Published for SISSA by Springer

Received: *March 20, 2014*
Accepted: *March 30, 2014*
Published: *May 6, 2014*

$\mathcal{N}=4$ supersymmetric Yang-Mills theories in AdS_3

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Abstract: For all types of $\mathcal{N}=4$ anti-de Sitter (AdS) supersymmetry in three dimensions, we construct manifestly supersymmetric actions for Abelian vector multiplets and explain how to extend the construction to the non-Abelian case. Manifestly $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) actions are explicitly given in the cases of (2,2) and critical (4,0) AdS supersymmetries. The $\mathcal{N}=4$ vector multiplets and the corresponding actions are then reduced to (2,0) AdS superspace, in which only $\mathcal{N}=2$ supersymmetry is manifest. Using the off-shell structure of the $\mathcal{N}=4$ vector multiplets, we provide complete $\mathcal{N}=4$ SYM actions in (2,0) AdS superspace for all types of $\mathcal{N}=4$ AdS supersymmetry. In the case of (4,0) AdS supersymmetry, which admits a Euclidean counterpart, the resulting $\mathcal{N}=2$ action contains a Chern-Simons term proportional to $\boldsymbol{q}/r$, where r is the radius of AdS_3 and $\boldsymbol{q}$ is the R-charge of a chiral scalar superfield. The R-charge is a linear inhomogeneous function of X, an expectation value of the $\mathcal{N}=4$ Cotton superfield. Thus our results explain the mysterious structure of $\mathcal{N}=4$ supersymmetric Yang-Mills theories on S^3 discovered in arXiv:1401.7952. In the case of (3,1) AdS supersymmetry, which has no Euclidean counterpart, the SYM action contains both a Chern-Simons term and a chiral mass-like term. In the case of (2,2) AdS supersymmetry, which admits a Euclidean counterpart, the SYM action has no Chern-Simons and chiral mass-like terms.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Superspaces, Supergravity Models

ArXiv ePrint: 1402.3961

doi:10.1007/JHEP05(2014)018

Contents

1 Introduction

Recently, Samsonov and Sorokin [1] have constructed $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theories on S^3, both in terms of $\mathcal{N}=2$ superfields and component fields. In the $\mathcal{N}=2$ superspace setting, such a theory describes coupling of the vector multiplet to a chiral scalar multiplet of R-charge $\boldsymbol{q}$, with $\boldsymbol{q}$ arbitrary.[1] The case $\boldsymbol{q}=1$ was actually considered at the component level three years earlier by Hama, Hosomichi and Lee [2], although the structure of extended supersymmetry transformations was not clarified by these authors. The remarkable feature of the $\mathcal{N}=4$ SYM theories on S^3 constructed in [1, 2] is the fact that the $\mathcal{N}=4$ supersymmetry requires the action to include a Chern-Simons term proportional to $\boldsymbol{q}/r$, where r denotes the radius of S^3. The presence of such a Chern-Simons term calls for an explanation within a manifestly $\mathcal{N}=4$ supersymmetric formulation of the theory.

Supersymmetric theories on S^3 may naturally be obtained from those defined on three-dimensional (3D) anti-de Sitter space, AdS_3, by Wick rotation.[2] There are three types of 3D $\mathcal{N}=4$ AdS superspaces [3], in accordance with the existence of several versions of $\mathcal{N}$-extended AdS supergravity in three dimensions, the (p,q) AdS supergravity theories [4], where $p+q=\mathcal{N}$ and $p\geq q$. These are the (4,0), (3,1) and (2,2) AdS superspaces. Furthermore, there exist three inequivalent versions of (4,0) AdS superspace [3]:

$$X=0\,; \tag{1.1a}$$

$$X\neq 0\,, \qquad |X|\neq 2S\,; \tag{1.1b}$$

$$|X|=2S\,. \tag{1.1c}$$

Here the paramaters X and S are constant AdS values of the $\mathcal{N}=4$ Cotton superfield and one of the superspace torsion components respectively [5, 6].[3] The Cotton superfield automatically vanishes, $X=0$, for the (3,1) and (2,2) AdS superspaces [3]. The $\mathcal{N}=4$ AdS superspaces are conformally flat if and only if $X=0$ [3]. The (3,1) AdS superspace has no Euclidean analogue.

Building on the off-shell formulation for general 3D $\mathcal{N}=4$ supergravity-matter systems given in [6], ref. [7] provided a powerful formalism[4] (off-shell supermultiplets, manifestly supersymmetric action principles etc.) to construct off-shell supersymmetric theories in all the $\mathcal{N}=4$ AdS superspaces, as well as to reduce these theories to $\mathcal{N}=2$ AdS superspaces [11]. However, the analysis in [7] was restricted to the case of the most general $\mathcal{N}=4$ supersymmetric nonlinear sigma models, due to their remarkably rich geometric structure and diverse physical properties associated with the different types of $\mathcal{N}=4$

[1] As argued in [1], a natural bound on the values of the R-charge emerges, $0\leq \boldsymbol{q}\leq 2$, if the spectrum of the theory is required to be free of negative energy states.

[2] Since S^3 and AdS_3 have different topologies, Wick rotation is a rather formal procedure, and some additional care is required in order to make it well defined.

[3] It is natural to think of S as a superspace analogue of the square root of the scalar curvature. However, there is no obvious spacetime interpretation for X.

[4] This formalism is inspired by the projective-superspace approach to 4D $\mathcal{N}=2$ supersymmetric theories in Minkowski space [8–10].

JHEP05(2014)018

AdS supersymmetry. In the present note we apply the formalism of [7] to construct SYM theories in the $\mathcal{N}=4$ AdS superspaces.

This paper is organized as follows. In section 2 we review the geometry of the various 3D $\mathcal{N}=4$ AdS superspaces. In section 3 we review the main results concerning the $\mathcal{N}=4$ vector multiplets in conformal supergravity and construct a new family of composite linear multiplets in $\mathcal{N}=4$ AdS superspaces. The latter result is used in section 4 to construct the $\mathcal{N}=4$ vector multiplet actions in all the $\mathcal{N}=4$ AdS superspaces. Sections 5 and 6 are devoted to the reduction of the results obtained to (2,0) AdS superspace. Concluding comments are given in section 7. The paper contains five technical appendices. In appendix A we discuss the projective-superspace formulation for a 3D $\mathcal{N}=4$ Yang-Mills multiplet in conformal supergravity. The isometries of $\mathcal{N}=4$ AdS superspaces are reviewed in appendix B. The fundamentals of (2,0) AdS superspace are collected in appendix C. In appendix D we present complete $\mathcal{N}=4$ SYM actions in (2,0) AdS superspace for all types of $\mathcal{N}=4$ AdS supersymmetry. Finally, in appendix E we relate the $\mathcal{N}=4$ and $\mathcal{N}=2$ superspace formulations for $\mathcal{N}=4$ SYM theories in AdS$_3$.

2 $\mathcal{N}=4$ AdS superspaces

In this section we review the salient points of the geometry of the various $\mathcal{N}=4$ AdS superspaces constructed in [3].

According to the on-shell supergravity analysis of [4], there are three types of $\mathcal{N}=4$ AdS supersymmetry in three dimensions. This implies the existence of three inequivalent maximally symmetric and conformally flat (p,q) AdS superspaces

$$\mathrm{AdS}_{(3|p,q)} = \frac{\mathrm{OSp}(p|2;\mathbb{R}) \times \mathrm{OSp}(q|2;\mathbb{R})}{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(p) \times \mathrm{SO}(q)}\,, \qquad p+q=4\,, \qquad p \geq q\,. \tag{2.1}$$

In accordance with the more recent analysis of [3], which was based on the use of the off-shell formulation for 3D $\mathcal{N}=4$ conformal supergravity [5, 6], there exist two more inequivalent versions of (4,0) AdS superspace. These superspaces are not conformally flat and correspond to the choices (1.1b) and (1.1c). Their existence is due to the fact that for $\mathcal{N} \geq 4$ there exist more general AdS supergroups in the case $p-\mathcal{N}=q=0$, than those considered by Achúcarro and Townsend [4].

All the $\mathcal{N}=4$ AdS superspace geometries may be described using covariant derivatives of the general form:

$$\mathcal{D}_A = \left(\mathcal{D}_a, \mathcal{D}_\alpha^{i\bar{i}}\right) = E_A{}^M \partial_M + \frac{1}{2}\Omega_A{}^{cd}\mathcal{M}_{cd} + \Phi_A{}^{kl}\mathbf{L}_{kl} + \Phi_A{}^{\bar{k}\bar{l}}\mathbf{R}_{\bar{k}\bar{l}}\,. \tag{2.2}$$

Here the operators $\mathbf{L}_{kl}$ and $\mathbf{R}_{\bar{k}\bar{l}}$ generate the R-symmetry group $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ and act on the covariant derivatives as

$$\left[\mathbf{L}^{kl}, \mathcal{D}_\alpha^{i\bar{i}}\right] = \varepsilon^{i(k}\mathcal{D}_\alpha^{l)\bar{i}}\,, \qquad \left[\mathbf{R}^{\bar{k}\bar{l}}, \mathcal{D}_\alpha^{i\bar{i}}\right] = \varepsilon^{\bar{i}(\bar{k}}\mathcal{D}_\alpha^{i\bar{l})}\,. \tag{2.3}$$

For each of the $\mathcal{N}=4$ AdS superspaces, the covariant derivatives obey (anti-)commutation relations of the form [3]:

$$\begin{aligned}
\left\{\mathcal{D}_\alpha^{i\bar{i}}, \mathcal{D}_\beta^{j\bar{j}}\right\} &= 2\mathrm{i}\,\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}\left(\mathcal{S}^{ij\bar{i}\bar{j}} + \varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{S}\right)\mathcal{M}_{\alpha\beta} \\
&\quad + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}(2\mathcal{S}+X)\mathbf{L}^{ij} - 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\mathcal{S}^{kl\bar{i}\bar{j}}\mathbf{L}_{kl} \\
&\quad + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}(2\mathcal{S}-X)\mathbf{R}^{\bar{i}\bar{j}} - 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}\mathcal{S}^{ij\bar{k}\bar{l}}\mathbf{R}_{\bar{k}\bar{l}}\,, \qquad (2.4a) \\
\left[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma^{k\bar{k}}\right] &= -2\left(\delta_l^k\delta_{\bar{l}}^{\bar{k}}\mathcal{S} + \mathcal{S}^k{}_l{}^{\bar{k}}{}_{\bar{l}}\right)\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}^{l\bar{l}}\,, \qquad (2.4b) \\
[\mathcal{D}_a, \mathcal{D}_b] &= -4\,S^2\mathcal{M}_{ab}\,, \qquad (2.4c)
\end{aligned}$$

where the real tensor $\mathcal{S}^{ij\bar{i}\bar{j}} = \mathcal{S}^{(ij)(\bar{i}\bar{j})}$ is covariantly constant, and the real scalars $\mathcal{S}$, X and S are constant. The parameter S determines the curvature of AdS$_3$. Depending on the superspace type, the parameters $\mathcal{S}$, $\mathcal{S}^{ij\bar{i}\bar{j}}$ and X have the following explicit form [3]:

$$\begin{aligned}
&(4,0)\text{ AdS}: \quad \mathcal{S} = S\,, \quad \mathcal{S}^{ij\bar{i}\bar{j}} = 0\,, \quad X \text{ arbitrary}\,; \qquad (2.5a) \\
&(3,1)\text{ AdS}: \quad \mathcal{S} = \frac{1}{2}S\,, \quad \mathcal{S}^{ij\bar{i}\bar{j}} = \frac{1}{2}\left(\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}} - 2w^{i\bar{i}}w^{j\bar{j}}\right)S\,, \quad X = 0\,; \qquad (2.5b) \\
&(2,2)\text{ AdS}: \quad \mathcal{S} = 0\,, \quad \mathcal{S}^{ij\bar{i}\bar{j}} = l^{ij}r^{\bar{i}\bar{j}}\,S\,, \quad X = 0\,. \qquad (2.5c)
\end{aligned}$$

In the (3,1) case, the covariantly constant tensor $w^{i\bar{i}}$ is real, $\overline{w^{i\bar{i}}} = w_{i\bar{i}} = \varepsilon_{ij}\varepsilon_{\bar{i}\bar{j}}w^{j\bar{j}}$, and normalized as

$$w^{i\bar{k}}w_{j\bar{k}} = \delta^i{}_j\,, \qquad w^{k\bar{i}}w_{k\bar{j}} = \delta^{\bar{i}}{}_{\bar{j}}\,. \qquad (2.6)$$

In the (2,2) case, the real iso-triplets $l^{ij} = l^{ji}$ and $r^{\bar{i}\bar{j}} = r^{\bar{j}\bar{i}}$ are covariantly constant and normalized as

$$l^{ik}l_{kj} = \delta^i{}_j\,, \qquad r^{\bar{i}\bar{k}}r_{\bar{k}\bar{j}} = \delta^{\bar{i}}{}_{\bar{j}}\,. \qquad (2.7)$$

We emphasize that X can appear in the algebra only in the (4,0) case. For general values of X, the tangent space group of the (4,0) AdS supergeometry is the full R-symmetry group $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$. For the two critical values, $X=2S$ and $X=-2S$, the $\mathrm{SU}(2)_{\mathrm{R}}$ or $\mathrm{SU}(2)_{\mathrm{L}}$ group, respectively, can be gauged away.

In the non-critical case, $|X| \neq 2S$, the isometry group of (4,0) AdS superspace is isomorphic to[5]

$$\mathrm{D}(2,1;\alpha)\times\mathrm{SL}(2,\mathbb{R})\,, \qquad \alpha \neq -1, 0\,, \qquad (2.8)$$

for some $\alpha\in\mathbb{R}$. Here $\mathrm{D}(2,1;\alpha)$ is one of the exceptional simple supergroups, see, e.g., [12] for a review.[6] As is known, not all values of the real parameter α lead to distinct supergroups. The point is that there is a finite group G (of order 6) of fractional linear transformations of $\mathbb{R}P^1 = \mathbb{R}\cup\{\infty\}$, the compactified real line, with the property that any transformation $g\in G$ maps $\alpha\to\alpha' = g(\alpha)$ such that $\mathrm{D}(2,1;\alpha)$ and $\mathrm{D}(2,1;\alpha')$ are isomorphic [12]. The subset $\{-1,0,\infty\}$ proves to be fixed under the action of G. Up to an isomorphism, it suffices to restrict α to the range $0<\alpha\leq 1$. The case $\alpha=1$ corresponds to the conformally flat (4,0) AdS superspace, for which $X=0$. The isometry group of this

[5] We are grateful to Igor Samsonov and Dima Sorokin for this observation.

[6] Various supercosets based on the exceptional supergroup $\mathrm{D}(2,1;\alpha)$ were considered in [13].

superspace is $\mathrm{OSp}(4|2)\times\mathrm{SL}(2,\mathbb{R})$. In general, there is a correspondence between α and the (4,0) AdS parameter $\boldsymbol{q} = 1 + \frac{X}{2S}$, which will play an important role in this paper. These parameters may be identified in the domain $0 < \alpha \leq 1$. The choice $\alpha = 0$ corresponds to $\boldsymbol{q} = 0$, which is one of the two critical (4,0) AdS cases.[7] The isometry group of this (4,0) AdS superspace degenerates to $\mathrm{SU}(1,1|2)\times\mathrm{SL}(2,\mathbb{R})$, see also the discussion in [13].

For the (3,1) and (2,2) AdS geometries, the R-symmetry sector of the superspace holonomy group is a subgroup of $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$ [3]. For the (3,1) supergeometry, the relevant subgroup is $\mathrm{SU}(2)_{\mathcal{J}}$ generated by

$$\mathcal{J}_{kl} = \mathbf{L}_{kl} + w_k{}^{\bar{k}} w_l{}^{\bar{l}} \mathbf{R}_{\bar{k}\bar{l}}\,, \quad \text{or} \quad \mathcal{J}_{\bar{k}\bar{l}} = w^k{}_{\bar{k}} w^l{}_{\bar{l}} \mathbf{L}_{kl} + \mathbf{R}_{\bar{k}\bar{l}} = w^k{}_{\bar{k}} w^l{}_{\bar{l}} \mathcal{J}_{kl}\,. \tag{2.9}$$

The generators $\mathcal{J}_{kl}$ and $\mathcal{J}_{\bar{k}\bar{l}}$ leave $w^{i\bar{i}}$ invariant, $\mathcal{J}_{kl} w^{i\bar{i}} = \mathcal{J}_{\bar{k}\bar{l}} w^{i\bar{i}} = 0$. Since the R-symmetry curvature is spanned by the generators of $\mathrm{SU}(2)_{\mathcal{J}}$, it is possible to choose a gauge in which the R symmetry connection takes its values in the Lie algebra of $\mathrm{SU}(2)_{\mathcal{J}}$; in this gauge, the parameter $w^{i\bar{i}}$ is constant.

In the (2,2) case, the R-symmetry sector of the superspace holonomy group is the Abelian subgroup $\mathrm{U}(1)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{R}}$ of $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$ generated by

$$\mathbf{L} := l^{kl}\mathbf{L}_{kl}\,, \qquad \mathbf{R} := r^{\bar{k}\bar{l}}\mathbf{R}_{\bar{k}\bar{l}}\,. \tag{2.10}$$

This subgroup leaves invariant the covariantly constant parameters l^{kl} and $r^{\bar{k}\bar{l}}$. In the remainder of the paper, we choose a gauge in which only this subgroup appears in the (2,2) covariant derivatives. In this gauge the parameters l^{kl} and $r^{\bar{k}\bar{l}}$ are constant.

3 Vector multiplets in $\mathcal{N} = 4$ AdS superspaces

There are two inequivalent $\mathcal{N} = 4$ vector multiplets in three dimensions, left and right ones.[8] In a curved $\mathcal{N} = 4$ superspace [6], they may be described in terms of gauge-invariant field strengths, $W^{ij} = W^{ji} = \overline{W_{ij}}$ and $W^{\bar{i}\bar{j}} = W^{\bar{j}\bar{i}} = \overline{W_{\bar{i}\bar{j}}}$, which transform under the left and right subgroups of the supergravity R-symmetry group $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$, respectively, and obey the *inequivalent* analyticity constraints[9]

$$\mathcal{D}_\alpha^{(i\bar{i}} W^{kl)} = 0\,, \tag{3.1a}$$

$$\mathcal{D}_\alpha^{i(\bar{i}} W^{\bar{k}\bar{l})} = 0\,. \tag{3.1b}$$

A real symmetric isospinor W^{ij} under the constraint (3.1a) is called a left linear multiplet. Similarly, eq. (3.1b) defines a right linear multiplet.

The field strengths introduced may be interpreted as special examples of the covariant projective $\mathcal{N} = 4$ supermultiplets studied in [6]. Let us introduce left and right isospinor

[7] The two critical (4,0) AdS superspaces, which are characterized by the choices $\boldsymbol{q} = 0$ and $\boldsymbol{q} = 2$, have isomorphic isometry groups.

[8] The existence of two inequivalent 3D $\mathcal{N} = 4$ vector multiplets was first discussed by Brooks and Gates [14]. The modern off-shell formulation for these multiplets was given by Zupnik [15] in the rigid supersymmetric case. In the locally supersymmetric case, these multiplets were described in [6].

[9] Here we focus our attention on the Abelian vector multiplets. In appendix A we elaborate on the non-Abelian case.

variables, $v_{\rm L} := v^i \in \mathbb{C}^2 \setminus \{0\}$ and $v_{\rm R} := v^{\bar i} \in \mathbb{C}^2 \setminus \{0\}$, and use them to define two different subsets, $\mathcal{D}^{(1)\bar i}_\alpha$ and $\mathcal{D}^{(\bar 1)i}_\alpha$, in the set of spinor covariant derivatives $\mathcal{D}^{i\bar i}_\alpha$,

$$\mathcal{D}^{(1)\bar i}_\alpha := v_i \mathcal{D}^{i\bar i}_\alpha \,, \qquad \mathcal{D}^{(\bar 1)i}_\alpha := v_{\bar i} \mathcal{D}^{i\bar i}_\alpha \,, \tag{3.2}$$

as well as the index-free superfields

$$W^{(2)}_{\rm L} := v_i v_j W^{ij} \equiv W^{(2)} \,, \qquad W^{(2)}_{\rm R} := v_{\bar i} v_{\bar j} W^{\bar i\bar j} \equiv W^{(\bar 2)} \tag{3.3}$$

associated with the left and the right linear multiplets, respectively. Now, the constraints (3.1a) and (3.1b) turn into the generalized chirality conditions

$$\mathcal{D}^{(1)\bar i}_\alpha W^{(2)}_{\rm L} = 0 \,, \tag{3.4a}$$

$$\mathcal{D}^{(\bar 1)i}_\alpha W^{(2)}_{\rm R} = 0 \,. \tag{3.4b}$$

The superfield $W^{(2)}_{\rm L}(v_{\rm L})$ is called a left $\mathcal{O}(2)$ multiplet. Similarly, $W^{(2)}_{\rm R}(v_{\rm R})$ is called a right $\mathcal{O}(2)$ multiplet.

All results concerning left vector multiplets may be related to the right ones by applying the so-called mirror map [6, 15]. Therefore we restrict our analysis to the case of left vector multiplets.

As shown in [6], the constraint (3.1a) may be solved in terms of an unconstrained gauge prepotential that is a right weight-zero tropical multiplet $V_{\rm R}(v_{\rm R})$. The most general solution to the analyticity constraint (3.1a) is

$$W^{ij} = \frac{\rm i}{4}\left(\mathcal{D}^{ij\bar i\bar j} - 4{\rm i}\mathcal{S}^{ij\bar i\bar j}\right) \oint_\gamma \frac{(v_{\rm R}, {\rm d}v_{\rm R})}{2\pi} \frac{u_{\bar i} u_{\bar j}}{(v_{\rm R}, u_{\rm R})^2} V_{\rm R}(v_{\rm R}) \,, \qquad (v_{\rm R}, u_{\rm R}) := v^{\bar i} u_{\bar i} \,, \tag{3.5}$$

where we have defined

$$\mathcal{D}^{ij\bar i\bar j} := \mathcal{D}^{\alpha(i(\bar i}\mathcal{D}^{j)\bar j)}_\alpha \,. \tag{3.6}$$

The right-hand side of (3.5) involves a constant isospinor $u_{\rm R} = u^{\bar i}$ constrained only by the condition $(v_{\rm R}, u_{\rm R}) \neq 0$ which must hold along the closed integration contour γ. It can be shown that (3.5) is invariant under an arbitrary infinitesimal variation of $u_{\rm R}$, which may be represented as $\delta u_{\rm R} = \alpha u_{\rm R} + \beta v_{\rm R}$, with $\alpha, \beta \in \mathbb{C}$. Thus W^{ij} is independent of $u_{\rm R}$. The right-hand side of (3.5) is invariant under gauge transformations

$$\delta V_{\rm R} = {\rm i}\left(\breve\lambda_{\rm R} - \lambda_{\rm R}\right) \,, \tag{3.7}$$

where the gauge parameter $\lambda_{\rm R}(v_{\rm R})$ is a right arctic weight-zero multiplet, see [6] for more details.

It is important to point out that the field strength of the left vector multiplet, $W^{(2)}_{\rm L}(v_{\rm L})$, is a left projective multiplet. However, its gauge prepotential, $V_{\rm R}(v_{\rm R})$, is a right projective multiplet.

All previous results in this section hold for the general curved $\mathcal{N} = 4$ superspace as defined in [6]. The specific feature of the AdS geometries is that a composite right $\mathcal{O}(2)$

JHEP05(2014)018

multiplet may be constructed starting from the left vector multiplet. Consider the tensor superfield

$$\boldsymbol{W}^{\bar{i}\bar{j}} := -\frac{\mathrm{i}}{12}\mathcal{D}^{ij\bar{i}\bar{j}}W_{ij}\,, \tag{3.8}$$

which can equivalently be realized as the right isotwistor superfield

$$\boldsymbol{W}^{(2)}_{\mathrm{R}}(v_{\mathrm{R}}) := v_{\bar{i}}v_{\bar{j}}\boldsymbol{W}^{\bar{i}\bar{j}} \equiv \boldsymbol{W}^{(\bar{2})}\,. \tag{3.9}$$

Making use of the algebra

$$\begin{aligned}\left\{\mathcal{D}^{(\bar{1})i}_{\alpha},\mathcal{D}^{(\bar{1})j}_{\beta}\right\} &= -4\mathrm{i}\mathcal{S}^{(\bar{2})ij}\mathcal{M}_{\alpha\beta} - 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\mathcal{S}^{(\bar{2})kl}\mathbf{L}_{kl} + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}(2\mathcal{S}-X)\mathbf{R}^{(\bar{2})}\,, &(3.10\text{a})\\ \mathcal{S}^{(\bar{2})ij} &:= v_{\bar{i}}v_{\bar{j}}\mathcal{S}^{ij\bar{i}\bar{j}}\,, \qquad \mathbf{R}^{(\bar{2})} := v_{\bar{i}}v_{\bar{j}}\mathbf{R}^{\bar{i}\bar{j}} &(3.10\text{b})\end{aligned}$$

in conjunction with the equations

$$\left[\mathbf{R}^{(\bar{2})},\mathcal{D}^{(\bar{1})i}_{\alpha}\right] = 0\,, \quad \mathbf{R}^{(\bar{2})}W^{ij} = 0\,, \quad \mathcal{D}^{i\bar{i}}_{\alpha}W^{jk} = \frac{2}{3}\varepsilon^{i(j}\mathcal{D}^{\bar{i}}_{\alpha l}W^{k)l}\,, \tag{3.11}$$

it is a short calculation to prove that

$$\mathcal{D}^{(\bar{1})i}_{\alpha}\boldsymbol{W}^{(2)}_{\mathrm{R}} = 0 \quad\Longleftrightarrow\quad \mathcal{D}^{i(\bar{i}}_{\alpha}\boldsymbol{W}^{\bar{j}\bar{k})} = 0\,. \tag{3.12}$$

Therefore, $\boldsymbol{W}^{\bar{i}\bar{j}}$ is a right linear multiplet.[10] This superfield and its mirror image will be our crucial building blocks to construct $\mathcal{N}=4$ SYM actions in AdS_3.

It is possible to express $\boldsymbol{W}^{(2)}_{\mathrm{R}}$ in terms of the gauge prepotential V_{R}. The result is

$$\boldsymbol{W}^{(2)}_{\mathrm{R}}(v_{\mathrm{R}}) = \Delta^{(4)}_{\mathrm{R}}\oint\frac{(\hat{v}_{\mathrm{R}},\mathrm{d}\hat{v}_{\mathrm{R}})}{2\pi\,(v_{\mathrm{R}},\hat{v}_{\mathrm{R}})^2}V_{\mathrm{R}}(\hat{v}_{\mathrm{R}})\,. \tag{3.13}$$

Here we have introduced the right analyticity projection operator

$$\Delta^{(4)}_{\mathrm{R}} = \frac{1}{48}\mathcal{D}^{(\bar{2})kl}\left(\mathcal{D}^{(\bar{2})}_{kl} - 4\mathrm{i}\mathcal{S}^{(\bar{2})}_{kl}\right) = \frac{1}{48}\left(\mathcal{D}^{(\bar{2})kl} - 4\mathrm{i}\mathcal{S}^{(\bar{2})kl}\right)\mathcal{D}^{(\bar{2})}_{kl}\,. \tag{3.14}$$

It is obtained from the projection operator $\Delta^{(4)}_{\mathrm{R}}$ defined in the curved-superspace case [6] by switching off those torsion tensors which vanish in the AdS superspaces.[11]

For any supergravity background, there is an alternative procedure [16] to construct a composite right linear multiplet, $\boldsymbol{G}^{\bar{i}\bar{j}}$, from the left vector multiplet:

$$\boldsymbol{G}^{\bar{i}\bar{j}} = \frac{\mathrm{i}}{4}\left(\mathcal{D}^{ij\bar{i}\bar{j}} + 8\mathrm{i}\mathcal{S}^{ij\bar{i}\bar{j}}\right)\left(\frac{W_{ij}}{W_{\mathrm{L}}}\right)\,, \qquad W_{\mathrm{L}} := \sqrt{W^{ij}W_{ij}}\,. \tag{3.15}$$

It is applicable only in the case when W_{L} is nowhere vanishing, $W_{\mathrm{L}} \neq 0$. The superfield $\boldsymbol{G}^{\bar{i}\bar{j}}$ proves to be primary under the super-Weyl transformations [16]. Unlike $\boldsymbol{G}^{\bar{i}\bar{j}}$, our composite linear multiplet (3.8) exists only in the AdS superspaces. Its definition does not require W^{ij} to be nowhere vanishing. These results are similar to those derived many years ago by Siegel for the 4D $\mathcal{N}=2$ tensor multiplets [17].

[10] Starting from $\boldsymbol{W}^{\bar{i}\bar{j}}$, we can construct a left linear multiplet, and so on and so forth. As a result, we have a procedure to generate higher-derivative left and right linear multiplets.

[11] The fundamental property of $\Delta^{(4)}_{\mathrm{R}}$ is that $Q^{(n)}_{\mathrm{R}} := \Delta^{(4)}_{\mathrm{R}}T^{(n-4)}_{\mathrm{R}}$ is a right weight-n projective multiplet for any right isotwistor superfield $T^{(n-4)}_{\mathrm{R}}(v_{\mathrm{R}})$, see [6] for more details.

4 SYM actions in $\mathcal{N}=4$ AdS superspaces

To start with, we recall the locally supersymmetric action principle in $\mathcal{N}=4$ matter-coupled supergravity [6]. In general, the $\mathcal{N}=4$ supersymmetric action may be presented as a sum of two terms, the left $S_{\rm L}$ and right $S_{\rm R}$ ones,

$$S = S_{\rm L} + S_{\rm R}\,. \tag{4.1}$$

The right action has the form

$$S_{\rm R} = \frac{1}{2\pi}\oint_{\gamma_{\rm R}} (v_{\rm R}, {\rm d}v_{\rm R}) \int {\rm d}^3x\,{\rm d}^8\theta\, E\, C_{\rm R}^{(-4)}\mathcal{L}_{\rm R}^{(2)}\,, \qquad E^{-1} = {\rm Ber}\left(E_A{}^M\right), \tag{4.2}$$

where the Lagrangian $\mathcal{L}_{\rm R}^{(2)}(v_{\rm R})$ is a real right projective multiplet of weight 2. The action involves a *model-independent* primary isotwistor superfield $C_{\rm R}^{(-4)}(v_{\rm R})$ defined to be real with respect to the smile-conjugation and obey the differential equation[12]

$$\Delta_{\rm R}^{(4)} C_{\rm R}^{(-4)} = 1\,, \tag{4.3}$$

with $\Delta_{\rm R}^{(4)}$ the covariant right projection operator. In AdS superspace, $\Delta_{\rm R}^{(4)}$ is given by eq. (3.14).

To describe the dynamics of an Abelian left vector multiplet in a given $\mathcal{N}=4$ AdS superspace, it suffices to make use of the right action only, such that $S_{\rm L}=0$. We choose

$$\mathcal{L}_{\rm R}^{(2)} = \frac{1}{2} V_{\rm R} \boldsymbol{W}_{\rm R}^{(2)} = -\frac{\rm i}{24} V_{\rm R} \mathcal{D}^{(\bar 2)ij} W_{ij} = -\frac{\rm i}{24}\mathcal{D}^{(\bar 2)ij}\Big(V_{\rm R} W_{ij}\Big)\,, \tag{4.4}$$

where the composite right $\mathcal{O}(2)$ multiplet is given by (3.9). The action defined by eqs. (4.2) and (4.4) is manifestly invariant under all the isometries of the $\mathcal{N}=4$ AdS superspace under consideration.

By applying the relations (3.13) and (4.3), the action defined by eqs. (4.2) and (4.4) may be rewritten in the form:

$$S[V_{\rm R}] = \frac{1}{8\pi^2}\oint (v_{\rm R}, {\rm d}v_{\rm R}) \oint (\hat v_{\rm R}, {\rm d}\hat v_{\rm R}) \int {\rm d}^3x\,{\rm d}^8\theta\, E\, \frac{1}{(v_{\rm R}, \hat v_{\rm R})^2} V_{\rm R}(v_{\rm R}) V_{\rm R}(\hat v_{\rm R})\,. \tag{4.5}$$

This is similar to the action for the Abelian $\mathcal{N}=2$ vector multiplet in four dimensions constructed first in the rigid supersymmetric case [18] (see also [19]) and later in supergravity [20].

The action defined by eqs. (4.2) and (4.4) is valid for all the $\mathcal{N}=4$ AdS superspaces. It turns out that alternative forms for the supersymmetric action exist in two special cases: (i) the (2,2) AdS superspace; and (ii) the critical (4,0) AdS superspace with $2S+X=0$.

In the case of (2,2) AdS superspace, the theory can be described using a left action only, such that $S_{\rm R}=0$. The left Lagrangian is

$$\mathcal{L}_{\rm L}^{(2)} = \frac{1}{2}\frac{W_{\rm L}^{(2)} W_{\rm L}^{(2)}}{d_{\rm L}^{(2)}}\,, \qquad d_{\rm L}^{(2)}(v_{\rm L}) := d^{ij} v_i v_j\,, \qquad \mathcal{D}_\gamma^{k\bar k} d^{ij} = 0\,, \tag{4.6}$$

[12] In conformal supergravity, the field $C_{\rm R}^{(-4)}(v_{\rm R})$ has to be primary of weight -2 under the super-Weyl transformations [6].

for some background covariantly constant real symmetric spinor d^{ij}. The covariant constancy of d^{ij} implies that $d^{ij} \propto l^{ij}$, with l^{ij} one of the parameters of (2,2) AdS superspace, see eq. (2.5c). Without loss of generality, d^{ij} and l^{ij} may be identified. The Lagrangian (4.6) admits a trivial extension to the non-Abelian case:

$$\mathcal{L}^{(2)}_{\rm L} = \frac{1}{2d^{(2)}_{\rm L}}\,\mathrm{tr}\Big(W^{(2)}_{\rm L}W^{(2)}_{\rm L}\Big)\,. \tag{4.7}$$

In the case of (4,0) AdS superspace with $2S + X = 0$, the $\mathrm{SU}(2)_{\rm L}$ curvature vanishes, according to eqs. (2.4) and (2.5a), and there exists a covariantly constant real symmetric spinor d^{ij} such that $\mathcal{D}^{k\bar{k}}_{\gamma}d^{ij} = 0$. As a result, in this case we can again use Lagrangians (4.6) or (4.7) to describe SYM theories.

In conclusion, we comment on two different schemes to extend our results to the non-Abelian case for any $\mathcal{N} = 4$ AdS superspace. Similar to the 5D discussion in [21], a SYM action may be defined by its variation[13]

$$\delta S_{\rm SYM}[V] = \frac{1}{2\pi}\oint(v_{\rm R}, {\rm d}v_{\rm R})\int {\rm d}^3x\,{\rm d}^8\theta\,E\,C^{(-4)}_{\rm R}\,\mathrm{tr}\Big(\Delta V\cdot \boldsymbol{W}^{(\bar{2})}_{+}\Big)\,, \tag{4.8}$$

where we have defined

$$\Delta V := {\rm e}^{-V}\delta{\rm e}^{V}\,, \qquad \boldsymbol{W}^{(\bar{2})}_{+} := {\rm e}^{-\Omega_+}\boldsymbol{W}^{(\bar{2})}{\rm e}^{\Omega_+}\,, \qquad \boldsymbol{W}^{(\bar{2})} = -\frac{\rm i}{12}\mathfrak{D}^{(\bar{2})}_{ij}\mathfrak{W}^{ij}\,. \tag{4.9}$$

Here $\mathfrak{W}^{ij}$ denotes the non-Abelian field strength, and $\delta{\rm e}^{V}$ an arbitrary variation of the non-Abelian tropical prepotential. For more details, including the definition of Ω_+, the reader should consult appendix A. The projective superfields ΔV and $\boldsymbol{W}^{(\bar{2})}_{+}$ take their values in the Lie algebra of the gauge group and transform only under the λ-group as follows:

$$\Delta V' = {\rm e}^{{\rm i}\lambda}\Delta V{\rm e}^{-{\rm i}\lambda}\,, \qquad \Big(\boldsymbol{W}^{(\bar{2})}_{+}\Big)' = {\rm e}^{{\rm i}\lambda}\boldsymbol{W}^{(\bar{2})}_{+}{\rm e}^{-{\rm i}\lambda}\,. \tag{4.10}$$

Now, making use of (A.33) and the expression for the analyticity projection operator (3.14), we obtain

$$\boldsymbol{W}^{(\bar{2})}_{+} = -\frac{\rm i}{12}\mathcal{D}^{(\bar{2})}_{ij}\mathfrak{W}^{ij}_{+} = -{\rm i}\Delta^{(\bar{4})}_{\rm R}\Big({\rm e}^{-\Omega_+}\partial^{(-\bar{2})}{\rm e}^{\Omega_+}\Big)\,. \tag{4.11}$$

As a result, the variation (4.8) can be rewritten in the form

$$\delta S_{\rm SYM}[V] = -\frac{\rm i}{2\pi}\oint(v_{\rm R}, {\rm d}v_{\rm R})\int {\rm d}^3x\,{\rm d}^8\theta\,E\,\mathrm{tr}\Big[\Delta V\,{\rm e}^{-\Omega_+}\partial^{(-\bar{2})}{\rm e}^{\Omega_+}\Big]\,. \tag{4.12}$$

It may be shown that this variation (4.8) is integrable. The action $S_{\rm SYM}[V]$ is gauge invariant, since an infinitesimal gauge transformation (A.14) corresponds to the choice[14]

$$\Delta V = {\rm i}\Big(\check{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\Big)\,, \qquad \check{\boldsymbol{\lambda}} := {\rm e}^{-V}\check{\lambda}{\rm e}^{V}\,, \qquad \boldsymbol{\lambda} := \lambda\,, \tag{4.13}$$

for which the variation (4.8) proves to vanish. However, we have not yet been able to integrate it in a closed form in terms of $\mathcal{N} = 4$ superfields. An alternative approach to obtaining a closed-form expression for the SYM action is to make use of the superform construction, see [16] and references therein. In appendix E, we explicitly integrate the variation (4.8) upon its reduction to (2,0) AdS superspace.

[13]To avoid cluttering of the equations, here we use notation V for the right tropical prepotential $V_{\rm R}$.

[14]The transformations (4.10) and (4.13) are classical and quantum realizations of the gauge transformation within the background-quantum splitting, see e.g. [22].

5 Reduction to (2,0) AdS superspace

Suppose there is a rigid supersymmetric field theory formulated in a given $\mathcal{N}=4$ AdS superspace. As shown in [7], such a dynamical system can always be reformulated as a supersymmetric theory realized in $(2,0)$ AdS superspace, with two supersymmetries hidden. In this section we give a brief review of the superspace reduction $\mathcal{N}=4$ AdS $\longrightarrow$ (2,0) AdS, concentrating mainly on the decomposition of the $\mathcal{N}=4$ AdS isometries into the (2,0) AdS isometries and additional non-manifest symmetries. In the next section the reduction procedure will be applied to reformulate the $\mathcal{N}=4$ theories in (2,0) AdS superspace.

We start by reminding the reader that the algebra of $\mathcal{N}=4$ AdS covariant derivatives, eq. (2.4), involves a covariantly constant tensor $\mathcal{S}^{ij\bar{i}\bar{j}}=\mathcal{S}^{(ij)(\bar{i}\bar{j})}$. Its explicit form is given by (2.5). For all the $\mathcal{N}=4$ AdS superspaces, it may be seen that applying an R-symmetry transformation allows us to choose several components of this tensor to vanish,

$$\mathcal{S}^{11\bar{1}\bar{2}}=\mathcal{S}^{12\bar{1}\bar{1}}=\mathcal{S}^{11\bar{1}\bar{1}}=\mathcal{S}^{22\bar{2}\bar{2}}=0\,, \tag{5.1}$$

as well as to have the property

$$\mathcal{S}+\mathcal{S}^{12\bar{1}\bar{2}}=S\,. \tag{5.2}$$

The proof of these claims was given in [7], and it will be reiterated below. In this gauge, the operators $\mathcal{D}_a$, $\mathcal{D}^{1\bar{1}}_\alpha$ and $\left(-\mathcal{D}^{2\bar{2}}_\alpha\right)$ form an algebra[15] which is isomorphic to that of (2,0) AdS superspace, eq. (C.2), provided the $\mathrm{U}(1)_R$ generator is identified with

$$\mathcal{J}:=\hat{\mathcal{J}}+\frac{X}{2S}\hat{\mathcal{Z}}\,, \tag{5.3}$$

where we have defined the operators

$$\hat{\mathcal{J}}:=\left(\mathbf{L}^{12}+\mathbf{R}^{\bar{1}\bar{2}}\right)\,,\qquad \hat{\mathcal{Z}}:=\left(\mathbf{L}^{12}-\mathbf{R}^{\bar{1}\bar{2}}\right)\,,\qquad \left[\hat{\mathcal{J}},\hat{\mathcal{Z}}\right]=0 \tag{5.4}$$

with the properties

$$\left[\hat{\mathcal{J}},\mathcal{D}^{1\bar{1}}_\alpha\right]=\mathcal{D}^{1\bar{1}}_\alpha\,,\qquad \left[\hat{\mathcal{J}},\left(-\mathcal{D}^{2\bar{2}}_\alpha\right)\right]=-\left(-\mathcal{D}^{2\bar{2}}_\alpha\right)\,, \tag{5.5a}$$

$$\left[\hat{\mathcal{Z}},\mathcal{D}^{1\bar{1}}_\alpha\right]=0\,,\qquad \left[\hat{\mathcal{Z}},\left(-\mathcal{D}^{2\bar{2}}_\alpha\right)\right]=0\,. \tag{5.5b}$$

The generator $\mathcal{J}$ defined by (5.3) coincides with $\hat{\mathcal{J}}$ for all conformally flat $\mathcal{N}=4$ AdS superspaces.

Given an $\mathcal{N}=4$ tensor superfield $U(x,\theta_{\imath\bar{\jmath}})$, we define its projection to (2,0) AdS superspace by

$$U|:=U(x,\theta_{\imath\bar{\jmath}})|_{\theta_{1\bar{2}}=\theta_{2\bar{1}}=0}\,. \tag{5.6}$$

By definition, $U|$ depends on the Grassmann coordinates $\theta^\mu:=\theta^\mu_{1\bar{1}}$ and their complex conjugates, $\bar{\theta}^\mu=\theta^\mu_{2\bar{2}}$. We will refer to $U|$ as the bar-projection of U. For the $\mathcal{N}=4$ AdS

[15] Given a tensor superfield U of Grassmann parity $\epsilon(U)$, the operation of complex conjugation maps $\mathcal{D}^{1\bar{1}}_\alpha U$ to $\overline{\mathcal{D}^{1\bar{1}}_\alpha U}=-(-1)^{\epsilon(U)}\mathcal{D}_{\alpha 1\bar{1}}\bar{U}=-(-1)^{\epsilon(U)}\mathcal{D}^{2\bar{2}}_\alpha\bar{U}$.

covariant derivatives (2.2) the bar-projection is defined as[16]

$$\mathcal{D}_A| = E_A{}^M|\partial_M + \frac{1}{2}\Omega_A{}^{bc}|\mathcal{M}_{bc} + \Phi_A{}^{kl}|\mathbf{L}_{kl} + \Phi_A{}^{\bar{k}\bar{l}}|\mathbf{R}_{\bar{k}\bar{l}}\,. \tag{5.7}$$

Since the algebra of operators $\left(\mathcal{D}_a,\, \mathcal{D}_\alpha^{1\bar{1}},\, -\mathcal{D}_\alpha^{2\bar{2}}\right)$ is isomorphic to that of the (2,0) AdS superspace, eq. (C.2), the freedom to perform general coordinate, local Lorentz and R-symmetry transformations may be used to choose a gauge in which

$$\mathcal{D}_\alpha^{1\bar{1}}| = \mathcal{D}_\alpha\,, \qquad -\mathcal{D}_\alpha^{2\bar{2}}| = \bar{\mathcal{D}}_\alpha\,, \tag{5.8}$$

where $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ are the spinor covariant derivatives of (2,0) AdS superspace (C.1).

In the coordinate system defined by (5.8), the operators $\mathcal{D}_\alpha^{1\bar{1}}|$ and $\mathcal{D}_\alpha^{2\bar{2}}|$ involve no partial derivative with respect to $\theta_{1\bar{2}}$, $\theta_{2\bar{1}}$. Therefore, for any positive integer k, it holds that $\left(\mathcal{D}_{\hat{\alpha}_1}\cdots\mathcal{D}_{\hat{\alpha}_k}U\right)| = \mathcal{D}_{\hat{\alpha}_1}|\cdots\mathcal{D}_{\hat{\alpha}_k}|U|$, where $\mathcal{D}_{\hat{\alpha}} := \left(\mathcal{D}_\alpha^{1\bar{1}}, -\mathcal{D}_\alpha^{2\bar{2}}\right)$ and U is a tensor superfield. This also implies that $\mathcal{D}_a|$ coincides with the vector covariant derivative of (2,0) AdS superspace. The latter will be denoted by the same symbol $\mathcal{D}_a$. We hope that no notational confusion will occur for the reader.

Let us fix an $\mathcal{N}=4$ AdS superspace and consider its Killing vector field ξ specified by eqs. (B.1)–(B.3). We introduce the bar-projections of the parameters involved:

$$\tau^a := \xi^a|\,, \quad \tau^\alpha := \xi^\alpha_{\underline{1\bar{1}}}|\,, \quad \bar{\tau}^\alpha = \xi^\alpha_{\underline{2\bar{2}}}|\,, \quad t := \mathrm{i}\left(\Lambda^{12} + \Lambda^{\bar{1}\bar{2}}\right)| = \bar{t}\,, \quad t^{ab} := \Lambda^{ab}|\,; \tag{5.9a}$$

$$\varepsilon^\alpha := -\xi^\alpha_{\underline{1\bar{2}}}|\,, \quad \bar{\varepsilon}^\alpha = \xi^\alpha_{\underline{2\bar{1}}}|\,, \quad \hat{\sigma} := \mathrm{i}\left(\Lambda^{12} - \Lambda^{\bar{1}\bar{2}}\right)| = \bar{\hat{\sigma}}\,; \tag{5.9b}$$

$$\bar{\varepsilon}_{\rm L} := -\frac{1}{4S}\Lambda^{11}|\,, \quad \varepsilon_{\rm L} = -\frac{1}{4S}\Lambda^{22}|\,, \quad \bar{\varepsilon}_{\rm R} = -\frac{1}{4S}\Lambda^{\bar{1}\bar{1}}|\,, \quad \varepsilon_{\rm R} = -\frac{1}{4S}\Lambda^{\bar{2}\bar{2}}|\,. \tag{5.9c}$$

The parameters $\left(\tau^a,\, \tau^\alpha,\, \bar{\tau}_\alpha,\, t,\, t^{ab}\right)$ describe the infinitesimal isometries of (2,0) AdS superspace. This may be proved by computing the bar-projection of the equations (B.2a)–(B.2e).

The parameters $(\varepsilon^\alpha,\, \bar{\varepsilon}_\alpha,\, \hat{\sigma},\, \varepsilon_{\rm L},\, \bar{\varepsilon}_{\rm L},\, \varepsilon_{\rm R},\, \bar{\varepsilon}_{\rm R})$ generate those $\mathcal{N}=4$ AdS isometries which are not manifest in the (2,0) AdS setting. These include two rigid supersymmetries and the residual R-symmetry transformations. Depending on the $\mathcal{N}=4$ AdS superspace chosen, these parameters obey different constraints. Let us spell out these constraints for the various cases.

5.1 AdS superspace reduction (4,0) → (2,0)

For the reduction from (4,0) to (2,0) AdS superspace, we find a set of differential relations between ε_α, $\varepsilon_{\rm L}$, $\varepsilon_{\rm R}$ and their complex conjugates:

$$\mathcal{D}_\alpha\bar{\varepsilon}_\beta = 4S\varepsilon_{\alpha\beta}\,\bar{\varepsilon}_{\rm L}\,, \qquad \bar{\mathcal{D}}_\alpha\varepsilon_\beta = -4S\varepsilon_{\alpha\beta}\,\varepsilon_{\rm L}\,, \tag{5.10a}$$

$$\mathcal{D}_\alpha\varepsilon_\beta = -4S\varepsilon_{\alpha\beta}\,\bar{\varepsilon}_{\rm R}\,, \quad \bar{\mathcal{D}}_\alpha\bar{\varepsilon}_\beta = 4S\varepsilon_{\alpha\beta}\,\varepsilon_{\rm R}\,, \tag{5.10b}$$

$$\bar{\mathcal{D}}_\alpha\varepsilon_{\rm L} = \bar{\mathcal{D}}_\alpha\varepsilon_{\rm R} = 0\,, \quad \mathcal{D}_\alpha\varepsilon_{\rm L} = \mathrm{i}\,\varepsilon_\alpha\left(1 + \frac{X}{2S}\right)\,, \quad \mathcal{D}_\alpha\varepsilon_{\rm R} = -\mathrm{i}\,\bar{\varepsilon}_\alpha\left(1 - \frac{X}{2S}\right)\,. \tag{5.10c}$$

[16] Depending on the choice of parameters $\mathcal{S}$, $\mathcal{S}^{ij\bar{i}\bar{j}}$ and X, the R-symmetry connection may take its values in a subgroup of $\mathrm{SU}(2)_{\rm L}\times\mathrm{SU}(2)_{\rm R}$. This point was discussed earlier.

The action of the $\mathrm{U}(1)_R$ generator (5.3) on these parameters is

$$\mathcal{J}\varepsilon_\alpha = -\frac{X}{2S}\varepsilon_\alpha\,, \qquad \mathcal{J}\varepsilon_{\rm L} = -\Big(1+\frac{X}{2S}\Big)\varepsilon_{\rm L}\,, \qquad \mathcal{J}\varepsilon_{\rm R} = -\Big(1-\frac{X}{2S}\Big)\varepsilon_{\rm R}\,. \tag{5.11}$$

The real parameter $\hat\sigma$, corresponding to one of the residual R-symmetries, can be shown to obey

$$\left(\hat\sigma - \frac{X}{2S}t\right) = {\rm const}\,. \tag{5.12}$$

A finite U(1) transformation generated by the constant parameter $(\hat\sigma - tX/2S)$ does not act on the (2,0) AdS superspace. It will be more convenient to parametrize this transformation using the constant parameter

$$\sigma := \left(\hat\sigma - \frac{X}{2S}t\right)\,, \tag{5.13}$$

such that $t+\hat\sigma = \Big(1+\frac{X}{2S}\Big)t+\sigma$.

In the critical cases, $|X| = 2S$, the parameters are further constrained as follows:

$$X = 2S: \qquad \Lambda^{\bar k \bar l} = \varepsilon_{\rm R} = 0\,, \qquad \mathcal{D}_\alpha\varepsilon_\beta = \bar{\mathcal{D}}_\alpha\bar\varepsilon_\beta = 0\,; \tag{5.14a}$$
$$X = -2S: \qquad \Lambda^{kl} = \varepsilon_{\rm L} = 0\,, \qquad \bar{\mathcal{D}}_\alpha\varepsilon_\beta = \mathcal{D}_\alpha\bar\varepsilon_\beta = 0\,. \tag{5.14b}$$

5.2 AdS superspace reduction (3,1) → (2,0)

In order to carry out reduction from (3,1) to (2,0) AdS superspace, a local R-symmetry transformation can be applied to choose $w^{i\bar i}$ of the form:

$$w^{1\bar 1} = w^{2\bar 2} = 0\,, \quad w^{1\bar 2} = 1\,, \quad w^{2\bar 1} = -\overline{(w^{1\bar 2})} = -1\,. \tag{5.15}$$

As a result, the conditions (5.1) and (5.2) hold. In the gauge chosen we have

$$\Lambda^{\bar k\bar l} = \delta_k^{\bar k}\delta_l^{\bar l}\Lambda^{kl}\,, \qquad \varepsilon_{\rm L} = \varepsilon_{\rm R} := \varepsilon\,. \tag{5.16}$$

Computing the bar-projection of (B.2a)–(B.2e) gives

$$\mathcal{D}_\alpha\bar\varepsilon_\beta = -\mathcal{D}_\alpha\varepsilon_\beta = 4S\varepsilon_{\alpha\beta}\,\bar\varepsilon\,, \qquad \bar{\mathcal{D}}_\alpha\varepsilon_\beta = -\bar{\mathcal{D}}_\alpha\bar\varepsilon_\beta = -4S\varepsilon_{\alpha\beta}\,\varepsilon\,, \tag{5.17a}$$
$$\bar{\mathcal{D}}_\alpha\varepsilon = 0\,, \qquad \mathcal{D}_\alpha\varepsilon = \frac{\rm i}{2}\big(\varepsilon_\alpha - \bar\varepsilon_\alpha\big)\,. \tag{5.17b}$$

These imply

$$\mathcal{D}_\alpha\left(\varepsilon_\beta + \bar\varepsilon_\beta\right) = \bar{\mathcal{D}}_\alpha\left(\varepsilon_\beta+\bar\varepsilon_\beta\right) = 0\,. \tag{5.18}$$

The real parameter $\hat\sigma$ proves to vanish.

5.3 AdS superspace reduction (2,2) → (2,0)

In order to carry out reduction from (2,2) to (2,0) AdS superspace, a local R-symmetry transformation can be applied to bring l^{ij} and $r^{\bar i\bar j}$ to the form:

$$l^{11} = l^{22} = 0\,, \quad r^{\bar 1\bar 1} = r^{\bar 2\bar 2} = 0\,, \quad l^{12} = -{\rm i}\,, \quad r^{\bar 1\bar 2} = {\rm i}\,. \tag{5.19}$$

As a result, the conditions (5.1) and (5.2) hold. We then have

$$\varepsilon_{\rm L} = \Lambda^{22} = 0\,, \quad \varepsilon_{\rm R} = \Lambda^{\bar{2}\bar{2}} = 0\,. \tag{5.20}$$

Computing the bar-projection of (B.2a)–(B.2e) gives

$$\mathcal{D}_\alpha \varepsilon_\beta = \bar{\mathcal{D}}_\alpha \varepsilon_\beta = 0\,. \tag{5.21}$$

The real parameter $\hat{\sigma}$ is constant.

6 $\mathcal{N} = 4$ vector multiplet theories in $(2, 0)$ AdS superspace

In this section we reduce all results, which were obtained in sections 3 and 4 within the manifestly $\mathcal{N} = 4$ AdS supersymmetric setting, to (2,0) AdS superspace.

6.1 The field strength

We recall that the left vector multiplet is described by the gauged-invariant field strength W^{ij}, which is a left linear multiplet. It can equivalently be described by the left $\mathcal{O}(2)$ multiplet $W^{(2)}(v_{\rm L}) := v_i v_j W^{ij}$, with v^i the homogeneous complex coordinates for $\mathbb{C}P^1$. It is useful to introduce an inhomogeneous complex coordinate $\zeta_{\rm L}$ for $\mathbb{C}P^1$ by the rule

$$\zeta_{\rm L} := \frac{v^2}{v^1} \in \mathbb{C}\,, \tag{6.1}$$

which is defined in the north chart of $\mathbb{C}P^1$. Then we can represent the (2,0) AdS projection of $W^{(2)}(v_{\rm L})$ as

$$W^{(2)}(v_{\rm L})| = {\rm i}\zeta_{\rm L}\left(v^1\right)^2 W^{[2]}(\zeta_{\rm L})|\,, \qquad W^{[2]}(\zeta_{\rm L})| = -\frac{\rm i}{\zeta_{\rm L}}\Phi + G - {\rm i}\zeta_{\rm L}\bar{\Phi}\,, \tag{6.2}$$

where we have introduced the $\mathcal{N} = 2$ superfields

$$\Phi := W^{22}|\,, \qquad G := 2{\rm i}\, W^{12}|\,, \qquad \bar{\Phi} = W^{11}|\,. \tag{6.3}$$

By projecting the analyticity constraint $\mathcal{D}_\alpha^{(i\bar{i}} W^{jk)} = 0$ to (2,0) AdS superspace, it is not difficult to prove that Φ is chiral and $G = \bar{G}$ is a real linear superfield,

$$\bar{\mathcal{D}}_\alpha \Phi = \mathcal{D}_\alpha \bar{\Phi} = 0\,, \qquad \mathcal{D}^2 G = \bar{\mathcal{D}}^2 G = 0\,. \tag{6.4}$$

The generators of the R-symmetry group $\mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(2)_{\rm R}$ act on W^{ij} by the rule

$$\mathbf{L}^{ij} W^{kl} = \varepsilon^{k(i} W^{j)l} + \varepsilon^{l(i} W^{j)k}\,, \qquad \mathbf{R}^{\bar{i}\bar{j}} W^{kl} = 0\,. \tag{6.5}$$

Bar-projecting the first relation to (2,0) superspace gives

$$\mathbf{L}^{11}\Phi = {\rm i}G\,, \qquad \mathbf{L}^{12}\Phi = -\Phi\,, \quad \mathbf{L}^{22}\Phi = 0\,, \tag{6.6a}$$

$$\mathbf{L}^{11}G = -2{\rm i}\bar{\Phi}\,, \quad \mathbf{L}^{12}G = 0\,, \qquad \mathbf{L}^{22}G = 2{\rm i}\Phi\,. \tag{6.6b}$$

The fields Φ and G are neutral under the right R-symmetry group $\mathrm{SU}(2)_{\mathrm{R}}$. This observation tells us that the $\mathrm{U}(1)_R$ generator of (2,0) AdS superspace, eq. (5.3), acts on the superfields introduced as follows:

$$\mathcal{J}\Phi = -\boldsymbol{q}\Phi\,, \qquad \mathcal{J}\bar{\Phi} = \boldsymbol{q}\bar{\Phi}\,, \qquad \mathcal{J}G = 0\,, \tag{6.7}$$

where we have defined

$$\boldsymbol{q} := 1 + \frac{X}{2S}\,. \tag{6.8}$$

Given an isometry transformation of the $\mathcal{N}=4$ AdS superspace (see appendix B for the details), the transformation laws of the field strength W^{ij} is

$$\delta_{\mathcal{K}} W^{ij} = \mathcal{K}W^{ij} = \xi^a \mathcal{D}_a W^{ij} + \xi^\alpha_{k\bar{k}} \mathcal{D}^{k\bar{k}}_\alpha W^{ij} + 2\Lambda^{(i}{}_k W^{j)k}\,. \tag{6.9}$$

We now project this transformation law to (2,0) superspace. Using the analyticity condition $\mathcal{D}^{(i\bar{i}}_\alpha W^{jk)} = 0$ and the results of section 5, we obtain

$$\delta_{\mathcal{K}}\Phi = (\tau + \mathrm{i}t\mathcal{J})\,\Phi + \mathrm{i}\left(\varepsilon^\alpha \bar{\mathcal{D}}_\alpha - 4S\varepsilon_{\mathrm{L}}\right) G - \mathrm{i}\sigma\Phi\,, \tag{6.10a}$$

$$\begin{aligned}\delta_{\mathcal{K}} G &= \tau G - \mathrm{i}\left(\bar{\varepsilon}^\alpha \mathcal{D}_\alpha - 8S\bar{\varepsilon}_{\mathrm{L}}\right)\Phi - \mathrm{i}\left(\varepsilon^\alpha \bar{\mathcal{D}}_\alpha - 8S\varepsilon_{\mathrm{L}}\right)\bar{\Phi} \\ &= \tau G + \mathrm{i}\mathcal{D}_\alpha\left(\bar{\varepsilon}^\alpha\Phi\right) + \mathrm{i}\bar{\mathcal{D}}_\alpha\left(\varepsilon^\alpha\bar{\Phi}\right)\,. \end{aligned} \tag{6.10b}$$

We recall that the parameters $\tau = \tau^a\mathcal{D}_a + \tau^\alpha\mathcal{D}_\alpha + \bar{\tau}_\alpha\bar{\mathcal{D}}^\alpha$ and t describe the isometry of (2,0) AdS superspace. The relations (6.10) are universal in the sense that they hold for all the $\mathcal{N}=4$ AdS superspaces. All information about a concrete $\mathcal{N}=4$ AdS superspace is encoded in the Killing parameters ε_{L}, ε_α and σ, which satisfy different constraints as described in the previous section.

6.2 The tropical prepotential

The left field strength W^{ij} is constructed in terms of the right weight-zero tropical prepotential $V_{\mathrm{R}}(v_{\mathrm{R}})$ according to eq. (3.5). We introduce an inhomogeneous complex coordinate ζ_{R} for $\mathbb{C}P^1$ by the rule

$$v^{\bar{i}} = v^{\bar{1}}(1, \zeta_{\mathrm{R}})\,, \qquad \zeta_{\mathrm{R}} := \frac{v^{\bar{2}}}{v^{\bar{1}}} \in \mathbb{C}\,. \tag{6.11}$$

We also choose the isospinor $u_{\bar{i}}$ in (3.5) to be

$$u_{\bar{i}} = (1, 0)\,. \tag{6.12}$$

Then the relation (3.5) becomes

$$W^{ij} = \frac{\mathrm{i}}{4}\oint \frac{\mathrm{d}\zeta_{\mathrm{R}}}{2\pi}\left(\mathcal{D}^{ij\bar{1}\bar{1}} - 4\mathrm{i}\mathcal{S}^{ij\bar{1}\bar{1}}\right) V_{\mathrm{R}}(\zeta_{\mathrm{R}})\,. \tag{6.13}$$

Here the right weight-zero tropical prepotential is described by the Laurent series

$$V_{\mathrm{R}}(v_{\mathrm{R}}) = \sum_{k=-\infty}^{+\infty} (\zeta_{\mathrm{R}})^k V_k\,, \qquad \bar{V}_k = (-1)^k V_{-k}\,. \tag{6.14}$$

The analyticity constraint (A.11) projected to (2,0) AdS implies

$$\mathcal{D}^{1\bar{2}}_\alpha V(\zeta_{\rm R})| = \zeta_{\rm R}\mathcal{D}_\alpha V(\zeta_{\rm R})| \,, \qquad \mathcal{D}^{2\bar{1}}_\alpha V(\zeta_{\rm R})| = -\frac{1}{\zeta_{\rm R}}\bar{\mathcal{D}}_\alpha V(\zeta_{\rm R})| \,. \tag{6.15}$$

The $\mathcal{N}=4$ AdS transformation law of the tropical prepotential [6] is

$$\delta_\mathcal{K} V_{\rm R} = \Big(\xi^a\mathcal{D}_a + \xi^\alpha_{k\bar{k}}\mathcal{D}^{k\bar{k}}_\alpha + \Lambda^{\bar{i}\bar{j}}\mathbf{R}_{\bar{i}\bar{j}}\Big)V_{\rm R} \,, \tag{6.16}$$

where

$$\Lambda^{\bar{i}\bar{j}}\mathbf{R}_{\bar{i}\bar{j}}V_{\rm R} = -\Lambda^{(\bar{2})}\partial^{(-\bar{2})}V_{\rm R} \,, \qquad \Lambda^{(\bar{2})} = \Lambda^{\bar{i}\bar{j}}v_{\bar{i}}v_{\bar{j}} \,, \tag{6.17}$$

and the operator $\partial^{(-\bar{2})}$ is defined according to (A.26). Projecting the transformation law (6.16) to (2,0) AdS superspace gives

$$\begin{aligned}\delta_\mathcal{K} V_{\rm R}(\zeta_{\rm R})| &= \big(\tau + {\rm i}t\mathcal{J}\big)V_{\rm R}(\zeta_{\rm R})| - \zeta_{\rm R}\varepsilon^\alpha\mathcal{D}_\alpha V_{\rm R}(\zeta_{\rm R})| + \frac{1}{\zeta_{\rm R}}\bar{\varepsilon}_\alpha\bar{\mathcal{D}}^\alpha V_{\rm R}(\zeta_{\rm R})| \\ &\quad + {\rm i}\left[4{\rm i}S\varepsilon_{\rm R}\frac{1}{\zeta_{\rm R}} - \sigma + 4{\rm i}S\bar{\varepsilon}_{\rm R}\zeta_{\rm R}\right]\zeta_{\rm R}\partial_{\zeta_{\rm R}}V_{\rm R}(\zeta_{\rm R})| \,.\end{aligned} \tag{6.18}$$

Note that the action of $\mathcal{J}$ on $V(\zeta_{\rm R})$ is

$$\mathcal{J}V_{\rm R}(\zeta_{\rm R}) = (2-\boldsymbol{q})\mathbf{R}^{\bar{1}\bar{2}}V_{\rm R}(\zeta_{\rm R}) = (2-\boldsymbol{q})\zeta_{\rm R}\partial_{\zeta_{\rm R}}V_{\rm R}(\zeta_{\rm R}) \,. \tag{6.19}$$

For the coefficients in the Laurent series expansion of $V_{\rm R}|$, eq. (6.14), the transformation law (6.18) leads to

$$\begin{aligned}\delta_\mathcal{K} V_k| &= \big(\tau + {\rm i}t\mathcal{J}\big)V_k| - {\rm i}\sigma k V_k| \\ &\quad - \Big(\varepsilon^\alpha\mathcal{D}_\alpha + 4(k-1)S\bar{\varepsilon}_{\rm R}\Big)V_{k-1}| + \Big(\bar{\varepsilon}_\alpha\bar{\mathcal{D}}^\alpha - 4(k+1)S\varepsilon_{\rm R}\Big)V_{k+1}| \,.\end{aligned} \tag{6.20}$$

Evaluating the contour integral in (6.13) and making use of the analyticity condition (6.15), it is possible to obtain the expression for Φ, $\bar{\Phi}$ and G in terms of $V_k|$. The results are:

$$\Phi = \frac{1}{4}\bar{\mathcal{D}}^2 V_1| \,, \quad \bar{\Phi} = -\frac{1}{4}\mathcal{D}^2 V_{-1} = \frac{1}{4}\mathcal{D}^2\bar{V}_1| \,, \quad G = \frac{\rm i}{2}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha V_0| \,. \tag{6.21}$$

These relations show that the components of the gauge-invariant field strength are constructed in terms of only three components of the tropical prepotential: V_1, $\bar{V}_1$ and V_0. It is easy to see that the other components of the tropical prepotential, $V_2, V_3, \dots$, are purely gauge degrees of freedom.

Let us first compute the isometry transformation of $V_1|$ by applying (6.20):

$$\delta_\mathcal{K} V_1| = \big(\tau + {\rm i}t\mathcal{J}\big)V_1| - \varepsilon^\alpha\mathcal{D}_\alpha V_0| + \big(\bar{\varepsilon}_\alpha\bar{\mathcal{D}}^\alpha - 8S\varepsilon_{\rm R}\big)V_2| - {\rm i}\sigma V_1| \,. \tag{6.22}$$

This is equivalent to

$$\delta_\mathcal{K} V_1| = \big(\tau + {\rm i}t\mathcal{J}\big)V_1| - \varepsilon^\alpha\mathcal{D}_\alpha V_0| - {\rm i}\sigma V_1| + \bar{\mathcal{D}}_\alpha\big(\bar{\varepsilon}^\alpha V_2|\big) \,. \tag{6.23}$$

The last term is a pure gauge transformation that does not contribute to $\Phi = \frac{1}{4}\bar{\mathcal{D}}^2 V_1|$. From eq. (6.23) we deduce

$$\delta_{\mathcal{K}}\Phi = \big(\tau + \mathrm{i}t\mathcal{J}\big)\Phi - \frac{1}{4}\bar{\mathcal{D}}^2(\varepsilon^\alpha \mathcal{D}_\alpha V_0|) - \mathrm{i}\sigma\Phi\,. \tag{6.24}$$

This may be seen to be equivalent to (6.10a).

Next we compute the isometry transformation of $V_0|$ using (6.20)

$$\delta_{\mathcal{K}} V_0| = \tau V_0| + \big(\bar{\varepsilon}_\alpha \bar{\mathcal{D}}^\alpha - 4S\varepsilon_{\mathrm{R}}\big)V_1| + \big(\varepsilon^\alpha \mathcal{D}_\alpha - 4S\bar{\varepsilon}_{\mathrm{R}}\big)\bar{V}_1|\,. \tag{6.25}$$

This can be rewritten, with the aid of the identities

$$\varepsilon_{\mathrm{R}} = -\frac{1}{8S}\bar{\mathcal{D}}_\alpha \bar{\varepsilon}^\alpha\,, \qquad \bar{\varepsilon}_{\mathrm{R}} = -\frac{1}{8S}\mathcal{D}^\alpha \varepsilon_\alpha\,, \tag{6.26}$$

as follows

$$\delta_{\mathcal{K}} V_0| = \tau V_0| + \left\{\bar{\varepsilon}_\alpha \bar{\mathcal{D}}^\alpha V_1| + \frac{1}{2}\left(\bar{\mathcal{D}}_\alpha \bar{\varepsilon}^\alpha\right) V_1| + \text{c.c.}\right\}. \tag{6.27}$$

This transformation law is valid for all the $\mathcal{N} = 4$ AdS superspaces.

6.2.1 (4,0) AdS supersymmetry

In the case of (4,0) AdS supersymmetry with $\boldsymbol{q} \neq 0$, the following relations hold:

$$\varepsilon_\alpha = -\frac{\mathrm{i}}{\boldsymbol{q}}\mathcal{D}_\alpha \varepsilon_{\mathrm{L}}\,, \quad \bar{\varepsilon}_\alpha = \frac{\mathrm{i}}{\boldsymbol{q}}\bar{\mathcal{D}}_\alpha \bar{\varepsilon}_{\mathrm{L}}\,. \tag{6.28}$$

Then the transformation of V_0, eq. (6.27), can be rewritten as

$$\delta_{\mathcal{K}} V_0| = \tau V_0| - \frac{2\mathrm{i}}{\boldsymbol{q}}\left(\bar{\varepsilon}_{\mathrm{L}}\Phi - \varepsilon_{\mathrm{L}}\bar{\Phi}\right) + \frac{\mathrm{i}}{2\boldsymbol{q}}\bar{\mathcal{D}}^2\left(\bar{\varepsilon}_{\mathrm{L}} V_1|\right) - \frac{\mathrm{i}}{2\boldsymbol{q}}\mathcal{D}^2\left(\varepsilon_{\mathrm{L}}\bar{V}_1|\right)\,. \tag{6.29}$$

The last two terms generate a pure gauge transformation and can be omitted.

In the case of critical (4,0) AdS supersymmetry, $X + 2S = 2S\boldsymbol{q} = 0$, we have $\varepsilon_{\mathrm{L}} = 0$. Here ε_α can be expressed in terms of a real parameter ρ such that

$$\varepsilon_\alpha = -\mathrm{i}\mathcal{D}_\alpha \rho\,, \quad \mathcal{D}^2\rho = 8\mathrm{i}S\bar{\varepsilon}_{\mathrm{R}}\,, \quad \bar{\rho} = \rho\,. \tag{6.30}$$

The parameter ρ is defined up to an arbitrary constant shift of the form

$$\rho \;\rightarrow\; \rho + \psi\,, \qquad \mathcal{J}\psi = 0\,, \qquad \bar{\psi} = \psi = \text{const}\,. \tag{6.31}$$

Using ρ, we can rewrite the transformation of $V_0|$, eq. (6.27), as

$$\delta_{\mathcal{K}} V_0| = \tau V_0| - 2\mathrm{i}\rho\big(\Phi - \bar{\Phi}\big) + \frac{1}{2}\Big\{\mathrm{i}\left(\bar{\mathcal{D}}^2 \rho V_1|\right) + \text{c.c.}\Big\}\,. \tag{6.32}$$

The last term generates a pure gauge transformation of $V_0|$, and so does the shift (6.31).

JHEP05(2014)018

6.2.2 (3,1) AdS supersymmetry

In the case of (3,1) AdS supersymmetry, we can introduce a complex parameter ρ such that

$$\mathcal{D}_\alpha \rho = \frac{\mathrm{i}}{2}(\varepsilon_\alpha + \bar{\varepsilon}_\alpha)\,, \qquad \mathcal{J}\rho = -\rho\,, \qquad \mathcal{D}^2\rho = 0\,. \tag{6.33}$$

The existence of this representation follows from eqs. (5.17a) and (5.18). The parameter ρ is defined modulo arbitrary shifts of the form

$$\rho \;\to\; \rho + \bar{\psi}\,, \qquad \mathcal{J}\bar{\psi} = -\bar{\psi}\,, \qquad \mathcal{D}_\alpha \bar{\psi} = 0\,. \tag{6.34}$$

Due to eqs. (5.17b) and (6.33), the spinor parameter ε_α can now be expressed in the form

$$\varepsilon_\alpha = -\mathrm{i}\mathcal{D}_\alpha(\varepsilon + \rho) = -\mathrm{i}\bar{\mathcal{D}}_\alpha(\bar{\varepsilon} - \bar{\rho})\,, \qquad \bar{\varepsilon}_\alpha = \mathrm{i}\bar{\mathcal{D}}_\alpha(\bar{\varepsilon} + \bar{\rho}) = \mathrm{i}\mathcal{D}_\alpha(\varepsilon - \rho)\,. \tag{6.35}$$

Using this representation allows us to rewrite the transformation law (6.27) as

$$\begin{aligned}\delta_{\mathcal{K}} V_0| = \tau V_0| - 2\mathrm{i}(\bar{\varepsilon} + \bar{\rho})\Phi + 2\mathrm{i}(\varepsilon + \rho)\bar{\Phi}\\ + \left\{\frac{\mathrm{i}}{2}\bar{\mathcal{D}}^2\Big((\bar{\varepsilon} + \bar{\rho})V_1|\Big) + \text{c.c.}\right\}.\end{aligned} \tag{6.36}$$

The expression in the second line generates a pure gauge transformation and can be omitted. It should be pointed out that any shift of ρ defined by (6.34) leads to a pure gauge transformation of $V_0|$.

6.2.3 (2,2) AdS supersymmetry

It remains to consider the case of the (2,2) AdS supersymmetry. In accordance with (5.21), we can introduce a complex parameter ρ such that

$$\varepsilon_\alpha = -\mathrm{i}\mathcal{D}_\alpha \rho\,, \qquad \mathcal{J}\rho = -\rho\,, \qquad \mathcal{D}^2\rho = 0\,. \tag{6.37}$$

As in the (3,1) case, this parameter is defined modulo arbitrary antichiral shifts of the form (6.34). Then the transformation law (6.27) can be rewritten as

$$\delta_{\mathcal{K}} V_0| = \tau V_0| - 2\mathrm{i}\left(\bar{\rho}\,\Phi - \rho\bar{\Phi}\right) + \left\{\frac{\mathrm{i}}{2}\bar{\mathcal{D}}^2\left(\bar{\rho}V_1|\right) + \text{c.c.}\right\}. \tag{6.38}$$

Here the third term generates a pure gauge transformation and can be omitted.

6.3 The composite right linear multiplet

One of the main aims of the present section is to reduce the action for $\mathcal{N} = 4$ SYM to (2,0) AdS. It involves the composite right $\mathcal{O}(2)$ multiplet $\boldsymbol{W}_{\rm R}^{(2)}$, which is defined by (3.9) and can be represented as

$$\boldsymbol{W}_{\rm R}^{(2)} = \mathrm{i}\zeta_{\rm R}\left(v^{\bar{1}}\right)^2 \boldsymbol{W}^{[2]}\,, \qquad \boldsymbol{W}^{[2]}(\zeta_{\rm R}) = -\frac{\mathrm{i}}{2\zeta_{\rm R}}\boldsymbol{W}^{\bar{2}\bar{2}} + 2\mathrm{i}\boldsymbol{W}^{\bar{1}\bar{2}} - \mathrm{i}\zeta_{\rm R}\boldsymbol{W}^{\bar{1}\bar{1}}\,. \tag{6.39}$$

Computing the bar-projection of the superfields on the right gives

$$\boldsymbol{W}^{\bar{1}\bar{1}}| = -\frac{\mathrm{i}}{4}\mathcal{D}^2\Phi + \mathcal{S}^{22\bar{1}\bar{1}}\bar{\Phi}\,, \qquad \boldsymbol{W}^{\bar{2}\bar{2}}| = \frac{\mathrm{i}}{4}\bar{\mathcal{D}}^2\bar{\Phi} + \mathcal{S}^{11\bar{2}\bar{2}}\Phi\,, \tag{6.40a}$$

$$\boldsymbol{W}^{\bar{1}\bar{2}}| = -\frac{1}{4}\Big(\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha + 4\mathrm{i}\boldsymbol{q}\mathcal{S}\Big)G\,. \tag{6.40b}$$

The values of $\mathcal{S}^{22\bar{1}\bar{1}}$ and $\mathcal{S}^{11\bar{2}\bar{2}}$ corresponding to the various types of $\mathcal{N}=4$ AdS supersymmetry are:

$$(4,0)\ \text{AdS:}\quad \mathcal{S}^{22\bar{1}\bar{1}} = \mathcal{S}^{11\bar{2}\bar{2}} = 0\,; \tag{6.41a}$$

$$(3,1)\ \text{AdS:}\quad \mathcal{S}^{22\bar{1}\bar{1}} = \mathcal{S}^{11\bar{2}\bar{2}} = -S\,; \tag{6.41b}$$

$$(2,2)\ \text{AdS:}\quad \mathcal{S}^{22\bar{1}\bar{1}} = \mathcal{S}^{11\bar{2}\bar{2}} = 0\,. \tag{6.41c}$$

6.4 The $\mathcal{N}=4$ vector multiplet actions

It was proven in [7] that the reduction of the right action, eq. (4.2), to (2,0) AdS superspace is given by

$$S_{\rm R} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\oint_C \frac{\mathrm{d}\zeta_{\rm R}}{2\pi\mathrm{i}\zeta_{\rm R}}\,\mathcal{L}^{[2]}_{\rm R}(\zeta_{\rm R})|\,, \qquad E^{-1} := \mathrm{Ber}\left(E_A{}^M\right)\,, \tag{6.42}$$

where $\mathcal{L}^{[2]}_{\rm R}$ is related to the original Lagrangian by the rule $\mathcal{L}^{(2)}_{\rm R}(v_{\rm R}) = \mathrm{i}\zeta_{\rm R}\left(v^{\bar{1}}\right)^2\mathcal{L}^{[2]}_{\rm R}(\zeta_{\rm R})$. In the $\mathcal{N}=4$ SYM case, the Lagrangian is given by (4.4). Its reduction to (2,0) AdS superspace is

$$\mathcal{L}^{[2]}_{\rm R}| = \frac{1}{2}\Big(V(\zeta_{\rm R})\boldsymbol{W}^{[2]}_{\rm R}(\zeta_{\rm R})\Big)|\,. \tag{6.43}$$

It is now a simple exercise to compute the contour integral in the action defined by (6.42) and (6.43). We obtain

$$S = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\bigg[\frac{1}{8}V_1\Big(\bar{\mathcal{D}}^2\bar{\Phi} - 4\mathrm{i}\mathcal{S}^{11\bar{2}\bar{2}}|\Phi\Big) - \frac{1}{4}V_0\big(\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha - 4\mathcal{S}\boldsymbol{q}\big)G$$
$$+\frac{1}{8}\bar{V}_1\Big(\mathcal{D}^2\Phi + 4\mathrm{i}\mathcal{S}^{22\bar{1}\bar{1}}|\bar{\Phi}\Big)\bigg]\,. \tag{6.44}$$

This is equivalent to

$$S = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\left[\bar{\Phi}\Phi - \frac{1}{2}G^2 + \mathcal{S}\boldsymbol{q}V_0G\right]$$
$$+\frac{\mathrm{i}}{2}\int \mathrm{d}^3x\,\mathrm{d}^2\theta\,\mathcal{E}\,\mathcal{S}^{11\bar{2}\bar{2}}\Phi^2 - \frac{\mathrm{i}}{2}\int \mathrm{d}^3x\,\mathrm{d}^2\bar{\theta}\,\bar{\mathcal{E}}\,\mathcal{S}^{22\bar{1}\bar{1}}\bar{\Phi}^2\,. \tag{6.45}$$

We see that the action involves a Chern-Simons term for $\boldsymbol{q}\neq 0$ and a mass-like chiral term for $\mathcal{S}^{11\bar{2}\bar{2}}\neq 0$.

In the (2,2) AdS case, the SYM theory can equivalently be described by the left Lagrangian (4.7). Let us prove this claim in the Abelian case by comparing the two different actions upon their reduction to (2,0) AdS superspace. Upon this reduction, the action associated with (4.7) becomes

$$S^{(2,2)} = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E \oint_C \frac{\mathrm{d}\zeta_{\rm L}}{2\pi \mathrm{i} \zeta_{\rm L}} \frac{1}{2d^{[2]}_{\rm L}} \Big(W^{[2]}_{\rm L} W^{[2]}_{\rm L} \Big) \Big| \,. \tag{6.46}$$

Here $W^{[2]}_{\rm L}(\zeta_{\rm L})|$ is given by eq. (6.2). We remind the reader that d^{ij} is proportional to l^{ij} and the latter has the only non-zero component $l^{12} = -\mathrm{i}$. By choosing

$$d^{ij} = -\frac{1}{2} l^{ij}\,, \qquad d^{[2]} = -1\,, \tag{6.47}$$

it is trivial to compute the contour integral in (6.46). The resulting action is

$$S^{(2,2)} = \int \mathrm{d}^3x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \left[\bar{\Phi}\Phi - \frac{1}{2} G^2 \right] . \tag{6.48}$$

This coincides with the action (6.45) in the (2,2) AdS case.

7 Concluding comments

In this paper we have constructed the pure $\mathcal{N} = 4$ SYM theories in three dimensions for all types of $\mathcal{N} = 4$ AdS supersymmetry.[17] In the Abelian case, these theories were described within the manifestly $\mathcal{N} = 4$ supersymmetric setting as well as in (2,0) AdS superspace where only $\mathcal{N} = 2$ supersymmetry is manifest. In all the $\mathcal{N} = 4$ AdS superspaces, the vector multiplet action has the universal form given by eqs. (4.2) and (4.4). This is an example of the right linear multiplet action involving a special composite linear multiplet, eq. (3.9).[18] All specific details of the theory are encoded in the type of $\mathcal{N} = 4$ AdS supersymmetry chosen. These differences become explicit when the theory is reformulated in (2,0) AdS superspace in which the $\mathcal{N} = 4$ vector multiplet decomposes into the $\mathcal{N} = 2$ vector multiplet described by a real linear superfield G and the chiral scalar Φ and its conjugate $\bar{\Phi}$. The latter multiplets are equivalently described by unconstrained gauge prepotentials $\mathcal{X} := V_1|$ and $\mathcal{V} := V_0| = \bar{\mathcal{V}}$ such that $\Phi = \frac{1}{4}\bar{\mathcal{D}}^2\mathcal{X}$ and $G = \frac{\mathrm{i}}{2}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{V}$. Let us now summarize the key properties of the theory for all types of $\mathcal{N} = 4$ AdS supersymmetry.

In the case of (4,0) AdS supersymmetry with $\boldsymbol{q} = 1 + X/2S \neq 0$, the non-manifest supersymmetry transformations are[19]

$$\delta_\varepsilon \Phi = \mathrm{i}\big(\varepsilon^\alpha \bar{\mathcal{D}}_\alpha - 4S\varepsilon_{\rm L}\big) G = -\frac{1}{4}\bar{\mathcal{D}}^2(\varepsilon^\alpha \mathcal{D}_\alpha \mathcal{V}) = -\frac{1}{2(2-\boldsymbol{q})}\bar{\mathcal{D}}^2\left(\bar{\varepsilon}_{\rm R} G\right) , \tag{7.1a}$$

$$\delta_\varepsilon \mathcal{V} = -\frac{2\mathrm{i}}{\boldsymbol{q}}\left(\bar{\varepsilon}_{\rm L}\Phi - \varepsilon_{\rm L}\bar{\Phi}\right) , \tag{7.1b}$$

[17] A brief discussion of off-shell $\mathcal{N} = 4$ hypermultiplets coupled to the vector multiplet is given in appendix A.

[18] The left and right linear multiplet actions [6] are known to be universal in the sense that the action of any off-shell $\mathcal{N} = 4$ supergravity-matter system may be realized as a sum of left and right linear multiplet actions [16].

[19] In the critical case with $X = 2S$ and $\boldsymbol{q} = 2$, the last expression in (7.1a) is not defined. In this case $\delta_\varepsilon\Phi$ can be represented as $\delta_\varepsilon\Phi = \frac{1}{2}\bar{\mathcal{D}}^2(\omega G)$. Here the real parameter ω is such that $\varepsilon_\alpha = -\mathrm{i}\bar{\mathcal{D}}_\alpha\omega$ and $\bar{\mathcal{D}}^2\omega = -8\mathrm{i}S\varepsilon_{\rm L}$.

JHEP05(2014)018

and therefore

$$\delta_\varepsilon G = -\mathrm{i}\big(\bar\varepsilon^\alpha \mathcal{D}_\alpha - 8S\bar\varepsilon_{\rm L}\big)\Phi + \text{c.c.} = \mathrm{i}\mathcal{D}_\alpha\left(\bar\varepsilon^\alpha\Phi\right) + \text{c.c.} = \frac{\mathrm{i}}{2}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\,\delta_\varepsilon\mathcal{V}\,. \tag{7.1c}$$

The parameters $\varepsilon_{\rm L}$ and ε_α are defined in section 5.1. The invariant action is

$$S^{(4,0)} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\left[\bar\Phi\Phi - \frac{1}{2}G^2 + S\boldsymbol{q}\mathcal{V}G\right]. \tag{7.2}$$

This is exactly the AdS analogue of the Abelian $\mathcal{N}=4$ SYM theory on S^3 recently constructed in [1]. The Chern-Simons term in (7.2) is generated due to the non-zero curvature of AdS_3. It disappears in the flat-superspace limit. Thus our results explain the mysterious structure of $\mathcal{N}=4$ supersymmetric Yang-Mills theories on S^3 discovered in [1].

In the case of critical (4,0) AdS supersymmetry with $X+2S = 2S\boldsymbol{q} = 0$, we have $\varepsilon_{\rm L}=0$ and the non-manifest supersymmetry transformations are

$$\delta_\varepsilon\Phi = \mathrm{i}\varepsilon^\alpha\bar{\mathcal{D}}_\alpha G = -\frac{1}{4}\bar{\mathcal{D}}^2(\varepsilon^\alpha\mathcal{D}_\alpha\mathcal{V})\,, \tag{7.3a}$$

$$\delta_\varepsilon\mathcal{V} = -2\mathrm{i}\rho\big(\Phi-\bar\Phi\big)\,, \tag{7.3b}$$

and therefore

$$\delta_\varepsilon G = -\mathrm{i}\bar\varepsilon^\alpha\mathcal{D}_\alpha\Phi + \text{c.c.} = \mathrm{i}\mathcal{D}_\alpha(\bar\varepsilon^\alpha\Phi) + \text{c.c.} = \frac{\mathrm{i}}{2}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\delta_\varepsilon\mathcal{V}\,. \tag{7.3c}$$

The action is given by (7.2) with $\boldsymbol{q}=0$.

In the case of (3,1) AdS supersymmetry, the non-manifest supersymmetry transformations are

$$\delta_\varepsilon\Phi = \mathrm{i}\big(\varepsilon^\alpha\bar{\mathcal{D}}_\alpha - 4S\varepsilon\big)G = -\frac{1}{4}\bar{\mathcal{D}}^2(\varepsilon^\alpha\mathcal{D}_\alpha\mathcal{V}) = -\frac{1}{2}\bar{\mathcal{D}}^2\big((\bar\varepsilon-\bar\rho)G\big)\,, \tag{7.4a}$$

$$\delta_\varepsilon\mathcal{V} = -2\mathrm{i}(\bar\varepsilon+\bar\rho)\Phi + 2\mathrm{i}(\varepsilon+\rho)\bar\Phi\,, \tag{7.4b}$$

and therefore

$$\delta_\varepsilon G = -\mathrm{i}\big(\bar\varepsilon^\alpha\mathcal{D}_\alpha - 8S\bar\varepsilon\big)\Phi + \text{c.c.} = \mathrm{i}\mathcal{D}_\alpha(\bar\varepsilon^\alpha\Phi) + \text{c.c.} = \frac{\mathrm{i}}{2}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\,\delta_\varepsilon\mathcal{V}\,. \tag{7.4c}$$

The invariant action is

$$S^{(3,1)} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\left[\bar\Phi\Phi - \frac{1}{2}G^2 + \frac{1}{2}S\mathcal{V}G\right] - \frac{1}{2}S\left\{\mathrm{i}\int\mathrm{d}^3x\,\mathrm{d}^2\theta\,\mathcal{E}\,\Phi^2 + \text{c.c.}\right\}. \tag{7.5}$$

This theory possesses a chiral mass-like term, which is a new feature as compared with the $\mathcal{N}=4$ SYM on S^3 [1].

In the case of (2,2) AdS supersymmetry, the non-manifest supersymmetry transformations are[20]

$$\delta_\varepsilon\Phi = \mathrm{i}\varepsilon^\alpha\bar{\mathcal{D}}_\alpha G = \frac{1}{2}\bar{\mathcal{D}}^2\big(\bar\rho G\big)\,, \tag{7.6a}$$

$$\delta_\varepsilon G = -\mathrm{i}\bar\varepsilon^\alpha\mathcal{D}_\alpha\Phi + \text{c.c.} = \mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\big(\bar\rho\,\Phi - \rho\,\bar\Phi\big)\,. \tag{7.6b}$$

[20] In this case we do not need to consider the transformation of $\mathcal{V}$ since it does not appear in the action.

JHEP05(2014)018

Here the parameter ρ is defined by (6.37). The invariant action is

$$S^{(2,2)} = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E \left[\bar{\Phi}\Phi - \frac{1}{2}G^2\right] . \tag{7.7}$$

No Chern-Simons term shows up because the theory is formulated entirely in term of the field strength W^{ij} in $\mathcal{N} = 4$ AdS superspace, see eq. (4.7).

The chiral mass-like term appears only in the (3,1) AdS case. In all other cases it is prohibited by the rigid U(1) symmetry

$$\delta\Phi = -\mathrm{i}\sigma\Phi . \tag{7.8}$$

In the non-Abelian case, we defined the $\mathcal{N} = 4$ SYM action by its variation, eqs. (4.8) and (4.9), induced by an arbitrary variation of the tropical prepotential.[21] So far we have not yet been able to integrate this variation in a closed form in terms of $\mathcal{N} = 4$ superfields. However, there are three obvious ways to obtain closed-form expressions for the $\mathcal{N} = 4$ SYM action for all types of $\mathcal{N} = 4$ AdS supersymmetry. Firstly, it may be achieved using the superform construction in complete analogy with the Chern-Simons results of [16]. Secondly, we may start with the $\mathcal{N} = 4$ tropical prepotential $V_{\mathrm{R}}(v_{\mathrm{R}})$ and the gauge covariant field strength $\mathfrak{W}^{ij}$ and then reduce their $\mathcal{N} = 4$ isometry transformations to (2,0) AdS superspace. Using the explicit structure of the non-manifest supersymmetry transformations derived, there is a standard procedure to reconstruct a closed-form expression for the $\mathcal{N} = 4$ SYM action in $\mathcal{N} = 2$ superspace.[22] This procedure is explicitly implemented in appendix D in which we present complete $\mathcal{N} = 4$ SYM actions in (2,0) AdS superspace for all types of $\mathcal{N} = 4$ AdS supersymmetry. Thirdly, we may reduce the variation of the SYM action, eqs. (4.8) and (4.9), to (2,0) AdS superspace, in which the variation may be readily integrated. This is explicitly done in appendix E.

In this paper we have focused our attention on the $\mathcal{N} = 4$ left vector multiplet. Analogous results for the right vector multiplet can be obtained by applying the mirror map [6, 15].

In this paper we have studied the $\mathcal{N} = 4$ SYM theories. In [3, 7] the off-shell formalism, which was developed for general 3D $\mathcal{N} = 3$ supergravity-matter systems [6], was applied to the (3,0) and (2,1) AdS cases. Using these techniques allows one to construct $\mathcal{N} = 3$ SYM theories for both types of $\mathcal{N} = 3$ AdS supersymmetry, along the same lines as in the present paper.

Acknowledgments

SMK is grateful to Igor Samsonov and Dima Sorokin for showing him a preliminary draft of their paper [1] and for asking questions that stimulated the research presented in this note. We thank Joseph Novak for reading the manuscript. The work of SMK was supported in part by the ARC Discovery projects DP1096372 and DP140103925. The work of GT-M was supported by the Australian Research Council's Discovery Early Career Award (DECRA) No. DE120101498 and by the ARC Discovery project DP140103925.

[21] We constructed the manifestly $\mathcal{N} = 4$ SYM actions in the cases of (2,2) and critical (4,0) AdS supersymmetries.

[22] Unlike [1], here we do not have to guess the structure of two non-manifest supersymmetry transformations, we derive them from first principles.

A $\mathcal{N} = 4$ SYM and projective superspace

In this appendix we consider a left $\mathcal{N} = 4$ Yang-Mills supermultiplet in a conformal supergravity background [6] and uncover the origin of a tropical prepotential $V_{\rm R}(v_{\rm R})$. Our consideration is similar to that given by Lindström and Roček in the case of 4D $\mathcal{N} = 2$ SYM theory [10]. Only right projective supermultiplets appear in this section. For this reason we consistently avoid using a subscript 'R' and simply denote $V_{\rm R}$ by V etc.

A.1 Tropical prepotential

To describe a left Yang-Mills supermultiplet, we introduce gauge covariant derivatives

$$\mathfrak{D}_A = \mathcal{D}_A + {\rm i}\,\mathfrak{A}_A\,, \tag{A.1}$$

where $\mathcal{D}_A$ denotes the $\mathcal{N} = 4$ supergravity covariant derivatives [6], and the connection $\mathfrak{A}_A(z)$ takes values in the Lie algebra of the gauge group. The fact that we are dealing with the left vector multiplet, is encoded in the anti-commutation relation:

$$\left\{\mathfrak{D}^{i\bar{i}}_\alpha, \mathfrak{D}^{j\bar{j}}_\beta\right\} = \cdots + 2\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}\mathfrak{W}^{ij}\,, \tag{A.2}$$

where the ellipsis denotes the purely supergravity terms. The SYM field strength $\mathfrak{W}^{ij} = \mathfrak{W}^{ji}$ is Hermitian, $(\mathfrak{W}^{ij})^\dagger = \mathfrak{W}_{ij}$, and obeys the Bianchi identity

$$\mathfrak{D}^{(i\bar{i}}_\gamma\mathfrak{W}^{jk)} = 0\,. \tag{A.3}$$

Under the gauge group (to be referred to as the τ-group), the covariant derivatives and any covariant matter superfield multiplet $U(z)$ transform as follows

$$\mathfrak{D}'_A = {\rm e}^{{\rm i}\tau}\mathfrak{D}_A e^{-{\rm i}\tau}\,, \qquad U' = {\rm e}^{{\rm i}\tau}U\,, \qquad \tau = \tau^\dagger\,, \tag{A.4}$$

with the Lie-algebra-valued gauge parameters $\tau(z)$ being Hermitian and otherwise unconstrained. In particular, the field strength transforms as

$$\mathfrak{W}^{ij\prime} = {\rm e}^{{\rm i}\tau}\mathfrak{W}^{ij}{\rm e}^{-{\rm i}\tau}\,. \tag{A.5}$$

Using an isospinor $v := v^{\bar{i}} \in \mathbb{C}^2 \setminus \{0\}$, which provides homogeneous coordinates for $\mathbb{C}P^1$, we introduce gauge covariant operators

$$\mathfrak{D}^{(\bar{1})i}_\alpha := v_{\bar{i}}\mathfrak{D}^{i\bar{i}}_\alpha\,, \tag{A.6}$$

in complete analogy with (3.2). It is easy to see that the anti-commutator $\left\{\mathfrak{D}^{(\bar{1})i}_\alpha, \mathfrak{D}^{(\bar{1})j}_\beta\right\}$ coincides with the right-hand side of (3.10a), i.e. it does not involve the gauge field. This means that we may represent $\mathfrak{D}^{(\bar{1})i}_\alpha$ in the form:

$$\mathfrak{D}^{(\bar{1})i}_\alpha = {\rm e}^{\Omega_+}\mathcal{D}^{(\bar{1})i}_\alpha{\rm e}^{-\Omega_+}\,, \tag{A.7}$$

where we have introduced a Lie-algebra-valued bridge superfield

$$\Omega_+(\zeta) = \sum_{n=0}^{\infty}\Omega_n\zeta^n\,, \qquad \zeta := \frac{v^{\bar{2}}}{v^{\bar{1}}}\,. \tag{A.8}$$

Another representation for $\mathfrak{D}^{(\bar{1})i}_\alpha$ follows by applying the smile-conjugation to (A.7) (see, e.g., [6] for the definition of the smile-conjugation). The result is

$$\mathfrak{D}^{(\bar{1})i}_\alpha = \mathrm{e}^{-\Omega_-}\mathcal{D}^{(\bar{1})i}_\alpha \mathrm{e}^{\Omega_-}\,, \qquad \Omega_-(\zeta) = \sum_{n=0}^{\infty}(-1)^n\,\Omega_n^\dagger\,\frac{1}{\zeta^n}\,. \tag{A.9}$$

Introduce a Lie-algebra-valued superfield $V(\zeta)$ defined by

$$\mathrm{e}^V := \mathrm{e}^{\Omega_-}\mathrm{e}^{\Omega_+}\,, \qquad V(\zeta) = \sum_{n=-\infty}^{\infty} V_n\zeta^n\,, \qquad V_n^\dagger = (-1)^n V_{-n}\,. \tag{A.10}$$

It may be seen from (A.7) and (A.9) that V is a covariant projective multiplet,

$$\mathcal{D}^{(\bar{1})i}_\alpha V = 0\,. \tag{A.11}$$

It follows from (A.4) and (A.7) that the gauge transformation law of Ω_+ is

$$\mathrm{e}^{\Omega'_+(\zeta)} = \mathrm{e}^{\mathrm{i}\tau}\mathrm{e}^{\Omega_+(\zeta)}\mathrm{e}^{-\mathrm{i}\lambda(\zeta)}\,, \tag{A.12}$$

where the new gauge parameter $\lambda(\zeta)$ is a covariant weight-zero arctic multiplet

$$\mathcal{D}^{(\bar{1})i}_\alpha \lambda = 0\,, \qquad \lambda(\zeta) = \sum_{n=0}^{\infty}\lambda_n\zeta^n\,. \tag{A.13}$$

The gauge transformation law of the tropical prepotential is

$$\mathrm{e}^{V'} = \mathrm{e}^{\mathrm{i}\breve{\lambda}}\mathrm{e}^{V}\mathrm{e}^{-\mathrm{i}\lambda}\,. \tag{A.14}$$

We see that V transforms under the λ-group only.

A.2 Polar hypermultiplets

$\mathcal{N} = 4$ supersymmetric matter may be described in terms of gauge-covariantly arctic multiplets and their smile-conjugate antarctic multiplets.

A gauge-covariantly arctic multiplet of weight n, $\boldsymbol{\Upsilon}^{(\bar{n})}(v)$, is defined by

$$\mathfrak{D}^{(\bar{1})i}_\alpha \boldsymbol{\Upsilon}^{(\bar{n})} = 0\,, \qquad \boldsymbol{\Upsilon}^{(\bar{n})}(v) = \left(v^{\bar{1}}\right)^n \sum_{k=0}^{\infty}\boldsymbol{\Upsilon}_k\zeta^k\,. \tag{A.15}$$

It can be represented in the form

$$\boldsymbol{\Upsilon}^{(n)}(v) = \mathrm{e}^{\Omega_+(\zeta)}\Upsilon^{(n)}(v)\,, \tag{A.16}$$

where $\Upsilon^{(n)}(v)$ is an ordinary covariant arctic multiplet of weight n (see [6] for more details),

$$\mathcal{D}^{(\bar{1})i}_\alpha \Upsilon^{(\bar{n})} = 0\,, \qquad \Upsilon^{(\bar{n})}(v) = \left(v^{\bar{1}}\right)^n \sum_{k=0}^{\infty}\Upsilon_k\zeta^k\,. \tag{A.17}$$

A gauge-covariantly antarctic multiplet of weight n, $\breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v)$, is defined by

$$\breve{\boldsymbol{\Upsilon}}^{(\bar{n})} \overleftarrow{\mathfrak{D}}^{(\bar{1})i}_{\alpha} = 0\,, \qquad \breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v) = \left(v^{\bar{2}}\right)^n \sum_{k=0}^{\infty} (-1)^k \boldsymbol{\Upsilon}^{\dagger}_k \frac{1}{\zeta^k}\,. \tag{A.18}$$

It can be represented in the form

$$\breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v) = \breve{\Upsilon}^{(\bar{n})}(v) \mathrm{e}^{\Omega_-(\zeta)}\,, \tag{A.19}$$

where $\breve{\Upsilon}^{(\bar{n})}(v)$ is an ordinary antarctic multiplet

$$\mathcal{D}^{(\bar{1})i}_{\alpha} \breve{\Upsilon}^{(\bar{n})} = 0\,, \qquad \breve{\Upsilon}^{(\bar{n})}(v) = \left(v^{\bar{2}}\right)^n \sum_{n=0}^{\infty} (-1)^n \Upsilon^{\dagger}_n \frac{1}{\zeta^n}\,. \tag{A.20}$$

The gauge-covariantly arctic multiplet of weight n, $\boldsymbol{\Upsilon}^{(\bar{n})}(v)$, and its smile-conjugate antarctic one, $\breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v)$, constitute the gauge-covariantly polar multiplet of weight n. The gauge transformation laws of $\boldsymbol{\Upsilon}^{(\bar{n})}(v)$ and $\breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v)$ are

$$\boldsymbol{\Upsilon}^{(\bar{n})\prime}(v) = \mathrm{e}^{\mathrm{i}\tau} \boldsymbol{\Upsilon}^{(\bar{n})}(v)\,, \qquad \breve{\boldsymbol{\Upsilon}}^{(\bar{n})\prime}(v) = \breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v) \mathrm{e}^{-\mathrm{i}\tau}\,. \tag{A.21}$$

The gauge transformation laws of $\Upsilon^{(\bar{n})}(v)$ and $\breve{\Upsilon}^{(\bar{n})}(v)$ are

$$\Upsilon^{(\bar{n})}(v) = \mathrm{e}^{\mathrm{i}\lambda(\zeta)} \Upsilon^{(\bar{n})}(v)\,, \qquad \breve{\Upsilon}^{(\bar{n})\prime}(v) = \breve{\Upsilon}^{(\bar{n})}(v) \mathrm{e}^{-\mathrm{i}\lambda(\zeta)}\,. \tag{A.22}$$

In the case of weight $n = 1$, a gauge invariant hypermultiplet Lagrangian can be constructed. It is

$$\mathcal{L}^{(\bar{2})} = \mathrm{i} \breve{\boldsymbol{\Upsilon}}^{(\bar{1})} \boldsymbol{\Upsilon}^{(\bar{1})} = \mathrm{i} \breve{\Upsilon}^{(\bar{1})} \mathrm{e}^{V} \Upsilon^{(\bar{1})}\,. \tag{A.23}$$

A.3 Arctic and antarctic representations

Here we show that the SYM gauge connection $\mathfrak{A}_A$ may be expressed in terms of the tropical prepotential $V(\zeta)$, modulo the τ-gauge freedom. Our analysis in this subsection is inspired by the famous paper by Zupnik [23].

Let us introduce a new isospinor $u_{\bar{i}} \in \mathbb{C}^2 \setminus \{0\}$, which is only required to obey the inequality $(v,u) := v^{\bar{i}} u_{\bar{i}} \neq 0$. Since $v^{\bar{i}}$ and $u^{\bar{i}}$ are linearly independent vectors, we can construct a new basis for the gauge covariant spinor derivatives that includes $\mathfrak{D}^{(\bar{1})i}_{\alpha}$ and the following operators:

$$\mathfrak{D}^{(-\bar{1})i}_{\alpha} := \frac{1}{(v,u)} u_{\bar{i}} \mathfrak{D}^{i\bar{i}}_{\alpha}\,. \tag{A.24}$$

It can be seen that

$$\left\{ \mathfrak{D}^{(\bar{1})i}_{\alpha}, \mathfrak{D}^{(-\bar{1})j}_{\beta} \right\} = \cdots - 2\varepsilon_{\alpha\beta} \mathfrak{W}^{ij}\,, \tag{A.25}$$

where the ellipsis denotes the purely supergravity terms.

We introduce the first-order differential operators

$$\partial^{(\bar{2})} := (v,u)\, v^{\bar{i}} \frac{\partial}{\partial u^{\bar{i}}}\,, \qquad \partial^{(-\bar{2})} := \frac{1}{(v,u)} u^{\bar{i}} \frac{\partial}{\partial v^{\bar{i}}} \tag{A.26}$$

such that

$$\left[\partial^{(\bar{2})}, \partial^{(-\bar{2})}\right] = v^{\bar{i}}\frac{\partial}{\partial v^{\bar{i}}} - u^{\bar{i}}\frac{\partial}{\partial u^{\bar{i}}} \equiv \partial^{(\bar{0})} . \tag{A.27}$$

These operators are invariant under the τ-group. It is easy to see that

$$\left[\partial^{(-\bar{2})}, \mathfrak{D}^{(\bar{1})i}_{\alpha}\right] = \mathfrak{D}^{(-\bar{1})i}_{\alpha} . \tag{A.28}$$

When dealing with polar hypermultiplets, it is useful to introduce an arctic representation defined by the transformation

$$\hat{\mathcal{O}} \to \hat{\mathcal{O}}_{+} := \mathrm{e}^{-\Omega_{+}}\hat{\mathcal{O}}\,\mathrm{e}^{\Omega_{+}} , \qquad U \to U_{+} := \mathrm{e}^{-\Omega_{+}}U \tag{A.29}$$

applied to any operator $\hat{\mathcal{O}}$ and covariant superfield U. In the arctic representation, any gauge-covariantly arctic multiplet $\boldsymbol{\Upsilon}^{(\bar{n})}(v)$ becomes the ordinary arctic one, $\Upsilon^{(\bar{n})}(v)$,

$$\boldsymbol{\Upsilon}^{(\bar{n})}(v) \to \Upsilon^{(\bar{n})}(v) , \qquad \breve{\boldsymbol{\Upsilon}}^{(\bar{n})}(v) \to \breve{\Upsilon}^{(\bar{n})}(v)\mathrm{e}^{V(\zeta)} . \tag{A.30}$$

The gauge covariant derivatives $\mathfrak{D}^{(\bar{1})i}_{\alpha}$ turn into the AdS spinor covariant derivatives,

$$\mathfrak{D}^{(\bar{1})i}_{\alpha} \to \mathcal{D}^{(\bar{1})i}_{\alpha} . \tag{A.31}$$

The important point is that the projective derivative $\partial^{(-\bar{2})}$ turns into the operator

$$\partial^{(-\bar{2})} \to \mathfrak{D}^{(-\bar{2})} := \partial^{(-\bar{2})} + \mathrm{e}^{-\Omega_{+}}\left(\partial^{(-\bar{2})}\mathrm{e}^{\Omega_{+}}\right) , \tag{A.32}$$

which transforms as a covariant derivative under the λ-group. Then making use of (A.25) in conjunction with $\left[\mathfrak{D}^{(-\bar{2})}, \mathfrak{D}^{(\bar{1})i}_{\alpha}\right] = \mathfrak{D}^{(-\bar{1})i}_{\alpha}$, we read off

$$\mathfrak{W}^{ij}_{+} = \frac{1}{4}\left(\mathcal{D}^{(\bar{2})ij} - 4\mathrm{i}\mathcal{S}^{(\bar{2})ij}\right)\left(\mathrm{e}^{-\Omega_{+}}\partial^{(-\bar{2})}\mathrm{e}^{\Omega_{+}}\right) . \tag{A.33}$$

It may be seen that $\mathfrak{W}^{ij}_{+}$ is independent of $u^{\bar{i}}$, $\partial^{(\bar{2})}\mathfrak{W}^{ij}_{+} = 0$. It also satisfies the property

$$\mathfrak{D}^{(-\bar{2})}\mathfrak{W}^{ij}_{+} = 0 , \tag{A.34}$$

since in the original representation $\mathfrak{W}^{ij}$ is independent of $v^{\bar{i}}$. The field strength obeys the Bianchi identity

$$\mathfrak{D}^{\alpha(i\bar{i}}_{+}\mathfrak{W}^{jk)}_{+} = 0 . \tag{A.35}$$

If the gauge group is Abelian, then $\mathfrak{W}^{ij} = \mathfrak{W}^{ij}_{+}$ and (A.33) turns into

$$\mathfrak{W}^{ij} = \frac{1}{4}\left(\mathcal{D}^{(\bar{2})ij} - 4\mathrm{i}\mathcal{S}^{(\bar{2})ij}\right)\partial^{(-\bar{2})}\Omega_{+} . \tag{A.36}$$

Since Ω_{+} is a homogeneous function of v_{R} of degree zero, we have $\Omega_{+}(v_{\mathrm{R}}) = \Omega_{+}(\zeta)$ and

$$\partial^{(-\bar{2})}\Omega_{+}(v_{\mathrm{R}}) = -\frac{1}{(v^{\bar{1}})^{2}}\partial_{\zeta}\Omega_{+}(\zeta) . \tag{A.37}$$

Taking into account the fact that $\mathfrak{W}^{ij}$ is independent of ζ, we end up with the expression

$$\mathfrak{W}^{ij} = -\frac{1}{4}\left(\mathcal{D}^{ij\bar{2}\bar{2}} - 4\mathrm{i}\mathcal{S}^{ij\bar{2}\bar{2}}\right)\Omega_{1} = \frac{1}{4}\left(\mathcal{D}^{ij\bar{1}\bar{1}} - 4\mathrm{i}\mathcal{S}^{ij\bar{1}\bar{1}}\right)\Omega_{-1} . \tag{A.38}$$

This expression may be shown to be equivalent to (6.13).

JHEP05(2014)018

In complete analogy with the arctic representation, eq. (A.29), one can introduce the antarctic representation defined by

$$\hat{\mathcal{O}} \to \hat{\mathcal{O}}_- := \mathrm{e}^{\Omega_-} \hat{\mathcal{O}}\, \mathrm{e}^{-\Omega_-} \,, \qquad U \to U_- := \mathrm{e}^{\Omega_-} U \,. \tag{A.39}$$

In this representation, the SYM field strength takes the form

$$\mathfrak{W}_-^{ij} = \frac{1}{4}\Big(\mathcal{D}^{(\bar{2})ij} - 4\mathrm{i}\mathcal{S}^{(\bar{2})ij}\Big)\Big(\mathrm{e}^{\Omega_-}\partial^{(-\bar{2})}\mathrm{e}^{-\Omega_-}\Big) \,. \tag{A.40}$$

Comparing the above with (A.33) gives

$$\mathfrak{W}_-^{ij} = \mathrm{e}^{V}\mathfrak{W}_+^{ij}\mathrm{e}^{-V} \,. \tag{A.41}$$

B Isometries of $\mathcal{N} = 4$ AdS superspaces

In this appendix we review the structure of the Killing vector fields of a given $\mathcal{N} = 4$ AdS superspace following [7].

Given a particular $\mathcal{N} = 4$ AdS superspace, its isometry group is generated by Killing vector fields, $\xi = \xi^a \mathcal{D}_a + \xi^\alpha_{i\bar{i}} \mathcal{D}_\alpha^{i\bar{i}}$, obeying the Killing equation

$$0 = [\mathcal{K}, \mathcal{D}_A] \,, \qquad \mathcal{K} := \xi + \frac{1}{2}\Lambda^{\gamma\delta}\mathcal{M}_{\gamma\delta} + \Lambda^{kl}\mathbf{L}_{kl} + \Lambda^{\bar{k}\bar{l}}\mathbf{R}_{\bar{k}\bar{l}} \,. \tag{B.1}$$

This equation is equivalent to

$$\mathcal{D}_\alpha^{i\bar{i}}\xi_{\beta\gamma} = 4\mathrm{i}\varepsilon_{\alpha(\beta}\xi_{\gamma)}^{i\bar{i}} \,, \tag{B.2a}$$

$$\mathcal{D}_\alpha^{i\bar{i}}\xi_\beta^{j\bar{j}} = \xi_{\alpha\beta}\Big(\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{S} + \mathcal{S}^{ij\bar{i}\bar{j}}\Big) + \frac{1}{2}\Lambda_{\alpha\beta}\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}} + \Lambda^{ij}\varepsilon^{\bar{i}\bar{j}}\varepsilon_{\alpha\beta} + \Lambda^{\bar{i}\bar{j}}\varepsilon^{ij}\varepsilon_{\alpha\beta} \,, \tag{B.2b}$$

$$\mathcal{D}_\alpha^{i\bar{i}}\Lambda_{\beta\gamma} = 8\mathrm{i}\varepsilon_{\alpha(\beta}\xi_{\gamma)j\bar{j}}\left(\mathcal{S}^{ij\bar{i}\bar{j}} + \varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{S}\right) \,, \tag{B.2c}$$

$$\mathcal{D}_\alpha^{i\bar{i}}\Lambda^{kl} = -2\mathrm{i}\varepsilon^{i(k}\xi_\alpha{}^{l)\bar{i}}(2\mathcal{S} + X) - 2\mathrm{i}\xi_\alpha{}^i{}_{\bar{j}}\mathcal{S}^{kl\bar{i}\bar{j}} \,, \tag{B.2d}$$

$$\mathcal{D}_\alpha^{i\bar{i}}\Lambda^{\bar{k}\bar{l}} = -2\mathrm{i}\varepsilon^{\bar{i}(\bar{k}}\xi_\alpha{}^{i\bar{l})}(2\mathcal{S} - X) - 2\mathrm{i}\xi_{\alpha j}{}^{\bar{i}}\mathcal{S}^{ij\bar{k}\bar{l}} \,, \tag{B.2e}$$

and

$$\mathcal{D}_a\xi_b = \Lambda_{ab} \,, \tag{B.3a}$$

$$\mathcal{D}_a\xi_{j\bar{j}}^\beta = -\left(\mathcal{S}\xi_{j\bar{j}}^\gamma + \mathcal{S}_{jk\bar{j}\bar{k}}\xi^{\gamma k\bar{k}}\right)(\gamma_a)_\gamma{}^\beta \,, \tag{B.3b}$$

$$\mathcal{D}_a\Lambda^{bc} = 4S^2\left(\delta_a^b\xi^c - \delta_a^c\xi^b\right) \,, \tag{B.3c}$$

$$\mathcal{D}_a\Lambda^{kl} = \mathcal{D}_a\Lambda^{\bar{k}\bar{l}} = 0 \,. \tag{B.3d}$$

Some useful implications of the above equations are

$$\mathcal{D}_{(\alpha}^{i\bar{i}}\xi_{\beta\gamma)} = \mathcal{D}_{(\alpha}^{i\bar{i}}\Lambda_{\beta\gamma)} = 0 \,, \tag{B.4a}$$

$$\mathcal{D}^{\beta i\bar{i}}\xi_{\alpha\beta} = 6\mathrm{i}\xi_\alpha^{i\bar{i}} \,, \qquad \mathcal{D}^{\beta i\bar{i}}\Lambda_{\alpha\beta} = 12\mathrm{i}\xi_{\alpha j\bar{j}}\left(\mathcal{S}^{ij\bar{i}\bar{j}} + \varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{S}\right) \,, \tag{B.4b}$$

$$\mathcal{D}_{(\alpha}^{(i\bar{i}}\xi_{\beta)\bar{i}}^{j)} = \mathcal{D}_{(\alpha}^{i(\bar{i}}\xi_{\beta)i}{}^{\bar{j})} = 0 \,, \quad \mathcal{D}_{(\alpha}^{(i(\bar{i}}\xi_{\beta)}^{j)\bar{j})} = \xi_{\alpha\beta}\mathcal{S}^{ij\bar{i}\bar{j}} \,, \quad \mathcal{D}_{(\alpha}^{i\bar{i}}\xi_{\beta)i\bar{i}} = 4\xi_{\alpha\beta}\mathcal{S} + 2\Lambda_{\alpha\beta} \,, \tag{B.4c}$$

$$\mathcal{D}^{\alpha i\bar{i}}\xi_{\alpha i\bar{i}} = \mathcal{D}^{\alpha(i(\bar{i}}\xi_\alpha^{j)\bar{j})} = 0 \,, \quad \mathcal{D}^{\alpha(i\bar{i}}\xi_\alpha^{j)}{}_{\bar{i}} = -4\Lambda^{ij} \,, \quad \mathcal{D}^{\alpha i(\bar{i}}\xi_{\alpha i}{}^{\bar{j})} = -4\Lambda^{\bar{i}\bar{j}} \,. \tag{B.4d}$$

Here we have written the results in a form valid for the (4,0), (3,1) and (2,2) cases. Depending on the $\mathcal{N}=4$ AdS superspace under consideration, $\mathcal{S}$, X and $\mathcal{S}^{ij\bar{i}\bar{j}}$ are constrained by (2.5a)–(2.5c), while the $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ parameters Λ^{kl} and $\Lambda^{\bar{k}\bar{l}}$ are restricted by

$$(4,0) \quad \text{with} \quad X = \ \ 2S: \qquad \Lambda^{\bar{k}\bar{l}} = 0\,; \tag{B.5a}$$
$$(4,0) \quad \text{with} \quad X = -2S: \qquad \Lambda^{kl} = 0\,; \tag{B.5b}$$
$$(3,1): \qquad \Lambda^{\bar{k}\bar{l}} = w_k{}^{\bar{k}} w_l{}^{\bar{l}} \Lambda^{kl}\,; \tag{B.5c}$$
$$(2,2): \qquad \Lambda^{kl} = l^{kl}\Lambda_{\mathrm{L}}\,, \qquad \overline{(\Lambda_{\mathrm{L}})} = \Lambda_{\mathrm{L}}\,, \qquad \Lambda^{\bar{k}\bar{l}} = r^{\bar{k}\bar{l}}\Lambda_{\mathrm{R}}\,, \quad \overline{(\Lambda_{\mathrm{R}})} = \Lambda_{\mathrm{R}}\,. \tag{B.5d}$$

C Geometry of (2,0) AdS superspace

In this appendix we collect the main results concerning the (2,0) AdS superspace following [7, 11].

The geometry of (2,0) AdS superspace is encoded in its covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A{}^M \partial_M + \frac{1}{2}\Omega_A{}^{cd}\mathcal{M}_{cd} + \mathrm{i}\,\Phi_A \mathcal{J} \tag{C.1}$$

obeying the following (anti-)commutation relations:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0\,, \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0\,, \tag{C.2a}$$
$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2\mathrm{i}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}\,S\,\varepsilon_{\alpha\beta}\mathcal{J} + 4\mathrm{i}\,S\,\mathcal{M}_{\alpha\beta}\,, \tag{C.2b}$$
$$[\mathcal{D}_a, \mathcal{D}_\beta] = S\,(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma\,, \qquad [\mathcal{D}_a, \bar{\mathcal{D}}_\beta] = S\,(\gamma_a)_\beta{}^\gamma \bar{\mathcal{D}}_\gamma\,, \tag{C.2c}$$
$$[\mathcal{D}_a, \mathcal{D}_b] = -4\,S^2\,\mathcal{M}_{ab}\,. \tag{C.2d}$$

The generator $\mathcal{J}$ in (C.2) corresponds to the gauged R-symmetry group, $\mathrm{U}(1)_R$, and acts on the covariant derivatives as

$$[\mathcal{J}, \mathcal{D}_\alpha] = \mathcal{D}_\alpha\,, \qquad [\mathcal{J}, \bar{\mathcal{D}}_\alpha] = -\bar{\mathcal{D}}_\alpha\,. \tag{C.3}$$

The isometries of the (2,0) AdS superspace are described by Killing vector fields, $\tau = \tau^a\mathcal{D}_a + \tau^\alpha\mathcal{D}_\alpha + \bar{\tau}_\alpha\bar{\mathcal{D}}^\alpha$, obeying the equation

$$\left[\tau + \mathrm{i}t\mathcal{J} + \frac{1}{2}t^{bc}\mathcal{M}_{bc}, \mathcal{D}_A\right] = 0\,, \tag{C.4}$$

for some parameters t and t^{ab}. Choosing $\mathcal{D}_A = \mathcal{D}_a$ in (C.4) gives

$$\mathcal{D}_a t = 0\,, \tag{C.5a}$$
$$\mathcal{D}_a \tau_b = t_{ab}\,, \tag{C.5b}$$
$$\mathcal{D}_a \tau^\beta = -S\tau^\gamma(\gamma_a)_\gamma{}^\beta\,, \tag{C.5c}$$
$$\mathcal{D}_a t^{bc} = 4S^2\left(\delta_a{}^b\tau^c - \delta_a{}^c\tau^b\right)\,. \tag{C.5d}$$

Eq. (C.5b) implies the standard Killing equation

$$\mathcal{D}_a\tau_b + \mathcal{D}_b\tau_a = 0\,, \tag{C.6}$$

while (C.5c) is a Killing spinor equation. From (C.5b) and (C.5d) it follows that

$$\mathcal{D}_a\mathcal{D}_b\tau_c = 4S^2(\eta_{ab}\tau_c - \eta_{ac}\tau_b)\,. \tag{C.7}$$

Next, choosing $\mathcal{D}_A = \mathcal{D}_\alpha$ in (C.4) gives

$$\mathcal{D}_\alpha\bar{\tau}_\beta = 0\,, \tag{C.8a}$$
$$\mathcal{D}_\alpha t = 4S\bar{\tau}_\alpha\,, \tag{C.8b}$$
$$\mathcal{D}_\alpha t^{\beta\gamma} = -4\mathrm{i}S\left(\delta_\alpha{}^\beta\bar{\tau}^\gamma + \delta_\alpha{}^\gamma\bar{\tau}^\beta\right)\,, \tag{C.8c}$$
$$\mathcal{D}_\alpha\tau^{\beta\gamma} = -2\mathrm{i}\left(\delta_\alpha{}^\beta\bar{\tau}^\gamma + \delta_\alpha{}^\gamma\bar{\tau}^\beta\right)\,, \tag{C.8d}$$
$$\mathcal{D}_\alpha\tau^\beta = \frac{1}{2}t_\alpha{}^\beta + S\tau_\alpha{}^\beta + \mathrm{i}\delta_\alpha{}^\beta t\,. \tag{C.8e}$$

These equations have a number of nontrivial implications including the following:

$$\mathcal{D}_{(\alpha}\tau_{\beta\gamma)} = \mathcal{D}_{(\alpha}t_{\beta\gamma)} = 0\,, \tag{C.9a}$$
$$\mathcal{D}_{(\alpha}\tau_{\beta)} = -\bar{\mathcal{D}}_{(\alpha}\bar{\tau}_{\beta)} = \frac{1}{2}t_{\alpha\beta} + S\tau_{\alpha\beta}\,, \tag{C.9b}$$
$$\tau_\alpha = \frac{\mathrm{i}}{6}\bar{\mathcal{D}}^\beta\tau_{\alpha\beta} = \frac{\mathrm{i}}{12S}\bar{\mathcal{D}}^\beta t_{\alpha\beta}\,, \tag{C.9c}$$
$$\mathcal{D}_\gamma\tau^\gamma = -\bar{\mathcal{D}}^\gamma\bar{\tau}_\gamma = 2\mathrm{i}t\,. \tag{C.9d}$$

It follows from the above equations that the Killing superfields τ^α, t and t^{ab} are given in terms of the vector parameter τ^a. Its components defined by $\tau^a|_{\theta=0}$ and $\left(-\mathcal{D}^b\tau^a\right)|_{\theta=0}$ describe the isometries of AdS$_3$. The other isometry transformations of the (2,0) AdS superspace are contained not only in τ^a but also in, e.g., the real scalar t subject to the following equations:

$$\mathcal{D}^2 t = \bar{\mathcal{D}}^2 t = 0\,, \qquad \left(\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha - 8S\right)t = 0\,, \qquad \mathcal{D}_a t = 0\,. \tag{C.10}$$

At the component level, t contains the real constant parameter $t|_{\theta=0}$ and the complex Killing spinor $\mathcal{D}_\alpha t|_{\theta=0}$, which generate the R-symmetry and supersymmetry transformations of the (2,0) AdS superspace respectively.

D $\mathcal{N} = 4$ SYM theories in (2,0) AdS superspace

In this appendix we provide complete $\mathcal{N} = 4$ SYM actions in (2,0) AdS superspace for all types of $\mathcal{N} = 4$ AdS supersymmetry. These actions are natural extensions of the $\mathcal{N} = 4$ vector multiplet models derived in section 6. We start by recalling the structure of the $\mathcal{N} = 2$ Yang-Mills supermultiplet [24, 25] as formulated in (2,0) AdS superspace.

D.1 $\mathcal{N} = 2$ SYM multiplet

To describe a Yang-Mills supermultiplet in (2,0) AdS superspace, we introduce gauge covariant derivatives[23]

$$\mathfrak{D}_A = \mathcal{D}_A + \mathrm{i}\mathfrak{A}_A\,, \tag{D.1}$$

[23]We use one and the same symbol, $\mathfrak{D}_A$, to denote $\mathcal{N} = 2$ and $\mathcal{N} = 4$ gauge covariant derivatives, the latter have been introduced in appendix A. We hope no confusion may occur, since only the $\mathcal{N} = 2$ operators are used in the present appendix.

where $\mathcal{D}_A$ stands for the covariant derivatives of (2,0) AdS superspace, and the gauge connection $\mathfrak{A}_A(z)$ takes values in the Lie algebra of the gauge group. The anti-commutators of two spinor gauge covariant derivatives are constrained [24, 25] by

$$\{\mathfrak{D}_\alpha, \mathfrak{D}_\beta\} = 0\,, \qquad \{\bar{\mathfrak{D}}_\alpha, \bar{\mathfrak{D}}_\beta\} = 0\,, \tag{D.2a}$$

$$\{\mathfrak{D}_\alpha, \bar{\mathfrak{D}}_\beta\} = \cdots + \mathrm{i}\varepsilon_{\alpha\beta}\mathfrak{G}\,, \tag{D.2b}$$

where the ellipsis denotes the right-hand side of (C.2b). The SYM field strength $\mathfrak{G}$ is Hermitian, $\mathfrak{G}^\dagger = \mathfrak{G}$, and obeys the Bianchi identity

$$0 = \mathfrak{D}^2\mathfrak{G} = \bar{\mathfrak{D}}^2\mathfrak{G}\,. \tag{D.3}$$

The gauge group is defined to act on the covariant derivatives and any matter multiplet $U(z)$ as follows

$$\mathfrak{D}'_A = \mathrm{e}^{\mathrm{i}\tau}\mathfrak{D}_A\mathrm{e}^{-\mathrm{i}\tau}\,, \qquad U' = \mathrm{e}^{\mathrm{i}\tau}U\,, \qquad \tau = \tau^\dagger\,, \tag{D.4}$$

where the Lie-algebra-valued gauge parameters $\tau(z)$ is only constrained to be Hermitian. The field strength transforms in the adjoint representation,

$$\mathfrak{G}' = \mathrm{e}^{\mathrm{i}\tau}\mathfrak{G}\mathrm{e}^{-\mathrm{i}\tau}\,. \tag{D.5}$$

The gauge group will be referred to as the τ-group.

The constraints (D.2a) are solved in complete analogy with the 4D $\mathcal{N} = 1$ case (see, e.g., [22]) as follows:

$$\bar{\mathfrak{D}}_\alpha = \mathrm{e}^{\Omega}\bar{\mathcal{D}}_\alpha\mathrm{e}^{-\Omega}\,, \qquad \mathfrak{D}_\alpha = \mathrm{e}^{-\Omega^\dagger}\mathcal{D}_\alpha\mathrm{e}^{\Omega^\dagger}\,. \tag{D.6}$$

Here $\Omega(z)$ is an unconstrained complex Lie-algebra-valued bridge superfield. Its gauge freedom is larger than the τ-group:

$$\mathrm{e}^{\Omega'} = \mathrm{e}^{\mathrm{i}\tau}\mathrm{e}^{\Omega}\mathrm{e}^{-\mathrm{i}\lambda}\,, \qquad \bar{\mathcal{D}}_\alpha\lambda = 0\,. \tag{D.7}$$

Under the λ-transformation introduced, the gauge covariant derivatives (D.6) remain unchanged.

Let $\mathbf{\Phi}$ be a gauge-covariantly chiral scalar superfield, $\bar{\mathfrak{D}}_\alpha\mathbf{\Phi} = 0$, transforming in the adjoint representation of the gauge group. It may be represented in the form

$$\mathbf{\Phi} = \mathrm{e}^{\Omega}\Phi\mathrm{e}^{-\Omega}\,, \qquad \bar{\mathcal{D}}_\alpha\Phi = 0\,. \tag{D.8}$$

Here the chiral scalar Φ is independent of the gauge field. It is inert under the τ-transformations and changes under the λ-transformations by the rule

$$\Phi' = \mathrm{e}^{\mathrm{i}\lambda}\Phi\mathrm{e}^{-\mathrm{i}\lambda}\,. \tag{D.9}$$

The Hermitian conjugate of $\mathbf{\Phi}$ is a gauge-covariantly antichiral superfield $\bar{\mathbf{\Phi}}$ constrained by $\mathfrak{D}_\alpha\bar{\mathbf{\Phi}} = 0$. Its explicit form is

$$\bar{\mathbf{\Phi}} := \mathbf{\Phi}^\dagger = \mathrm{e}^{-\Omega^\dagger}\Phi^\dagger\mathrm{e}^{\Omega^\dagger}\,, \qquad \mathcal{D}_\alpha\Phi^\dagger = 0\,. \tag{D.10}$$

JHEP05(2014)018

It is often advantageous to use a chiral representation defined by the transformation

$$\hat{\mathcal{O}} \to \mathrm{e}^{-\Omega}\hat{\mathcal{O}}\mathrm{e}^{\Omega}\,, \qquad U \to \mathrm{e}^{-\Omega}U\,, \tag{D.11}$$

which has to be applied to any operator $\hat{\mathcal{O}}$ and covariant superfield U. In this representation, the gauge covariant spinor derivatives look like

$$\mathfrak{D}_\alpha = \mathrm{e}^{-\mathcal{V}}\mathcal{D}_\alpha \mathrm{e}^{\mathcal{V}}\,, \qquad \bar{\mathfrak{D}}_\alpha = \bar{\mathcal{D}}_\alpha\,, \tag{D.12}$$

and the adjoint multiplets $\mathbf{\Phi}$ and $\bar{\mathbf{\Phi}}$ turn into

$$\mathbf{\Phi} = \Phi\,, \qquad \bar{\mathbf{\Phi}} = \mathrm{e}^{-\mathcal{V}}\Phi^\dagger \mathrm{e}^{\mathcal{V}}\,. \tag{D.13}$$

Here we have introduced the Hermitian Lie-algebra-valued prepotential $\mathcal{V}$ defined by

$$\mathrm{e}^{\mathcal{V}} := \mathrm{e}^{\Omega^\dagger}\mathrm{e}^{\Omega}\,, \qquad \mathcal{V}^\dagger = \mathcal{V}\,. \tag{D.14}$$

The virtue of the chiral representation is that the gauge field is described in terms of a single prepotential, $\mathcal{V}$, with the gauge transformation law

$$\mathrm{e}^{\mathcal{V}'} = \mathrm{e}^{\mathrm{i}\lambda^\dagger}\mathrm{e}^{\mathcal{V}}\mathrm{e}^{-\mathrm{i}\lambda}\,. \tag{D.15}$$

The τ-group is completely gauged away in this representation.

In the chiral representation, the constraint (D.2b) is solved as follows:

$$\mathfrak{G} = \frac{\mathrm{i}}{2}\bar{\mathcal{D}}^\alpha \left(\mathrm{e}^{-\mathcal{V}}\mathcal{D}_\alpha \mathrm{e}^{\mathcal{V}}\right)\,. \tag{D.16}$$

The field strength $\mathfrak{G}$ is no longer Hermitian. It obeys the modified reality condition $\mathfrak{G}^\dagger = \mathrm{e}^{\mathcal{V}}\mathfrak{G}\mathrm{e}^{-\mathcal{V}}$.

In the remainder of this section, we will work in the chiral representation.

D.2 $\mathcal{N} = 2$ Chern-Simons and SYM actions

When dealing with the non-Abelian $\mathcal{N} = 2$ vector supermultiplet, it is useful to introduce a covariant variation, $\Delta\mathcal{V}$, of the prepotential $\mathcal{V}$ following [22]. It is defined by

$$\Delta\mathcal{V} := \mathrm{e}^{-\mathcal{V}}\delta \mathrm{e}^{\mathcal{V}}\,, \tag{D.17}$$

and hence $\delta \mathrm{e}^{\mathcal{V}} = \mathrm{e}^{\mathcal{V}}\Delta\mathcal{V}$ and $\delta \mathrm{e}^{-\mathcal{V}} = -\Delta\mathcal{V}\mathrm{e}^{-\mathcal{V}}$. Varying the field strength gives

$$\delta\mathfrak{G} = [\mathfrak{G}, \Delta\mathcal{V}] + \frac{\mathrm{i}}{2}\bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha\Delta\mathcal{V} - \frac{\mathrm{i}}{2}\left\{\bar{\mathcal{D}}^\alpha\Delta\mathcal{V}, \mathrm{e}^{-\mathcal{V}}\mathcal{D}_\alpha \mathrm{e}^{\mathcal{V}}\right\}\,, \tag{D.18a}$$

which is equivalent to

$$\delta\mathfrak{G} = \frac{\mathrm{i}}{2}\bar{\mathfrak{D}}^\alpha\mathfrak{D}_\alpha\Delta\mathcal{V} = \frac{\mathrm{i}}{2}\mathfrak{D}^\alpha\bar{\mathfrak{D}}_\alpha\Delta\mathcal{V}\,. \tag{D.18b}$$

The $\mathcal{N} = 2$ SYM action in (2,0) AdS superspace is a minimal extension of the one in Minkowski space [25],

$$S^{(2,0)}_{\mathrm{SYM}} = -\frac{1}{2g^2}\int \mathrm{d}^3x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathrm{tr}\left[\mathfrak{G}^2\right]\,, \tag{D.19}$$

with g the coupling constant. Its variation is given by

$$\delta S^{(2,0)}_{\rm SYM} = -\frac{\rm i}{2g^2}\int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,{\rm tr}\big[\Delta\mathcal{V}\,\bar{\mathfrak{D}}^\alpha\mathfrak{D}_\alpha\mathfrak{G}\big]\,. \tag{D.20}$$

We now turn to introducing a supersymmetric Chern-Simons (SCS) action in (2,0) AdS superspace. In the case of Poincaré supersymmetry, the $\mathcal{N}=2$ SCS action was constructed by Zupnik and Pak [25], and a few years later by Ivanov [26] in a more general form. Here we follow Ivanov's construction. Let us consider a one-parameter family of superfields $\mathcal{V}(t)$, with $t\in[0,1]$, such that $\mathcal{V}(0)=0$ and $\mathcal{V}(1)=\mathcal{V}$. Up to an overall constant, the SCS action is

$$S^{(2,0)}_{\rm SCS} = \int_0^1{\rm d}t\int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,{\rm tr}\Big[\mathfrak{G}(t){\rm e}^{-\mathcal{V}(t)}\partial_t{\rm e}^{\mathcal{V}(t)}\Big]\,. \tag{D.21}$$

In the Abelian case this reduces to

$$S^{(2,0)}_{\text{SCS-Abelian}} = \frac{1}{2}\int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,{\rm tr}\big[\mathcal{V}\mathfrak{G}\big]\,. \tag{D.22}$$

Zupnik and Pak [25] used the specific parametrization, $\mathcal{V}(t)=t\mathcal{V}$.

It follows from the definition (D.17) that

$$\delta({\rm e}^{-\mathcal{V}(t)}\partial_t{\rm e}^{\mathcal{V}(t)}) = \Big[{\rm e}^{-\mathcal{V}(t)}\partial_t{\rm e}^{\mathcal{V}(t)},\Delta\mathcal{V}(t)\Big]+\partial_t\Delta\mathcal{V}(t)\,. \tag{D.23}$$

Making also use of (D.18a), we compute the variation of the SCS action (D.21) to be

$$\delta S^{(2,0)}_{\rm SCS} = \int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,{\rm tr}\big[\Delta\mathcal{V}\,\mathfrak{G}\big]\,. \tag{D.24}$$

D.3 $\mathcal{N}=4$ SYM theory with (4,0) AdS supersymmetry

We are now in a position to present $\mathcal{N}=2$ superspace formulations for all $\mathcal{N}=4$ SYM actions which correspond to the different types of $\mathcal{N}=4$ AdS supersymmetry. The R-charge of $\mathbf{\Phi}$ is universally defined by $\mathcal{J}\mathbf{\Phi}=-\boldsymbol{q}\mathbf{\Phi}$, where $\boldsymbol{q}=1+\frac{X}{2S}$. In the cases of (3,1) and (2,2) AdS supersymmetries, X is equal to zero and $\boldsymbol{q}=1$.

In the case of (4,0) AdS supersymmetry with $\boldsymbol{q}\neq 0$, the non-manifest supersymmetry transformations are

$$\delta_\varepsilon\mathbf{\Phi} = {\rm i}\big(\varepsilon^\alpha\bar{\mathfrak{D}}_\alpha-4S\varepsilon_{\rm L}\big)\mathfrak{G} = -\frac{1}{2(2-\boldsymbol{q})}\bar{\mathfrak{D}}^2(\bar\varepsilon_{\rm R}\mathfrak{G})\,, \tag{D.25a}$$

$$\Delta_\varepsilon\mathcal{V} = -\frac{2{\rm i}}{\boldsymbol{q}}\left(\bar\varepsilon_{\rm L}\mathbf{\Phi}-\varepsilon_{\rm L}\bar{\mathbf{\Phi}}\right)\,, \tag{D.25b}$$

$$\delta_\varepsilon\mathfrak{G} = -{\rm i}\big(\bar\varepsilon^\alpha\mathfrak{D}_\alpha-8S\bar\varepsilon_{\rm L}\big)\mathbf{\Phi}+\text{h.c.} = {\rm i}\mathfrak{D}_\alpha(\bar\varepsilon^\alpha\mathbf{\Phi})+\text{h.c.} = \frac{\rm i}{2}\mathfrak{D}^\alpha\bar{\mathfrak{D}}_\alpha\Delta_\varepsilon\mathcal{V}\,. \tag{D.25c}$$

These transformation laws are non-Abelian extensions of (7.1).

The $\mathcal{N}=4$ SYM action is

$$S^{(4,0)}_{\rm SYM} = \frac{1}{g^2}\int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,{\rm tr}\left[\bar{\mathbf{\Phi}}\mathbf{\Phi}-\frac{1}{2}\mathfrak{G}^2+2S\boldsymbol{q}\int_0^1{\rm d}t\,\mathfrak{G}(t){\rm e}^{-\mathcal{V}(t)}\partial_t{\rm e}^{\mathcal{V}(t)}\right]\,. \tag{D.26}$$

It is invariant under the transformations (D.25). The action reduces to (7.2) in the Abelian limit.

JHEP05(2014)018

Our $\mathcal{N}=4$ SYM action (D.26) is analogous to the one given in the Euclidean case in [1]. There is, however, a minor technical difference. The point is that $\mathcal{V}(t)$ was chosen in [1] to be of the form $\mathcal{V}(t)=t\mathcal{V}$. In our approach $\mathcal{V}(t)$ is an arbitrary function modulo the boundary conditions $\mathcal{V}(0)=0$ and $\mathcal{V}(1)=\mathcal{V}$.

In the case of critical (4,0) AdS supersymmetry with $X+2S=2S\boldsymbol{q}=0$, we have $\varepsilon_{\rm L}=0$ and the non-manifest supersymmetry transformations are

$$\delta_\varepsilon \boldsymbol{\Phi} = {\rm i}\varepsilon^\alpha \bar{\mathfrak{D}}_\alpha \mathfrak{G} = -\frac{1}{4}\bar{\mathfrak{D}}^2(\bar{\varepsilon}_{\rm R}\mathfrak{G})\,, \tag{D.27a}$$

$$\Delta_\varepsilon \mathcal{V} = -2{\rm i}\rho\big(\boldsymbol{\Phi}-\bar{\boldsymbol{\Phi}}\big)\,, \tag{D.27b}$$

$$\delta_\varepsilon \mathfrak{G} = -{\rm i}\bar{\varepsilon}^\alpha \mathfrak{D}_\alpha \boldsymbol{\Phi} + {\rm h.c.} = {\rm i}\mathfrak{D}_\alpha(\bar{\varepsilon}^\alpha \boldsymbol{\Phi}) + {\rm h.c.} = \frac{\rm i}{2}\mathfrak{D}^\alpha \bar{\mathfrak{D}}_\alpha \Delta_\varepsilon \mathcal{V}\,. \tag{D.27c}$$

These transformation laws are non-Abelian extensions of (7.3). The corresponding $\mathcal{N}=4$ SYM action is given by (D.26) with $\boldsymbol{q}=0$. It is an instructive exercise to show that the action is invariant under (D.27).

D.4 $\mathcal{N}=4$ SYM theory with (3,1) AdS supersymmetry

In the case of (3,1) AdS supersymmetry, the non-manifest supersymmetry transformations are

$$\delta_\varepsilon \boldsymbol{\Phi} = {\rm i}\big(\varepsilon^\alpha \bar{\mathfrak{D}}_\alpha - 4S\varepsilon\big)\mathfrak{G} = -\frac{1}{2}\bar{\mathfrak{D}}^2\big((\bar{\varepsilon}-\bar{\rho})\mathfrak{G}\big)\,, \tag{D.28a}$$

$$\Delta_\varepsilon \mathcal{V} = -2{\rm i}(\bar{\varepsilon}+\bar{\rho})\boldsymbol{\Phi} + 2{\rm i}(\varepsilon+\rho)\bar{\boldsymbol{\Phi}}\,, \tag{D.28b}$$

$$\delta_\varepsilon \mathfrak{G} = {\rm i}\mathfrak{D}_\alpha(\bar{\varepsilon}^\alpha \boldsymbol{\Phi}) + {\rm h.c.} = \frac{\rm i}{2}\mathfrak{D}^\alpha \bar{\mathfrak{D}}_\alpha \Delta_\varepsilon \mathcal{V}\,. \tag{D.28c}$$

These transformation laws are non-Abelian extensions of (7.4)

The $\mathcal{N}=4$ SYM action is

$$S^{(3,1)}_{\rm SYM} = \frac{1}{g^2}\int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar{\theta}\,E\,{\rm tr}\left[\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} - \frac{1}{2}\mathfrak{G}^2 + S\int_0^1 {\rm d}t\,\mathfrak{G}(t){\rm e}^{-\mathcal{V}(t)}\partial_t {\rm e}^{\mathcal{V}(t)}\right] - \frac{S}{g^2}\left\{\frac{\rm i}{2}\int {\rm d}^3x\,{\rm d}^2\theta\mathcal{E}\,{\rm tr}\left[\Phi^2\right] + {\rm c.c.}\right\}\,. \tag{D.29}$$

It is invariant under (D.28) and reduces to (7.5) in the Abelian limit.

D.5 $\mathcal{N}=4$ SYM theory with (2,2) AdS supersymmetry

In the case of (2,2) AdS supersymmetry, the non-manifest supersymmetry transformations of $\boldsymbol{\Phi}$ and $\mathfrak{G}$ are

$$\delta_\varepsilon \boldsymbol{\Phi} = {\rm i}\varepsilon^\alpha \bar{\mathfrak{D}}_\alpha \mathfrak{G} = \frac{1}{2}\bar{\mathfrak{D}}^2\left(\bar{\rho}\mathfrak{G}\right)\,, \tag{D.30a}$$

$$\delta_\varepsilon \mathfrak{G} = -{\rm i}\bar{\varepsilon}^\alpha \mathfrak{D}_\alpha \boldsymbol{\Phi} + {\rm h.c.} = \mathfrak{D}^\alpha \bar{\mathfrak{D}}_\alpha\left(\bar{\rho}\,\boldsymbol{\Phi} - \rho\,\bar{\boldsymbol{\Phi}}\right)\,, \tag{D.30b}$$

where the parameter ρ is defined by (6.37). These transformation laws are non-Abelian extensions of (7.6).

The $\mathcal{N}=4$ SYM action is

$$S^{(2,2)}_{\rm SYM} = \frac{1}{g^2}\int {\rm d}^3x\,{\rm d}^2\theta{\rm d}^2\bar{\theta}\,E\,{\rm tr}\left[\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} - \frac{1}{2}\mathfrak{G}^2\right]\,. \tag{D.31}$$

It is invariant under the transformations (D.30).

E Relating the $\mathcal{N}=4$ and $\mathcal{N}=2$ superspace formulations for $\mathcal{N}=4$ SYM theories in AdS$_3$

In appendix A, we described the projective-superspace formulation for the $\mathcal{N}=4$ SYM multiplet in a curved superspace of $\mathcal{N}=4$ supergravity. Here we will specify the background superspace to be one of the $\mathcal{N}=4$ AdS superspaces and show how to reduce the formulation of appendix A to (2,0) AdS superspace. Using the results obtained, we will integrate the variation (4.8) in (2,0) AdS superspace.

E.1 Relating the bridge superfields

In the $\mathcal{N}=4$ SYM formulation given in appendix A, the fundamental role is played by the bridge Ω_+, eqs. (A.7) and (A.8), and its smile-conjugate Ω_- defined by (A.9). It is possible to represent

$$\mathrm{e}^{\Omega_+(\zeta_{\rm R})}=\mathrm{e}^{\Omega_0}\mathrm{e}^{\hat\Omega_+(\zeta_{\rm R})}\,,\qquad \mathrm{e}^{\Omega_-(\zeta_{\rm R})}=\mathrm{e}^{\hat\Omega_-(\zeta_{\rm R})}\mathrm{e}^{\Omega_0^\dagger}\,,\tag{E.1}$$

where

$$\hat\Omega_+(\zeta_{\rm R})=\sum_{n=1}^{\infty}(\zeta_{\rm R})^n\hat\Omega_n\,,\qquad \hat\Omega_-(\zeta_{\rm R})=\sum_{n=1}^{\infty}(-1)^n\frac{1}{(\zeta_{\rm R})^n}\hat\Omega_n^\dagger\,.\tag{E.2}$$

It may be shown that $\hat\Omega_1$, the leading coefficient in the Taylor series for $\hat\Omega_+(\zeta_{\rm R})$, is related to Ω_1, the next-to-leading in the Taylor series for $\Omega_+(\zeta_{\rm R})$, as follows:

$$\hat\Omega_1=\int_0^1 \mathrm{d}\tau\mathrm{e}^{-\tau\Omega_0}\Omega_1\mathrm{e}^{\tau\Omega_0}\,.\tag{E.3}$$

The gauge covariant spinor derivative $\mathfrak{D}_\alpha^{(\bar1)i}$ defined by eq. (A.6) may be represented as $\mathfrak{D}_\alpha^{(\bar1)i}=v^{\bar1}\mathfrak{D}_\alpha^{[\bar1]i}$, where $\mathfrak{D}_\alpha^{[\bar1]i}=\mathrm{e}^{\Omega_0}\mathrm{e}^{\hat\Omega_+}\mathcal{D}_\alpha^{[\bar1]i}\mathrm{e}^{-\hat\Omega_+}\mathrm{e}^{-\Omega_0}$ is such that

$$\mathfrak{D}_\alpha^{[\bar1]i}=\mathfrak{D}_\alpha^{i\bar2}-\zeta_{\rm R}\mathfrak{D}_\alpha^{i\bar1}=\mathrm{e}^{\Omega_0}\mathrm{e}^{\hat\Omega_+(\zeta_{\rm R})}\mathcal{D}_\alpha^{i\bar2}\mathrm{e}^{-\hat\Omega_+(\zeta_{\rm R})}\mathrm{e}^{-\Omega_0}-\zeta_{\rm R}\mathrm{e}^{\Omega_0}\mathrm{e}^{\hat\Omega_+(\zeta_{\rm R})}\mathcal{D}_\alpha^{i\bar1}\mathrm{e}^{-\hat\Omega_+(\zeta_{\rm R})}\mathrm{e}^{-\Omega_0}\,.\tag{E.4}$$

We see that the $\zeta_{\rm R}$-independent part of $\mathfrak{D}_\alpha^{[\bar1]i}$ is

$$\mathfrak{D}_\alpha^{i\bar2}=\mathrm{e}^{\Omega_0}\mathcal{D}_\alpha^{i\bar2}\mathrm{e}^{-\Omega_0}\,.\tag{E.5}$$

Choosing here $i=2$ and projecting to (2,0) AdS superspace gives

$$-\mathfrak{D}_\alpha^{2\bar2}|=-\mathrm{e}^{\Omega_0|}\mathcal{D}_\alpha^{2\bar2}|\mathrm{e}^{-\Omega_0|}=\mathrm{e}^{\Omega}\bar{\mathcal{D}}_\alpha\mathrm{e}^{-\Omega}=\bar{\mathfrak{D}}_\alpha\,,\quad \Omega:=\Omega_0|\,.\tag{E.6}$$

Here $\bar{\mathfrak{D}}_\alpha$ denotes one of the $\mathcal{N}=2$ gauge covariant derivative, eq. (D.6). Taking the Hermitian conjugate of (E.6) leads to

$$\mathfrak{D}_\alpha^{1\bar1}|=\mathfrak{D}_\alpha=\mathrm{e}^{-\Omega^\dagger}\mathcal{D}_\alpha\mathrm{e}^{\Omega^\dagger}\,.\tag{E.7}$$

JHEP05(2014)018

E.2 Relating the SYM field strengths

Our next task is to reduce the $\mathcal{N}=4$ SYM field strength to (2,0) AdS superspace. Making use of (A.33), it may be shown that

$$\mathfrak{W}^{ij} = \mathrm{e}^{\Omega_0}\,\mathfrak{W}_+^{ij}(\zeta_{\mathrm{R}}=0)\,\mathrm{e}^{-\Omega_0} = -\frac{1}{4}\mathrm{e}^{\Omega_0}\left[\left(\mathcal{D}^{ij\bar{2}\bar{2}} - 4\mathrm{i}\mathcal{S}^{ij\bar{2}\bar{2}}\right)\hat{\Omega}_1\right]\mathrm{e}^{-\Omega_0}\,. \tag{E.8}$$

It is convenient to represent the bar-projection of $\mathfrak{W}^{ij}$ in terms of the left projective superfield $\mathfrak{W}^{(2)}(v_{\mathrm{L}}) = \mathfrak{W}^{ij}v_iv_j$, where $v^i = v^1(1,\zeta_{\mathrm{L}})$. It follows that

$$\mathfrak{W}^{(2)}(v_{\mathrm{L}})| = \mathrm{i}\zeta_{\mathrm{L}}(v^1)^2\mathfrak{W}^{[2]}(\zeta_{\mathrm{L}})|\,, \qquad \mathfrak{W}^{[2]}(\zeta_{\mathrm{L}})| = -\frac{\mathrm{i}}{\zeta_{\mathrm{L}}}\boldsymbol{\Phi} + \mathfrak{G} - \mathrm{i}\zeta_{\mathrm{L}}\bar{\boldsymbol{\Phi}}\,, \tag{E.9}$$

where we have introduced the following $\mathcal{N}=2$ superfields:

$$\boldsymbol{\Phi} := \mathfrak{W}^{22}|\,, \qquad \bar{\mathfrak{D}}_\alpha\boldsymbol{\Phi} = 0\,, \tag{E.10a}$$
$$\mathfrak{G} := 2\mathrm{i}\,\mathfrak{W}^{12}|\,, \qquad \mathfrak{D}^2\mathfrak{G} = 0\,, \qquad \bar{\mathfrak{D}}^2\mathfrak{G} = 0\,, \tag{E.10b}$$
$$\bar{\boldsymbol{\Phi}} = \mathfrak{W}^{11}|\,, \qquad \mathfrak{D}_\alpha\bar{\boldsymbol{\Phi}} = 0\,. \tag{E.10c}$$

The constraints on $\boldsymbol{\Phi}$ and $\mathfrak{G}$ are direct consequences of the Bianchi identity obeyed by $\mathfrak{W}^{ij}$. Since the reduction to (2,0) AdS superspace is characterized by the conditions (5.1), from (E.8) we deduce that

$$\boldsymbol{\Phi} = \frac{1}{4}\mathrm{e}^{\Omega}\left(\bar{\mathcal{D}}^2\hat{\Omega}_1|\right)\mathrm{e}^{-\Omega} = \frac{1}{4}\bar{\mathfrak{D}}^2\boldsymbol{\mathcal{X}}\,, \qquad \boldsymbol{\mathcal{X}} := \mathrm{e}^{\Omega}\hat{\Omega}_1|\mathrm{e}^{-\Omega}\,. \tag{E.11}$$

This is the non-Abelian extension of the first expression in (6.21).

Let us now express the covariantly real linear superfield $\mathfrak{G} := 2\mathrm{i}\,\mathfrak{W}^{12}|$ in terms of prepotentials. In this case it is simpler to work in the $\mathcal{N}=2$ chiral representation defined by eqs. (D.11)–(D.14). Using (E.8), a short calculation gives

$$\mathrm{e}^{-\Omega}\mathfrak{G}\mathrm{e}^{\Omega} = \frac{\mathrm{i}}{2}\bar{\mathcal{D}}^\gamma\mathcal{D}_\gamma^{1\bar{2}}\hat{\Omega}_1|\,. \tag{E.12}$$

Note that the $\mathcal{N}=4$ analyticity condition $0 = \mathcal{D}_\alpha^{[1]i}\mathrm{e}^{V(\zeta_{\mathrm{R}})} = \left(-\zeta_{\mathrm{R}}\mathcal{D}_\alpha^{i\bar{1}} + \mathcal{D}_\alpha^{i\bar{2}}\right)\mathrm{e}^{V(\zeta_{\mathrm{R}})}$ implies the following constraint on Ω_+ and Ω_-:

$$\mathrm{e}^{-\Omega_-}\left(\mathcal{D}_\alpha^{1\bar{2}}\mathrm{e}^{\Omega_-}\right) - \zeta_{\mathrm{R}}\mathrm{e}^{-\Omega_-}\left(\mathcal{D}_\alpha^{1\bar{1}}\mathrm{e}^{\Omega_-}\right) = \mathrm{e}^{\Omega_+}\left(\mathcal{D}_\alpha^{1\bar{2}}\mathrm{e}^{-\Omega_+}\right) - \zeta_{\mathrm{R}}\mathrm{e}^{\Omega_+}\left(\mathcal{D}_\alpha^{1\bar{1}}\mathrm{e}^{-\Omega_+}\right)\,. \tag{E.13}$$

Picking the linear in ζ_{R} term in the Laurent expansion of (E.13) and then bar-projecting to (2,0) AdS superspace, we obtain the constraint

$$\mathcal{D}_\alpha^{1\bar{2}}\hat{\Omega}_1| = \mathrm{e}^{-\mathcal{V}}\mathcal{D}_\alpha\mathrm{e}^{\mathcal{V}}\,. \tag{E.14}$$

Here the right-hand side is expressed in terms of the $\mathcal{N}=2$ SYM prepotential $\mathcal{V}$ defined by $\mathrm{e}^{\mathcal{V}} = \mathrm{e}^{\Omega^\dagger}\mathrm{e}^{\Omega}$. Now we can make use of (E.14) in (E.12) to obtain

$$\mathrm{e}^{-\Omega}\mathfrak{G}\mathrm{e}^{\Omega} = \frac{\mathrm{i}}{2}\bar{\mathcal{D}}^\gamma\left(\mathrm{e}^{-\mathcal{V}}\mathcal{D}_\gamma\mathrm{e}^{\mathcal{V}}\right)\,, \tag{E.15}$$

which is the $\mathcal{N}=2$ SYM field strength in the chiral representation.

E.3 Integrating the variation of the SYM action

To conclude this appendix, let us consider the (2,0) AdS reduction of the variation of the $\mathcal{N}=4$ SYM action, eq. (4.8). The bar-projection of (4.8) turns out to be

$$\delta S_{\rm SYM} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\oint_C \frac{\mathrm{d}\zeta_{\rm R}}{2\pi\mathrm{i}\zeta_{\rm R}}\,\mathrm{tr}\Big\{\Big[\mathrm{e}^{\Omega_0}\mathrm{e}^{\hat\Omega_+}\Delta\hat\Omega_+\mathrm{e}^{-\hat\Omega_+}\mathrm{e}^{-\Omega_0} + \mathrm{e}^{-\Omega_0^\dagger}\mathrm{e}^{-\hat\Omega_-}\Delta\hat\Omega_-\mathrm{e}^{\hat\Omega_-}\mathrm{e}^{\Omega_0^\dagger} \\ + \mathrm{e}^{\Omega_0}\Delta\mathcal{V}\mathrm{e}^{-\Omega_0}\Big]\boldsymbol{W}^{[\bar2]}\Big\}\Big| \,, \tag{E.16}$$

where we have defined

$$\Delta\hat\Omega_+ := \mathrm{e}^{-\hat\Omega_+}\delta\mathrm{e}^{\hat\Omega_+}\,, \tag{E.17a}$$

$$\Delta\hat\Omega_- := \Big(\delta\mathrm{e}^{\hat\Omega_-}\Big)\,\mathrm{e}^{-\hat\Omega_-}\,, \tag{E.17b}$$

$$\Delta\mathcal{V} =: \mathrm{e}^{-\mathcal{V}}\delta\mathrm{e}^{\mathcal{V}} = \mathrm{e}^{-\Omega}\delta\mathrm{e}^{\Omega} + \mathrm{e}^{-\mathcal{V}}\Big(\delta\mathrm{e}^{\Omega^\dagger}\Big)\,\mathrm{e}^{-\Omega^\dagger}\mathrm{e}^{\mathcal{V}}\,. \tag{E.17c}$$

We recall that the non-Abelian composite superfield $\boldsymbol{W}^{(\bar2)}$ is defined by (4.9). The superfield $\boldsymbol{W}^{[\bar2]}$ in (E.16) is

$$\boldsymbol{W}^{[\bar2]}(\zeta_{\rm R}) = -\frac{\mathrm{i}}{2\zeta_{\rm R}}\boldsymbol{W}^{\bar2\bar2} + 2\mathrm{i}\boldsymbol{W}^{\bar1\bar2} - \mathrm{i}\zeta_{\rm R}\boldsymbol{W}^{\bar1\bar1}\,. \tag{E.18}$$

Computing the bar-projection of the superfields on the right gives

$$\boldsymbol{W}^{\bar1\bar1}| = -\frac{\mathrm{i}}{4}\mathfrak{D}^2\boldsymbol{\Phi} + \mathcal{S}^{22\bar1\bar1}\bar{\boldsymbol{\Phi}}\,, \qquad \boldsymbol{W}^{\bar2\bar2}| = \frac{\mathrm{i}}{4}\bar{\mathfrak{D}}^2\bar{\boldsymbol{\Phi}} + \mathcal{S}^{11\bar2\bar2}\boldsymbol{\Phi}\,, \tag{E.19a}$$

$$\boldsymbol{W}^{\bar1\bar2}| = -\frac{1}{4}(\mathfrak{D}^\alpha\bar{\mathfrak{D}}_\alpha + 4\mathrm{i}\boldsymbol{q}\mathcal{S})\mathfrak{G}\,. \tag{E.19b}$$

Upon evaluation of the contour integral in (E.16) we derive

$$\delta S_{\rm SYM} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\mathrm{tr}\bigg\{\frac14\delta\boldsymbol{\mathcal{X}}\bar{\mathfrak{D}}^2\bar{\boldsymbol{\Phi}} + \frac14\delta\boldsymbol{\mathcal{X}}^\dagger\mathfrak{D}^2\boldsymbol{\Phi} - \mathrm{e}^{\Omega}\Delta\mathcal{V}\mathrm{e}^{-\Omega}\Big(\frac{\mathrm{i}}{2}\mathfrak{D}^\alpha\bar{\mathfrak{D}}_\alpha\mathfrak{G}\Big) \\ + 2\mathcal{S}\boldsymbol{q}\mathrm{e}^{\Omega}\Delta\mathcal{V}\mathrm{e}^{-\Omega}\mathfrak{G} - \mathrm{i}\mathcal{S}^{11\bar2\bar2}\delta\boldsymbol{\mathcal{X}}\boldsymbol{\Phi} + \mathrm{i}\mathcal{S}^{22\bar1\bar1}\delta\boldsymbol{\mathcal{X}}^\dagger\bar{\boldsymbol{\Phi}}\bigg\}\,. \tag{E.20}$$

It may be seen that this variation is generated by the action

$$S_{\rm SYM} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\mathrm{tr}\left[\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} - \frac12\mathfrak{G}^2 + 2\mathcal{S}\boldsymbol{q}\int_0^1\mathrm{d}t\,\mathfrak{G}(t)\mathrm{e}^{-\mathcal{V}(t)}\partial_t\mathrm{e}^{\mathcal{V}(t)}\right] \\ + \left\{\frac{\mathrm{i}}{2}\mathcal{S}^{11\bar2\bar2}\int\mathrm{d}^3x\,\mathrm{d}^2\theta\mathcal{E}\,\mathrm{tr}\left[\boldsymbol{\Phi}^2\right] + \mathrm{c.c.}\right\}\,. \tag{E.21}$$

This is indeed the correct action for all the types of $\mathcal{N}=4$ AdS supersymmetry, as discussed in the previous appendix. Action (E.21) is the non-Abelian extension of (6.45).

JHEP05(2014)018

References

[1] I.B. Samsonov and D. Sorokin, *Superfield theories on S^3 and their localization*, arXiv:1401.7952 [INSPIRE].

[2] N. Hama, K. Hosomichi and S. Lee, *Notes on SUSY Gauge Theories on Three-Sphere*, *JHEP* **03** (2011) 127 [arXiv:1012.3512] [INSPIRE].

[3] S.M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, *Three-dimensional (p,q) AdS superspaces and matter couplings*, *JHEP* **08** (2012) 024 [arXiv:1205.4622] [INSPIRE].

[4] A. Achucarro and P.K. Townsend, *A Chern-Simons Action for Three-Dimensional anti-de Sitter Supergravity Theories*, *Phys. Lett.* **B 180** (1986) 89 [INSPIRE].

[5] P.S. Howe, J.M. Izquierdo, G. Papadopoulos and P.K. Townsend, *New supergravities with central charges and Killing spinors in (2+1)-dimensions*, *Nucl. Phys.* **B 467** (1996) 183 [hep-th/9505032] [INSPIRE].

[6] S.M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, *Off-shell supergravity-matter couplings in three dimensions*, *JHEP* **03** (2011) 120 [arXiv:1101.4013] [INSPIRE].

[7] D. Butter, S.M. Kuzenko and G. Tartaglino-Mazzucchelli, *Nonlinear σ-models with AdS supersymmetry in three dimensions*, *JHEP* **02** (2013) 121 [arXiv:1210.5906] [INSPIRE].

[8] A. Karlhede, U. Lindström and M. Roček, *Selfinteracting Tensor Multiplets in $N = 2$ Superspace*, *Phys. Lett.* **B 147** (1984) 297 [INSPIRE].

[9] U. Lindström and M. Roček, *New HyperKähler Metrics and New Supermultiplets*, *Commun. Math. Phys.* **115** (1988) 21 [INSPIRE].

[10] U. Lindström and M. Roček, *$N = 2$ Super Yang-Mills Theory in Projective Superspace*, *Commun. Math. Phys.* **128** (1990) 191 [INSPIRE].

[11] S.M. Kuzenko and G. Tartaglino-Mazzucchelli, *Three-dimensional $N = 2$ (AdS) supergravity and associated supercurrents*, *JHEP* **12** (2011) 052 [arXiv:1109.0496] [INSPIRE].

[12] B. DeWitt, *Supermanifolds*, Cambridge University Press, Cambridge, U.K. (1992).

[13] I.A. Bandos, E. Ivanov, J. Lukierski and D. Sorokin, *On the superconformal flatness of AdS superspaces*, *JHEP* **06** (2002) 040 [hep-th/0205104] [INSPIRE].

[14] R. Brooks and S.J. Gates Jr., *Extended supersymmetry and superBF gauge theories*, *Nucl. Phys.* **B 432** (1994) 205 [hep-th/9407147] [INSPIRE].

[15] B. Zupnik, *Harmonic superpotentials and symmetries in gauge theories with eight supercharges*, *Nucl. Phys.* **B 554** (1999) 365 [*Erratum ibid.* **B 644** (2002) 405] [hep-th/9902038] [INSPIRE].

[16] S.M. Kuzenko and J. Novak, *Supergravity-matter actions in three dimensions and Chern-Simons terms*, arXiv:1401.2307 [INSPIRE].

[17] W. Siegel, *Chiral Actions for $N = 2$ Supersymmetric Tensor Multiplets*, *Phys. Lett.* **B 153** (1985) 51 [INSPIRE].

[18] F. Gonzalez-Rey, *Feynman rules in $N = 2$ projective superspace. 3. Yang-Mills multiplet*, hep-th/9712128 [INSPIRE].

[19] S.M. Kuzenko, *Projective superspace as a double punctured harmonic superspace*, *Int. J. Mod. Phys.* **A 14** (1999) 1737 [hep-th/9806147] [INSPIRE].

JHEP05(2014)018

[20] S.M. Kuzenko, *On $N = 2$ supergravity and projective superspace: Dual formulations*, Nucl. Phys. **B 810** (2009) 135 [arXiv:0807.3381] [INSPIRE].

[21] S.M. Kuzenko, *Five-dimensional supersymmetric Chern-Simons action as a hypermultiplet quantum correction*, Phys. Lett. **B 644** (2007) 88 [hep-th/0609078] [INSPIRE].

[22] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, Front. Phys. **58** (1983) 1 [hep-th/0108200] [INSPIRE].

[23] B.M. Zupnik, *The Action of the Supersymmetric $N = 2$ Gauge Theory in Harmonic Superspace*, Phys. Lett. **B 183** (1987) 175 [INSPIRE].

[24] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *HyperKähler Metrics and Supersymmetry*, Commun. Math. Phys. **108** (1987) 535 [INSPIRE].

[25] B.M. Zupnik and D.G. Pak, *Superfield Formulation of the Simplest Three-dimensional Gauge Theories and Conformal Supergravities*, Theor. Math. Phys. **77** (1988) 1070 [*Teor. Mat. Fiz.* **77** (1988) 97] [INSPIRE].

[26] E.A. Ivanov, *Chern-Simons matter systems with manifest $N = 2$ supersymmetry*, Phys. Lett. **B 268** (1991) 203 [INSPIRE].

JHEP05(2014)018