Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 2293–2299
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Characterizations of maximum fractional (g, f)-factors of graphs[☆]

Guizhen Liu[a,*], Lanju Zhang[b]

[a] School of Mathematics and System Science, Shandong University, Jinan, Shandong, 250100, China
[b] Biostatistics and Data Management, Medimmune Inc., Gaithersburg, MD 20878, USA

Received 22 November 2005; received in revised form 15 October 2007; accepted 23 October 2007
Available online 26 December 2007

Abstract

In this paper a characterization of maximum fractional (g, f)-factors of a graph is presented. The properties of the maximum fractional (g, f)-factors and fractional (g, f)-factors with the minimum of edges are also given, generalizing the results given in [William Y.C. Chen, Maximum (g, f)-factors of a general graph, Discrete Math. 91 (1991) 1–7] and [Edward R. Scheinerman, Daniel H. Ullman, Fractional Graph Theory, John Wiley and Sonc, Inc., New York, 1997]. Furthermore, some new results on fractional factors are obtained which may be used in the design of networks. A polynomial time algorithm can be obtained for actually finding such maximum fractional (g, f)-factors in a graph from the proof.

Keywords: Fractional (g, f)-factor; Fractional matching; Augmenting chain

1. Introduction

Many physical structures can be conveniently modeled by networks. Examples include a communication network with the nodes and links modeling cities and communication channels, respectively; and a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout and so on. In particular, a wide variety of systems can be described using complex networks. Such systems include: the cell, where we model the chemicals by nodes and their interactions by edges; the World Wide Web, which is a virtual network of Web pages connected by hyperlinks; and food chain webs, the networks by which human diseases spread, human collaboration networks etc [7]. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term "*graph*" instead of "*network*".

We study the fractional factor problem in graphs, which can be considered as a relaxation of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved

☆ This work is supported by NSFC (60673047) and RSDP (20040422004) of China.
* Corresponding author.
E-mail address: gzliu@sdu.edu.cn (G. Liu).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.10.016

if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

The graphs considered in this paper will be finite undirected graphs which may have multiple edges but no loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex x of G, the degree of x in G is denoted by $d_G(x)$. Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G satisfying $g(x) \le d_F(x) \le f(x)$ for all $x \in V(G)$. If $g(x) = f(x)$ for all $x \in V(G)$, then a (g, f)-factor is called an f-factor. If $f(x) = k$ for some integer k and all $x \in V(G)$, then an f-factor is called a k-factor. A fractional (g, f)-indicator function is a function h that assigns to each edge of a graph G a fractional number in the interval $[0, 1]$ so that for each vertex x we have $g(x) \le h(E_x) \le f(x)$, where $E_x = \{e|e = xy \in E(G)\}$ and $h(E_1) = \sum_{e\in E_1} h(e)$ for any $E_1 \subseteq E(G)$. When $g(x) = 0$ and $f(x) = 1$ for all $x \in V(G)$, a fractional (g, f)-indicator function is a an indicator function of a fractional matching [6]. When $g(x) = f(x) = 1$ for every $x \in V(G)$, a fractional (g, f)-indicator function is an indicator function of a fractional perfect matching or a fractional 1-factor [6,7]. Let h be a fractional (g, f)-indicator function of a graph G. Set $E_h = \{e|e \in E(G)$ and $h(e) \ne 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional (g, f)-factor of G. h is also called the indicator function of G_h. If $h(e) \in \{0, 1\}$ for every e, then G_h is just a (g, f)-factor of G. Let $F = G_h$ be a fractional (g, f)-factor of graph G. Define the (g, f)-defect of F as follows:

$$\mathrm{def}(F) = \sum_{x\in U} (f(x) - h(E_x))$$

where $U = \{x|x \in V(G) \text{ and } h(E_x) < f(x)\}$. Next, define the (g, f)-defect of G to be

$$\mathrm{def}(G) = \min\{\mathrm{def}(F)|F \text{ is a fractional } (g, f)\text{-factor of } G\}.$$

Let F be a fractional (g, f)-factor such that $\mathrm{def}(F) = \mathrm{def}(G)$. Then F is called *a maximum fractional* (g, f)*-factor of* G. Pulleyblank studied the properties of fractional matchings [8]. Anstee gave a necessary and sufficient condition for a graph to have a fractional (g, f)-factor [1]. Chen discusses the characterization of maximum (g, f)-factors in a graph [2]. Liu and Zhang studied the properties of fractional factors in [3,4,11]. Other results on fractional factors can be found in [5,6,10]. In this paper the characterizations of maximum fractional (g, f)-factors of a graph are presented; the properties of maximum fractional (g, f)-factors and fractional (g, f)-factors with the minimum number of edges are obtained. Furthermore, a polynomial time algorithm can be deduced for actually finding such a maximum fractional (g, f)-factors in a graph from the proof.

2. The characterization of maximum fractional (g, f)-factors

In this section a necessary and sufficient condition for a subgraph to be a maximum fractional (g, f)-factor is given. Therefore some results on maximum matchings [8,9] and maximum (g, f)-factors [2] are generalized.

Let G be a graph. For a subset S of $V(G)$, we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with edges incident with vertices in S. For $E' \subseteq E(G)$, the subgraph induced by E' is denoted by $G[E']$. Let S and T be two disjoint subsets of $V(G)$; we write $E_G(S, T) = \{xy|xy \in E(G), x \in S \text{ and } y \in T\}$ and $e_G(S, T) = h(E_G(S, T))$. If f is any real function on set S, we let $f(S) = \sum_{x\in S} f(x)$ and $f(\emptyset) = 0$.

Anstee gave a necessary and sufficient condition for a graph to have a fractional (g, f)-factor as follows.

Theorem A ([1]). *Let G be a graph. Then G has a fractional (g, f)-factor if and only if for any $S \subseteq V(G)$*

$$g(T) - d_{G-S}(T) \le f(S)$$

where $T = \{x|x \in V(G) \setminus S \text{ and } d_{G-S}(x) \le g(x)\}$.

When $g(x) = f(x)$ for all $x \in V(G)$, the following result is immediately obtained from Theorem A.

Corollary B. *Let G be a graph. Then G has a fractional f-factor if and only if for any $S \subseteq V(G)$*

$$f(T) - d_{G-S}(T) \le f(S)$$

where $T = \{x|x \in V(G) \setminus S \text{ and } d_{G-S}(x) \le f(x)\}$.

In the following we always assume that g and f are two integer-valued functions defined on $V(G)$ and $0 \le g(x) \le f(x)$ for every $x \in V(G)$. Suppose that G has a fractional (g, f)-factor. Let $F = G_h$ be a fractional (g, f)-factor of a graph G with indicator fractional function h. An x-alternating path with respect to h is a sequence of vertices of G, $P(x_1, x) = \{x_1, x_2, \ldots, x_k = x\}$, such that for each i, $1 \le i \le k-1$, $x_i x_{i+1} \in E(G)$ and $h(x_{2i-1}x_{2i}) < 1$ and $h(x_{2i}x_{2i+1}) > 0$. The path $P(x_1, x)$ is even or odd according as k is odd or even. We denote an odd (even) x-alternating path from x_1 to x by $P_o(x_1, x)$ ($P_e(x_1, x)$). We define an augmenting path in G with respect to h to be an odd x-alternating path $P_o(x_1, x)$ with $h(E_{x_1}) < f(x_1)$ and $h(E_x) < f(x)$. Similarly, we define an augmenting chain in G with respect to h to be an odd x-alternating chain $C_o(x_1, x) = \{x_1, x_2, \ldots, x_{2k} = x\}$ with $h(x_{2i-1}x_{2i}) < 1$ and $h(x_{2i}x_{2i+1}) > 0$, $h(E_{x_1}) < f(x_1)$ and $h(E_x) < f(x)$, where a chain $\{x_1, x_2, \ldots, x_k\}$ means that $x_i x_{i+1} \in E(G)$ and the edge may be used at most two times. Note that $x_1 = x$ is allowed.

In the following we give a necessary and sufficient condition for a subgraph to be a maximum fractional (g, f)-factor, which is a fractional analogue of Theorem 2.1 in [2]

Theorem 2.1. *Let G be a graph and $F = G_h$ be a fractional (g, f)-factor. Then F is a maximum fractional (g, f)-factor if and only if there are no augmenting chains with respect to h in G.*

Proof. If $F = G_h$ is a maximum fractional (g, f)-factor, then $\mathrm{def}(G) = \mathrm{def}(F) = \sum_{x \in U}(f(x) - h(E_x))$, that is, F minimizes the quantity

$$\mathrm{def}(F) = \sum_{x \in U}(f(x) - h(E_x))$$

where $U = \{x | x \in V(G), h(E_x) < f(x)\}$.

We show that there are no augmenting chains in G. Otherwise, if there is an augmenting chain $C_o(x_1, x) = \{x_1, x_2, \ldots, x_{2k} = x\}$ such that $\{x_1, x\} \subseteq U$, set

$$\varepsilon_1 = \min\{1 - h(x_{2i-1}x_{2i})\}$$

and

$$\varepsilon_2 = \min\{h(x_{2i}x_{2i+1})\}.$$

If $x_1 \neq x$, then set

$$\varepsilon_3 = \min\{f(x_1) - h(E_{x_1}), f(x) - h(E_x)\}.$$

If $x_1 = x$, then set

$$\varepsilon_3 = \frac{1}{2}\min\{f(x_1) - h(E_{x_1}), f(x) - h(E_x)\}.$$

We set

$$\varepsilon = \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

Let $h'(x_{2i-1}x_{2i}) = h(x_{2i-1}x_{2i}) + \varepsilon$, $h'(x_{2i}x_{2i+1}) = h(x_{2i}x_{2i+1}) - \varepsilon$, and $h'(e) = h(e)$ when $e \notin C_o(x_1, x)$. Then

$$\mathrm{def}(G_{h'}) < \mathrm{def}(F),$$

a contradiction.

Conversely, if there are no augmenting chains with respect to h in G, we show that $F = G_h$ is a maximum fractional (g, f)-factor of G. By the definition, we only need to prove that

$$\mathrm{def}(G) = \mathrm{def}(F).$$

If F is a fractional f-factor, then $\mathrm{def}(G) = \mathrm{def}(F) = 0$. Otherwise, set

$$U = \{x | x \in V(G) \text{ and } h(E_x) < f(x)\}.$$

We have $U \neq \emptyset$ and $\mathrm{def}(F) > 0$. Define

$$S^* = \{x |\text{ there is an odd } x\text{-alternating path } P_o(x_1, x) \text{ in } G \text{ and } x_1 \in U\}$$

and

$$D = \{x | x \notin U, \text{there exists an even } x\text{-alternating path } P_e(x_1, x) \text{ in } G \text{ and } x_1 \in U\}.$$

Since there are no augmenting paths with respect to h in G, for $x \in S^* \cup D$, $x \notin U$ or $h(E_x) = f(x)$. It is easy to see that if $xy \in E(G)$ and $h(xy) > 0$, then $x \in S^*$ implies $y \in D \cup U$. If $xy \in E(G)$ and $h(xy) < 1$, then $x \in D \cup U$ implies $y \in S^*$. Now we prove that $S^* \cap D = \emptyset$. If $x \in D \cap S^*$, then there exists an odd alternating path $P_o(x_1, x) = \{x_1, x_2, \dots, x_{2k} = x\}$ and an even alternating path $p_e(y_1, x) = \{y_1, y_2, \dots, y_{2k+1} = x\}$. Thus $P_o(x_1, x) \cup P_e(y_1, x)$ is an augmenting chain with respect to h, a contradiction.

Now for any $S \subseteq V(G)$, set

$$\delta(S) = f(T) - d_{G-S}(T) - f(S),$$

where $T = \{x | x \in V(G) \setminus S \text{ and } d_{G-S}(x) \le f(x)\}$.

For any $S \subseteq V(G)$ and any fractional (g, f)-factor H, we have $e_H(T, \{x\}) \le h(E_x) \le f(x)$ for each $x \in S$. Therefore

$$\sum_{x \in T} h(E_x) \le d_{G-S}(x) + e_H(T, S) \le d_{G-S}(x) + f(S).$$

We have $\delta(S) = f(T) - d_{G-S}(T) - f(S) \le \sum_{x \in T}(f(x) - h(E_x)) \le \sum_{x \in V(G)}(f(x) - h(E_x)) = \mathrm{def}(H)$.

Thus for any $S \subseteq V(G)$ and any fractional (g, f)-factor H, we have

$$\delta(S) \le \mathrm{def}(H). \tag{2.1}$$

In particular,

$$\delta(S^*) \le \mathrm{def}(F). \tag{2.2}$$

In the following we will show that $\delta(S^*) \ge \mathrm{def}(F)$. Let $E_1 = \{e | e \in E(F) \text{ and } h(e) = 1\}$ and $F_1 = G[E_1]$. Then for $x \in D \cup U$

$$d_{G-S^*}(x) = d_{F_1}(x) - e_{F_1}(\{x\}, S^*) \le h(E_x) - e_{F_1}(\{x\}, S^*) \le f(x).$$

Set $T^* = \{x | x \in V(G) \setminus S^* \text{ and } d_{G-S^*}(x) \le f(x)\}$. Then $D \cup U \subseteq T^*$. Thus

$$\begin{aligned} f(T^*) - d_{G-S^*}(T^*) &\ge f(D \cup U) - d_{G-S^*}(D \cup U) \\ &= f(D \cup U) - d_{F_1}(D \cup U) + e_{F_1}(D \cup U, S^*) \\ &\ge f(U) - d_{F_1}(U) + f(D) - d_{F_1}(D) + e_{F_1}(D \cup U, S^*). \end{aligned} \tag{2.3}$$

Note that if $x \in S^*$ and $h(xy) > 0$, then $y \in D \cup U$. Therefore

$$\begin{aligned} \sum_{x \in S^*} h(E_x) &= h(E_G(S^*, V(G) \setminus S^*)) = f(S^*) = h(E_F(S^*, V(G) \setminus S^*)) \\ &= h(E_F(S^*, D \cup U)) = e_F(S^*, D \cup U). \end{aligned} \tag{2.4}$$

Let $E_2 = \{e | e \in E(F) \text{ and } h(e) < 1\}$ and $F_2 = G[E_2]$. Then by (2.3) and (2.4) we have

$$\begin{aligned} f(T^*) - d_{G-S^*}(T^*) &\ge f(D \cup U) - d_{F_1}(D \cup U) + e_{F_1}(D \cup U, S^*) \\ &= f(D \cup U) - d_{F_1}(D \cup U) + e_{F_2}(D \cup U, S^*) - e_{F_2}(D \cup U, S^*) + e_{F_1}(D \cup U, S^*) \\ &= f(D \cup U) - d_{F_1}(D \cup U) + e_F(D \cup U, S^*) - e_{F_2}(D \cup U, S^*) \\ &= f(D) - d_{F_1}(D) - e_{F_2}(D, S^*) + f(U) - d_{F_1}(U) - e_{F_2}(U, S^*) + f(S^*) \\ &= f(D) - \sum_{x \in D} h(E_x) + f(U) - \sum_{x \in U} h(E_x) + f(S^*) \\ &\ge \mathrm{def}(F) + f(S^*). \end{aligned} \tag{2.5}$$

Note that $f(D) - \sum_{x \in D} h(E_x) \ge 0$, $f(U) - \sum_{x \in U} h(E_x) = \mathrm{def}(F)$ and $e_F(D \cup U, S^*) = f(S^*)$. So (2.5) holds. Thus

$$\delta(S^*) = f(T^*) - d_{G-S^*}(T^*) - f(S^*) \ge \mathrm{def}(F). \tag{2.6}$$

Thus by (2.2), (2.6) and (2.1) it follows that

$$\delta(S^*) = \text{def}(F) = \text{def}(G).$$

The proof of the theorem is completed. □

3. Properties of maximum fractional (g, f)-factors with the minimum number of edges

In the following if $P = \{x_0, x_1, \dots, x_k\}$ is a path with edge set $\{e_1, e_2, \dots, e_k\}$ where $e_i = v_{i-1}v_i$, $1 \le i \le k$, then we also write $P = \{e_1, e_2, \dots, e_k\}$. Let $F = G_h$ be a fractional (g, f)-factor of G with indicator function h. Let $E'_h = \{e | 0 < h(e) < 1\}$. Set

$$\mathcal{F} = \{F | F \text{ is a maximum fractional } (g, f)\text{-factor of } G\}.$$

To obtain our main result in this section, we first need the following lemma.

Lemma 3.1. *Let G be a graph and $F = G_h \in \mathcal{F}$ with $|E'_h|$ minimum. Then subgraph $H = G[E'_h]$ is a disjoint union of odd cycles.*

Proof. Let $F = G_h \in \mathcal{F}$ with $|E'_h|$ minimum. Then we have the following claims.

Claim 1. H has no even cycles.

Otherwise, suppose that H has an even cycle $C = \{e_1e_2 \cdots e_{2l}\}$. Set

$$\varepsilon_1 = \min_{1\le i\le 2l}\{h(e_i)\},$$
$$\varepsilon_2 = \min_{1\le i\le 2l}\{1 - h(e_i)\}$$

and

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}.$$

When $\varepsilon = \varepsilon_1$, without loss of generality, assume $h(e_1) = \varepsilon$. Let g be a function defined on $E(G)$ which takes alternately the values -1 and 1 on edges in $E(C)$ with $g(e_1) = -1$ and takes the value 0 on edges in $E(G) \setminus E(C)$. Let $h' = h + \varepsilon g$. Then $F' = G_{h'} \in \mathcal{F}$ with $|E_{h'}| < |E_h|$, a contradiction.

When $\varepsilon = \varepsilon_2$, similarly, without loss of generality, let $1 - h(e_1) = \varepsilon$. Let g be a function defined on $E(G)$ which takes alternately the values 1 and -1 on edges in $E(C)$ with $g(e_1) = 1$ and takes the value 0 on edges in $E(G) \setminus E(C)$. Let $h' = h + \varepsilon g$. Then $F' = G_{h'} \in \mathcal{F}$ with $|E_{h'}| < |E_h|$, a contradiction again. Thus Claim 1 holds.

Claim 2. $d_H(x) \ge 2$ for any $x \in V(H)$.

Otherwise, suppose that there is a vertex $x_0 \in V(H)$ such that $d_H(x_0) = 1$. Then $h(E_{x_0}) < f(x_0)$. Let $P(x_0, x_k) = \{e_1, e_2, \dots, e_k\}$ be a longest path in H from x_0 to x_k. If k is even, we have $d_H(x_k) = 1$. Set $\varepsilon = \min\{h(e_1), h(e_2), \cdots, h(e_k)\}$. Without loss of generality, assume that $h(e_{i_0}) = \varepsilon$. Let $h'(e_i) = h(e_i) - \varepsilon$, $i \equiv i_0 \pmod k$, $h'(e_i) = h(e_i) + \varepsilon$, $i \equiv i_0 + 1 \pmod k$, and $h'(e) = h(e)$ for any other $e \in E(G)$. Then $F' = G_{h'} \in \mathcal{F}$ with $|E_{h'}| < |E_h|$, contradicting the definition of h. So k must be odd. Since $F \in \mathcal{F}$, there are no augmenting chains with respect to F by Theorem 2.1. Hence, $h(E_{x_k}) = f(x_k)$. This implies that there are two vertices of the path adjacent to x_k and therefore an odd cycle is formed by Claim 1. Then the chain, which traverses the edges on the cycle once and the other edges of the path two times is an augmenting chain with respect to h, contradicting $F \in \mathcal{F}$ and Theorem 2.1. Thus Claim 2 holds.

Claim 3. If H has an odd cycle C, then C must be a component of H.

Otherwise, let $C = \{e_1e_2 \cdots e_{2l+1}\}$ be an odd cycle of H with a vertex $x \in V(C)$ of degree larger than 2 in H. Suppose that e'_1 is an edge not in $E(C)$ and it incident with x. Starting from e'_1, we find a longest path in $H \setminus E(C)$. By Claim 1, the path cannot return to a vertex of $V(C) \setminus \{x\}$. By Claim 2, we finally must get another cycle C'(also odd) which is connected to cycle C by a path P (possibly of length 0). Set

$$\varepsilon_1 = \min\{h(e) | e \in E(C \cup C')\},$$
$$\varepsilon_2 = \min\{1 - h(e) | e \in E(C \cup C')\},$$
$$\varepsilon_3 = \min\{h(e) | e \in E(P)\},$$
$$\varepsilon_4 = \min\{1 - h(e) | e \in E(P)\}$$

and

$$\varepsilon = \min\left\{\varepsilon_1, \varepsilon_2, \frac{1}{2}\varepsilon_3, \frac{1}{2}\varepsilon_4\right\}.$$

If $\varepsilon = \varepsilon_1$, we may assume that $h(e_1) = \varepsilon$, without loss of generality. Let g be a function defined on $E(G)$ which takes alternately the values -1 and 1 on edges in $E(C \cup C')$ with $g(e_1) = -1$, takes the value -2 and 2 on edges in $E(P)$ and takes the value 0 on edges in $E(G) \backslash E(C \cup C' \cup P)$ such that $g(E(G)) = 0$. (It is easy to check that such a function is feasible because both C and C' are odd cycles.) Let $h' = h + \varepsilon g$. Then $F' = G_{h'} \in \mathcal{F}$ with $|E_{h'}| < |E_h|$, contradicting the definition of h. Similarly, we can also obtain a contradiction if $\varepsilon = \varepsilon_2$, ε_3 or ε_4. Thus Claim 3 holds.

By Claim 1, Claim 2 and Claim 3, it follows that every component of H is an odd cycle. □

Theorem 3.2. *Suppose that G has fractional (g, f)-factors. Then there is a maximum fractional (g, f)-factor $F = G_h$ such that $h(e) \in \{0, \frac{1}{2}, 1\}$ for any $e \in E(G)$.*

Proof. Let $F = G_h$ be the maximum fractional (g, f)-factor of G as defined in Lemma 3.1. By Lemma 3.1, it is easy to see that for any $e \in E'_h$, $h(e) = \frac{1}{2}$. And our conclusion follows. □

In particular, let $F = G_h$ be a maximum fractional matching of G. Then from Theorem 3.2 we obtain the following result.

Corollary 3.3 ([7]). *For any graph G, there exists a maximum fractional matching $F = G_h$ of G such that $h(e) \in \{0, \frac{1}{2}, 1\}$ for any $e \in E(G)$.*

Now we have our main result in this section.

Theorem 3.4. *Suppose that graph G has fractional (g, f)-factors. Then the maximum fractional (g, f)-factor described as in Lemma 3.1 is a maximum fractional (g, f)-factor of G with the minimum number of edges.*

Proof. Let $F = G_h$ be the maximum fractional (g, f)-factor of G described as in Lemma 3.1. By Theorem 3.2 $h(E(G))$ is an integer. Let $m = h(E(G))$. Set

$$m_1 = |\{e \in E(G) | h(e) = 1\}|$$

and

$$m_2 = \left|\left\{e \in E(G) | h(e) = \frac{1}{2}\right\}\right|.$$

We have

$$|F| = m_1 + m_2$$

and

$$m = m_1 + \frac{1}{2}m_2.$$

Hence

$$|F| = m + \frac{1}{2}m_2.$$

By the definition of m_2, the desired conclusion follows. □

Remark. It is easy to see that each search for an augmenting chain can be performed by breadth first search in time $O(|E|)$ and the corresponding augmentation lowers the value $\max\{0, f(x) - h(E_x)\}$ for at least one vertex. In [4] a polynomial algorithm for finding a fractional (g, f)-factor is given. Therefore a polynomial algorithm for finding a maximum fractional (g, f)-factor is given from the proof of Theorem 2.1. An polynomial algorithm for finding a maximum fractional (g, f)-factor with the minimum number of edges from a maximum (g, f)-factor is given from the proofs of Lemma 3.1, Theorems 3.2 and 3.4.

Acknowledgments

The authors are indebted to the anonymous referees for pointing out a gap in the proof of Theorem 2.1, and valuable comments and suggestions.

References

[1] R.P. Anstee, Simplified existence theorems for (g, f)-factors, Discrete Appl. Math. 27 (1990) 29–38.
[2] William Y.C. Chen, Maximum (g, f)-factors of a general graph, Discrete Math. 91 (1991) 1–7.
[3] G. Liu, L. Zhang, Maximum fractional $(0, f)$-factors of graphs, Math. Appl. 13 (1) (2000) 31–35.
[4] G. Liu, L. Zhang, Fractional (g, f)-factors of graphs, Acta Math. Sci. 21 B (4) (2001) 541–545.
[5] G. Liu, L. Zhang, Properties of fractional k-factors of graphs, Acta Math. Sci. B 25 (2) (2005) 301–304.
[6] Y. Ma, G. Liu, Some results on fractional k-extendable graphs, Chinese J. Eng. Math. 21 (4) (2004) 567–573.
[7] M.E.J. Newman, The structure and function of complex networks, SIAM Rev. 45 (2) (2003) 167–236.
[8] W.R. Pulleyblank, Fractional matchings and the Edmonds–Gallai theorem, Discrete Appl. Math. 16 (1987) 51–58.
[9] Edward R. Scheinerman, Daniel H. Ullman, Fractional Graph Theory, John Wiley and Sonc, Inc., New York, 1997.
[10] J. Yu, G. Liu, B. Cao, Connected factors of graphs, OR Trans. 9 (1) (2005) 49–57.
[11] L. Zhang, G. Liu, Fractional k-factors of graphs, J. Syst. Sci. Math. Sci. 21 (1) (2001) 88–92.