The Solutions to an Infinite Family of Matrix Inequalities Involving *ZME*-matrices

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ABSTRACT

Solutions for a certain infinite system of matrix inequalities are determined. In these inequalities, the coefficient matrices are *ZME*-matrices, which were introduced by Friedland, Hershkowitz, and Schneider. The solutions are shown to have a simple form under certain restrictions on the magnitudes of the minimal eigenvalues of the coefficient matrices. Finally, solutions to the system of inequalities are used to study the structure of reducible *ZM*-matrices.

1. INTRODUCTION

The subject of this paper is the solution of an infinite system of matrix inequalities involving *ZME*-matrices. While interesting in its own right, the study of these inequalities is motivated by the fundamental role that the solutions play in the structure of reducible *ZM*- and *MM*-matrices.

Let $\mathscr{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices over the real numbers. A matrix A in $\mathscr{M}_n(\mathbb{R})$ is called a *Z-matrix* if each off-diagonal entry of A is nonpositive. A Z-matrix is called an *M-matrix* if its spectrum lies in the closed right half plane. A matrix each of whose positive-integer powers is a Z-matrix (*M*-matrix) is called a *ZM-matrix* (*MM-matrix*). A *ZM*-matrix each of whose positive-odd-integer powers is irreducible is called a *ZME-matrix*. An *MM*-matrix each of whose positive powers is irreducible is called an *MMA-matrix*. The properties of *ZME*- and *MMA*-matrices were investigated in a recent paper by Friedland, Hershkowitz, and Schneider [2].

A matrix (vector) is called a *nonnegative matrix* (*nonnegative vector*) if each of its entries is nonnegative. A matrix (vector) is called a *strictly positive*

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matrix (strictly positive vector) *if each of its entries is positive. If A is a nonnegative matrix or nonnegative vector, this will be denoted by $A \geqslant 0$.*

Suppose that A is in $\mathcal{M}_n(\mathbb{R})$ and B is in $\mathcal{M}_m(\mathbb{R})$. Let X be an $n \times m$ matrix of indeterminates. For $k \geqslant 0$, define the matrix polynomial $\Pi_k(A, B; X)$ by

$$\Pi_k(A, B; X) = \sum_{j=0}^{k} A^{k-j} X B^{j}. \tag{1.1}$$

Observe that $\Pi_k(A, B; X)$ is linear in X and that $\Pi_0(A, B; X) = X$.

One of the principal goals of this paper is to determine all solutions over the real numbers to

$$\Pi_k(A, B; X) \geqslant 0 \qquad \text{for all} \quad k \geqslant 0 \tag{1.2}$$

when A and B are *ZME*-matrices. These solutions are characterized in Section 5. The relations between these solutions and the structure of reducible *ZM*-matrices is developed in Section 7, and the main results are Theorems 7.2, 7.5, and 7.6.

2. THE GEOMETRY OF THE SOLUTION SET

LEMMA 2.1. *Let A be in $\mathcal{M}_n(\mathbb{R})$ and B be in $\mathcal{M}_m(\mathbb{R})$. Then the solution set to (1.1) is either the single matrix $X = 0$, or else a nonempty, closed, proper, convex cone in $\mathbb{R}^{nm}$.*

Proof. Since $\Pi_k(A, B; 0) = 0$ for all $k \geqslant 0$, $X = 0$ is always in the solution set. Since $\Pi_0(A, B; X) = X$, all solutions must lie in the closed first orthant of $\mathbb{R}^{nm}$. For each $k \geqslant 0$, the inequality $\Pi_k(A, B; X) \geqslant 0$ is actually a finite system of closed linear inequalities in nm indeterminates. Thus for each k, the solution set is a closed, convex, polyhedral cone. The solution set for (1.2) is the intersection of the solution sets for each k; hence it has the desired properties. ■

For each nonnegative integer k, and for each pair of complex numbers λ and ν, define $H_k(\lambda, \nu)$ by $H_0(\lambda, \nu) = 1$, and for $k > 0$,

$$H_k(\lambda, \nu) = \sum_{i=0}^{k} \lambda^{i} \nu^{k-i}. \tag{2.2}$$

LEMMA 2.3. *Let A be in $\mathcal{M}_n(\mathbb{R})$ and B be in $\mathcal{M}_m(\mathbb{R})$. Suppose that the matrix A has a column eigenvector e corresponding to an eigenvalue λ, and suppose that B has a row eigenvector f^t corresponding to an eigenvalue μ. Suppose that c is a real number. Then*

$$\Pi_k(A, B; cef^t) = H_k(\lambda, \mu) cef^t$$

for each $k \geqslant 0$.

Proof. For $k = 0$, the result is clear. For $k > 0$, choose i with $1 \leqslant i \leqslant k$. Then

$$A^i(cef^t)B^{k-i} = c(A^i e)(f^t B^{k-i}) = c(\lambda^i e)(\nu^{k-i} f^t) = \lambda^i \nu^{k-i} cef^t.$$

The result now follows by summing over i. ∎

3. DELTA BASES

In light of the preceding lemma, it appears to be desirable to express solutions to (1.2) in terms of eigenvectors for A and B. This will in fact be the case. Consequently, we now construct a special type of basis for $\mathbb{R}^{nm}$.

Let the set $\{v_1, v_2, \dots, v_n\}$ be a basis for $\mathbb{R}^n$. Let the set $\{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n\}$ also be a basis for $\mathbb{R}^n$. The sets $\{v_1, v_2, \dots, v_n\}$ and $\{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n\}$ are called a *delta basis pair* if $[v_i]^t[\tilde{v}_j] = \delta_{ij}$ (the Kronecker delta) for each i and j. A delta basis pair is called a *positive* delta basis pair if there is an index i such that both v_i and $\tilde{v}_i$ are strictly positive. By convention, the elements of each basis in a positive delta basis pair will be indexed so that the unique index i for the strictly positive vector in each of the bases is always $i = 1$.

LEMMA 3.1. *Let $\{u_1, u_2, \dots, u_n\}$ and $\{\bar{u}_1, \tilde{u}_2, \dots, \tilde{u}_n\}$ be a delta basis pair for $\mathbb{R}^n$. Let $\{v_1, v_2, \dots, v_m\}$ and $\{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_m\}$ be a delta basis pair for $\mathbb{R}^m$. Then*

$$\mathscr{B} = \left\{ u_i [v_j]^t : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \right\}$$

is a basis for $\mathbb{R}^{nm}$. Further, if both delta basis pairs are positive, then $u_1[v_1]^t$ is the unique strictly positive element of $\mathscr{B}$.

Proof. That $\mathscr{B}$ is a basis follows immediately from two facts. First, $u_i[v_j]^t$ is the matrix version of the tensor product of the vectors u_i and v_j. Second, the tensor product of two real vector spaces is a real vector space whose basis is the set of tensor products of the vectors in the bases of the original spaces.

If u_1 and $\tilde{u}_1$ are strictly positive vectors, then every element of $\{u_2,\dots,u_n\}$ must have both positive and negative entries. Similarly for $\{v_2,\dots,v_m\}$ if v_1 and $\tilde{v}_1$ are strictly positive vectors. Suppose that both delta basis pairs are positive. Clearly $u_1v_1^t$ is strictly positive. Assume that $u_iv_j^t$ is also strictly positive, where at least one of i and j is not equal to one. Since $u_iv_j^t$ is strictly positive, and since $\tilde{u}_1$ and $\tilde{v}_1$ are strictly positive, $\tilde{u}_1^t(u_iv_j^t)\tilde{v}_1 = \delta_{1i}\delta_{j1} = 0$ is strictly positive, a contradiction. ■

LEMMA 3.2. *Let $\{u_1, u_2,\dots,u_n\}$ and $\{\tilde{u}_1,\tilde{u}_2,\dots,\tilde{u}_n\}$ be a positive delta basis pair for $\mathbb{R}^n$. Let $\{v_1,v_2,\dots,v_m\}$ and $\{\tilde{v}_1,\tilde{v}_2,\dots,\tilde{v}_m\}$ be a positive delta basis pair for $\mathbb{R}^m$. Let $W=\sum_{i,j}c_{ij}u_iv_j^t$, where the c_{ij} are real. Suppose that $W \geqslant 0$. Then:*

(i) $c_{11} \geqslant 0$;
(ii) $c_{11}=0$ *implies* $c_{ij}=0$ *for all i and j.*

Thus $W \geqslant 0$ and $c_{11}=0$ together imply $W=0$.

Proof. Since $\tilde{u}_1$ and $\tilde{v}_1$ are strictly positive vectors, $W \geqslant 0$ implies $\tilde{u}_1^tW\tilde{v}_1 \geqslant 0$, with the equality if and only if $W=0$. Observe that

$$\tilde{u}_1^tW\tilde{v}_1 = \sum_{i,j} c_{ij}\left(\tilde{u}_1^tu_i\right)\left(v_j^t\tilde{v}_1\right) = c_{11}$$

by the orthogonality of the vectors in each of the delta basis pairs. Thus $W \geqslant 0$ implies $c \geqslant 0$. If $c_{11}=0$, then the computation above shows that $W=0$. Since the vectors $u_iv_j^t$ form a basis, $W=0$ if and only if $c_{ij}=0$ for every i and j. ■

LEMMA 3.3. *Let A be an $n\times n$ ZME-matrix with eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$. Then the eigenvalues of A can be labeled so that $\lambda_1<\lambda_2\leqslant \cdots \leqslant \lambda_n$. Further, there exists a positive delta basis pair for $\mathbb{R}^n$ which is composed of two bases: the basis $\{e_1,e_2,\dots,e_n\}$ consisting of transposes of row eigenvectors e_i^t for A such that e_i corresponds to λ_i for each i, and the basis $\{f_1,f_2,\dots,f_n\}$ consisting of column eigenvectors f_i for A such that f_i corresponds to λ_i for each i.*

Proof. By Lemma 3.1 of [2], the spectrum of A is real and the multiplicity of λ_1 is one, so it can be labeled so as to satisfy the given inequalities. By Corollary 6.25 of [2], the matrix A is diagonalizable. Thus $A = S^{-1}DS$ where D is the diagonal matrix with $D_{ii} = \lambda_i$ for each i. Let x_i be the ith standard column vector for each i. Then $(x_i^t S)A = (x_i^t S)S^{-1}DS = \lambda_i(x_i^t S)$ for each i. Similarly, $A(S^{-1}x_i) = \lambda_i(S^{-1}x_i)$ for each i. Since the x_i form a basis for $\mathbb{R}^n$, the $(x_i^t S)^t$ form a basis for $\mathbb{R}^n$, and so do the $S^{-1}x_i$. Note that $(x_i^t S)(S^{-1}x_j) = x_i^t x_j = \delta_{ij}$ for all i and j. For each i, let $e_i = (x_i^t S)^t$ and let $f_i = S^{-1}x_i$. Then the sets $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ are a delta basis pair for $\mathbb{R}^n$. Since λ_1 has multiplicity one, the corresponding row and column eigenvectors are unique up to scalar multiples. By Theorem 4.16 of [1], there are scalar multiples of the eigenvectors corresponding to λ_1 which are strictly positive. Thus there exist scalars r and s such that re_1 and sf_1 are strictly positive vectors. Then $(re_1)^t f_j = e_j^t(sf_1) = 0$ for $j \neq 1$, and $(re_1)(sf_1)^t = rs$. Replace e_1 by $r(|rs|)^{-1/2}e_1$, and replace f_1 by $s(|rs|)^{-1/2}f_1$. The resultant sets are a positive delta basis pair for $\mathbb{R}^n$. (Consequently, it can be assumed that $r = s = 1$.) ∎

Convention 3.4. For the duration of this paper, A will be an $n \times n$ ZME-matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$, and such that $\{e_1, e_2, \dots, e_n\}$ and $\{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n\}$ will be a positive delta basis pair for $\mathbb{R}^n$ where for each i, $\tilde{e}_i^t$ is a row eigenvector for A corresponding to λ_i, and e_i is a column eigenvector for A corresponding to λ_i. Similarly, B will be an $m \times m$ ZME-matrix with eigenvalues $\mu_1, \mu_2, \dots, \mu_m$ such that $\mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_m$, and such that $\{f_1, f_2, \dots, f_n\}$ and $\{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n\}$ will be a positive delta basis pair for $\mathbb{R}^m$ where for each i, $\tilde{f}_i$ is a column eigenvector for B corresponding to μ_i, and f_i^t is a row eigenvector for B corresponding to μ_i. Finally, if $n = 1$, then set $\lambda_2 = +\infty$, and if $m = 1$, then set $\mu_2 = +\infty$.

4. PROPERTIES OF $H_k(\lambda, \mu)$

Lemma 4.1. *Let k be a nonnegative integer. Let x and y be complex numbers. Then*

$$H_k(x,y) = \begin{cases} x^k(k+1) & \textit{if} \quad x = y, \\ (x-y)^{-1}(x^{k+1} - y^{k+1}) & \textit{if} \quad x \neq y, \end{cases} \tag{4.2}$$

$$H_k(x,y) = H_k(y,x) = \begin{cases} x^k H_k(1, y/x) & \textit{if} \quad x \neq 0, \\ y^k H_k(1, x/y) & \textit{if} \quad y \neq 0. \end{cases} \tag{4.3}$$

Proof. If $x = y$, then (4.2) is clear. If $x \neq y$, then without loss of generality, $y \neq 0$. Then $H_k(x, y) = y^k \sum_{j=0}^{k} (x/y)^j$. Then (4.2) follows from the standard results for geometric series. Note also, that $y^k \sum_{j=0}^{k} (x/y)^j = y^k H_k(x/y, 1)$. Finally, by the symmetry of $H_k(x, y)$, (4.3) holds. ∎

LEMMA 4.4. *Let x and y be complex numbers with $|x| < |y|$. Then for each integer k,*

$$|H_k(x, y)| \leqslant H_k(|x|, |y|) \leqslant \frac{|y|^{k+1}}{|y| - |x|}.$$

Proof. By the generalized triangle inequality,

$$|H_k(x, y)| = \left| \sum_{j=0}^{k} x^j y^{k-j} \right| \leqslant \sum_{j=0}^{k} |x|^j |y|^{k-j}.$$

The last summation is exactly $H_k(|x|, |y|)$. Since $|y| > |x| \geqslant 0$, (4.2) implies

$$H_k(|x|, |y|) = (|y| - |x|)^{-1} (|y|^{k+1} - |x|^{k+1}).$$

Since $|y| > |x|$, and since $k \geqslant 0$,

$$|y|^{k+1} \geqslant \left[|y|^{k+1} - |x|^{k+1} \right] > 0.$$

Thus the final inequality holds. ∎

LEMMA 4.5. *Let x and y be complex numbers. For each nonnegative integer k,*

$$0 \leqslant \max\{|x|^k, |y|^k\} \leqslant H_k(|x|, |y|) \leqslant (k+1) \max\{|x|^k, |y|^k\}.$$

Further, if $k > 0$, then $H_k(|x|, |y|)$ is strictly monotonic increasing with each of $|x|$ and $|y|$.

Proof. If $k = 0$, then $|x|^0 = |y|^0 = 1$ and $H_k(|x|, |y|) = 1$, so the inequalities hold. Suppose that $k > 0$. It suffices to check the inequalities for when x and y are nonnegative, real numbers. Since $H_k(x, y) = x^k + y^k + \sum_{j=1}^{k-1} x^j y^{k-j}$, the inequalities are clear. Since x and y are nonnegative, the monotonicity in x and y is also clear. ■

LEMMA 4.6. *Let x and y be real numbers. Then $H_k(x, y) \geqslant 0$ for all nonnegative integers k if and only if $x + y \geqslant 0$. Further, $H_k(x, y) > 0$ for all nonnegative integers k if and only if $x + y > 0$.*

Proof. Since $H_1(x, y) = x + y$, one direction is clear for both the strict and the weak inequality cases.

Suppose, conversely, that $x + y \geqslant 0$. If both x and y are nonnegative, then by Lemma 4.5, $H_k(x, y) \geqslant 0$. So suppose that one of x and y is negative: without loss of generality, $y < 0$. Since $x + y \geqslant 0$, $x \geqslant |y| > 0$. For each nonnegative k, $x^k \geqslant |y|^k$. Then $x^{k+1} - y^{k+1} \geqslant x^{k+1} - |y|^{k+1} \geqslant 0$. Since $x - y = x + |y| > 0$, Equation (4.2) implies $H_k(x, y) \geqslant 0$.

The proof of the strict inequality mirrors that of the weak inequality. Note that $x + y > 0$ implies either that $x = y > 0$, or else that one of $x > |y|$ and $y > |x|$ holds. If $x = y$, then use (4.2). If not, assume without loss of generality that $x > |y|$. For each nonnegative k, $x^k > |y|^k$. Then $x^{k+1} - y^{k+1} \geqslant x^{k+1} - |y|^{k+1} > 0$. By (4.2), $H_k(x, y) > 0$. ■

5. SOLUTIONS TO (1.2)

LEMMA 5.1. *Let A and B be ZME-matrices satisfying Convention 3.4. Suppose that $X = \sum_{i,j} c_{ij} e_i f_j^t$ is a solution to (1.2) for all $k \geqslant 0$. Then*

(i) $c_{11} \geqslant 0$,
(ii) $c_{11} = 0$ *if and only if* $X = 0$,
(iii) $c_{11} > 0$ *implies* $\lambda_1 + \mu_1 \geqslant 0$,
(iv) $c_{ij} = 0$ *whenever* $\max\{\lambda_i, \mu_j\} > \max\{|\lambda_1|, |\mu_1|\}$.

Proof. Since X solves (1.2) for each nonnegative integer k, the $k = 0$ case implies $X \geqslant 0$. Thus Lemma 3.2 applies. If $c_{11} = 0$, the results are clear. So suppose that $c_{11} > 0$. For convenience, let $\theta = \max\{|\lambda_1|, |\mu_1|\} \geqslant 0$.

Let τ be a real number with $\tau > \theta$. For each nonnegative integer k, define $W_k = \tau^{-k}\Pi_k(A, B; X)$. Then for $k \geqslant 0$, $W_k \geqslant 0$. Pre- and postmultiplying W_k by the strictly positive vectors $\tilde{e}_1$ and $\tilde{f}_1$,

$$0 \leqslant \tilde{e}_1^t W_k \tilde{f}_1 = \tau^{-k} H_k(\lambda_1, \mu_1) c_{11}$$

for each $k \geqslant 0$. Hence $\tau^{-k} H_k(\lambda_1, \mu_1) \geqslant 0$. Since τ is positive, it follows that

$H_k(\lambda_1, \mu_1) \geqslant 0$ for all $k \geqslant 0$. By Lemma 4.6, $\lambda_1 + \mu_1 \geqslant 0$. By Lemmas 4.4 and 4.5,

$$\left|\tau^{-k} H_k(\lambda_1, \mu_1)\right| \leqslant \tau^{-k}(k+1)\theta^k.$$

Since $0 \leqslant \theta < \tau$,

$$\lim_{k \to \infty} \tau^{-k}(k+1)\theta^k = 0.$$

Observe that the quantity within the limit is an upper bound on the coefficient of $e_1[f_1]^t$ in W_k. Since W_k is nonnegative for all k, the coefficient of $e_1[f_1]^t$ must be nonnegative, and hence converges to zero. That is, $\tilde{e}_1^t W_k \tilde{f}_1$ converges to zero. Note $\tilde{e}_1^t W_k \tilde{f}_1$ is just the positively weighted sum of the entries of W_k. Hence if W_k does not converge to the zero matrix, then there is a subsequence of the nonnegative W_k matrices for which $\tilde{e}_1^t W_k \tilde{f}_1$ is bounded away from zero, a contradiction. So W_k converges to the zero matrix. By Lemma 3.2, this implies each of the coefficients of W_k must converge to zero:

$$\lim_{k \to \infty} \tau^{-k} H_k(\lambda_i, \mu_j) c_{ij} = 0 \tag{5.2}$$

for every i and j.

Choose the number τ so that $\tau > \theta$, and so that τ satisfies the following two conditions:

(1) If $\lambda_n > \theta$, then $\theta < \tau < \min\{\lambda_i : \lambda_i > \theta\}$, and
(2) If $\mu_m > \theta$, then $\theta < \tau < \min\{\mu_i : \mu_i > \theta\}$.

Suppose that i and j are such that $\max\{\lambda_i, \mu_j\} > \theta$. Without loss of generality, suppose that $\lambda_i = \max\{\lambda_i, \mu_j\} > \theta$. There are two cases: (i) $\mu_j \geqslant 0$, and (ii) $\mu_j < 0$. In the first case, $|\tau^{-k} H_k(\lambda_i, \mu_j)| \geqslant \tau^{-k}\lambda_i^k$ by Lemma 4.5. Note that $\lambda_i/\tau > 1$. Thus the absolute value diverges to infinity. Then (5.2) implies $c_{ij} = 0$. In the second case, $\mu_j = \mu_1$, since B is a *ZME*-matrix, and by Lemma 3.1 of [2], a *ZME*-matrix has at most one negative eigenvalue. Thus $|\mu_j/\lambda_i| < 1$. Then

$$\begin{aligned} \tau^{-k} H_k(\lambda_i, \mu_j) &= \tau^{-k}\lambda_i^k H_k(1, \mu_j/\lambda_1) \\ &= \tau^{-k}\lambda_i^k\left(1 - \frac{\mu_j}{\lambda_i}\right)^{-1}\left[1 - \left(\frac{\mu_j}{\lambda_1}\right)^{k+1}\right] \end{aligned}$$

by Lemma 4.1. Further, this last expression is positive for all k, and as k goes to infinity, the expression behaves as $\tau^{-k}\lambda_i^k[1-(\mu_j/\lambda_i)]^{-1}$, which diverges to infinity. Thus (5.2) implies $c_{ij}=0$. ■

COROLLARY 5.3. *Let A and B be ZME-matrices satisfying Convention 3.4. If $\lambda_1+\mu_1<0$, then $X=0$ is the unique matrix which satisfies* (1.2) *for every nonnegative integer k.*

Proof. This is immediate from the preceding lemma. ■

THEOREM 5.4. *Let A and B be ZME-matrices satisfying convention* 3.4. *Suppose that $\lambda_1+\mu_1\geqslant 0$. If*

$$\min\{\lambda_2,\mu_2\}\geqslant\max\{|\lambda_1|,|\mu_1|\},$$

then the complete set of solutions satisfying (1.2) *for all nonnegative integers k is $\{X=c_{11}e_1f_1^t: c_{11}\geqslant 0\}$.*

Proof. First, observe from Lemma 2.3 that if $X=c_{11}e_1f_1^t$, then $\Pi_k(A,B;X)=H_k(\lambda_1,\mu_1)c_{11}e_1f_1^t$. Since e_1 and f_1 are strictly positive, X satisfies (1.2) for all $k\geqslant 0$ precisely when $H_k(\lambda_1,\mu_1)c_{11}\geqslant 0$ for all $k\geqslant 0$. For $k=0$, this reduces to $1\cdot c_{11}\geqslant 0$, so $c_{11}\geqslant 0$. Since $\lambda_1+\mu_1\geqslant 0$, $H_k(\lambda_1,\mu_1)\geqslant 0$ by Lemma 4.6. Thus the set of solutions given in the statement of the theorem must be contained in the complete set of solutions.

Suppose that $\min\{\lambda_2,\mu_2\}>\max\{|\lambda_1|,|\mu_1|\}$. If $i\geqslant 2$, then $\lambda_i\geqslant\lambda_2$. Similarly, if $j\geqslant 2$, then $\mu_j\geqslant\mu_2$. Thus $\max\{\lambda_i,\mu_j\}\geqslant\min\{\lambda_2,\mu_2\}>\max\{|\lambda_1|,|\mu_1|\}$ when either $i\geqslant 2$ or $j\geqslant 2$. Let $X=\sum_{i,j}c_{ij}e_if_j^t$ be a solution to (1.2) for all $k\geqslant 0$. Then by Lemma 5.1, $c_{ij}=0$ whenever $i\geqslant 2$ or $j\geqslant 2$. Thus $X=c_{11}e_1f_1^t$.

Suppose $\min\{\lambda_2,\mu_2\}=\max\{|\lambda_1|,|\mu_1|\}$. Without loss of generality, $\lambda_2\geqslant\mu_2$. There are two cases: (i) $\lambda_2=\mu_2$, and (ii) $\lambda_2>\mu_2$. Since $\lambda_1+\mu_1\geqslant 0$, one of $\lambda_1\geqslant|\mu_1|$ and $\mu_1\geqslant|\lambda_1|$ holds. In the first case, this implies either $\lambda_2=\lambda_1$ or $\mu_2=\mu_1$, both of which contradict the fact that A and B are ZME-matrices. Thus case (i) cannot occur.

Suppose that $\lambda_2>\mu_2$. Then $\mu_2=\max\{|\lambda_1|,|\mu_1|\}$, $\mu_2>\mu_1$, and $\lambda_1+\mu_1\geqslant 0$ together imply $\mu_2=\lambda_1\geqslant|\mu_1|$. There are two possibilities to consider: first, that there is an r such that $\mu_{r+1}>\mu_r=\mu_2$; and second, that $\mu_m=\mu_2$ (in which case set $r=m$). In the both cases, $\max\{\lambda_i,\mu_j\}\leqslant\max\{|\lambda_1|,|\mu_1|\}$ implies $i=1$ and $1\leqslant j\leqslant r$. By Lemma 5.1, if X is a solution for (1.2) for all

$k \geqslant 0$, then

$$X = \sum_{j=1}^{r} c_{1j} e_1 f_j^t.$$

Since $\mu_2 = \lambda_1$, it follows that for $2 \leqslant j \leqslant r$,

$$H_k(\lambda_1, \mu_j) = H_k(\lambda_1, \lambda_1) = (k+1)\lambda_1^k.$$

Now compute $H_k(\lambda_1, \mu_1)$. Suppose $\lambda_1 > |\mu_1|$; then by Lemma 4.4, $|H_k(\lambda_1, \mu_1)| \leqslant \lambda_1^k(\lambda_1 - |\mu_1|)^{-1}$. Thus

$$\lim_{k \to \infty} \lambda_2^{-k}(k+1)^{-1} H_k(\lambda_1, \mu_1) = 0. \tag{5.5}$$

Suppose $\lambda_1 = |\mu_1|$. Then either $\lambda_1 = \mu_1$ or else $\lambda_1 = -\mu_1$. In the first case, $H_k(\lambda_1, \mu_1) = (k+1)\lambda_1^k$, and in the second case, $H_k(\lambda_1, \mu_1) = \frac{1}{2}\lambda_1^k[1 + (-1)^k]$. In both of these cases, the limit in (5.5) is again zero.

Let $\hat{W} = \sum_{j=2}^{r} c_{1j} e_1 f_j^t$. For each $k \geqslant 0$, let $W_k = \lambda_2^{-k}(k+1)^{-1}\Pi_k(A, B; X)$. Note that by the preceding statements and Lemma 2.3,

$$W_k = \lambda_2^{-k}(k+1)^{-1} H_k(\lambda_1, \mu_1) c_{11} e_1 f_1^t + \hat{W}.$$

Since X is a solution to (1.2) for each $k \geqslant 0$, $W_k \geqslant 0$ for each $k \geqslant 0$. Since $\lim_{k \to \infty} W_k = \hat{W}$, it follows that $\hat{W} \geqslant 0$. Since the coefficient of $e_1 f_1^t$ in $\hat{W}$ is zero, it follows by Lemma 3.2 that $\hat{W} = 0$ and that $c_{1j} = 0$ for $2 \leqslant j \leqslant r$. Thus $X = c_{11} e_1 f_1^t$. ■

THEOREM 5.6. *Let A and B be ZME-matrices satisfying Convention* 3.4. *Suppose that* $\lambda_1 + \mu_1 \geqslant 0$. *If*

$$\min\{\lambda_2, \mu_2\} < \max\{|\lambda_1|, |\mu_1|\},$$

then the complete set of solutions satisfying (1.2) *for all nonnegative integers k properly contains the set* $\{X = c_{11} e_1 f_1^t : c_{11} \geqslant 0\}$. *Further, every other solution is of the form* $c_{11} e_1 f_1^t + Y$ *with* $c_{11} > 0$, *where Y is in the span of* $\{e_1 f_j^t : j \neq 1 \text{ and } \mu_j < \lambda_1\}$ *when* $|\lambda_1| = \max\{|\lambda_1|, |\mu_1|\}$, *and where Y is in the span of* $\{e_i f_1^t : i \neq 1 \text{ and } \lambda_i < \mu_1\}$ *when* $|\mu_1| = \max\{|\lambda_1|, |\mu_1|\}$.

Proof. As argued in the proof of the preceding theorem, $\{X = c_{11}e_1f_1^t : c_{11} \geqslant 0\}$ is contained in the complete set of solutions to (1.2) for all $k \geqslant 0$.

Suppose that X is a solution to (1.2) for all $k \geqslant 0$, but that $X \neq c_{11}e_1f_1^t$ for $c_{11} \geqslant 0$. From Lemma 5.1, $X = \Sigma_{i,j}c_{ij}e_if_j^t$ such that $c_{11} > 0$, and such that $c_{ij} = 0$ when $\max\{\lambda_i, \mu_j\} > \max\{|\lambda_1|, |\mu_1|\}$. Thus $X = c_{11}e_1f_1^t + Y$, where Y is in the span of $\{e_if_j^t : (i,j) \neq (1,1)$ and $\max\{\lambda_i, \mu_j\} \leqslant \max\{|\lambda_1|, |\mu_1|\}\}$. Since $\lambda_1 + \mu_1 \geqslant 0$, it may be supposed without loss of generality that $\max\{|\lambda_1|, |\mu_1|\} = |\lambda_1| = \lambda_1$. Since $\lambda_i > \lambda_1$ for $i \geqslant 2$, Lemma 5.1 implies Y is in the span of $\{e_1f_j^t : j \neq 1$ and $\mu_j \leqslant \lambda_1\}$.

Suppose that $\mu_1 < \cdots < \mu_r = \cdots = \mu_s = \lambda_1$, and that either $\lambda_1 < \mu_{s+1}$, or else B is $s \times s$. Then for $1 \leqslant j < r$,

$$\begin{aligned}[][H_K(\lambda_1, \mu_r)]^{-1}[H_k(\lambda_1, \mu_j)] &= (k+1)^{-1}H_k(1, \mu_j/\mu_r)\\ &= (k+1)^{-1}\left(1 - \frac{\mu_j}{\mu_r}\right)^{-1}\left[1 - \left(\frac{\mu_j}{\mu_r}\right)^{k+1}\right],\end{aligned}$$

which goes to zero as k goes to infinity. Let $W = \Sigma_{j=r}^{s}c_{1j}e_1f_j^t$, and let $W_k = [H_K(\lambda_1, \mu_r)]^{-1}\Pi_k(A, B; X)$ for each $k \geqslant 0$. Thus $W_k \geqslant 0$ for each k. Now apply the argument from the final paragraph of the proof of the preceding theorem to show that $W = 0$, hence $c_{ij} = 0$ for $r \leqslant j \leqslant s$. Thus Y must lie in the span of $\{e_1f_j^t : j \neq 1$ and $\mu_j < \lambda_1\}$.

It remains to show that there are solutions with $Y \neq 0$. Note that $\lambda_1 > 0$, and that $\lambda_1 > \mu_2 \geqslant |\mu_1|$. Thus by Lemma 4.6, $H_k(\lambda_1, \mu_1) > 0$ for $k \geqslant 0$. The following construction produces solutions of the form $X = c_{11}e_1f_1^t + c_{12}e_1f_2^t$, that is, solutions with $Y = c_{12}e_1f_2^t$. For this X, the inequalities (1.2) for $k \geqslant 0$ are equivalent to

$$H_k(\lambda_1, \mu_1)c_{11}e_1f_1^t + H_k(\lambda_1, \mu_2)c_{12}e_1f_2^t \geqslant 0 \qquad \text{for} \quad k \geqslant 0.$$

Since e_1 and f_1 are strictly positive, this system can be rewritten as

$$c_{11} + \frac{H_k(\lambda_1, \mu_2)}{H_k(\lambda_1, \mu_1)}c_{12}\frac{(f_2)_i}{(f_1)_i} \geqslant 0 \qquad \text{for} \quad k \geqslant 0 \text{ and } 1 \leqslant i \leqslant m, \tag{5.7}$$

where B is $m \times m$. Note that the only possibly nonpositive terms are c_{12} and $(f_2)_i$. Thus the values of i such that $c_{12}\cdot(f_2)_i < 0$ determine the nontrivial cases. Also,

$$0 < \frac{H_k(\lambda_1, \mu_2)}{H_k(\lambda_1, \mu_1)} = \frac{[\lambda_1 - \mu_1]\left[1 - (\mu_2/\lambda_1)^{k+1}\right]}{[\lambda_1 - \mu_2]\left[1 - (\mu_1/\lambda_2)^{k+1}\right]}$$

$$\leqslant \frac{\lambda_1 - \mu_1}{\lambda_1 - \mu_2}\left[1 - \left(\frac{\mu_1}{\lambda_1}\right)^{k+1}\right]^{-1}.$$

The maximum value of the final expression depends on the sign of μ_1. Define M to be the positive number given by

$$M = \frac{\lambda_1 - \mu_1}{\lambda_1 - \mu_2} \cdot \begin{cases} \left[1 - (\mu_1/\lambda_1)^2\right]^{-1} & \text{if} \quad \mu_1 < 0, \\ \left[1 - (\mu_1/\lambda_1)\right]^{-1} & \text{if} \quad \mu_1 \geqslant 0. \end{cases} \tag{5.8}$$

Then

$$0 < \frac{H_k(\lambda_1, \mu_2)}{H_k(\lambda_1, \mu_1)} \leqslant M \qquad \text{for} \quad k \geqslant 0.$$

Thus the infinite system (5.7) can be replaced with a possibly stronger, *finite* system:

$$c_{11} + Mc_{12}\frac{(f_2)_i}{(f_1)_i} \geqslant 0 \qquad \text{for} \quad 1 \leqslant i \leqslant m. \tag{5.9}$$

Again, this holds trivially unless

$$c_{12}(f_2)_i = -|c_{12}||(f_2)_i| < 0.$$

Let $F = \max\{|(f_2)_i|/(f_1)_i : 1 \leqslant i \leqslant m\}$. Then (5.9) can be replaced with a single, possibly slightly stronger inequality: $c_{11} \geqslant MF|c_{12}|$. Whenever c_{11} and c_{12} satisfy this last inequality, $X = c_{11}e_1 f_1^t + c_{12}e_1 f_2^t$ is a solution to (1.2) for all $k \geqslant 0$. Thus there are an infinite number of solutions which are not of the form $c_{11}e_1 f_1^t$. ■

SUMMARY 5.10. *Let A and B be ZME-matrices which satisfy convention 3.4. Let $\mathscr{P}$ be the set of solutions to* (1.2) *for all $k \geqslant 0$. Then:*

(i) *If $\lambda_1 + \mu_1 < 0$, then $\mathscr{P} = \{0\}$.*
(ii) *If $\lambda_1 + \mu_1 \geqslant 0$, and if $\min\{\lambda_2, \mu_2\} \geqslant \max\{|\lambda_1|, |\mu_2|\}$, then $\mathscr{P} = \{ce_1 f_1^t : c \geqslant 0\}$.*
(iii) *If $\lambda_1 + \mu_1 \geqslant 0$, and if $\min\{\lambda_2, \mu_2\} < \max\{|\lambda_1|, |\mu_1|\}$, then $\mathscr{P} \supsetneqq \{ce_1 f_1^t : c \geqslant 0\}$, and the remaining solutions depend not only on the values of the λ_i and μ_j, but also on the entries of the vectors e_i and f_j.*

6. AN EXAMPLE

The following is an example of the third case of the preceding summary, that is, of solutions which are not multiples of $e_1 f_1^t$.

Let $\mathscr{E}$ be the pair of orthogonal, idempotent matrices $\{E_1, E_2\}$ where

$$E_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let $A = 2E_1 + 4E_2$ and let $B = 6E_1 + 8E_2$. Let $e_1 = \tilde{e}_1 = (\sqrt{2})^{-1}(1,1)^t$ and let $e_2 = \tilde{e}_2 = (\sqrt{2})^{-1}(1, -1)^t$. Then A and B are MMA-matrices which satisfy Convention 3.4 such that A has spectrum $\{2,4\}$ with a corresponding positive delta basis pair $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$, and such that B has spectrum $\{6,8\}$ with a corresponding positive delta basis pair $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$. Since $\lambda_1 + \mu_1 = 2 + 6 \geqslant 0$, and since $4 = \min\{\lambda_2, \mu_2\} < \max\{|\lambda_1|, |\mu_2|\} = 6$, it follows that every solution of (1.2) for all $k \geqslant 0$ must be of the form $X = \alpha e_1 e_1^t + \beta e_2 e_1^t$ where $\alpha \geqslant 0$. In fact, (1.2) is equivalent to

$$6^k\left[\tfrac{3}{2}\left[1 - \left(\tfrac{1}{3}\right)^{k+1}\right]\alpha e_1 + 3\left[1 - \left(\tfrac{2}{3}\right)^{k+1}\right]\beta e_2\right] e_1^t \geqslant 0.$$

It is easily checked that this holds for all $k \geqslant 0$ if and only if at least one of the following systems holds:

(i) $\beta \leqslant 0$ and $\alpha + 3\beta \geqslant 0$,
(ii) $\beta \geqslant 0$ and $\alpha - 3\beta \geqslant 0$.

That is, X solves (1.2) for all $k \geqslant 0$ if and only if $\alpha \geqslant 3|\beta|$. Further, by Theorem 5.6, these are the only solutions. Finaly, note that $\alpha \geqslant 3|\beta|$ is precisely $c_{11} \geqslant MF|c_{21}|$, where M and F are as in the proof of Theorem 5.6.

7. REDUCIBLE ZM-MATRICES

A matrix N with index of reducibility s has a Frobenius normal form with block order s. That is, N is permutation similar to a matrix $\hat{N}$ which is block upper triangular with s diagonal blocks, each of which is a square, irreducible matrix. Explicitly, $\hat{N}$ can be written in partitioned form

$$\hat{N} = \begin{bmatrix} A_1 & * & * & \cdots & * \\ & A_2 & * & \cdots & * \\ & & A_3 & & * \\ & 0 & & \ddots & \vdots \\ & & & & A_s \end{bmatrix}, \tag{7.1}$$

where the A_i are square, irreducible matrices. For convenience, denote the i, j block of a matrix with this partitioning by a subscript $\langle i, j\rangle$.

THEOREM 7.2. *Let N be a matrix with Frobenius normal form $\hat{N}$ given by* (7.1). *Then the matrix N is a ZM-matrix (MM-matrix) if and only if $\hat{N}$ is a ZM-matrix (MM-matrix). Further, $\hat{N}$, and hence N, are ZM-matrices (MM-matrices) if and only if both of condition (ii) and the following condition hold*:

(i) *for* $1 \leqslant i \leqslant s$, A_i *is a ZME-matrix (MMA-matrix)*;
(ii) *for* $1 \leqslant i < j \leqslant s$, *and for all* $k \geqslant 1$, $[\hat{N}^k]_{\langle i, j\rangle} \leqslant 0$.

Finally, for $j = i + 1$, *condition* (ii) *becomes*

$$\Pi_k\left(A_i, A_{i+1}; -\hat{N}_{\langle i, i+1\rangle}\right) \geqslant 0 \qquad \text{for all} \quad k \geqslant 0.$$

Proof. Since permutation similarity transformations act as permutations on the set of diagonal entries of a matrix, permutation similarity sends Z-matrices to Z-matrices. Since permutation similarity preserves the spectrum of a matrix, and since it commutes with the raising of a matrix to an integer power, it follows that N is a ZM-matrix (MM-matrix) if and only if $\hat{N}$ is a ZM-matrix (MM-matrix).

Since $\hat{N}$ given by (7.1) is a block upper triangular, it follows for $k \geqslant 1$ that $\hat{N}^k$ is block upper triangular with diagonal blocks A_i^k. It is clear that $\hat{N}$ is a ZM-matrix if and only if both of condition (*ii*) and the following condition hold:

(i′) Each matrix A_i is a ZM-matrix.

Since the A_i are irreducible, it follows from Theorem 7.7 of [2] that (i′) is equivalent to the A_i being *ZME*-matrices. Thus (i) and (i′) are equivalent. Note that if each A_i is an *MMA*-matrix, then $\hat{N}^k$ is clearly an *M*-matrix, and thus $\hat{N}$ is an *MM*-matrix. Conversely, if $\hat{N}$ is an *M*-matrix, then so is each A_i. By Theorem 7.7 of [2], it follows that each A_i is an *MMA*-matrix.

Since $\hat{N}$ is block upper triangular, an easy induction shows that $[\hat{N}^k]_{\langle i,i+1\rangle} = -\Pi_{k-1}(A_i, A_{i+1}; \hat{N}_{\langle i,i+1\rangle})$ for $k \geqslant 1$. ∎

Corollary 7.3 (Index of reducibility two). *Let N be a matrix with Frobenius normal form*

$$\hat{N} = \left[\begin{array}{c|c} A & -X \\ \hline 0 & B \end{array}\right] \tag{7.4}$$

where A and B are irreducible. Then

$$\hat{N}^k = \left[\begin{array}{c|c} A^k & -\Pi_{k-1}(A,B;X) \\ \hline 0 & B^k \end{array}\right] \quad \textit{for} \quad k \geqslant 1.$$

Further, $\hat{N}$, and hence N, are ZM-matrices (MM-matrices) if and only if both of the following hold:

(i) *A and B are ZME-matrices (MMA-matrices);*
(ii) *$\Pi_k(A,B;X) \geqslant 0$ for all $k \geqslant 0$.*

Proof. In view of the preceding theorem, it suffices to note that (i) and (ii) are clearly sufficient conditions for $\hat{N}$ to be a *ZM*-matrix. ∎

Thus the study of block-order-two *ZM*-matrices is equivalent to the study of the solutions of $\Pi_k(A,B;X) \geqslant 0$ for all $k \geqslant 0$ where A and B are *ZME*-matrices. Since this was addressed in the preceding sections, the following results are immediate.

Theorem 7.5. *Let N be a matrix with Frobenius normal form given by (7.4). Suppose A and B are ZME-matrices.*

(i) *If $\lambda_1 + \mu_1 < 0$, then N is a ZM-matrix if and only if $X = 0$. That is, N is a ZM-matrix if and only if N is permutation similar to $A \oplus B$.*

(ii) *If* $\min\{\lambda_2, \mu_2\} \geqslant \max\{|\lambda_1|, |\mu_1|\}$ *and if* $\lambda_1 + \mu_1 \geqslant 0$, *then* N *is a ZM-matrix if and only if* $X = ce_1 f_1^t$ *where* $c \geqslant 0$.

(iii) *If* $\min\{\lambda_2, \mu_2\} < \max\{|\lambda_1|, |\mu_1|\}$ *and if* $\lambda_1 + \mu_1 \geqslant 0$, *then* N *is a ZM-matrix only if* $X = ce_1 f_1^t + Y$ *where* $c \geqslant 0$ *and where* Y *lies in the span of*

$$\left\{ e_i f_j^t : (i,j) \neq (1,1) \quad \textit{and} \quad \min\left\{ \lambda_i, \mu_j \right\} < \max\left\{ \lambda_i, \mu_j \right\} = \max\{ |\lambda_1|, |\mu_1| \} \right\}.$$

There always exist nonzero matrices Y *such that* N *is a ZM-matrix.*

THEOREM 7.6. *Let* N *be a matrix with Frobenius normal form given by* (7.4). *Suppose* A *and* B *are MMA-matrices.*

(i) *If* $\min\{\lambda_2, \mu_2\} \geqslant \max\{\lambda_1, \mu_1\}$, *then* N *is an MM-matrix if and only if* $X = ce_1 f_1^t$ *where* $c \geqslant 0$.

(ii) *If* $\min\{\lambda_2, \mu_2\} \leqslant \max\{\lambda_1, \mu_1\}$, *then* N *is an MM-matrix only if* $X = ce_1 f_1^t + Y$ *where* $c \geqslant 0$ *and where* Y *lies in the span of*

$$\left\{ e_i f_j^t : (i,j) \neq (1,1) \quad \textit{and} \quad \min\left\{ \lambda_i, \mu_j \right\} < \max\left\{ \lambda_i, \mu_j \right\} = \max\{ \lambda_1, \mu_1 \} \right\}.$$

There always exist nonzero matrices Y *such that* N *is a MM-matrix.*

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