m_0,n_0}(\lambda^*, \ell)) = 0, \quad \text{det } (M_{m_0,n_0}(\lambda^*, \ell)) > 0. \tag{2.11}
\]

If we consider the length $\ell$ as the second-bifurcation parameter, mode interactions of (2.2) can be classified as follows.

A multiple bifurcation point on the trivial solution curve $\{(0, \lambda); \lambda \in \mathbb{R}\}$ for some $\lambda = \lambda_0$, $\ell = \ell_0$ correspond to the fact that zero is a multiple eigenvalues of $L$. This may happen if one of the following situations holds.

(i) If there exists a positive integer pair $(m, n)$ with $m \neq n$ such that
\[
\det(M_{m,n}(\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \text{det } (M_{n,m}(\lambda_0, \ell_0)) = 0, \tag{2.12}
\]
then $(0, \lambda_0)$ is a bifurcation of corank $\geq 2$.

**Lemma 1.** For $\ell_0 = 1$ and $m \neq n$, we have
\[
\text{det } (M_{m,n}(\lambda_0, \ell_0)) = 0, \quad \text{if and only if } \text{det } (M_{n,m}(\lambda_0, \ell_0)) = 0.
\]

**Proof.** From (2.8), we have
\[
\det(M_{m,n}(\lambda_0, \ell_0)) = d_1 d_2 \pi^4 \left( \frac{m^2}{\ell^2} + n^2 \right)^2
- \left( f_{u}^0 d_2 + g_{x}^0 d_1 \right) \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + f_{u}^0 g_{x}^0 - f_{v}^0 g_{y}^0, \tag{2.13}
\]
\[
\det(M_{n,m}(\lambda_0, \ell_0)) = d_1 d_2 \pi^4 \left( \frac{n^2}{\ell^2} + m^2 \right)^2
- \left( f_{u}^0 d_2 + g_{x}^0 d_1 \right) \left( \frac{n^2}{\ell^2} + m^2 \right) \pi^2 + f_{u}^0 g_{x}^0 - f_{v}^0 g_{y}^0. \tag{2.14}
\]
For $\ell_0 = 1$, the result follows immediately from (2.13) and (2.14).

Lemma 1 shows that system (2.4) preserves the multiplicity of the eigenvalues of the linear eigenvalue problem
\[
\Delta u + \lambda u = 0, \quad \text{in } \Omega = [0, 1]^2,
\]
\[
u = 0, \quad \text{on } \partial \Omega.
\]
Actually for $\ell_0 \neq 1$ if (2.12) holds, then
\[
d_1 d_2 \pi^2 \left[ \left( \frac{m^2}{\ell^2} + n^2 \right)^2 - \left( \frac{n^2}{\ell^2} + m^2 \right)^2 \right] = \left( f_0^0 d_2 + g_0^0 d_1 \right) \left( \left( \frac{m^2}{\ell^2} + n^2 \right) - \left( \frac{n^2}{\ell^2} + m^2 \right) \right),
\]
which implies that
\[
\left( \frac{1}{\ell^2} + 1 \right) \left( m^2 + n^2 \right) = \frac{f_0^0 d_2 + g_0^0 d_1}{d_1 d_2 \pi^2}.
\]
Thus, in general, $(0, \lambda_{m,n})$ is a simple bifurcation point. However, $(0, \lambda_{m,n})$ can be a multiple bifurcation point by choosing an appropriate length $\ell_0$.

(ii) Steady/steady state mode interactions of two simple bifurcations. If there exist $(m_1, m_1), (m_2, m_2) \in \mathbb{N}^2$ with $m_1 \neq m_2$, such that
\[
\det (M_{m_1, m_1} (\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{m_2, m_2} (\lambda_0, \ell_0)) = 0,
\]
then $(0, \lambda_0)$ is a bifurcation of corank $\geq 2$. We have
\[
d_1 d_2 m_1^4 \pi^4 \left( \frac{m_1^4}{\ell^2} + 1 \right)^2 - \left( f_0^0 d_2 + g_0^0 d_1 \right) \left( \frac{1}{\ell^2} + 1 \right) m_1^2 \pi^2 + f_0^0 g_0^0 - f_0^0 g_0^0 = 0, \tag{2.15a}
\]
\[
d_1 d_2 m_2^4 \pi^4 \left( \frac{m_2^4}{\ell^2} + 1 \right)^2 - \left( f_0^0 d_2 + g_0^0 d_1 \right) \left( \frac{1}{\ell^2} + 1 \right) m_2^2 \pi^2 + f_0^0 g_0^0 - f_0^0 g_0^0 = 0. \tag{2.15b}
\]
Equation (2.15a) together with (2.15b) imply that
\[
d_1 d_2 \pi^2 \left( \frac{1}{\ell^2} + 1 \right) \left( m_1^4 - m_2^4 \right) = \left( f_0^0 d_2 + g_0^0 d_1 \right) \left( \frac{1}{\ell^2} + 1 \right) \left( m_1^2 - m_2^2 \right)
\]
or
\[
\left( \frac{1}{\ell^2} + 1 \right) \left( m_1^2 + m_2^2 \right) = \frac{f_0^0 d_2 + g_0^0 d_1}{d_1 d_2 \pi^2}. \tag{2.16}
\]
In this case, we obtain a multiple bifurcation point of corank $\geq 2$ by choosing an appropriate length $\ell_0$ as well.

(iii) Steady/steady state mode interactions of one simple and one double bifurcation. If there exist $(m_1, m_1), (m_2, n_2) \in \mathbb{N}^2$ with $m_2 \neq n_2$, such that
\[
\det (M_{m_1, m_1} (\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{m_2, n_2} (\lambda_0, \ell_0)) = 0,
\]
then $(0, \lambda_0)$ is a bifurcation of corank $\geq 2$. In particular, for $\ell_0 = 1$ it follows from Lemma 1 that $(0, \lambda_0)$ is of corank $\geq 3$.

(iv) Steady/steady state mode interactions of two double-bifurcations. If there exist $(m_1, n_1), (m_2, n_2) \in \mathbb{N}^2$ with $m_1 \neq n_1, m_2 \neq n_2$, and $(m_2, n_2) \neq (n_1, n_1)$ such that
\[
\det (M_{m_1, n_1} (\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{m_2, n_2} (\lambda_0, \ell_0)) = 0,
\]
then $(0, \lambda_0)$ is a bifurcation of corank $\geq 2$. In particular, for $\ell_0 = 1$ $(0, \lambda_0)$ is of corank $\geq 4$.

The Hopf/steady state mode interactions occur if there are $\lambda_0, \ell_0$ and $(m_1, n_1), (m_2, n_2) \in \mathbb{N}^2$ such that
\[
\text{trace} (M_{m_1, n_1} (\lambda_0, \ell_0)) = 0, \quad \det (M_{m_1, n_1} (\lambda_0, \ell_0)) > 0, \quad \text{and} \quad \det (M_{m_2, n_2} (\lambda_0, \ell_0)) = 0.
\]
We have
\[
\begin{align*}
\frac{d_1}{\ell^2} \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 + f_u^0 + \left( -d_2 \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 + g_v^0 \right) &= 0, \\
\frac{d_1}{\ell^2} \left( \frac{m_2^2}{\ell^2} + n_2^2 \right) \pi^2 + f_u^0 + \left( -d_2 \left( \frac{m_2^2}{\ell^2} + n_2^2 \right) \pi^2 + g_v^0 \right) &= 0.
\end{align*}
\]
Equation (2.17a) implies that
\[
\left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 = \frac{f_u^0 + g_v^0}{d_1 + d_2}.
\]
Substituting the above equation into (2.17b), we obtain
\[
\left( \frac{f_u^0 + g_v^0}{d_1 + d_2} \right)^2 - \left( \frac{f_u^0 + g_v^0}{d_1 + d_2} \right) \left( \frac{f_u^0 + g_v^0}{d_1 + d_2} \right) + \left( \frac{f_u^0 g_v^0 - f_v^0 g_u^0}{d_1 d_2} \right) > 0.
\]
Thus a Hopf/steady state mode interactions occur at \( u = 0, \lambda = \lambda_0, \ell = \ell_0 \), if (2.17c) and (2.18) hold.

The Hopf/Hopf mode interactions occur if there are \( \lambda_0, \ell_0 \) and \((m_1, n_1), (m_2, n_2) \in \mathbb{N}^2 \) such that
\[
\text{trace} \left( M_{m_1, n_1} (\lambda_0, \ell_0) \right) = 0, \quad \det \left( M_{m_1, n_1} (\lambda_0, \ell_0) \right) > 0,
\]
and
\[
\text{trace} \left( M_{m_2, n_2} (\lambda_0, \ell_0) \right) = 0, \quad \det \left( M_{m_2, n_2} (\lambda_0, \ell_0) \right) > 0.
\]
We have
\[
\begin{align*}
\frac{d_1}{\ell^2} \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 + f_u^0 + \left( -d_2 \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 + g_v^0 \right) &= 0, \\
\frac{d_1}{\ell^2} \left( \frac{m_2^2}{\ell^2} + n_2^2 \right) \pi^2 + f_u^0 + \left( -d_2 \left( \frac{m_2^2}{\ell^2} + n_2^2 \right) \pi^2 + g_v^0 \right) &= 0.
\end{align*}
\]
Equations (2.19a) and (2.19c) imply that
\[
\left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 = \frac{f_u^0 + g_v^0}{d_1 + d_2}, \quad \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 = \frac{f_u^0 + g_v^0}{d_1 + d_2},
\]
which can be simplified as
\[
\frac{1}{\ell^2} \left( \frac{m_1^2}{\ell^2} - n_1^2 \right) = n_2^2 - n_1^2.
\]
Thus, a Hopf/Hopf mode interactions occur at \( u = 0, \lambda = \lambda_0, \ell = \ell_0 \) if there exist \((m_1, n_1), (m_2, n_2) \in \mathbb{N}^2 \) such that (2.19b), (2.19d), and (2.20) are satisfied. Finally, we remark here that the bifurcations of (1.1) possess reflection symmetry of the domain, see, e.g., [4,12]. However, as was pointed out by Duncan and Eilbeck [14], symmetry-breaking bifurcations do exist in reaction-diffusion systems. Our numerical results in Section 5 verify this fact, too. Thus, it would be difficult to study the reduced system of (1.1) on a fundamental domain of \( \Omega \), see, e.g., [15,16]. And last but not the least, it would be interesting to study how many solution curves branching from a multiple bifurcation. This may be executed by investigating the algebraic bifurcation equations (ABE) associated with system (1.1), see [17,18] for more detailed discussions. The ideal of ABE has been extensively studied in [8,15,16] for the one-dimensional counterpart of (1.1), and for a class of second-order semilinear elliptic problems, respectively. In order to keep this paper from becoming too lengthy, we skip similar discussions here.
2.2. The Brusselator Equations

The discussion in the previous section for general systems may be carried over to the Brusselator. Since one of our aims in this paper is to study the bifurcations of the Brusselator by using the numerical continuation methods, it is reasonable to also discuss this special example.

The Brusselator equations have a uniform steady-state solution \((u_0, v_0) = (\alpha, \lambda/\alpha)\). After shifting \((u_0, v_0)\) to the origin 0, we obtain

\[
f(u, v, \lambda) = (\lambda - 1)u + \alpha^2 v + \frac{\lambda}{\alpha} u^2 + 2\alpha uv + u^2 v,
\]

\[
g(u, v, \lambda) = -\lambda u - \alpha^2 v - \frac{\lambda}{\alpha} u^2 + 2\alpha uv + u^2 v.
\]

Statement (2.6) becomes

\[
M_{m,n}(\lambda, \ell) = \begin{pmatrix}
-d_1 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 + \lambda - 1 & \alpha^2 \\
-\lambda & -d_2 \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 - \alpha^2
\end{pmatrix}.
\]

Hence,

\[
\text{trace } (M_{m,n}(\lambda, \ell)) = - (d_1 + d_2) \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 - \alpha^2 - 1 + \lambda,
\]

\[
det (M_{m,n}(\lambda, \ell)) = d_1 d_2 \pi^4 \left( \frac{m^2}{\ell^2} + n^2 \right)^2
\]

\[
+ \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 \left[ (1 - \lambda) d_2 + \alpha^2 d_1 \right] + \alpha^2.
\]

Since the trace of \(M_{m,n}\) changes sign at

\[
\lambda = 1 + \alpha^2 + (d_1 + d_2) \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2,
\]

a Hopf bifurcation occurs when \(\lambda\) is chosen as (2.25) for some \(m, n \in \mathbb{N}\).

A stationary bifurcation corresponds to \(\text{det}(M_{m,n}) = 0\) for some \(m, n \in \mathbb{N}\), which is satisfied by

\[
\lambda_{m,n} := 1 + d_1 \pi^2 \left( \frac{m^2}{\ell^2} + n^2 \right) + \alpha^2 \frac{d_1}{d_2} + \frac{\alpha^2}{\pi^2 d_2} \left( \frac{\ell^2}{m^2 + \ell^2 n^2} \right) > 0.
\]

Furthermore, if \(\lambda > \lambda_{m,n}\), then \(\text{det}(M_{m,n}) < 0\). In this case, the two eigenvalues of \(M_{m,n}\) are real and have opposite signs. Consequently, the uniform steady-state solution is unstable for \(\lambda > \lambda_{m,n}, m, n \in \mathbb{N}\). In conclusion, whenever

\[
\lambda > \min_{m, n \in \mathbb{N}} \left\{ \lambda_{m,n}, 1 + \alpha^2 + (d_1 + d_2) \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2 \right\},
\]

at which either a Hopf or a stationary bifurcation takes place, the uniform steady-state solution is unstable.

At a Hopf bifurcation point \(\lambda_0\), there exist \(m, n \in \mathbb{N}\) such that \(\text{trace}(M_{m,n}) = 0\), i.e.,

\[
\lambda_0 = 1 + \alpha^2 + (d_1 + d_2) \left( \frac{m^2}{\ell^2} + n^2 \right) \pi^2.
\]
Moreover, if we set $c := \frac{m^2}{\ell^2} + n^2$, then from (2.24)

$$\det (M_{m,n} (\lambda_0, \ell)) = d_1 d_2 \pi^4 c^2 + c \pi^2 \left[ (1 - \lambda) d_2 + \alpha^2 d_1 \right] + \alpha^2$$

$$= -d_2^2 \left[ c \pi^2 - \frac{1}{2} \left( \frac{\alpha}{d_2} \right)^2 (d_1 - d_2) + \frac{1}{2} \sqrt{4 \left( \frac{\alpha}{d_2} \right)^2 + \left( \frac{\alpha}{d_2} \right)^4 (d_1 - d_2)^2} \right] \times \left[ c \pi^2 - \frac{1}{2} \left( \frac{\alpha}{d_2} \right)^2 (d_1 - d_2) - \frac{1}{2} \sqrt{4 \left( \frac{\alpha}{d_2} \right)^2 + \left( \frac{\alpha}{d_2} \right)^4 (d_1 - d_2)^2} \right]$$

$$> 0.$$

Since

$$c \pi^2 - \frac{1}{2} \left( \frac{\alpha}{d_2} \right)^2 (d_1 - d_2) + \frac{1}{2} \sqrt{4 \left( \frac{\alpha}{d_2} \right)^2 + \left( \frac{\alpha}{d_2} \right)^4 (d_1 - d_2)^2} \geq c \pi^2 > 0,$$

we must have

$$c \pi^2 - \frac{1}{2} \left( \frac{\alpha}{d_2} \right)^2 (d_1 - d_2) - \frac{1}{2} \sqrt{4 \left( \frac{\alpha}{d_2} \right)^2 + \left( \frac{\alpha}{d_2} \right)^4 (d_1 - d_2)^2} < 0$$

or

$$\frac{m^2 \pi^2}{\ell^2} < \frac{1}{2} \left[ \left( \frac{\alpha}{d_2} \right)^2 (d_1 - d_2) + \sqrt{4 \left( \frac{\alpha}{d_2} \right)^2 + \left( \frac{\alpha}{d_2} \right)^4 (d_1 - d_2)^2 - 2n^2 \pi^2} \right].$$

Thus, $\ell$ has to satisfy

$$\ell^2 > \frac{2m^2 \pi^2 d_2^2}{\alpha^2 (d_1 - d_2) + \sqrt{4 \alpha^2 d_2^2 + \alpha^4 (d_1 - d_2)^2 - 2n^2 d_2^2 \pi^2}} > 0. \tag{2.27}$$

In other words, Hopf bifurcations occur only when the domain is large enough, say, at least satisfies (2.27).

A multiple bifurcation of the Brusselator occurs on the trivial solution curve $\{(0, \lambda); \lambda \in \mathbb{R}\}$ if one of the following cases holds.

(i) If there exists an integer pair $(m, n) \in \mathbb{N}^2$ with $m \neq n$ such that

$$\det (M_{m,n} (\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{n,m} (\lambda_0, \ell_0)) = 0,$$

then $(\lambda_0, \ell_0)$ is a bifurcation point of corank $\geq 2$ with

$$\lambda_0 = 1 + \frac{d_1}{d_2} \alpha^2 + d_1 \left( \frac{1}{\ell_0^2} + 1 \right) (m^2 + n^2) \pi^2. \tag{2.28}$$

In particular, for $\ell_0 = 1$ one obtains

$$\lambda_0 = 1 + \frac{d_1}{d_2} \alpha^2 + d_1 \pi^2 (m^2 + n^2) + \frac{\alpha^2}{\pi^2 d_2 (m^2 + n^2)}. \tag{2.29}$$

(ii) Steady/steady state mode interactions of two simple bifurcations. If there exist $(m_1, m_1)$, $(m_2, m_2) \in \mathbb{N}^2$ with $m_1 \neq m_2$ such that

$$\det (M_{m_1,m_1} (\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{m_2,m_2} (\lambda_0, \ell_0)) = 0,$$

then $(0, \lambda_0)$ is at least a double bifurcation point. The two conditions imply that

$$d_1 d_2 \pi^4 \left( \frac{m_1^2}{\ell^2} + m_1^2 \right)^2 + \left( \frac{m_2^2}{\ell^2} + m_2^2 \right) \pi^2 [(1 - \lambda_0) d_2 + \alpha^2 d_1] + \alpha^2 = 0 \quad \tag{2.30a}$$
and
\[ d_1 d_2 \pi^4 \left( \frac{m_1^2}{\ell^2} + m_2^2 \right)^2 + \left( \frac{m_1^2}{\ell^2} + m_2^2 \right) \pi^2 \left[ (1 - \lambda_0)d_2 + \alpha^2 d_1 \right] + \alpha^2 = 0. \]  
(2.30b)

The equations (2.30a) together with (2.30b) imply that
\[ \left( \frac{1}{\ell^2} + 1 \right) (m_1^2 + m_2^2) = \frac{(\lambda_0 - 1)d_2 - \alpha^2 d_1}{d_1 d_2 \pi^2}. \]  
(2.31)

Thus, the multiple bifurcation occurs at
\[ \lambda_0 = 1 + \frac{d_1}{d_2} \alpha^2 + d_1 \left( \frac{1}{\ell^2} + 1 \right) (m_1^2 + m_2^2) \pi^2. \]  
(2.32)

(iii) Steady/steady state mode interactions of one simple and one double-bifurcations. If there exist \((m_1, m_1), (m_2, n_2) \in \mathbb{N}^2\) with \(m_2 \neq n_2\) such that
\[ \det (M_{m_1, m_1}(\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{m_2, n_2}(\lambda_0, \ell_0)) = 0, \]
then \((0, \lambda_0)\) is a multiple bifurcation point of corank \(\geq 2\). In particular, if \(\ell_0 = 1\) and \(m_1, m_2, n_2\) satisfy
\[ 2 \cdot m_1^2 = m_2^2 + n_2^2, \]
it follows from Lemma 1 that \((0, \lambda_0)\) is a bifurcation point of corank \(\geq 3\).

(iv) Steady/steady state mode interactions of two double-bifurcations. If there exist \((m_1, n_1), (m_2, n_2) \in \mathbb{N}^2\) with \(m_1 \neq n_1, m_2 \neq n_2, (m_2, n_2) \neq (n_1, m_1)\) such that
\[ \det (M_{m_1, n_1}(\lambda_0, \ell_0)) = 0 \quad \text{and} \quad \det (M_{m_2, n_2}(\lambda_0, \ell_0)) = 0, \]
then \((0, \lambda_0)\) is a multiple bifurcation point of corank \(\geq 2\). In particular, if \(\ell_0 = 1\) and \(m_1, m_2, n_2\) satisfy
\[ m_1^2 + n_1^2 = m_2^2 + n_2^2, \]
then \((0, \lambda_0)\) is a bifurcation point of corank \(\geq 4\).

The Hopf/steady state mode interactions occur if there are \(\lambda_0, \ell_0, (m_1, n_1), (m_2, n_2) \in \mathbb{N}^2\) such that
\[ \text{trace} (M_{m_1, n_1}(\lambda_0, \ell_0)) = -(d_1 + d_2) \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 - \alpha^2 - 1 + \lambda = 0, \]  
(2.33a)

\[ \det (M_{m_1, n_1}(\lambda_0, \ell_0)) = d_1 d_2 \pi^4 \left( \frac{m_1^2}{\ell^2} + n_1^2 \right)^2 \]
\[ + \left( \frac{m_1^2}{\ell^2} + n_1^2 \right) \pi^2 \left[ (1 - \lambda)d_2 + \alpha^2 d_1 \right] + \alpha^2 > 0, \]  
(2.33b)

\[ \det (M_{m_2, n_2}(\lambda_0, \ell_0)) = d_1 d_2 \pi^4 \left( \frac{m_2^2}{\ell^2} + n_2^2 \right)^2 \]
\[ + \left( \frac{m_2^2}{\ell^2} + n_2^2 \right) \pi^2 \left[ (1 - \lambda)d_2 + \alpha^2 d_1 \right] + \alpha^2 = 0 \]  
(2.33c)

hold, then a Hopf/steady state mode interactions occur at \(u = 0, \lambda = \lambda_0, \ell = \ell_0\). The equations in (2.33) make up a generically solvable system for \(\lambda_0, \ell_0\).

Finally, a Hopf/Hopf mode interactions occur at \(u = 0, \lambda = \lambda_0, \ell = \ell_0\) if there exist \((m_1, n_1), (m_2, n_2) \in \mathbb{N}^2\), such that the following equations:
\[ \text{trace} (M_{m_1, n_1}(\lambda_0, \ell_0)) = 0, \quad \det (M_{m_1, n_1}(\lambda_0, \ell_0)) > 0 \]
and
\[ \text{trace} (M_{m_2, n_2}(\lambda_0, \ell_0)) = 0, \quad \det (M_{m_2, n_2}(\lambda_0, \ell_0)) > 0 \]
hold.
3. CENTRAL DIFFERENCE APPROXIMATIONS

In this section, we discuss discretizations of system (1.1) with boundary conditions (1.2) by using the central difference method, and bifurcations of the discrete problem. In particular, we will show how multiple bifurcations of (1.1) may be preserved after discretizations. Similar discussions can be found in [8,15].

3.1. General Systems

Let

\[ \Omega_h = \{(x_i, y_j); 0 < x_1 < \cdots < x_N < 1, \ 0 < y_1 < \cdots < y_N < 1, \ x_i = i \cdot h, \ y_j = j \cdot h\} \]

be a uniform grid on the unit square with meshsize \( h = 1/(N + 1) \). At the discrete points, the functions \( u(x,y) \) and \( v(x,y) \) are approximated by the net functions \( U(x_i, y_j) := u_{i,j} \) and \( V(x_i, y_j) := v_{i,j} \), \( i,j = 1, \ldots, N \). Let \( U, V \in \mathbb{R}^{N^2} \) be defined by

\[ U := (U_{1,1}, \ldots, U_{N,1}, U_{1,2}, \ldots, U_{N,2}, \ldots, U_{1,N}, \ldots, U_{N,N})^T \]

and

\[ V := (V_{1,1}, \ldots, V_{N,1}, V_{1,2}, \ldots, V_{N,2}, \ldots, V_{1,N}, \ldots, V_{N,N})^T. \]

Discretizing \( \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2} \) with the central difference schemes, and with \( F, G \) denoting the net functions associated with the functions \( f \) and \( g \), respectively, we obtain the discretization of \( (u, \lambda) \equiv 0 \), namely,

\[ H(Z, \lambda) := \begin{pmatrix} d_1 A U \\ d_2 A V \end{pmatrix} - \ell^2 h^2 \begin{pmatrix} F(Z, \lambda) \\ G(Z, \lambda) \end{pmatrix} = 0, \quad Z := (U, V)^T, \quad (3.1) \]

where

\[ A = \begin{pmatrix} B_N + 2\ell^2 I_N & \ell^2 I_N \\ \ell^2 I_N & B_N + 2\ell^2 I_N \end{pmatrix} \in \mathbb{R}^{N^2 \times N^2}, \quad (3.2) \]

with

\[ B_N = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ & & \ddots & \ddots & \ddots \\ & & & -1 \\ & & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{N \times N} \]

and

\[ F(Z, \lambda) = (f_{1,1}, \ldots, f_{N,1}, f_{1,2}, \ldots, f_{N,2}, \ldots, f_{N,N})^T, \]

\[ G(Z, \lambda) = (g_{1,1}, \ldots, g_{N,1}, g_{1,2}, \ldots, g_{N,2}, \ldots, g_{N,N})^T. \]

Evidently, \( H(0, \lambda) \equiv 0 \), for all \( \lambda \in \mathbb{R} \). The Jacobian matrix corresponding to (3.1) is

\[ DH(Z, \lambda) = \begin{pmatrix} d_1 A - h^2 \ell^2 D_U F & -\ell^2 h^2 D_U G \\ -\ell^2 h^2 D_V F & d_2 A - h^2 \ell^2 D_V G \end{pmatrix}, \]

where \( D_U F, D_V F, D_\lambda F, D_U G, D_V G, D_\lambda G \) are given by

\[ D_U F(Z, \lambda) = \text{diag} \ (f_u(U_{1,1}, V_{1,1}, \lambda), \ldots, f_u(U_{N,N}, V_{N,N}, \lambda)), \quad D_U F(0, \lambda) = f_u^0 I_{N^2}, \]

\[ D_V F(Z, \lambda) = \text{diag} \ (f_v(U_{1,1}, V_{1,1}, \lambda), \ldots, f_v(U_{N,N}, V_{N,N}, \lambda)), \quad D_V F(0, \lambda) = f_v^0 I_{N^2}, \]

\[ D_\lambda F(Z, \lambda) = [f_\lambda(U_{1,1}, V_{1,1}, \lambda), \ldots, f_\lambda(U_{N,N}, V_{N,N}, \lambda)]^T, \quad D_\lambda F(0, \lambda) = 0. \]

The partial derivatives \( D_U G(Z, \lambda), D_V G(Z, \lambda), D_\lambda G(Z, \lambda) \) are defined in a similar way.
Along the trivial solution curve $Z \equiv 0$, $\lambda \in \mathbb{R}$, we have

$$DH(0, \lambda) = \begin{pmatrix} d_1 A - h^2 2 f_u^0 I_{N^2} & -h^2 2 f_u^0 I_{N^2} & 0 \\ -h^2 2 g_u^0 I_{N^2} & d_2 A - h^2 2 g_v^0 I_{N^2} & 0 \end{pmatrix} := (L_h 0^T),$$

(3.3)

where

$$L_h := \begin{pmatrix} d_1 A - h^2 2 f_u^0 I_{N^2} & -h^2 2 f_u^0 I_{N^2} \\ -h^2 2 g_u^0 I_{N^2} & d_2 A - h^2 2 g_v^0 I_{N^2} \end{pmatrix}.$$

Let $R = (r_{i,j}) \in \mathbb{R}^{m \times n}$ and $S = (s_{i,j}) \in \mathbb{R}^{p \times q}$ be arbitrary. We recall that the matrix tensor product of $R$ and $S$, denoted by $R \otimes S$, is a $mp$ by $nq$ matrix and is defined by

$$R \otimes S = \begin{pmatrix} r_{11}S & r_{12}S & \ldots & r_{1n}S \\ r_{21}S & r_{22}S & \ldots & r_{2n}S \\ \vdots & \vdots & \ddots & \vdots \\ r_{mn}S & r_{m2}S & \ldots & r_{mn}S \end{pmatrix}.$$

We denote the eigenpairs of $A$ defined in (3.2) by $(\mu_{p,q}, V_{p,q})$.

**Lemma 2.** The eigenpairs of $A$ are

$$\mu_{p,q} = \mu_p + \ell^2 \mu_q = (2 + 2\ell^2) - 2 \left( \cos \frac{p\pi}{N+1} + \ell^2 \cos \frac{q\pi}{N+1} \right),$$

$$V_{p,q} = V_q \otimes V_p, \quad 1 \leq p, q \leq N,$$

(3.4)

where $(\mu_p, V_p)$ and $(\mu_q, V_q)$ are the eigenpairs of $B_N$.

**Proof.** Note that

$$A = I_N \otimes B_N + \ell^2 B_N \otimes I_N,$$

and the eigenpairs of $B_N$ are (see [19])

$$\mu_p = 2 - 2 \cos \frac{p\pi}{N+1}, \quad p = 1, \ldots, N,$$

$$V_p = \begin{pmatrix} \sin \frac{p\pi}{N+1} & \ldots & \sin \frac{p\pi}{N+1} \end{pmatrix}^T, \quad p = 1, \ldots, N.$$

By [20, p. 9], the result follows.

Let

$$X_{p,q}^h := \mathbb{R}^2 V_{p,q} = \left\{ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} : c_i \in \mathbb{R}, 1 \leq p, q \leq N \right\}.$$

We have

$$\mathbb{R}^{2N^2} = \bigoplus_{i,j=1}^N X_{i,j}^h.$$

(3.5)

The operator $L_h$ maps $X_{p,q}^h$ into itself, and for $c_1, c_2 \in \mathbb{R}$, $1 \leq p, q \leq N$, we have

$$L_h \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} V_{p,q} = \begin{pmatrix} d_1^{\mu_{p,q}} - h^2 2 f_u^0 & -h^2 2 f_u^0 \\ -h^2 2 g_u^0 & d_2^{\mu_{p,q}} - h^2 2 g_v^0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} V_{p,q}.$$

(3.6)

Consequently, under the decomposition (3.5) $L_h$ is a block diagonal matrix, i.e.,

$$L_h = -h^2 2 \ell^2 \text{diag} (M_{1,1}^h, M_{2,1}^h, \ldots, M_{N,1}^h, \ldots, M_{1,N}^h, \ldots, M_{N,N}^h),$$

(3.7)
where
\[ M_{i,j}^h(\lambda, \ell) := \begin{pmatrix} -\frac{d_1 \mu_{i,j}}{\ell^2} + f_u^0 & f_v^0 \\ g_u^0 & -\frac{d_2 \mu_{i,j}}{\ell^2} + g_v^0 \end{pmatrix}, \quad \ell < i, j < N. \] (3.8)

Hence, the spectrum of \( L_h \) is the union of the eigenvalues of \( M_{i,j}^h, 1 \leq i, j \leq N \).

Now suppose that \( \lambda = \lambda_0 \) is a simple bifurcation point of (2.1). Then there exist \( i, j \in \mathbb{N}, 1 \leq i, j \leq N \) such that \( \det(M_{i,j}(\lambda_0, \ell)) = 0 \). So we have
\[
\det(M_{i,j}(\lambda_0, \ell)) = \det(M_{i,j}(\lambda_0, t)) - \det (M_{i,j}(\lambda_0, t)) = 0 \quad (3.9)
\]

As expected, \( \lambda_0 \) is no longer a bifurcation point of the discrete problem. Nevertheless, for \( h > 0 \) sufficiently small, there exists \( \lambda_0^h \in \mathbb{R} \) such that \( \det(M_{i,j}^h(\lambda_0^h, 0)) = 0 \) and \( |\lambda_0^h - \lambda_0| = O(h^2) \). In other words, there exists a simple bifurcation point \((0, \lambda_0)\) of the discrete problem in the \( O(h^2) \) neighborhood of the bifurcation point of the continuous problem.

For \( \ell = \ell_0 \), let \((0, \lambda_0)\) be a double bifurcation point, and let \( M_{m_1,n_1}, M_{m_2,n_2} \) be the two matrices which have zero eigenvalues, respectively. It is easy verify that
\[
\det(M_{m_1,n_1}^h(\lambda_0, \ell_0)) = O(h^2), \quad \det(M_{m_2,n_2}^h(\lambda_0, \ell_0)) = O(h^2).
\]

We show that there exists \((\lambda_0^h, \ell_0^h)\) in the neighborhood of \((\lambda_0, \ell_0)\) such that
\[
\det(M_{m_1,n_1}^h(\lambda_0^h, \ell_0^h)) = \det(M_{m_2,n_2}^h(\lambda_0^h, \ell_0^h)) = 0,
\]
which corresponds to a double bifurcation of the discrete problem.

Define a map \( d : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by
\[
d(\lambda, \ell, h) := \begin{pmatrix} \det(M_{m_1,n_1}^h(\lambda_0, \ell_0)) \\ \det(M_{m_2,n_2}^h(\lambda_0, \ell_0)) \end{pmatrix} \begin{pmatrix} -\frac{d_1 \mu_{m_1,n_1}}{\ell^2} + f_u^0 \\ -\frac{d_2 \mu_{m_2,n_2}}{\ell^2} + g_u^0 \end{pmatrix} + \begin{pmatrix} f_v^0 \\ g_v^0 \end{pmatrix},
\]
and note that for \( h = 0 \),
\[
d(\lambda, \ell, 0) := \begin{pmatrix} \det(M_{m_1,n_1}(\lambda, \ell)) \\ \det(M_{m_2,n_2}(\lambda, \ell)) \end{pmatrix}.
\]

If \( f, g \) are smooth functions, then so is \( d \) for \( \ell \neq 0 \). Note that
\[
d(\lambda_0, \ell_0, 0) = 0.
\]

If the Jacobian
\[
\frac{\partial d(\lambda, \ell, 0)}{\partial(\lambda, \ell)} = \begin{pmatrix} \frac{\partial d(\lambda, \ell, 0)}{\partial \lambda} \\ \frac{\partial d(\lambda, \ell, 0)}{\partial \ell} \end{pmatrix}^T
\]
is nonsingular at \((\lambda_0, \ell_0, 0)\), then by the implicit function theorem for \( h > 0 \) small enough, there is a unique solution curve \((\lambda(h), \ell(h))\) such that
\[
d(\lambda(h), \ell(h), h) = 0
\]
and
\[
|\lambda(h) - \lambda_0| + |\ell(h) - \ell_0| \leq \gamma \cdot \|d(\lambda_0, \ell_0, 0)\| = O(h^2), \quad \text{for some constant } \gamma > 0.
\]
3.2. The Brusselator Equation

For the Brusselator equations, the function $f(u, v, \lambda), g(u, v, \lambda)$ are given in (2.1) and

$$M_{i,j}^h := \begin{pmatrix} -\frac{d_1 \mu_{i,j}}{h^2 \ell^2} + (\lambda - 1) & \alpha^2 \\ \alpha & -\frac{d_2 \mu_{i,j}}{h^2 \ell^2} - \alpha^2 \end{pmatrix}. \quad (3.10)$$

**Theorem 3.** The bifurcation points of the discrete Brusselator equations are

$$(0, \lambda^h_{m,n}) = \left(0, 1 + \left(d_1 + \frac{h^2 \ell^2}{\mu_{m,n}}\right) \frac{\alpha^2}{d_2} + \frac{d_1}{h^2 \ell^2 \mu_{m,n}}\right), \quad 1 \leq m, n \leq N. \quad (3.11)$$

Furthermore, the null vector $(U_{m,n}, V_{m,n})$ of $L_h$ in (3.3) can be chosen as

$$U_{m,n} = U_{m,m}, \quad V_{m,n} = -\left(\frac{d_1}{d_2} + \frac{h^2 \ell^2}{d_2 \mu_{m,n}}\right) U_{m,n}, \quad (3.12)$$

where $\mu_{m,n}$ is given as in (3.4).

**Proof.** Note that the operator $L_h$ is singular if and only if there exist two positive integers $m$ and $n$ with $1 \leq m, n \leq N$ such that $\det(M^h_{m,n}) = 0$. Solving it for $\lambda$, we obtain

$$\lambda^h_{m,n} = 1 + \left(d_1 + \frac{h^2 \ell^2}{\mu_{m,n}}\right) \frac{\alpha^2}{d_2} + \frac{d_1}{h^2 \ell^2 \mu_{m,n}}.$$

Recall the statements (3.6) and (3.7). We see the null vectors of $L_h$ are of the form $(c_1, c_2)^T$ with $(c_1, c_2)^T$ as a null vector of $M^h_{m,n}(\lambda^h_{m,n}, \ell)$, which can be chosen as $c_1 = 1, c_2 = -(d_1/d_2 + h^2 \ell^2/\mu_{m,n}d_2)$. The statement (3.12) follows immediately.

**Theorem 4.** $(0, \lambda^h_{m_1,n_1}) = (0, \lambda^h_{m_2,n_2})$ is a multiple bifurcation of the discrete Brusselator equations if and only if one of the following statements hold:

(i) $\mu_{m_1,n_1} = \mu_{m_2,n_2}$; or
(ii) $\mu_{m_1,n_1} \neq \mu_{m_2,n_2}$ and $\mu_{m_1,n_1} \cdot \mu_{m_2,n_2} = \alpha^2 \ell^2 h^4 / d_1 d_2$.

**Proof.**

(i) It follows from (3.11).

(ii) If $\mu_{m_1,n_1} \neq \mu_{m_2,n_2}$, then by Theorem 3, we have

$$\frac{d_1}{h^2 \ell^2} (\mu_{m_1,n_1} - \mu_{m_2,n_2}) = \alpha^2 \ell^2 h^2 \left(\frac{\mu_{m_1,n_1} - \mu_{m_2,n_2}}{\mu_{m_1,n_1} \cdot \mu_{m_2,n_2}}\right),$$

and the result follows immediately.

4. THE BCG METHOD AND BIFURCATION PROBLEMS

4.1. A Brief Review of the Continuation-BCG Method

We consider the following two linear systems:

$$Ax = b, \quad A^T \tilde{x} = \tilde{b},$$

where $A \in \mathbb{R}^{N \times N}$ is nonsymmetric and nonsingular, $A^T$ denotes the transpose of $A$, and $b, \tilde{b} \in \mathbb{R}^N$. Let $x_0$ and $\tilde{x}_0$ be the given initial guesses with corresponding residuals $r_0 = b - Ax_0$ and $\tilde{r}_0 = b - A^T \tilde{x}_0$. Since we are only interested in solving the first linear system, $\tilde{r}_0$ can be chosen arbitrarily.
Define the two Krylov subspaces $K_n(r_0, A)$ and $K_n(\tilde{r}_0, A^T)$ by

$$K_n(r_0, A) = \text{span} \{r_0, Ar_0, \ldots, A^{n-1}r_0\}$$

and

$$K_n(\tilde{r}_0, A^T) = \text{span} \{\tilde{r}_0, A^T\tilde{r}_0, \ldots, (A^T)^{n-1}\tilde{r}_0\},$$

respectively. Then the BCG iterates $x_n = x_0 + y_n$ which are characterized by the Galerkin conditions:

1. $y_n \in K_n(r_0, A)$;
2. $r_n \perp K_n(\tilde{r}_0, A^T)$, $\tilde{r}_n \perp K_n(r_0, A)$.

The following algorithm computes the BCG iterates $x_n$'s by an efficient and simple iteration [21].

**Algorithm 4.1. (BCG method.)**

1. Given $x_0$ and set $r_0 = b - Ax_0 = p_0$;
   Choose $\tilde{r}_0 \neq 0$ and set $\tilde{p}_0 = \tilde{r}_0, \rho_0 = \tilde{r}_0^T r_0$.
   For $n = 0, 1, 2, \ldots$ do
2. Set $\sigma_n = \tilde{p}_n^T A p_n$;
   If $\sigma_n = 0$: stop. Otherwise compute
   $\alpha_n = \rho_n / \sigma_n$;
   $x_{n+1} = x_n + \alpha_n p_n$;
   $r_{n+1} = r_n - \alpha_n A p_n$;
   $\tilde{r}_{n+1} = \tilde{r}_n - \alpha_n A^T \tilde{p}_n$.
3. If $\rho_n = 0$: stop. Otherwise compute
   $\rho_{n+1} = \tilde{r}_{n+1}^T r_{n+1}$;
   $\beta_{n+1} = \rho_{n+1} / \rho_n$;
   $p_{n+1} = r_{n+1} + \beta_{n+1} p_n$;
   $\tilde{p}_{n+1} = \tilde{r}_{n+1} + \beta_{n+1} \tilde{p}_n$.
4. If $r_n = 0$ or $\tilde{r}_n = 0$: stop.

Let $R_n = [r_0, r_1, \ldots, r_{n-1}], \tilde{R}_n = [\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_{n-1}]$, and similarly for $P_n$ and $\tilde{P}_n$. Note that $R_n, \tilde{R}_n, P_n, \tilde{P}_n \in \mathbb{R}^{n \times n}$. We denote the range of a matrix $A$ by $\text{R}(A)$. The following results can be found in [21,22].

**Theorem 5.** Suppose that the BCG algorithm runs successfully to step $n$ (i.e., $\sigma_i \neq 0, \rho_i \neq 0$, $i = 0, 1, \ldots, n - 1$), then the iterates satisfy:

1. $\text{R}(R_n) = \text{R}(P_n) = K_n(r_0, A), \text{R}(\tilde{R}_n) = \text{R}(\tilde{P}_n) = K_n(\tilde{r}_0, A^T)$,
2. $\tilde{R}_n^T R_n$ is diagonal (biorthogonality property),
3. $\tilde{P}_n^T A P_n$ is diagonal (biconjugacy property).

In our application, we will incorporate Algorithm 4.1 in the context of continuation methods [9,10] to trace the first few solution curves of (3.1) branching from the bifurcation point $(0, \lambda^*) \equiv y^*$. First of all, we will use it to solve the discretized linear systems associated with (3.1). Next, we will exploit it to detect $y^*$ along the trivial solution curve by approximating the condition number of the partial tridiagonalization $T_n$ generated by the BCG method. This may be explained as follows.

We recall in the unsymmetric Lanczos method [22,23], one builds two biorthogonal systems $W_n, V_n$ such that the matrix $W_n^T AV_n$ has tridiagonal form. We then use the extremum eigenvalues of $T_n$ to approximate the counterparts of $A$, see [24] for details. Since the BCG method is theoretically equivalent to the unsymmetric Lanczos method, it is possible to build two biorthogonal systems for $K_n(r_0, A)$ and $K_n(\tilde{r}_0, A^T)$. The idea is similar to the connections between the
Lanczos method and the conjugate gradient method [23]. To begin with, we define the upper bidiagonal matrix \( B_n \in \mathbb{R}^{n \times n} \) by

\[
B_n = \begin{pmatrix}
1 & -\beta_1 & 0 \\
1 & -\beta_2 & \\
& \ddots & -\beta_{n-1} \\
0 & 1 & 
\end{pmatrix}.
\]

From the equations \( p_{j+1} = r_{j+1} + \beta_{j+1} p_j, j = 0, \ldots, n-1, \) and \( p_0 = r_0 \), it follows that \( R_n = P_n B_n \). Similarly, we have \( \tilde{R}_n = \tilde{P}_n B_n \). Let

\[
D := \text{diag}(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}), \quad \tilde{D} := \text{diag}(\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n-1}),
\]

then the columns of \( R_n D^{-1} \) and \( \tilde{R}_n \tilde{D}^{-1} \) are biorthonormal.

By the biconjugacy property of Theorem 5, we have

\[
\left( \tilde{R}_n \tilde{D}^{-1} \right)^T A \left( R_n D^{-1} \right) = \tilde{D}^{-1} B_n^T \left( \tilde{P}_n^T A P_n \right) B_n D^{-1} = \tilde{D}^{-1} B_n^T \text{diag}(\tilde{p}_i^T A p_i) B_n D^{-1} := T_n,
\]

which is tridiagonal. The computation of \( T_n \) is readily available.

We remark here that in [24] the unsymmetric Lanczos was exploited to treat the von Kármán equations defined in the whole domain \([0, 1]^2\). However, our numerical experiments show that one cannot use the minimum eigenvalue of \( T_n \) to approximate the counterpart of \( A \) for the detection of the first simple bifurcation point. To overcome this difficulty, we suggest that one may detect the first bifurcation point by approximating the condition number of \( T_n \). A bifurcation point is monitored on the solution curve if \( T_n \) has relatively small condition number. This aspect has been verified by our numerical experiments. One may estimate the condition number of \( T_n \) as follows.

Suppose that we wish to solve

\[
T_n z = c, \quad c \in \mathbb{R}^n,
\]

in the corrector step of the continuation methods. Then we have

\[
z = T_n^{-1} c \rightarrow \|z\|_{\infty} \leq \|T_n^{-1}\|_{\infty} \cdot \|c\|_{\infty} \rightarrow \|z\|_{\infty} \leq \|T_n^{-1}\|_{\infty} \cdot \|c\|_{\infty}.
\]

The condition number of \( T_n \) in maximum norm, denoted by \( \kappa_{\infty}(T_n) \), is estimated by using

\[
\kappa_{\infty}(T_n) \approx \|T_n\|_{\infty} \cdot \frac{\|z\|_{\infty}}{\|c\|_{\infty}}.
\]

Note that the right-hand side of (4.3) is a lower bound of \( \kappa_{\infty}(T_n) \). We refer to [23] and the references cited therein for details.

### 4.2. Practical Implementations

For simplicity, we denote the augmented Jacobian of \( DH(0, \lambda) \) in equation (3.3) by \( M \), where

\[
M = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}.
\]
Here $A = D_2 H$, $b = D_\lambda H$ and $c \in \mathbb{R}^{2N^2}$, $d \in \mathbb{R}$ should be properly chosen, see, e.g., [9,10]. We recall that in predictor-corrector continuation methods the tangent vectors are obtained by solving linear systems of the form

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} u = \begin{bmatrix} \bar{0} \\ 1 \end{bmatrix},$$

(4.4)

where $\bar{0} \in \mathbb{R}^{2N^2}$. While in the Newton corrector one solves the following linear systems:

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} w = \begin{bmatrix} -H(x, \lambda) \\ 0 \end{bmatrix},$$

(4.5)

Similar to the Continuation-Lanczos algorithm described in [25], we have the following.
Algorithm 4.2. A Continuation-BCG algorithm for (3.1).

\textbf{Input:}
\begin{itemize}
    \item $y \in \mathbb{R}^{2N^2+1}$, such that $H(y) = 0$ \{approximate point on $H^{-1}(0)$\}
    \item $u$ \{initial tangent vector\}
    \item $t > 0$ \{initial step size\}
    \item $\varepsilon^* > 0$ \{an approximation of the zero eigenvalue\}
    \item $c_M$ \{a lower bound for $\kappa_\infty(T_n)$ where a bifurcation point is detected\}
\end{itemize}

\textbf{Step 1.} $v \equiv y + tu$ \{predictor step\}

\textbf{Step 2.}
\begin{itemize}
    \item (i) Solve (4.5) by Algorithm 4.1 until convergence, and obtain $y$. \{Newton corrector\}
    \item (ii) adapt stepsize. \{stepsie control\}
\end{itemize}
Figure 3. Bifurcation scenario at the double bifurcation point \((0, \lambda_{1,2}^4) = (0, \lambda_{2,1}^4) = (0, 58.167366)\).
Figure 4. Bifurcation scenario at the double bifurcation point \((0, \lambda_{1,3}) = (0, \lambda_{3,1}) = (0, 106.126545)\).

Step 3. Test for bifurcation point by estimating the condition number of \(T_n\) in maximum norm.

(i) If \(\kappa_{\infty}(T_n) > c_M\), then

- perform local perturbation (see [20]),
- else

   solve \((4.4)\) by Algorithm 4.1 to obtain \(u\), and go to Step 1.

Until traversing is stopped.

REMARK. One may use the block elimination (BE) algorithm of Chan [26] or the mixed block elimination (BEM) algorithm of Govaerts [27,28] to solve the bordered linear systems \((4.4)\) and \((4.5)\).

5. NUMERICAL RESULTS

We have performed extensive numerical experiments to trace the solution branches of the Brusselator via the predictor-corrector continuation methods, where a local perturbation technique is
used for branch-switching. Examples 1–7 show the results of this method with direct linear solver. Example 8 shows the results with the continuation-BCG Algorithm described in Section 4. All
Multiple Bifurcations

(a) Bifurcation diagram.

(b) Contours of the solution curves at $\lambda = 1033.6668, 1033.6391, 1033.6708, 1033.6391, 1033.6708, 1033.6685, 1033.6685, 1033.6321, 1033.6072$, respectively.

Figure 6. Bifurcation scenario at the triple $(0, \lambda_1^{1,1}, \lambda_1^{1,2}) = (0, \lambda_2^{1,1}) = (0, \lambda_2^{1,2}) = (0, 1035.01792), \alpha = 43.93898$.

computations of our numerical experiments were performed on an IBM SP machine with double precision arithmetic at National Chung-Hsing University. All of the examples were discretized by the central-difference formula with domain length $\ell = 1$, diffusion rates $d_1 = 1$, $d_2 = 2$, and
uniform meshsize $h = 0.05$ on the $x$- and $y$-axis, respectively. Moreover, in Examples 1–4, we choose $\alpha = 4$.

**Example 1. Simple Bifurcation, $(m, n) = (1, 1)$.** Figure 1a shows the first solution curve $U^{h}_{1,1}$ branching from $(0, \lambda_{1,1}^{h}) = (0, 29.1047741)$ has a "barely transcritical" form (cf. [4, p. 233]). Figure 1b shows the 3-D contours of the solution curve $U_{1,1}$ at $\lambda = 30.0837$.

**Example 2. Simple Bifurcation, $(m, n) = (2, 2)$.** Figure 2a shows the solution curve bifurcating at $(0, \lambda_{2,2}^{h}) = (0, 87.47325)$ is transcritical, while Figure 2b shows the contours of the solution curves 1a and 1b at $\lambda = 85.9533$, respectively. Note that the contour of 1b can be obtained from that of 1a by a rotation with 90 degree. Actually, this solution curve has reflection symmetry with respect to $x = y$ and $x + y = 1$.

Figure 7. Bifurcation scenario at the bifurcation point $(0, \lambda_{1,1}^{h}) = (0, \lambda_{2,1}^{h}) = (0, \lambda_{3,1}^{h}) = (0, 4902.6088), \alpha = 97.524977$. 
EXAMPLE 3. **MULTIPLE BIFURCATION**, \((m, n) = (1, 2)\). Figure 3a shows four solution curves branching from the double bifurcation point \((0, \lambda_{1,2}^1, \lambda_{1,2}^2) = (0, A_1, A_2) \approx (0, 5.8167366)\), where the solution curves 1, 2, and 3 are pitchfork, and the other is transcritical. We observe that the solution curves 1a, 2a and 1b, 2b have the same maximum norm. On the other hand, the solution curve 4a has the same norm as 4b. Moreover, the solution curves 1, 2, 3, and 4 have reflection symmetry with respect to \(x = 1/2, y = 1/2, x + y = 1, \) and \(x = y\), respectively.

EXAMPLE 4. **MULTIPLE BIFURCATION** \((m, n) = (1, 3)\). Figure 4a shows two solution curves bifurcating at \((0, \lambda_{1,3}^1, \lambda_{1,3}^2) = (0, A_{1,3}^1, A_{1,3}^2) = (0, 106.126545)\), where the solution curves 2a and 2b are barely transcritical (cf. [4, p. 233]), and the solution curves 1a and 1b are transcritical. Figure 4b shows the contours of these two solution curves at \(\lambda = 107.3492, 107.3492, 107.3592, \) and 107.3316, respectively. Note that the solution curve 1a has the same norm as 1b and has a turning point on the upper part. Moreover, the contours of the former can be obtained from those of the latter via a rotation of 90 degree, and vice versa. Actually, this solution curve has reflection symmetry with respect to \(x = 1/2\) and \(y = 1/2\).

EXAMPLE 5. **Steady/steady state mode interactions of two simple bifurcations**, where \((m_1, n_1) = (1, 1)\), and \((m_2, n_2) = (2, 2)\). By choosing \(c_0 = 55.544456\), we obtain \((0, \lambda_{1,1}^{1,1}, \lambda_{2,2}^{1,2}) = (0, 1641.601513)\) which is a double bifurcation point. Figure 5a shows that two solution curves branching from this double bifurcation point, where the solution curves 1a and 1b are transcritical, and the solution curves 2a and 2b are barely transcritical. It is interesting to see that the solution curves 2a, 2c, and 2b form a closed loop. More precisely, we can follow the solution curves via the paths 2a \(\rightarrow 2c \rightarrow 2b \rightarrow 1b \) or 2b \(\rightarrow 2c \rightarrow 2a \rightarrow 1b\). Figure 5b shows the contours of the solution curves 2a, 2b, and 1b at \(\lambda = 1640.2643, 1640.2884, \) and 1640.2291, respectively. Figure 5c shows how the contours of the solution curve 2c vary for different values of \(\lambda\).

EXAMPLE 6. **Steady/steady state mode interactions of one simple and one double-bifurcations**, where \((m_1, n_1) = (1, 1)\) and \((m_2, n_2) = (1, 2)\). If we choose \(c_0 = 43.938938\), then \((0, \lambda_{1,1}^{1,1}, \lambda_{1,2}^{1,2}) = (0, 1035.01792)\) is a bifurcation point of corank 3. Figure 6a shows that five solution curves branching from this triple bifurcation point, where the solution curves 1a, 1b and 4a, 4b are transcritical, and the other are barely transcritical. Figure 6b shows the contours of these solution curves. Note that the solution curves 2, 3, 4, and 5 have reflection symmetry with respect to \(x = 1/2, y = 1/2, x + y = 1, \) and \(x = y\), respectively.

(c) Contours of the solution curves at \(\lambda = 4901.2442, 4901.2048, 4901.2376, 4901.2396, 4901.2386, \) and 4901.2386, respectively. 

Figure 7. (cont.)
EXAMPLE 7. Steady/steady state mode interactions of two double-bifurcations, where \((m_1, n_1) = (1, 2)\) and \((m_2, n_2) = (1, 3)\). If we choose \(\alpha = 97.524977\), then \((0, \lambda_{1,2}^h, 0, \lambda_{1,3}^h) = (0, 4902.6088)\) is a bifurcation point of corank 4. Figure 7a shows that six solution curves branching from this bifurcation point, where the solution curves 3a, 3b are transcritical, 4a, 4b is pitchfork, and the others are barely transcritical. Figures 7b and 7c show the contours of these solution curves. Actually the solution curves 1, 2, 5, and 6 have reflection symmetry with respect to \(x = 1/2, y = 1/2, x = y, x + y = 1\), respectively, while the solution curve 3 has reflection symmetry with respect to \(x = 1/2\) and \(y = 1/2\).

EXAMPLE 8. We traced the first solution curve of the Brusselator by using Algorithm 4.2. For the meshsize \(h = 0.05\), \(D_u H\) is a matrix of order 722 × 722. Our sample numerical output is listed in Table 1.

The following notations are used in Table 1.

- \(\text{NCS}\) : order of continuation steps.
- \(\epsilon\) : accuracy tolerance in Newton corrector.
- \(\kappa_{\infty}\) : the maximum-norm condition number of \(T_j\).
- \(\text{tol}\) : stopping criterion for the BCG method in solving a linear system.
- \(h\) : uniform meshsize.
- \(\|d\|_{\infty}\) : maximum norm of the perturbation vector.
- \(\text{maxinorm}\) : maximum norm of the discrete solution curve.

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REFERENCES


