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Groetsch's representation of Moore–Penrose inverses and ill-posed problems in Hilbert C^*-modules

K. Sharifi

Department of Mathematics, Shahrood University of Technology, PO Box 3619995161-316, Shahrood, Iran

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ABSTRACT

The explicit representation $\lim_{\omega\to 0^+}(\omega 1+T^*T)^{-1}T^*$ for the Moore–Penrose inverse of an operator T between Hilbert spaces has been given by C.W. Groetsch (1975) [6]. We obtain his formula for the Moore–Penrose inverse of an unbounded operator between Hilbert C^*-modules. Ill-posed problems with unbounded operator between Hilbert C^*-modules are also discussed.

1. Introduction

A great many linear inverse problems of physics and engineering may be formulated in an abstract setting as linear operator equations of the form

$$Tx = y \tag{1.1}$$

which implicitly define the solution x of the given problem. The desired solution x is often given in terms of the Moore–Penrose generalized inverse $T^\dagger$ in the form $x = T^\dagger y$. In all interesting cases the Moore–Penrose generalized inverse is an unbounded operator and the challenge is then to provide approximations to the unknown solution $T^\dagger y$ that are stable with respect to perturbations in the data y. Problem (1.1) with an unbounded operator T between Hilbert spaces has been extensively studied in [7,8,15–17]. C.W. Groetsch in [6,7] gave the explicit representation $\lim_{\omega\to 0^+}(\omega 1+T^*T)^{-1}T^*$ for the Moore–Penrose inverse of an operator T between Hilbert spaces as an application of a general representation theorem. Later J.J. Koliha [10] gave the same formula for the Moore–Penrose inverse of an arbitrary element in unital C^*-algebras. In the present paper we give Groetsch's representation for the Moore–Penrose inverse of unbounded regular operator T between Hilbert C^*-modules and then we reconsider Eq. (1.1).

A Hilbert C^*-module obeys the same axioms as an ordinary Hilbert space except that the inner product, from which the geometry emerges, takes values in an arbitrary C^*-algebra $\mathcal{A}$ rather than $\mathbb{C}$. Some fundamental properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements must be given up. Hilbert C^*-modules play an important role in the modern theory of C^*-algebras and the study of locally compact quantum groups (see e.g. [19,20]).

Throughout this paper we assume $\mathcal{A}$ to be an arbitrary C^*-algebra (not necessarily unital). We deal with bounded and unbounded operators at the same time, so as a general rule, we will denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $Dom(\cdot)$, $Ker(\cdot)$ and $Ran(\cdot)$ for domain, kernel and range of operators, respectively.

E-mail addresses: sharifi.kamran@gmail.com, sharifi@shahroodut.ac.ir.

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An unbounded regular operator between Hilbert C^*-modules is an analogue of a closed operator on a Hilbert space. A closed and densely defined operator t from a Hilbert C^*-module E to another Hilbert C^*-module F is called regular if its adjoint t^* is also densely defined and if the range of $(1+t^*t)$ is dense in E. Moore–Penrose inverses of unbounded regular operators have been studied by the author and M. Frank in [5]. Suppose t and t^* have unique Moore–Penrose inverses which are adjoint to each other, $t^\dagger$ and $t^{*\dagger}$. We represent $t^\dagger$ as a limit of bounded operator, indeed, we show that $t^\dagger y = \lim_{\omega\to 0^+} \overline{(\omega 1 + t^*t)^{-1}t^*} y = \lim_{\omega\to 0^+} t^*(\omega 1 + tt^*)^{-1} y$ for all $y \in Dom(t^\dagger)$. This fact enables us to show that $t^\dagger$ is bounded if and only if t has closed range, if and only if the set $\{t^*(\omega 1 + tt^*)^{-1}\colon\ \omega \in \mathbb{R}^+\}$ is uniformly bounded.

Suppose $y \in Dom(t^\dagger) = Ran(t) \oplus Ker(t^*)$ and $x_\omega = t^*(\omega 1 + tt^*)^{-1} y$. Let x_* be any minimum point of $\{|tx - y|^2\colon\ x \in Dom(t) = Ran(t^\dagger) \oplus Ker(t)\}$ in which $|.|$ is the $\mathcal{A}$-valued 'norm' given by $|x| = \langle x, x\rangle^{1/2}$, then we obtain $|tx_* - y| = \lim_{\omega\to 0} |tx_\omega - y|$. In this situation, x_ω is also the solution of C^*-valued variational problem $\min\{|tx - y|^2 + \omega|x|^2\colon\ x \in Dom(t),\ \omega \in \mathbb{R}^+\}$. Since every C^*-algebra can be considered as a Hilbert C^*-module, the results are also relevant in the case of unbounded operators affiliated with C^*-algebras (see e.g. [19,20]).

2. Preliminaries

A (left) *pre-Hilbert C^*-module* over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E endowed with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}$, $(x, y) \mapsto \langle x, y\rangle$ which is linear in the first variable x (and conjugate-linear in the second variable y), satisfying the conditions

$$\langle x, y\rangle = \langle y, x\rangle^*, \qquad \langle ax, y\rangle = a\langle x, y\rangle \quad \text{for all } a \in \mathcal{A},$$

$$\langle x, x\rangle \geqslant 0 \quad \text{with equality if and only if } x = 0.$$

A pre-Hilbert $\mathcal{A}$-module E is called a *Hilbert $\mathcal{A}$-module* if E is a complete space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. As well as this scalar-valued norm, E has an $\mathcal{A}$-valued 'norm' given by $|x| := \langle x, x\rangle^{1/2}$ which is evaluated in the partially ordered set of all positive element of the C^*-algebra $\mathcal{A}$. The $\mathcal{A}$-valued norm needs to be handled with care, for example, it need not be the case that $|x + y| \leqslant |x| + |y|$ (see e.g. [1]). A pre-Hilbert $\mathcal{A}$-submodule E of a pre-Hilbert $\mathcal{A}$-module F is a direct orthogonal summand if $E \oplus E^\perp = F$, where $E^\perp := \{y \in F\colon\ \langle x, y\rangle = 0 \text{ for all } x \in E\}$ is the orthogonal complement of E in F. For the elementary theory of Hilbert C^*-modules we refer to the book by E.C. Lance [12] and the papers [3,13].

We denote by $B(E, F)$ the set of all adjointable operators from a Hilbert $\mathcal{A}$-module E to another Hilbert $\mathcal{A}$-module F, i.e. of all maps $T : E \to F$ such that there exists $T^* : F \to E$ with the property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E$, $y \in F$. $B(E, E)$ is abbreviated by $B(E)$.

Letting E, F be Hilbert $\mathcal{A}$-modules, we will use the notation $t : Dom(t) \subseteq E \to F$ to indicate that t is an $\mathcal{A}$-linear operator whose domain $Dom(t)$ is a dense submodule of E (not necessarily identical with E) and whose range is in F. Given $t : Dom(t) \subseteq E \to F$ and $s : Dom(t) \subseteq E \to F$, we write $s \subseteq t$ if $Dom(s) \subseteq Dom(t)$ and $s(x) = t(x)$ for all $x \in Dom(s)$. A densely defined operator $t : Dom(t) \subseteq E \to F$ is called *closed* if its graph $G(t) = \{(x, tx)\colon\ x \in Dom(t)\}$ is a closed submodule of the Hilbert $\mathcal{A}$-module $E \oplus F$. If t is closable, the operator $s : Dom(s) \subseteq E \to F$ with the property $G(s) = \overline{G(t)}$ is called the *closure* of t denoted by $s = \bar{t}$. The operator $\bar{t}$ is the smallest closed operator that contains t. An $\mathcal{A}$-linear operator $t : Dom(t) \subseteq E \to F$ is said to be *regular* if

(i) t is closed and densely defined with domain $Dom(t)$,
(ii) its adjoint t^* is also densely defined, and
(iii) the range of $1 + t^*t$ is dense in E.

If we set $\mathcal{A} = \mathbb{C}$, i.e. if we take E, F to be Hilbert spaces, then this is exactly the definition of a densely defined closed operator, except that in that case, both the second and third conditions follow from the first one. We denote the set of all regular operators from E to F by $R(E, F)$. It is well known that a densely defined operator t is regular if and only if its graph $G(t)$ is orthogonally complemented in the Hilbert $\mathcal{A}$-module $E \oplus F$ (see e.g. [4, Corollary 3.2]).

Corollary 2.1. *The operator $t : Dom(t) \subseteq E \to F$ is regular if and only if $\omega^{-1/2}t$ is regular for any positive real number ω.*

For the proof just recall that for an $\mathcal{A}$-linear densely defined operator $t : Dom(t) \subseteq E \to F$ and any positive real number ω, the graph of t is orthogonally complemented if and only if the graph of $\omega^{-1/2}t$ is orthogonally complemented (see also [4, Corollary 3.3]).

If t is regular then t^* and t^*t are regular, and $t = t^{**}$. Define $Q_t := (1 + t^*t)^{-1/2}$, and $F_t := t(1 + t^*t)^{-1/2} = tQ_t$, then $Ran(Q_t) = Dom(t)$, $0 \leqslant Q_t \leqslant 1$ in $B(E)$ and $F_t \in B(E, F)$, cf. [12, (10.4)]. The bounded operator F_t is called the *bounded transform* of the regular operator t. The map $t \to F_t$ defines a bijection

$$R(E, F) \to \bigl\{T \in B(E, F)\colon\ \|T\| \leqslant 1 \text{ and } Ran\bigl(1 - T^*T\bigr) \text{ is dense in } F\bigr\}.$$

This map is adjoint-preserving, i.e. $F_t^* = F_{t^*}$, cf. [12, Theorem 10.4]. We also define $R_t := (1 + t^*t)^{-1} = Q_t^2$, then $Ran(R_t) \subseteq Dom(t)$, $tR_t = F_tQ_t \in B(E, F)$ and $\|tR_t\| = \|F_tQ_t\| \leqslant \|F_t\|\|Q_t\| \leqslant 1$.

Recall that the composition of two densely defined operators t, s is the unbounded operator ts with $Dom(ts) = \{x \in Dom(s)\colon\ sx \in Dom(t)\}$ given by $(ts)(x) = t(sx)$ for all $x \in Dom(ts)$. The operator ts is not necessarily densely defined. Suppose two densely defined operators t, s are adjointable, then $s^*t^* \subseteq (ts)^*$. If T is a bounded adjointable operator, then $s^*T^* = (Ts)^*$. The equality implies that s^*T^* and $(Ts)^*$ actually have the same domains.

Lemma 2.2. *If $t \in R(E, F)$ is an unbounded regular operator then $R_t t^* \subseteq (tR_t)^* = t^* R_{t^*}$.*

Proof. The equality $F_t^* = F_{t^*}$ implies $(tQ_t)^* = t^*Q_{t^*}$. Applying Remark 2.2 of [5] to the regular operator t^*, we obtain $t^*Q_{t^*}^2 = Q_t t^* Q_{t^*}$. We therefore have $(tR_t)^* = (tQ_tQ_t)^* = Q_t^*(tQ_t)^* = Q_t t^* Q_{t^*} = t^*Q_{t^*}^2 = t^*R_{t^*}$. Since $R_t^* = R_t$, we have $R_t t^* \subseteq (tR_t)^* = t^*R_{t^*}$. $\square$

If ω is a positive real number, the properties of the bounded adjointable operators $R_{\omega^{-1/2}t}$, $\omega^{-1/2} t R_{\omega^{-1/2}t}$ now read as follows:

Lemma 2.3. *Suppose $t \in R(E, F)$ is a regular operator and ω is a positive real number. Then $(\omega 1 + t^*t)^{-1}$ and $t(\omega 1 + t^*t)^{-1}$ are bounded adjointable operators,*

$$0 \leqslant \omega\bigl(\omega 1 + t^*t\bigr)^{-1} \leqslant 1 \quad \text{in } B(E), \tag{2.1}$$

$$\bigl\| t\bigl(\omega 1 + t^*t\bigr)^{-1}\bigr\| \leqslant \omega^{-1/2}, \tag{2.2}$$

$$\bigl(\omega 1 + t^*t\bigr)^{-1} t^* \subseteq t^*\bigl(\omega 1 + tt^*\bigr)^{-1}, \quad \text{and} \tag{2.3}$$

$$\bigl(\omega 1 + tt^*\bigr)^{-1} t \subseteq t\bigl(\omega 1 + t^*t\bigr)^{-1}. \tag{2.4}$$

Moreover, the operator $(\omega 1 + t^*t)^{-1}t^*t$ has a bounded extension to $\overline{Dom(t)} = E$ which satisfies

$$0 \leqslant \bigl(\omega 1 + t^*t\bigr)^{-1} t^*t = 1 - \omega\bigl(\omega 1 + t^*t\bigr)^{-1} \leqslant 1 \quad \text{on } Dom(t). \tag{2.5}$$

Suppose $x \in Dom(t^*)$ and $\omega > 0$, then (2.3) implies that $(\omega 1 + t^*t)^{-1}t^*x = t^*(\omega 1 + tt^*)^{-1}x$. Consequently, the operator $(\omega 1 + t^*t)^{-1}t^*$ is bounded on the dense submodule $Dom(t^*)$, and its extension by continuity to F satisfies

$$\overline{\bigl(\omega 1 + t^*t\bigr)^{-1} t^*} = t^*\bigl(\omega 1 + tt^*\bigr)^{-1}. \tag{2.6}$$

Definition 2.4. Let $t \in R(E, F)$ be a regular operator between two Hilbert $\mathcal{A}$-modules E, F over some fixed C^*-algebra $\mathcal{A}$. A regular operator $t^\dagger \in R(F, E)$ is called the Moore–Penrose inverse of t if $tt^\dagger t = t$, $t^\dagger t t^\dagger = t^\dagger$, $(tt^\dagger)^* = \overline{tt^\dagger}$ and $(t^\dagger t)^* = \overline{t^\dagger t}$.

If a regular operator t has a Moore–Penrose inverse $t^\dagger$, then the above definition implies that $Ran(t) \subseteq Dom(t^\dagger)$ and $Ran(t^\dagger) \subseteq Dom(t)$. The reader should be aware of the fact that a (bounded or unbounded) module map between Hilbert C^*-modules generally does not have a Moore–Penrose inverse, see e.g. [5,9,18,21]. However, the author and M. Frank in Theorem 3.1 of [5] gave a necessary and sufficient condition as follows:

Theorem 2.5. *If E, F are arbitrary Hilbert $\mathcal{A}$-modules and $t \in R(E, F)$ denotes a regular operator then the following conditions are equivalent*:

(i) *t and t^* have unique Moore–Penrose inverses which are adjoint to each other, $t^\dagger$ and $t^{\dagger *}$.*
(ii) *$E = Ker(|t|) \oplus \overline{Ran(|t|)}$ and $F = Ker(t^*) \oplus \overline{Ran(t)}$.*

In this situation, $\overline{t^*t^{\dagger *}}$ and $\overline{tt^\dagger}$ are the projections onto $\overline{Ran(|t|)} = \overline{Ran(t^*)}$ and $\overline{Ran(t)}$, respectively.

This theorem and Proposition 2.2 of [4] show that each regular operator with closed range has a Moore–Penrose inverse.

Remark 2.6. By an arbitrary C^*-algebra of compact operators $\mathcal{A}$ we mean that $\mathcal{A} = c_0 - \bigoplus_{i \in I} \mathcal{K}(H_i)$, i.e. $\mathcal{A}$ is a c_0-direct sum of elementary C^*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i, $i \in I$ cf. [2, Theorem 1.4.5]. If $\mathcal{A}$ is an arbitrary C^*-algebra of compact operators then for every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : Dom(t) \subseteq E \to F$ is automatically regular and has a Moore–Penrose inverse, cf. [4,5,9].

Corollary 2.7. *Suppose $t \in R(E, F)$ is a regular operator and t and t^* possess the Moore–Penrose inverses $t^\dagger$ and $t^{\dagger *}$, respectively.*

(i) *$Ker(t^*) = Ker(t^\dagger)$ and $Ker(t) = Ker(t^{\dagger *})$.*
(ii) *$Dom(t^\dagger) = Ran(t) \oplus Ker(t^\dagger)$.*

(iii) $Dom(t^{\dagger *}) = Ran(t^*) \oplus Ker(t^{\dagger *})$.
(iv) $Dom(t) = Ran(t^\dagger) \oplus Ker(t)$.
(v) $Dom(t^*) = Ran(t^{\dagger *}) \oplus Ker(t^*)$.

Proof. To show (i), suppose $y \in Dom(t^\dagger)$ and $t^\dagger y = 0$. Since $tt^\dagger \subseteq \overline{tt^\dagger} = (tt^\dagger)^*$, for every $x \in Dom(t)$ we have

$$\langle tx, y\rangle = \langle tt^\dagger t(x), y\rangle = \langle tx, (tt^\dagger)^* y\rangle = \langle tx, (tt^\dagger) y\rangle = 0.$$

It follows $y \in Dom(t^*)$ and $t^* y = 0$, i.e., $Ker(t^\dagger) \subseteq Ker(t^*)$. Since t^* is the Moore–Penrose inverse of $t^{\dagger *}$, $Ker(t^*) \subseteq Ker(t^{\dagger **}) = Ker(t^\dagger)$. Similarly, we obtain $Ker(t) = Ker(t^{\dagger *})$.

We prove (iv). Let $x \in Dom(t)$, then $t(x) = tt^\dagger t(x)$. So every element of $Dom(t)$ can be written as the sum of two elements $(t^\dagger t)(x) \in Ran(t^\dagger)$ and $(1 - t^\dagger t)(x) \in Ker(t)$. If $x = a + t^\dagger(b)$ where $b \in Dom(t^\dagger) \cap \overline{Ran(t)}$ and $a \in Ker(t)$, then $(t^\dagger t)(x) = 0 + t^\dagger t t^\dagger(b) = t^\dagger(b)$ and $a = (1 - t^\dagger t)(x)$. Since $\overline{t^\dagger t}$ is an orthogonal projection onto $Ran(\overline{t^\dagger t}) = \overline{Ran(t^\dagger t)} = \overline{Ran(t^\dagger)}$, we find $\langle a, t^\dagger(b)\rangle = 0$, i.e. $Dom(t) = Ran(t^\dagger) \oplus Ker(t)$. The equalities (ii), (iii) and (v) are established in the same way. □

Theorem 2.8. *Suppose $t \in R(E, F)$ is a regular operator and t and t^* possess the Moore–Penrose inverses $t^\dagger$ and $t^{\dagger *}$.*

(i) $t^\dagger = \lim_{\omega \to 0^+} t^*(\omega 1 + tt^*)^{-1} = \lim_{\omega \to 0^+} \overline{(\omega 1 + t^*t)^{-1} t^*}$ *on* $Dom(t^\dagger)$.
(ii) $t^{\dagger *} = \lim_{\omega \to 0^+} t(\omega 1 + t^*t)^{-1} = \lim_{\omega \to 0^+} \overline{(\omega 1 + tt^*)^{-1} t}$ *on* $Dom(t^{\dagger *})$.

Proof. To prove (i), suppose $y \in Dom(t^\dagger)$ and $x = t^\dagger y \in Ran(t^\dagger) \subseteq Dom(t)$. Then $y = (tt^\dagger)y + (1 - tt^\dagger)y \in Ran(t) \oplus Ker(t^\dagger) = Dom(t^\dagger)$. Suppose ω is an arbitrary positive real number and $x_\omega = t^*(\omega 1 + tt^*)^{-1} y$, then

$$x_\omega = t^*(\omega 1 + tt^*)^{-1} tt^\dagger y + t^*(\omega 1 + tt^*)^{-1}(1 - tt^\dagger)y. \tag{2.7}$$

Using (2.3) and the fact that $(1 - tt^\dagger)y \in Ker(t^\dagger) = Ker(t^*)$, we get $t^*(\omega 1 + tt^*)^{-1}(1 - tt^\dagger)y = (\omega 1 + t^*t)^{-1} t^*(1 - tt^\dagger)y = 0$. We then find from (2.7), (2.4) and (2.5) that

$$x_\omega = t^*(\omega 1 + tt^*)^{-1} tt^\dagger y = t^*t(\omega 1 + t^*t)^{-1} t^\dagger y = \left(1 - \omega(\omega 1 + t^*t)^{-1}\right) t^\dagger y.$$

Hence $x_\omega - x = x_\omega - t^\dagger y = -\omega(\omega 1 + t^*t)^{-1} x$. On the other hand $x = t^\dagger y \in \overline{Ran(t^\dagger)} = (Ker(t^{\dagger *}))^\perp = (Ker(t))^\perp = \overline{Ran(t^*)} = \overline{Ran(t^*t)}$, where the last equality follows from [11, Proposition 4.18]. Given $\epsilon > 0$, there is an element $\tilde{x} \in Dom(t)$ such that $\|x - t^*t\tilde{x}\| \leqslant \epsilon$. Using (2.1) and (2.5), we obtain

$$\begin{aligned}
\|x - x_\omega\| &\leqslant \left\|\omega(\omega 1 + t^*t)^{-1} x - \omega(\omega 1 + t^*t)^{-1} t^*t\tilde{x}\right\| + \left\|\omega(\omega 1 + t^*t)^{-1} t^*t\tilde{x}\right\| \\
&\leqslant \left\|\omega(\omega 1 + t^*t)^{-1}\right\| \left\|x - t^*t\tilde{x}\right\| + \omega\left\|(\omega 1 + t^*t)^{-1} t^*t\right\| \|\tilde{x}\| \\
&\leqslant \epsilon + \omega\|\tilde{x}\|.
\end{aligned}$$

Therefore $t^\dagger y = \lim_{\omega \to 0^+} t^*(\omega 1 + tt^*)^{-1} y$ for every $y \in Dom(t^\dagger)$. The second equality of (i) follows from the first one and (2.6). The equalities of (ii) follow by noting that $t^{\dagger *} = t^{*\dagger}$ and interchanging the roles of t and t^* in the first part. □

Corollary 2.9. *Suppose $t \in R(E, F)$ is a regular operator and t and t^* possess the Moore–Penrose inverses $t^\dagger$ and $t^{\dagger *}$.*

(i) *If $y \in Dom(t^\dagger)$, $y_\omega \in F$, $\omega > 0$ and $\lim_{\omega \to 0^+} \omega^{-1/2}\|y - y_\omega\| = 0$, then*

$$\lim_{\omega \to 0^+} t^*(\omega 1 + tt^*)^{-1} y_\omega = t^\dagger y.$$

(ii) *If $y \in Dom(t^{\dagger *})$, $y_\omega \in E$, $\omega > 0$ and $\lim_{\omega \to 0^+} \omega^{-1/2}\|y - y_\omega\| = 0$, then*

$$\lim_{\omega \to 0^+} t(\omega 1 + t^*t)^{-1} y_\omega = t^{\dagger *} y.$$

Proof. For every $x \in Dom(tt^*)$ we have

$$\left|(\omega 1 + tt^*)x\right|^2 = \omega^2|x|^2 + \langle \omega x, tt^*x\rangle + \langle tt^*x, \omega x\rangle + \left|tt^*x\right|^2 \geqslant 2\omega\langle t^*x, t^*x\rangle = 2\omega\left|t^*x\right|^2.$$

Set $x = (\omega 1 + tt^*)^{-1} z$ where $z \in F$, then $x \in Dom(t^*)$ and $2\omega|t^*(\omega 1 + tt^*)^{-1} z|^2 \leqslant |z|^2$. Consequently, $\|t^*(\omega 1 + tt^*)^{-1} z\| \leqslant (2\omega)^{-1/2}\|z\|$ for every $z \in F$. Using the later inequality and Theorem 2.8, we have

$$\begin{aligned}
\left\|t^*(\omega 1 + tt^*)^{-1} y_\omega - t^\dagger y\right\| &\leqslant \left\|t^*(\omega 1 + tt^*)^{-1}(y_\omega - y)\right\| + \left\|t^*(\omega 1 + tt^*)^{-1} y - t^\dagger y\right\| \\
&\leqslant (2\omega)^{-1/2}\|y_\omega - y\| + \left\|t^*(\omega 1 + tt^*)^{-1} y - t^\dagger y\right\| \to 0, \quad \text{as } \omega \to 0.
\end{aligned}$$

This completes the proof of (i). The second assertion is easily proved by interchanging the roles of t and t^* in the first part. □

Corollary 2.10. *Suppose $t \in R(E,F)$ is a regular operator and t and t^* possess the Moore–Penrose inverse $t^\dagger$. Then the following assertions are equivalent*:

(i) *$t^\dagger : Dom(t^\dagger) \subseteq F \to E$ is bounded.*
(ii) *$Ran(t)$ is a closed submodule of F.*
(iii) *The set $\{t^*(\omega 1 + tt^*)^{-1}\colon\ \omega \in \mathbb{R}^+\}$ is uniformly bounded.*

Proof. (i) ⇔ (ii) According to Corollary 2.7 we have $Dom(t^\dagger) = Ran(t) \oplus Ker(t^*)$. The operator $t^\dagger$ is bounded if and only if $Dom(t^\dagger) = F$, if and only if $F = Dom(t^\dagger) = Ran(t) \oplus Ker(t^*)$, if and only if $\underline{Ran(t)}$ is closed.

(ii) ⇒ (iii) Suppose that $Ran(t)$ is a closed submodule of F, then $F = \overline{Ran(t)} \oplus Ker(t^\dagger) = Dom(t^\dagger)$. In view of Theorem 2.8, the net $\{t^*(\omega 1 + tt^*)^{-1} y\}_\omega$ converges for any $y \in F$ and hence, by the Principle of the Uniform Boundedness, $\{t^*(\omega 1 + tt^*)^{-1}\colon\ \omega \in \mathbb{R}^+\}$ is uniformly bounded.

(iii) ⇒ (i) Suppose $\{t^*(\omega 1 + tt^*)^{-1}\colon\ \omega \in \mathbb{R}^+\}$ is uniformly bounded. Since $t^*(\omega 1 + tt^*)^{-1}$, $\omega > 0$ are bounded operators and $\lim_{\omega \to 0^+} t^*(\omega 1 + tt^*)^{-1} y = t^\dagger y$ for all $y \in Dom(t^\dagger)$, $t^\dagger$ is a bounded operator on its domain $Dom(t^\dagger)$. The domain of $t^\dagger$ is dense in F and E is a Hilbert module, so $t^\dagger$ has a unique bounded $\mathcal{A}$-linear extension $\widetilde{t^\dagger} : F \to E$ which is defined by

$$\widetilde{t^\dagger} z = \lim_{n \to +\infty} t^\dagger y_n \quad \text{for all } z \in F,$$

where $\{y_n\}$ is a sequence in $Dom(t^\dagger)$ which converges to z in norm. Hence, for every $z \in F$ there exist a sequence $\{y_n\}$ in $Dom(t^\dagger)$ and an element $\widetilde{t^\dagger} z$ in E such that $y_n \to z$ and $t^\dagger y_n \to \widetilde{t^\dagger} z$. The closedness of $t^\dagger$ implies that $z \in Dom(t^\dagger)$ and $\widetilde{t^\dagger} z = t^\dagger z$, that is, $t^\dagger$ is everywhere defined and bounded. □

3. Ill-posed problems

The equation $tx = y$ where $t : Dom(t) \subseteq E \to F$ is an unbounded regular operator which has Moore–Penrose inverse $t^\dagger$, is called ill-posed if t is not boundedly invertible. Of course the equation has a solution if and only if $y \in Ran(t)$, in this situation, $x = t^\dagger y + (1 - t^\dagger t) z \in Ran(t^\dagger) \oplus Ker(t) = Dom(t)$ for some $z \in Dom(t)$. However, we can associate generalized solutions with any y in the dense submodule $Ran(t) \oplus Ker(t^*) = Ran(t) \oplus Ker(t^\dagger) = Dom(t^\dagger)$ of F. We begin our section with the following useful lemma.

Lemma 3.1. *Suppose a, b are self-adjoint elements in an arbitrary C^*-algebra $\mathcal{A}$ and $k^2 a^2 + kb \geqslant 0$ for any k in the set of real numbers $\mathbb{R}$, then $b = 0$.*

Proof. According to [14, Theorem 3.3.6] there exists a positive linear functional τ such that $\tau(b) = \|b\|$. Since $k^2 a^2 + kb \geqslant 0$ for all $k \in \mathbb{R}$, we get

$$k^2 \tau\left(a^2\right) + k\tau(b) = k^2 \tau\left(a^2\right) + k\|b\| \geqslant 0 \quad \text{for all } k \in \mathbb{R}. \tag{3.1}$$

Suppose first that $\tau(a^2) > 0$. Then the necessary and sufficient condition for the positivity of the quadratic form (3.1) in k is exactly $\|b\| \leqslant 0$, that is, $b = 0$. Now suppose that $\tau(a^2) = 0$, again by using (3.1) with $k = -1$, we find $b = 0$. □

Lemma 3.2. *Suppose $t : Dom(t) \subseteq E \to F$ is an unbounded regular operator and $y \in Ran(t) \oplus Ker(t^*)$. The equation*

$$t^*(tx - y) = 0 \tag{3.2}$$

and the C^*-valued variational problem

$$\min\left\{|tx - y|^2\colon\ x \in Dom(t)\right\} \tag{3.3}$$

are solvable if and only if y is in the submodule $Ran(t) \oplus Ker(t^*)$ of F.

Proof. For $h \in Dom(t)$

$$\delta(h) = \left|t(x+h) - y\right|^2 - |tx - y|^2 = \langle tx - y, th\rangle + \langle tx - y, th\rangle^* + |th|^2. \tag{3.4}$$

If x is a solution of (3.2), then $tx - y \in Ker(t^*) = Ran(t)^\perp$, which implies $\langle tx - y, th\rangle = \langle tx - y, th\rangle^* = 0$. Consequently, $\delta(h) = |th|^2 \geqslant 0$, that is, x is a solution of (3.3). Conversely, if x is a solution of (3.3), then $\delta(h) = |th|^2 + \langle tx - y, th\rangle + \langle tx - y, th\rangle^* \geqslant 0$

for all $h \in Dom(t)$. Thus, $k^2|th|^2 + k(\langle tx - y, th\rangle + \langle tx - y, th\rangle^*) \geqslant 0$ for all $h \in Dom(t)$ and $k \in \mathbb{R}$. Using Lemma 3.1, we have $\langle tx - y, th\rangle + \langle tx - y, th\rangle^* = 0$. Consequently,

$$\big\langle t^*(tx - y), h\big\rangle + \big\langle t^*(tx - y), h\big\rangle^* = 0 \quad \text{for all } h \in Dom(t). \tag{3.5}$$

Since $Dom(t)$ is a dense submodule of E, the equality (3.5) remains valid for each $h \in E$. In particular, for $h = t^*(tx - y)$ we obtain $2\langle t^*(tx - y), t^*(tx - y)\rangle = 0$, i.e. $t^*(tx - y) = 0$.

If $y \in Ran(t) \oplus Ker(t^*)$, there exists $x_0 \in Dom(t)$ such that $y - tx_0 \in Ker(t^*)$, i.e. x_0 is the solution of (3.2). Conversely, suppose $y \in F$ and (3.2) has a solution x, then $y = tx - (tx - y) \in Ran(t) \oplus Ker(t^*)$. □

Theorem 3.3. *Suppose $t \in R(E, F)$ is a regular operator and t and t^* possess the Moore–Penrose inverses $t^\dagger$ and $t^{\dagger *}$. Let $y \in Dom(t^\dagger) = Ran(t) \oplus Ker(t^*)$, then $x_\omega = t^*(\omega 1 + tt^*)^{-1} y$ is the unique solution of the C^*-valued variational problem*

$$\min\big\{|tx - y|^2 + \omega|x|^2\colon\ x \in Dom(t),\ \omega \in \mathbb{R}^+\big\}. \tag{3.6}$$

Moreover, if x_* is any solution of (3.3)*, then*

$$|tx_* - y| = \lim_{\omega \to 0^+} |tx_\omega - y|. \tag{3.7}$$

Proof. Let $H_\omega(x) = |tx - y|^2 + \omega|x|^2$, $x \in Dom(t)$. One has

$$\mu(h) = H_\omega(x + h) - H_\omega(x) = |th|^2 + \omega|h|^2 + \langle tx - y, th\rangle + \langle tx - y, th\rangle^* + \omega\langle x, h\rangle + \omega\langle x, h\rangle^*,$$

for $h \in Dom(t)$ and $y \in Dom(t^\dagger)$. Using $tt^*(\omega 1 + tt^*)^{-1} = 1 - \omega(\omega 1 + tt^*)^{-1}$, for $x_\omega \in Dom(t)$ we obtain

$$\begin{aligned}
\mu(h) = H_\omega(x_\omega + h) - H_\omega(x_\omega) &= |th|^2 + \omega|h|^2 + \big\langle tt^*\big(\omega 1 + tt^*\big)^{-1} y - y, th\big\rangle \\
&\quad + \big\langle tt^*\big(\omega 1 + tt^*\big)^{-1} y - y, th\big\rangle^* + \omega\big\langle t^*\big(\omega 1 + tt^*\big)^{-1} y, h\big\rangle + \omega\big\langle t^*\big(\omega 1 + tt^*\big)^{-1} y, h\big\rangle^* \\
&= |th|^2 + \omega|h|^2 \geqslant 0.
\end{aligned}$$

Consequently, $H_\omega(x_\omega) \leqslant H_\omega(x_\omega + h)$ for any $h \in Dom(t)$, that is, $H_\omega(.)$ attains a minimum on $x_\omega = t^*(\omega 1 + tt^*)^{-1} y$, $y \in Dom(t^\dagger)$. If $\tilde{x}$ is an another minimum point of $H_\omega(.)$ and $h = \tilde{x} - x_\omega$, then $H_\omega(x_\omega) = H_\omega(x_\omega + h)$, which implies $\mu(h) = |th|^2 + \omega|h|^2 = 0$. Hence $h = \tilde{x} - x_\omega = 0$, i.e. x_ω is the unique solution of (3.6).

Suppose x_* is any minimum point of $H_0(x) = |tx - y|^2$, $x \in Dom(t)$. Then

$$|tx_* - y|^2 \leqslant |tx_\omega - y|^2 \leqslant |tx_\omega - y|^2 + \omega|x_\omega|^2 \leqslant |tx_* - y|^2 + \omega|x_*|^2,$$

which yields

$$\big\| |tx_\omega - y|^2 - |tx_* - y|^2 \big\| \leqslant \big\| \omega|x_*|^2 \big\| = \omega\|x_*\|^2.$$

Hence, $|tx_* - y|^2 = \lim_{\omega \to 0^+} |tx_\omega - y|^2$. By continuity of the function $g(x) = \sqrt{x}$ on $[0, +\infty)$ we can deduce $|tx_* - y| = \lim_{\omega \to 0^+} |tx_\omega - y|$. □

We close the paper with the observation that we can reformulate our results in terms of densely defined closed operators on Hilbert C^*-modules over C^*-algebras of compact operators, since they automatically have Moore–Penrose inverses.

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