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Elastic spaces may snap under perfect maps

P.M. Gartside [a,*], P.J. Moody [b]

[a] St. Edmund Hall, Oxford University, Oxford, OX1 4AR, UK
[b] Trinity College, Oxford University, Oxford, OX1 3BH, UK

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Abstract

The perfect image of an elastic space need not be elastic. Other relevant examples are presented.

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1. Introduction

Elastic spaces were introduced by Tamano and Vaughan in [16] as a natural generalization of stratifiable spaces. It turns out that they share many properties: for example, every elastic space is paracompact [16] and monotonically normal [2]. Pope proved in [15] that every regular first countable space in which all but countably many points are isolated, is elastic. Hence, the well-known Michael line [13] is elastic. This provided the first example of an elastic space which is fundamentally different from a stratifiable or linearly stratifiable space [17]. Much more recently the authors have shown [8] that every proto-metrisable space is elastic. The class of proto-metrisable spaces is rather wide, containing, for example, the Michael line, and is essentially disjoint from the class of stratifiable spaces in the sense that every stratifiable proto-metrisable space is metrisable.

It is well known that the closed continuous image of a (linearly) stratifiable space is again (linearly) stratifiable, and that another class of spaces, very closely related to elastic spaces, the well ordered (F) spaces [7], are also invariant under the action of closed continuous maps. Thus it was hardly surprising that Tamano and Vaughan conjectured, in

* Corresponding author. E-mail: gartside@maths.ox.ac.uk.

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their original paper, that the class of elastic spaces was also closed under closed continuous maps. However, in this paper we will develop techniques for constructing spaces which fail—but only barely—to be elastic. One application of this construction demonstrates that the perfect image of an elastic space need not be elastic. Other examples answer a question of Borges.

In the next section, notation is fixed and the definitions of elastic and proto-metrisable spaces given. Section 3 introduces the two methods (the scattering process, and duplication) used in this paper for constructing spaces, and gives sufficient conditions for elasticity to be preserved by these processes. Then in Section 4 a "machine" is developed for destroying elasticity. Finally, the examples are presented in Section 5.

2. Elastic and proto-metrisable spaces

Suppose that X is a set and $P = \langle A, B\rangle$ is a pair of subsets of X. We shall denote A, the first element of the pair P, by P_1, and B by P_2. If $\mathbb{P}$ is a collection of pairs, then for $i = 1, 2$, $\mathbb{P}_i$ will denote $\{P_i\colon\ P \in \mathbb{P}\}$.

Notation concerning maps and pairs will be abused in the following way. If X, Y and Z are sets, and $f: X \to Y$ and $g: Y \to Z$ are functions, then if P is a pair of subsets of Y and $\mathbb{P}$ is a collection of pairs of subsets, of Y, then

$$f^{-1}(P) = \langle f^{-1}(P_1), f^{-1}(P_2)\rangle, \qquad g(P) = \langle g(P_1), g(P_2)\rangle,$$
$$f^{-1}(\mathbb{P}) = \{f^{-1}(P)\colon\ P \in \mathbb{P}\}, \qquad g(\mathbb{P}) = \{g(P)\colon\ P \in \mathbb{P}\}.$$

Suppose now that $\mathbb{P}$ is a collection of pairs of subsets of the set X, and $\sim$ is a relation on $\mathbb{P}$. The relation $\sim$ is said to be *framing* provided that P, P' are related by $\sim$ whenever $P_1 \cap P_1' \neq \emptyset$. We have to consider the situation where a topology is refined by adding a number of isolated points. For each $a \in X$ define $p(a)$ to be the pair $\langle\{a\}, \{a\}\rangle$. Now define the *point extension* of $\sim$ to be the relation $\approx$ on $\mathbb{P} \cup \{p(a)\colon\ a \in X\}$ which is the transitive closure of the relation $\sim^*$ obtained by taking the union of $\sim$ and all pairs $(P, p(a))$ with $a \in P_1$.

We are now in a position to give some of the main definitions.

Definition (Pair-base). If X is a space (i.e., a T_1 topological space), then $\mathbb{P}$ is a pair-base on X provided:

(1) the set $\mathbb{P}$ consists of pairs of subsets of X,
(2) every element of $\mathbb{P}_1$ is open, and
(3) if U is an open neighbourhood of a point x, then there is a $P \in \mathbb{P}$ such that

$$x \in P_1 \subseteq P_2 \subseteq U.$$

Definition (Elastic). If X is a space, $\mathbb{P}$ a pair-base on X and $\sim$ a relation on $\mathbb{P}$, then $\sim$ is an elastic relation provided:

(1) the relation $\sim$ is transitive and framing, and

(2) if $P \in \mathbb{P}$ and $\mathbb{P}' \subseteq \{P' \in \mathbb{P}\colon P' \sim P\}$, then

$$\overline{\bigcup\{P_1'\colon P' \in \mathbb{P}'\}} \subseteq \bigcup\{P_2'\colon P' \in \mathbb{P}'\}.$$

A space X is said to be elastic if there is a pair-base $\mathbb{P}$ with an elastic relation $\sim$ on $\mathbb{P}$.

We observe that the above definition differs from that given in [16]. Unfortunately the original definition of elasticity is ambiguous since it is not clear how one defines the "frame map" when there are distinct $P, P' \in \mathbb{P}$ such that $P_1 = P_1'$. Even if there is no problem defining the frame map for a space X, the same problem arises when attempting to show that subspaces of X are elastic. However, if one analyzes proofs of results involving elastic spaces (particularly Theorem 2 of [16]), then it is clear that the above is the intended definition.

There are many characterisations of proto-metrisable spaces. One (due to Gruenhage and Zenor [9]) is that a space, X, is proto-metrisable provided it has a pair-base $\mathbb{P}$ such that, if $P, P' \in \mathbb{P}$ and $P_1 \cap P_1' \neq \emptyset$, then either $P_1 \subseteq P_2'$ or $P_1' \subseteq P_2$. (Such a pair-base is called a "rank-1" pair-base). Another is formulated in terms of the scattering process (see the next section for details). The characterisation given below was obtained in [8] and is ideally suited for our current purposes.

Definition (Proto-metrisable). If X is a space, then $\langle \mathbb{P}, \sim \rangle$ is a unitary point extendable pair-base if:

(1) the set $\mathbb{P}$ is a pair-base on X,

(2) the relation $\sim$ on $\mathbb{P}$ is transitive and framing, and

(3) if $\approx$ is the point extension of $\sim$ and $\mathbb{P}' \subseteq \{P \in \mathbb{P}\colon P \approx p(a)\}$ for some $a \in X$, then there is a $P^M \in \mathbb{P}'$ such that

$$\overline{\bigcup\{P_1\colon P \in \mathbb{P}'\}} \subseteq P_2^M \quad \text{and} \quad a \in P_2^M.$$

A space is proto-metrisable if it has a unitary point extendable pair-base.

3. The scattering process and duplication

If $\mathcal{C}$ is a class of topological spaces, then we define $S(\mathcal{C})$ to be the class of spaces which are obtained by the following process: take any space in $\mathcal{C}$, isolate all the points of some subset, replace each such point by a space in $\mathcal{C}$, and repeat transfinitely, taking some subspace of the inverse limit at limit ordinals. Observe that $\mathcal{C} \subseteq S(\mathcal{C})$. We shall say that $\mathcal{C}$ is closed under the scattering process if $\mathcal{C} = S(\mathcal{C})$.

Nyikos [14] has shown that the class of proto-metrisable spaces is S(Metrizable).

The Alexandroff duplicate $\mathcal{D}(X)$ of a space X is, by definition, the set $X \times \{0, 1\}$, topologized so that $\langle x, 1\rangle$ is isolated for every $x \in X$, and so that a local base for the point $\langle x, 0\rangle$ $(x \in X)$ is $\{(U \times \{0,1\}) \setminus \{\langle x, 1\rangle\}\colon U$ open in X and $x \in U\}$.

Duplication and scattering are related processes, and lie at the heart of many constructions in topology (see, for example, [17]). It remains unclear whether elasticity is preserved by the scattering process. To give the most general positive result known to the authors, we require a further definition.

If X is a space, then define $\langle \mathbb{P}, \sim \rangle$ to be a *weakly point extendable pair-base* if:

(1) the set $\mathbb{P}$ is a pair-base on X,

(2) the relation $\sim$ on $\mathbb{P}$ is transitive and framing, and

(3) if $\approx$ is the point extension of $\sim$ and $\mathbb{P}' \subseteq \{P \in \mathbb{P}\colon\ P \approx p(a)\}$ for some $a \in X$, then

$$\overline{\bigcup \{P_1\colon\ P \in \mathbb{P}'\}} \subseteq \left[\bigcup \{P_2\colon\ P \in \mathbb{P}'\}\right] \cup \{a\}.$$

The relevance of this definition should be clear from the following pair of results.

Lemma 1. *If the space X has a weakly point extendable pair-base, then X is elastic.*

Proof. Let $\langle \mathbb{P}, \sim \rangle$ be a weakly point extendable pair-base for X. It suffices to show that $\sim$ is an elastic relation on $\mathbb{P}$. Suppose that $P \in \mathbb{P}$ and $\mathbb{P}' \subseteq \{P' \in \mathbb{P}\colon\ P' \sim P\}$. Note that if $a \in P_1$ then $\mathbb{P}' \subseteq \{P'\colon\ P' \approx p(a)\}$, and hence

$$\overline{\bigcup \{P'_1\colon\ P' \in \mathbb{P}'\}} \subseteq \left[\bigcup \{P'_2\colon\ P' \in \mathbb{P}'\}\right] \cup \{a\}. \tag{†}$$

If P_1 is the singleton $\{a\}$ say, then a is isolated and clearly

$$\overline{\bigcup \{P'_1\colon\ P' \in \mathbb{P}'\}} \subseteq \bigcup \{P'_2\colon\ P' \in \mathbb{P}'\}$$

in this case. If P_1 consists of more than one point, then by considering (†) for two different points of P_1 we again see that

$$\overline{\bigcup \{P'_1\colon\ P' \in \mathbb{P}'\}} \subseteq \bigcup \{P'_2\colon\ P' \in \mathbb{P}'\}$$

as required. □

Theorem 2 [8]. *The class of spaces with weakly point extendable pair-bases is closed under the scattering process.*

The next theorem shows that the duplicate of a proto-metrisable space is elastic. However, Examples 9 and 10 will show that duplication can kill elasticity.

Theorem 3. *If a space X has a unitary point extendable pair-base, then the space $\mathcal{D}(X)$ has a weakly point extendable pair-base.*

Proof. Let $\langle \mathbb{P}, \sim_{\mathbb{P}} \rangle$ be a unitary point extendable pair-base X. If $P \in \mathbb{P}$ and $x \in P_1$ then define

$$D\langle P, x\rangle = \Big\langle \big(P_1 \times \{0,1\}\big) \backslash \big\{\langle x, 1\rangle\big\}, \big(P_2 \times \{0,1\}\big) \backslash \big\{\langle x, 1\rangle\big\}\Big\rangle$$

and set

$$\mathbb{D} = \big\{D\langle P, x\rangle\colon\ P \in \mathbb{P} \text{ and } x \in P_1\big\} \cup \big\{p\big(\langle x, 1\rangle\big)\colon\ x \in X\big\}.$$

Observe that $\mathbb{D}$ is a pair-base for $\mathcal{D}(X)$. Let $\sim_{\mathbb{D}}$ be the minimal transitive relation on $\mathbb{D}$ satisfying:

(1) If $P, P' \in \mathbb{P}$ and $x \in P_1$, $y \in P'_1$, then $D\langle P, x\rangle \sim_{\mathbb{D}} D\langle P', y\rangle$ if and only if $P \sim_{\mathbb{P}} P'$.

(2) If $D \in \mathbb{D}$ and $x \in X$, then $D \sim_{\mathbb{D}} p(\langle x, 1\rangle)$ whenever $\langle x, 1\rangle \in D_1$.

Notice that $\sim_{\mathbb{D}}$ is framing. It can now be verified without difficulty that $\langle \mathbb{D}, \sim_{\mathbb{D}}\rangle$ is the required weakly point extendable pair-base on $\mathcal{D}(X)$. □

4. Destroying elasticity

In this section we demonstrate how our two processes of scattering and duplication can be used to destroy elasticity while preserving other strong properties such as monotone normality and paracompactness. We begin by introducing some notation for a simple case of the scattering process. Suppose that X is a space with some isolated points. Define X_0 to be X and inductively define X_{n+1} to be the space obtained from X_n by replacing all the isolated points of X_n by copies of X. Let X_ω be the inverse limit of the X_n's. Since the class of proto-metrisable spaces coincides with $\mathcal{S}$(Metrizable), it is clear that if X is proto-metrisable then X_ω is proto-metrisable and hence by Theorem 3, $\mathcal{D}(X_\omega)$ is elastic. Our aim is to show that if X is not proto-metrisable but is T_3 and has a dense subset of isolated points, then $\mathcal{D}(X_\omega)$ is not elastic. If X is elastic then, by the results of [7], $\mathcal{D}(X_\omega)$ has $\mathcal{W}$ satisfying well-ordered (F) and hence is monotonically normal and hereditarily paracompact. Indeed, it can be shown that $\mathcal{D}(X_\omega)$ is strongly stratonormal (see [3] for the definition). We proceed with a number of lemmas. The first establishes that if X is not proto-metrisable, but is T_3 with a dense subset of isolated points, and $\mathbb{P}$ is a pair-base on X with a framing relation $\sim$ on $\mathbb{P}$, then there is an isolated point x and $P, Q \in \mathbb{P}$ such that $P \sim Q$, $x \in Q_1$, but $x \notin P_2$.

Lemma 4. *Suppose that X is a T_3 space with a dense subset of isolated points. Further suppose that X has a pair-base $\mathbb{P}$ with a framing relation $\sim$ satisfying*

$$(P \sim Q,\ x \in Q_1,\ x \text{ isolated}) \Longrightarrow x \in \overline{P_2}.$$

Then X is proto-metrisable.

Proof. Since X is regular, $\{\langle P_1, \overline{P_2}\rangle\colon P \in \mathbb{P}\}$ is a pair-base for X. By Gruenhage and Zenor's result, it suffices to show that this is a rank-1 pair-base. Suppose that $P, Q \in \mathbb{P}$ and $P_1 \cap Q_1 \neq \emptyset$. The relation $\sim$ is framing and so we will assume that $P \sim Q$. Suppose that $Q_1 \not\subseteq \overline{P_2}$. If this is the case then $Q_1 \setminus \overline{P_2}$ is a nonempty open set and hence contains an isolated point x. Thus $P \sim Q$, $x \in Q_1$, x isolated, but $x \notin \overline{P_2}$ contradicting the hypotheses. Hence, X is proto-metrisable. □

For the next three lemmas X will be a T_3 space with a dense subset of isolated points, but X will not be proto-metrisable. If $\alpha \leqslant \beta \leqslant \omega$, then $j_{\beta\to\alpha} : X_\beta \to X_\alpha$ will be the usual map associated with the scattering process.

Lemma 5. *For $\alpha \leqslant \beta \leqslant \omega$ the map $j_{\beta\to\alpha}$ is an open map.*

Proof. The fact that $j_{m\to n}$ is open follows directly from the definition of the X_n's. To see that $j_{\omega\to n}$ is open, consider any open $O \subseteq X_\omega$. If $x \in j_{\omega\to n}(O)$ and x is not isolated, then by the definitions of the X_n's and the fact that X_ω is the inverse limit of the X_n's, there is an open U in X_n such that $x \in U$ and $j_{\omega\to n}^{-1}(U) \subseteq O$. Thus $j_{\omega\to n}(O)$ is open as required. □

Lemma 6. *Suppose that $\langle \mathbb{P}, \sim\rangle$ is an elastic pair-base on $\mathcal{D}(X_\omega)$, that x is an isolated point of X_n and $(j_{\omega\to n}^{-1}(x) \times \{0\}) \cap P_1 \neq \emptyset$ for some $P \in \mathbb{P}$. Then there is $m > n$ and open U in X_m such that:*

(1) $x \in j_{m\to n}(U)$,
(2) $j_{\omega\to m}^{-1}(U) \times \{0,1\} \subseteq P_1$, *and*
(3) $P \sim p(\langle y,1\rangle)$ *for every* $y \in j_{\omega\to m}^{-1}(U)$.

Proof. Pick $y = \langle y_m\rangle \in X_\omega$ such that $y_n = x$ and $\langle y,0\rangle \in P_1$. Since $\langle \mathbb{P}, \sim\rangle$ is an elastic pair-base

$$\langle y,0\rangle \notin \overline{\{\langle z,1\rangle\colon\ z \in X_\omega \text{ and } p(\langle z,1\rangle) \sim P\}}.$$

Hence there is an open subset A of X_ω for which

$$\langle y,0\rangle \in (A \times \{0,1\}) \setminus \{\langle y,1\rangle\} \subseteq P_1$$

and $P \sim p(\langle z,1\rangle)$ whenever $z \in A$ and $z \neq y$. Since x is isolated in X_n and is an element of $j_{\omega\to n}(A)$, there is $m > n$ and B open in X_m such that B is not a singleton, $y_m \in B$, $j_{m\to m-1}(B) = \{y_{m-1}\}$, and $j_{\omega\to m}^{-1}(B) \subseteq A$. Define $U = B\setminus\{y_m\}$. It is clear that U satisfies (1)–(3) as required. □

Lemma 7. *Suppose that $\mathbb{P}$ is a pair-base on X_ω with framing relation $\sim$ on $\mathbb{P}$. Further suppose that x is an isolated point of X_n and C is the clopen copy of X that replaces x in X_{n+1}. Then there is an isolated point y of C and $P, Q \in \mathbb{P}$ such that:*

(a) $P \sim Q$,
(b) $y \notin j_{\omega\to n+1}(P_2)$,
(c) $x \in j_{\omega\to n}(P_1)$, *and*
(d) $y \in j_{\omega\to n+1}(Q_1)$.

Proof. We begin with a definition. Suppose that O is an open subset of X_ω and $j_{\omega\to n+1}(O)$ is a subset of C that contains some nonisolated points. Recall that, if $a \in j_{\omega\to n+1}(O)$ is not isolated, then there is an open set U such that $a \in U \subseteq C$ and $j_{\omega\to n+1}^{-1}(U) \subseteq O$. Define

$$V(O) = \bigcup\{U\colon\ U \text{ open in } C \text{ and } j_{\omega\to n+1}^{-1}(U) \subseteq O\}$$

and notice that $j_{\omega\to n+1}^{-1}(V(O)) \subseteq O$. Now define $\mathbb{P}^*$ to be the set of all P in $\mathbb{P}$ such that $j_{\omega\to n+1}(P_2) \subseteq C$ and $j_{\omega\to n+1}(P_1)$ contains some nonisolated points. For each $P \in \mathbb{P}^*$, set $P^C = \langle V(P_1), j_{\omega\to n+1}(P_2)\rangle$ and define

$$\mathbb{E} = \{P^C\colon\ P \in \mathbb{P}^*\} \cup \{\langle\{a\},\{a\}\rangle\colon\ a \text{ isolated in } C\}.$$

We now show that $\mathbb{E}$ is a pair-base for C. Suppose that $a \in U \subseteq C$, a is not isolated and U is open. Pick $b \in j^{-1}_{\omega\to n+1}(a)$ and $P \in \mathbb{P}$ such that

$$b \in P_1 \subseteq P_2 \subseteq j^{-1}_{\omega\to n+1}(U).$$

Thus, $a \in V(P_1) \subseteq j_{\omega\to n+1}(P_1) \subseteq j_{\omega\to n+1}(P_2) \subseteq U$ and $P \in \mathbb{P}^*$. Hence $\mathbb{E}$ is indeed a pair-base for C. Our next aim is to define a framing relation $\sim_{\mathbb{E}}$ on $\mathbb{E}$. Suppose that E is an element of $\mathbb{E}$. If $E = \langle\{a\},\{a\}\rangle$ for some isolated point a, then pick $P^E \in \mathbb{P}$ such that $j_{\omega\to n+1}(P^E) = E$. If $E \in \mathbb{E}$ and E_1 contains some nonisolated points, then pick $P^E \in \mathbb{P}$ such that $E = \langle V(P^E_1), j_{\omega\to n+1}(P^E_2)\rangle$. Now define $\sim_{\mathbb{E}}$ by $E \sim_{\mathbb{E}} E'$ if $P^E \sim P^{E'}$. To see that $\sim_{\mathbb{E}}$ is framing recall that $\sim$ is framing and observe that if $a \in E_1$ and E_1 is not a singleton set, then $j^{-1}_{\omega\to n+1}(a) \subseteq P^E_1$.

Now, the space C is homeomorphic to X and hence is T_3 with a dense subset of isolated points, but C is not proto-metrisable. Thus by Lemma 4, there is an isolated point y of C and $E, F \in \mathbb{E}$ such that $E \sim_{\mathbb{E}} F$, $y \in F_1$, but $y \notin E_2$. Let P and Q denote P^E and P^F respectively. Observe that $P \sim Q$, $y \in F_1 \subseteq j_{\omega\to n+1}(Q_1)$ and $y \notin E_2 = j_{\omega\to n+1}(P_2)$. Finally notice that $x \in j_{n+1\to n}(E_1) \subseteq j_{\omega\to n}(P_1)$ as required. □

We are now in a position to prove the theorem.

Theorem 8. *Suppose that X is a T_3 space with a dense subset of isolated points, but that X is not proto-metrisable. Then $\mathcal{D}(X_\omega)$ is not elastic.*

Proof. Suppose for contradiction that $\langle\mathbb{P},\sim\rangle$ is an elastic pair-base on $\mathcal{D}(X_\omega)$. We will inductively define a point $x = \langle x\rangle$ of X_ω such that x_n is isolated in X_n for every n, and for each n in some infinite subset I of $\mathbb{N}$, we will define various sets, including an element Q^n of $\mathbb{P}$ such that $(j^{-1}_{\omega\to n}(x_n) \times \{0\}) \cap Q^n_1 \neq \emptyset$. The induction is begun by defining x_0 to be any isolated point of X_0 and Q^0 to be any element of $\mathbb{P}$ such that $(j^{-1}_{\omega\to 0}(x_0) \times \{0\}) \cap Q^0_1 \neq \emptyset$. The number 0 will be an element of I. Suppose that we have defined $x_0, \ldots, x_n$ so that:

(i) x_m is isolated in X_m for each m,
(ii) $j_{m+1\to m}(x_{m+1}) = x_m$ for each m, and that
(iii) $n \in I$ and we have defined $Q^n \in \mathbb{P}$ so that $(j^{-1}_{\omega\to n}(x_n) \times \{0\}) \cap Q^n_1 \neq \emptyset$.

By Lemma 6, there is $m > n$ and open $U = U_m$ in X_m such that:

(1) $x_n \in j_{m\to n}(U)$,
(2) $j^{-1}_{\omega\to m}(U) \times \{0,1\} \subseteq Q^n_1$, and
(3) $Q^n \sim p(\langle y,1\rangle)$ for every $y \in j^{-1}_{\omega\to m}(U)$.

Since the isolated points of X are dense in X, the isolated points of X_m are dense in X_m. Hence, since x_n is isolated in X_n, we can pick an isolated $x_m \in U$ such that $j_{m\to n}(x_m) = x_n$. For $n < r < m$, define $x_r = j_{m\to r}(x_m)$. The number $m+1$ will be the smallest element of I larger then n. In the construction of X_{m+1}, x_m is replaced by a clopen copy of X. Denote this copy by C. The pair-base $\langle\mathbb{P},\sim\rangle$ induces a framed pair-base on X_ω. Hence by Lemma 7, there is an isolated point x_{m+1} of C and $P^{m+1}, Q^{m+1} \in \mathbb{P}$ such that:

(a) $P^{m+1} \sim Q^{m+1}$,
(b) $(j^{-1}_{\omega\to m+1}(x_{m+1}) \times \{0\}) \cap P_2^{m+1} = \emptyset$,
(c) $(j^{-1}_{\omega\to m}(x_m) \times \{0\}) \cap P_1^{m+1} \neq \emptyset$, and
(d) $(j^{-1}_{\omega\to m+1}(x_{m+1}) \times \{0\}) \cap Q_1^{m+1} \neq \emptyset$.

To complete the induction, notice that $j_{m+1\to m}(x_{m+1}) = x_m$.

Let $I^* = I\backslash\{0\}$.

Claim 1. *If $n \in I^*$, then $P^n \sim p(\langle x, 1\rangle)$.*

To see this, suppose that $n \in I^*$ and $m+1$ is the next element of I. By construction, there is an open set U_m of X_m containing x_m such that $Q^n \sim p(\langle y, 1\rangle)$ whenever $y \in j^{-1}_{\omega\to m}(U_m)$, and thus $Q^n \sim p(\langle x, 1\rangle)$. Again by construction, $P^n \sim Q^n$ and hence by the transitivity of $\sim$, $P^n \sim p(\langle x, 1\rangle)$.

Claim 2. $\langle x, 0\rangle \in \overline{\bigcup\{P_1^n\colon n \in I^*\}}$.

Suppose that U is open in X_m and $x \in j^{-1}_{\omega\to m}(U)$ (recall sets of this type form a basis for X_ω). Pick an $n \in I^*$ such that $n > m$. By construction, $(j^{-1}_{\omega\to n-1}(x_{n-1}) \times \{0\}) \cap P_1^n \neq \emptyset$. Thus $(j^{-1}_{\omega\to m}(x_m) \times \{0\}) \cap P_1^n \neq \emptyset$ and hence $(j^{-1}_{\omega\to m}(U) \times \{0\}) \cap P_1^n \neq \emptyset$. Therefore $\langle x, 0\rangle \in \overline{\bigcup\{P_1^n\colon n \in I^*\}}$ as required.

Now $\langle \mathbb{P}, \sim\rangle$ is an elastic pair-base. So, by the two claims, it must be that case that $\langle x, 0\rangle \in \bigcup\{P_2^n\colon n \in I^*\}$. However, for each $n \in I^*$, $(j^{-1}_{\omega\to n}(x_n) \times \{0\}) \cap P_2^n = \emptyset$. Hence $\langle x, 0\rangle \notin \bigcup\{P_2^n\colon n \in I^*\}$, contradicting our assumption that $\mathcal{D}(X_\omega)$ is elastic. □

5. Examples

We finish with three applications of the above results. The first two demonstrate the limits of the theory from earlier sections. In addition, they provide counterexamples to a question of Borges [3] by giving examples of strongly stratonormal spaces that are not elastic. The final application establishes that elasticity is not preserved by perfect maps.

Example 9. The duplicate of a compact first countable elastic space need not be elastic. There is a compact first countable space which is strongly stratonormal but not elastic.

Proof. Let $X = \mathcal{D}([0,1])$ and $Y = X_\omega$. As [0,1] is metrisable, by Theorems 2 and 3, X_ω has a weakly point extendable pair-base. Y is easily seen to be first countable and is also a closed subset of the compact space $\prod_n X_n$, hence is compact. The duplicate of any first countable compact space is first countable and compact. In [7] it is shown that the class of well ordered (F) spaces is closed under the scattering and duplication processes, and that any elastic space is well ordered (F). We also remark that the duplicate of a strongly stratonormal space is strongly stratonormal. Finally, recall that compact proto-metrisable spaces are metrisable [14]. Hence, X is not proto-metrisable but is T_3 and has a dense subset of isolated points. Therefore, by Theorem 8, $\mathcal{D}(X_\omega)$ is not elastic. □

Example 10. There is a stratifiable space S such that $\mathcal{D}(S)$ is not elastic.

Proof. We make two observations: first, if $\mathcal{T}$ is a stratifiable topology on a set Y, then so is $\mathcal{T}'$ where $\mathcal{T}'$ is obtained by isolating a countable subset of points. Second, if Y is a stratifiable space with at most countably many isolated points, then the space Y' is also stratifiable if Y' is obtained from Y by replacing each isolated point of Y by a stratifiable space. Let B denote McAuley's bow-tie space [12] and define X to be the space obtained from B by isolating all points with both coordinates rational which are not on the x-axis. Finally set $S = X_\omega$. By the above observations X_n is stratifiable for each n. Thus $\prod_n X_n$ is a stratifiable space which contains S as a subspace. Hence S is stratifiable. The space X is stratifiable, but not metrisable and therefore not proto-metrisable [14]. However, X has a dense subset of isolated points and hence by Theorem 8, $\mathcal{D}(S)$ is not elastic. □

We now prove that in certain cases $\mathcal{D}(X_\omega)$ is the perfect image of an elastic space. The authors do not know whether the result can be extended to any $\mathcal{D}(X_\omega)$ where X is elastic.

Theorem 11. *Suppose that the spaces X and Y and the function $f : X \to Y$ are such that:*

(1) *X is proto-metrisable and Y has isolated points,*

(2) *f is a closed, continuous surjection and the fibers of f are finite,*

(3) *if y is an isolated point of Y, then $f^{-1}(y)$ consists of just one point, and*

(4) *if $x \in X$ then, x is isolated in X if and only if $f(x)$ is isolated in Y.*

Then $\mathcal{D}(X_\omega)$ is an elastic space and there is $g : \mathcal{D}(X_\omega) \to \mathcal{D}(Y_\omega)$ such that g is a closed, continuous surjection and the fibers of g are finite (hence g is a perfect map).

Proof. To see that $\mathcal{D}(X_\omega)$ is elastic, recall that X_ω is proto-metrisable and apply Lemma 1 and Theorem 3.

If $\alpha \leqslant \beta \leqslant \omega$ then let $i_{\beta\to\alpha} : X_\beta \to X_\alpha$ and $j_{\beta\to\alpha} : Y_\beta \to Y_\alpha$ denote the usual maps associated with the scattering process. We will inductively define maps $f_n : X_n \to Y_n$. Set $f_0 = f$ and suppose f_n has been defined so that x is isolated in X_n if and only if $f_n(x)$ is isolated in Y_n. Suppose that $x \in X_{n+1}$ and let $a = i_{n+1\to n}(x)$. If a is not isolated in X_n then $f_n(a)$ is not isolated in Y_n and $x = a$. In this case define $f_{n+1}(x) = f_n(a) \in Y_{n+1}$. Suppose now that a is isolated in X_n. Set $b = f_n(a)$ and let X_a and Y_a denote the clopen copies of X and Y which replace a and b in the construction of X_{n+1} and Y_{n+1}. Let $f_a : X_a \to Y_a$ be the function that corresponds to $f : X \to Y$, and define $f_{n+1}(x) = f_a(x)$. Notice that x is isolated in X_{n+1} if and only if $f_{n+1}(x)$ is isolated in Y_{n+1}.

We make the following observations concerning the above maps:

(1) If $m \geqslant n$, then the following diagram commutes:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ {\scriptstyle i_{m\to n}}\downarrow & & \downarrow{\scriptstyle j_{m\to n}} \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

(2) Suppose that $x \in X_n$, $y \in Y_n$ and $y' \in Y_{n+1}$ are such that $f_n(x) = y$ and $j_{n+1\to n}(y') = y$, then there is $x' \in X_{n+1}$ such that $f_{n+1}(x') = y'$ and $i_{n+1\to n}(x') = x$.

(3) For each n, f_n is surjective, has finite fibers, in continuous and closed.

(4) If y is an isolated point of Y_n, then $f_n^{-1}(y)$ consists of just one point.

Now define $f_\omega : X_\omega \to Y_\omega$ by

$$f_\omega(x) = \langle f_n(x_n)\rangle \quad \text{where } x = \langle x\rangle \in X_\omega.$$

By observation (1), $f_\omega(x)$ is indeed an element of Y_ω. From the definitions it is clear that the following diagram commutes:

$$\begin{array}{ccc} X_\omega & \xrightarrow{f_\omega} & Y_\omega \\ {\scriptstyle i_{\omega\to n}}\downarrow & & \downarrow{\scriptstyle j_{\omega\to n}} \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

We will prove that f_ω is a closed, continuous surjection with finite fibers.

Claim 1. *f_ω is a surjection and has finite fibers.*

Suppose $y = \langle y_n\rangle \in Y_\omega$. Pick $x_0 \in X_0$ such that $f_0(x_0) = y_0$. Assuming that x_n has been defined so that $f_n(x_n) = y_n$, observation (2) allows us to pick $x_{n+1} \in X_{n+1}$ such that $f_{n+1}(x_{n+1}) = y_{n+1}$ and $i_{n+1\to n}(x_{n+1}) = x_n$. Hence, if $x = \langle x_n\rangle$, then $x \in X_\omega$ and $f_\omega(x) = y$. Thus f_ω is a surjection. We now show that $f_\omega^{-1}(y)$ is finite. Notice that if y_n is isolated in Y_n, then there is a unique $x_n \in X_n$ such that $f_n(x_n) = y_n$. Hence, if y_n is isolated in Y_n for every n, then $f_\omega^{-1}(y)$ consists of just one point. So, suppose that y_n is not isolated in Y_n for some n. Let N be the least n such that y_n is not isolated. Suppose that $a \in f_N^{-1}(y_N)$. Since y_N is not isolated, a is not isolated. Hence, there is a unique $x \in X_\omega$ such that $x_N = a$. Notice that if $z \in f_\omega^{-1}(y)$, then $z_N \in f_N^{-1}(y_N)$. Thus, since $f_N^{-1}(y_N)$ is finite, $f_\omega^{-1}(y)$ is finite as required.

A direct proof that f_ω is closed and continuous is straightforward. Alternatively, Theorem 3.7.12 of [4] gives us the result since each function f_n is perfect.

Now define $g : \mathcal{D}(X_\omega) \to \mathcal{D}(Y_\omega)$ by

$$g(\langle x, i\rangle) = \langle f_\omega(x), i\rangle \quad \text{for } i = 0, 1.$$

Since f_ω is a surjection and has finite fibers, g is surjective and has finite fibers. We shall prove that g is closed and continuous.

Claim 2. *g is continuous.*

Since $g^{-1}(\langle y,1\rangle) = f_\omega^{-1}(y) \times \{1\}$ is open, it suffices to show that if O is open in Y_ω and $y \in O$, then $g^{-1}(U)$ is open, where $U = (O \times \{0,1\}) \backslash \{\langle y,1\rangle\}$. Notice that

$$g^{-1}(U) = \big(f_\omega^{-1}(O) \times \{0,1\}\big) \backslash \big(f_\omega^{-1}(y) \times \{1\}\big).$$

Now, $f_\omega^{-1}(O) \times \{0,1\}$ is open in $\mathcal{D}(X_\omega)$ since f_ω is continuous. Furthermore, $f_\omega^{-1}(y) \times \{1\}$ is finite and therefore closed. Hence $g^{-1}(U)$ is open in $\mathcal{D}(X_\omega)$ as required.

Claim 3. *g is closed.*

Suppose that $b \in \mathcal{D}(Y_\omega)$, U an open subset of $\mathcal{D}(X_\omega)$ and $g^{-1}(b) \subseteq U$. Since g is continuous it suffices, by [4, 1.4.13], to show there is an open V in $\mathcal{D}(Y_\omega)$ such that $b \in V$ and $g^{-1}(V) \subseteq U$. If b is isolated, then let $V = \{b\}$. Consider the case when $b = \langle y,0\rangle$ for some $y \in Y_\omega$. Notice that $g^{-1}(b) = f_\omega^{-1}(y) \times \{0\}$. Define

$$O = \bigcup \big\{T\colon T \text{ open in } X_\omega \text{ and there is } x \in f_\omega^{-1}(y) \text{ such that } (T \times \{0,1\}) \backslash \{\langle x,1\rangle\} \subseteq U\big\}.$$

Observe that O is open in X_ω, $f_\omega^{-1}(y) \subseteq O$, and

$$\big(O \times \{0,1\}\big) \backslash \big(f_\omega^{-1}(y) \times \{1\}\big) \subseteq U.$$

Now, f_ω is closed and so there is an open subset W of Y_ω such that $y \in W$ and $f_\omega^{-1}(W) \subseteq O$. Define $V = (W \times \{0,1\}) \backslash \{\langle y,1\rangle\}$ and notice that $b \in V$, V open in $\mathcal{D}(Y_\omega)$, and

$$\begin{aligned} g^{-1}(V) &= \big(f_\omega^{-1}(W) \times \{0,1\}\big) \backslash \big(f_\omega^{-1}(y) \times \{1\}\big) \\ &\subseteq \big(O \times \{0,1\}\big) \backslash \big(f_\omega^{-1}(y) \times \{1\}\big) \subseteq U \end{aligned}$$

as required. □

Example 12. The perfect image of an elastic space need not be elastic.

Proof. Let A be $\omega_1 + 1$ with the usual (order) topology refined so that α is isolated whenever $\alpha < \omega_1$. Let B be a copy of the space $\omega + 1$ which is disjoint from A. Let X be $A \oplus B$, the topological sum of A and B. Observe that A and B are proto-metrisable and hence X is proto-metrisable. Denote the point of B corresponding to ω by ω^*, and let E be the equivalence relation on X which identifies ω_1 and ω^* (and no other pair of distinct points). Define Y to be the quotient space X/E and denote the quotient map by f. Observe that X, Y and f satisfy the hypotheses of Theorem 11 and thus $\mathcal{D}(X_\omega)$ is an elastic space and there is a map $g\colon \mathcal{D}(X_\omega) \to \mathcal{D}(Y_\omega)$ such that g is a closed, continuous surjection and the fibers of g are finite. Observe that not every point of Y

has a linearly ordered local base; hence Y is not proto-metrisable. Thus, since Y is T_3 and has a dense subset of isolated points, we have, again by Theorem 8, that $\mathcal{D}(Y_\omega)$ is not elastic. □

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