Oscillation of delay difference equations with several delays

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Abstract

In this paper, we obtain some new oscillation criteria for the difference equation with several delays

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots, \]


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1. Introduction

In the past two decades, the oscillatory behavior for the delay difference equation

\[ x_{n+1} - x_n + p(n)x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \] \tag{1.1}

has been investigated extensively, and many interesting oscillation criteria have been obtained; see, for example, [1–16] and references therein. For the more general equation with several delays

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots, \] \tag{1.2}

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where \( p_i(n) \geq 0, \) \( m, k_i \) are positive integers, \( i = 1, 2, \ldots , m, \) the oscillation criteria are usually obtained by reducing Eq. (1.2) to a difference inequality with single delay \( k = \min \{k_1, k_2, \ldots , k_m \} \) and \( p(n) = \sum_{i=1}^{m} p_i(n) \) and by using the corresponding results for Eq. (1.1). Nevertheless, Tang and Yu [7] and Tang and Zhang [11] obtained two oscillation criteria for Eq. (1.2),

\[
\lim \inf_{n \to \infty} \sum_{i=1}^{m} \left( \frac{k_i + 1}{k_i} \right)^{k_i + 1} \sum_{s = n+1}^{n+k_i} p_i(s) > 1 \tag{1.3}
\]

and

\[
\lim \sup_{n \to \infty} \sum_{i=1}^{m} p_i(s) > 1, \tag{1.4}
\]

which formulated in the terms of the numbers \( k_i \) and \( p_i(n) \) directly. Very recently, Zhang and Zhou [16] established a new oscillation condition for Eq. (1.2) different from (1.3) and (1.4). That is

**Assertion 1.1** [16]. Let

\[
\lim \inf_{n \to \infty} p_i(n) = p_i, \quad i = 1, 2, \ldots , m, \tag{1.5}
\]

and let \( \lambda_a \) is the largest root of the following equation on \((0, 1]::\)

\[
\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0. \tag{1.6}
\]

Assume that

\[
\lim \sup_{n \to \infty} \sum_{i=1}^{m} p_i(n)\lambda_a^{-k_i} > \frac{1}{1 + \lambda_a}. \tag{1.7}
\]

Then every solution of Eq. (1.2) oscillates.

Nevertheless, Assertion 1.1 [16, Theorem 1] is false. To illustrate this, we consider delay difference equation

\[
x_{n+1} - x_n + p(n)x_{n-1} = 0, \quad n = 0, 1, 2, \ldots , \tag{1.8}
\]

where \( k = 1 \) and \( p(2n) = 0, \) \( p(2n + 1) = a > 0, \) \( n = 0, 1, 2, \ldots \). It is easy to see that \( p = \lim \inf_{n \to \infty} p(n) = 0, \) and so \( \lambda_a = 1. \) If \( 1/2 < a < 1, \) then

\[
\lim \sup_{n \to \infty} p(n)\lambda_a^{-k} = a > \frac{1}{2} = \frac{1}{1 + \lambda_a}.
\]

Hence, in view of Assertion 1.1, every solution of Eq. (1.8) oscillates. However, Eq. (1.8) with initial condition \( x_{-1} = x_0 = 1 \) has a positive \( \{x_n\}, \) where \( x_{2n} = x_{2n+1} = (1 - a)^n \) for \( n = 0, 1, 2, \ldots \).

In fact, in the proof of Theorem 1 in [16], the assertion that

\[
x_n \geq x_{n-1} \sum_{i=1}^{m} p_i(n)(\lambda_a + \epsilon_i)^{-k_i+1}
\]
implies
\[ \frac{x_{n+1}}{x_n} \geq \sum_{i=1}^{m} p_i(n)(\lambda_a + \epsilon_i)^{-k_i} + 1 \]
is incorrect. It results in the incorrect conclusion of Assertion 1.1 [16, Theorem 1]. By analyzing the proof of [16, Theorem 1], we can easy conclude the following correct version of Assertion 1.1.

**Theorem 1.2.** Assume that (1.5) holds and \( \lambda_a \) is the largest root of Eq. (1.6). Further assume that
\[ \limsup_{n \to \infty} \sum_{i=1}^{m} \left[p_i(n) + \lambda_a p_i(n + 1)\right]\lambda_a^{-k_i} > 1. \]  
Then every solution of Eq. (1.2) oscillates.

Obviously, conditions (1.4) and (1.9) are complementary. In this paper, we establish some new oscillation criteria for Eq. (1.2) to improve (1.4) and (1.9).

### 2. Some lemmas

In this section, we consider the inequality
\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-k_i} \leq 0, \quad n = 0, 1, 2, \ldots, \]  
and establish some lemmas on the positive solution of (2.1), which are useful in the proofs of our main results in next section. The first lemma is taken from [16], whose proof can be found in [16].

**Lemma 2.1** [16]. Assume that (1.5) holds and \( \lambda_a \) has positive roots. Let \( \{x_n\} \) be the eventually positive solution of (2.1). Then
\[ \liminf_{n \to \infty} \frac{x_n}{x_{n+1}} \geq \lambda_a^{-1}. \]  
Here and in the sequel, \( \lambda_a \) is the largest root of Eq. (1.6).

**Lemma 2.2.** Assume that
\[ \liminf_{n \to \infty} \sum_{s=n+1}^{n+k_i} p_i(s) = q_i, \quad i = 1, 2, \ldots, m, \]  
and
\[ \liminf_{n \to \infty} \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s) = d < 1. \]
Let \( \{x_n\} \) be the eventually positive solution of (2.1). Then
\[
\liminf_{n \to \infty} \frac{x_{n+1}}{x_n} \geq \mu_*. \tag{2.5}
\]
Here and in the sequel, \( \mu_* \in [0, 1) \) is the smallest root of the equation
\[
\mu \left( 1 - \sum_{i=1}^{m} \frac{q_i \mu^{k_i-1}}{1 - \mu^{k_i}} \right) = d. \tag{2.6}
\]

**Proof.** Let \( \theta \in (0, 1) \). Choose a positive integer \( n_0 \) such that
\[
x_{n-k} > 0, \quad \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s) \geq \theta d \quad \text{and} \quad \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s) \geq \theta q_i,
\]
for \( i = 1, 2, \ldots, m, \ n \geq n_0, \)
here and in the sequel, \( k = \max\{k_1, k_2, \ldots, k_m\} \). Summing (2.1) from \( n+1 \) to \( \infty \), we have
\[
x_{n+1} \geq \sum_{i=1}^{m} \sum_{s=n+1}^{\infty} p_i(s)x_{s-k_i} \geq x_n \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s) \geq \theta d x_n, \quad n \geq n_0 + k. \tag{2.7}
\]
It follows that
\[
\frac{x_{n+1}}{x_n} \geq \theta d := d_1, \quad n \geq n_0 + k.
\]
From (2.1) again and using the above, we have
\[
x_{n+1} \geq \sum_{i=1}^{m} \sum_{s=n+1}^{\infty} p_i(s)x_{s-k_i} = \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s)x_{s-k_i} + \sum_{i=1}^{m} \sum_{s=n+k_i+1}^{\infty} p_i(s)x_{s-k_i}
\]
\[
\geq \theta d x_n + x_{n+1} \sum_{i=1}^{m} \sum_{s=n+k_i+1}^{\infty} p_i(s)d_1^{s-n-k_i-1}
\]
\[
= d_1 x_n + x_{n+1} \sum_{i=1}^{m} \sum_{j=0}^{\infty} p_i(j+n+k_i+1)d_1^{j}
\]
\[
\geq d_1 x_n + x_{n+1} \sum_{i=1}^{m} \sum_{j=1}^{\infty} d_1^{j-1} \sum_{s=n+k_i+1}^{\infty} p_i(s)
\]
\[
\geq d_1 x_n + x_{n+1} \sum_{i=1}^{m} \theta q_i \sum_{j=1}^{\infty} d_1^{j-1}
\]
\[
\geq d_1 x_n + x_{n+1} \sum_{i=1}^{m} \theta q_i d_1^{k_i-1} \frac{1}{1-d_1^{k_i}}, \quad n \geq n_0 + 2k.
\]
which implies
\[
\frac{x_{n+1}}{x_n} \geq \frac{d_1}{1 - \sum_{i=1}^{m} \frac{\theta q_i d_{ki}}{1 - d_{ki}}}, \quad n \geq n_0 + 2k.
\]
Following this iterative procedure, we have
\[
\frac{x_{n+1}}{x_n} \geq \frac{d_1}{1 - \sum_{i=1}^{m} \frac{\theta q_i d_{ki}}{1 - d_{ki}}}, \quad n \geq n_0 + (j + 1)k, \quad j = 1, 2, \ldots.
\]
It is easy to see that \(0 \leq d_1 \leq d_2 \leq \cdots \leq 1\). Therefore, the limit \(\lim_{j \to \infty} d_j = d_\star(\theta)\) exists and satisfies
\[
d_\star\left(1 - \sum_{i=1}^{m} \frac{\theta q_i d_{ki}^{j-1}}{1 - d_{ki}^j}\right) = \theta d.
\] (2.8)
Furthermore,
\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n} \geq d_\star(\theta).
\] From the definition of \(\mu_\star\) and (2.8), it is easy to show that \(\lim_{\theta \to 1} d_\star(\theta) \geq \mu_\star\). Then (2.5) holds and the proof is complete. \(\square\)

Similarly, we have

**Lemma 2.3.** Assume that (1.5) and (2.4) hold. Let \(\{x_n\}\) be the eventually positive solution of (2.1). Then
\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n} \geq v_\star.
\] (2.9)
Here and in the sequel, \(v_\star \in [0, 1)\) is the smaller of two roots of the equation
\[
v\left(1 - \frac{1}{1 - v \sum_{i=1}^{m} p_i}\right) = d.
\] (2.10)
It is easy to see that
\[
v_\star = \frac{1}{2} \left[1 + d - \sum_{i=1}^{m} p_i - \sqrt{\left(1 + d - \sum_{i=1}^{m} p_i\right)^2 - 4d}\right].
\] (2.11)
Therefore, from Lemma 2.3, we have

**Corollary 2.1.** Assume that (1.5) and (2.4) hold. Let \(\{x_n\}\) be the eventually positive solution of (2.1). Then
\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n} \geq \frac{1}{2} \left[1 + d - \sum_{i=1}^{m} p_i - \sqrt{\left(1 + d - \sum_{i=1}^{m} p_i\right)^2 - 4d}\right].
\] (2.12)
3. Main results

**Theorem 3.1.** Assume that \( \sum_{j=1}^{m} p_j(n) < 1 \) for large \( n \), and that

\[
\limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \prod_{l=s-k_i}^{n-1} \left( 1 - \sum_{j=1}^{m} p_j(l) \right) \right. \\
+ \sum_{s=1}^{\infty} p_i(n+k_i+s) \prod_{r=1}^{s} \left( \sum_{j=1}^{m} \sum_{l=n+r}^{n+r+k_i-1} p_j(l) \right) \left. \right] > 1. \tag{3.1}
\]

Then every solution of Eq. (1.2) oscillates.

**Proof.** For the sake of contradiction, assume that Eq. (1.2) has an eventually positive solution \( \{x_n\} \). Then there exists a positive integer \( n_0 \) such that

\[ x_{n-k} > 0 \quad \text{and} \quad x_{n+1} - x_n \leq 0, \quad n \geq n_0. \]

Hence, from (1.2), we have

\[
x_n \geq x_{n+1} \left( 1 - \sum_{i=1}^{m} p_i(n) \right)^{-1}, \quad n \geq n_0 + k, \tag{3.2}
\]

\[
x_{n+1} \geq \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s)x_{s-k_i} \geq x_n \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s), \quad n \geq n_0 + k, \tag{3.3}
\]

and

\[
x_n \geq \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s)x_{s-k_i} + \sum_{s=1}^{\infty} p_i(n+k_i+s)x_{n+s} \right], \quad n \geq n_0. \tag{3.4}
\]

Substituting (3.2) and (3.3) into (3.4), we have

\[
x_n \geq x_n \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \prod_{l=s-k_i}^{n-1} \left( 1 - \sum_{j=1}^{m} p_j(l) \right) \right. \\
+ \sum_{s=1}^{\infty} p_i(n+k_i+s) \prod_{r=1}^{s} \left( \sum_{j=1}^{m} \sum_{l=n+r}^{n+r+k_i-1} p_j(l) \right) \left. \right], \quad n \geq n_0 + 2k,
\]

and so,

\[
1 \geq \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \prod_{l=s-k_i}^{n-1} \left( 1 - \sum_{j=1}^{m} p_j(l) \right) \right. \\
+ \sum_{s=1}^{\infty} p_i(n+k_i+s) \prod_{r=1}^{s} \left( \sum_{j=1}^{m} \sum_{l=n+r}^{n+r+k_i-1} p_j(l) \right) \left. \right], \quad n \geq n_0 + 2k.
\]
It follows that
\[
\limsup_{n \to \infty} \sum_{i=1}^{m} \left( \prod_{s=1}^{n-k_i} p_i(s) \frac{1 - \sum_{j=1}^{m} p_j(l)}{1 - \sum_{i=1}^{m} k_i p_j} \right)^{-1} \\
+ \sum_{s=1}^{\infty} p_i(n + k_i + s) \prod_{r=1}^{m} \left( \sum_{j=1}^{n+r+k_i-1} \sum_{l=n+r} p_j(l) \right) \leq 1,
\]
which contradicts (3.1) and so the proof is complete. \(\Box\)

Note that when \(\sum_{j=1}^{m} k_j p_j < 1\),
\[
\liminf_{n \to \infty} \sum_{s=1}^{\infty} p_i(n + k_i + s) \prod_{r=1}^{m} \left( \sum_{j=1}^{n+r+k_i-1} \sum_{l=n+r} p_j(l) \right) \leq \sum_{s=1}^{\infty} p_i(n + k_i + s) \prod_{r=1}^{m} \left( \sum_{j=1}^{n+r+k_i-1} \sum_{l=n+r} p_j(l) \right) \geq p_i \sum_{j=1}^{m} k_j p_j / (1 - \sum_{j=1}^{m} k_j p_j).
\]

Therefore, by Theorem 3.1, we have immediately

**Corollary 3.1.** Assume that (1.5) holds, \(\sum_{j=1}^{m} k_j p_j < 1\) and \(\sum_{j=1}^{m} p_j(n) < 1\) for large \(n\), and that
\[
\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{s=n}^{n+k_i} p_i(s) \prod_{l=n-k_i}^{n-1} \left( 1 - \sum_{j=1}^{m} p_j(l) \right)^{-1} > 1 - \sum_{i=1}^{m} p_i \frac{\sum_{j=1}^{m} k_j p_j}{1 - \sum_{j=1}^{m} k_j p_j}.
\]
Then every solution of Eq. (1.2) oscillates.

**Remark 3.1.** If there exists a sequence \(\{n_i\}\) of integers such that \(\sum_{j=1}^{m} p_j(n_i) > 1\) for \(i = 1, 2, \ldots\), it is easy to show that every solution of Eq. (1.2) oscillates. If \(\sum_{j=1}^{m} k_j p_j > 1\), then
\[
\liminf_{n \to \infty} \sum_{i=1}^{m} \sum_{s=n+1}^{n+k_i} p_i(s) \prod_{l=n+1}^{n+k_i} \left( 1 - \sum_{j=1}^{m} p_j(l) \right) > 1 - \sum_{i=1}^{m} p_i \frac{\sum_{j=1}^{m} k_j p_j}{1 - \sum_{j=1}^{m} k_j p_j}.
\]
In view of (1.3), every solution of Eq. (1.2) oscillates.

**Theorem 3.2.** Assume that (1.5), (2.3) and (2.4) hold, and that
\[
\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{s=n}^{n+k_i} p_i(s) k_s^{x-n-k_i} > 1 - \sum_{i=1}^{m} q_i \frac{\mu_i}{1 - \mu_i}.
\]
Then every solution of Eq. (1.2) oscillates.

**Proof.** For the sake of contradiction, assume that Eq. (1.2) has an eventually positive solution \(\{x_n\}\). Then there exists a positive integer \(n_0\) such that
\[
x_n - k > 0 \quad \text{and} \quad x_{n+1} - x_n \leq 0, \quad n > n_0,
\]
and (3.4) hold. From (3.4) and using Lemmas 2.1 and 2.2, we have

\[
1 \geq \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \frac{x_{s-k_i}}{x_n} + \sum_{s=1}^{\infty} p_i(n + k_i + s) \frac{x_{n+s}}{x_n} \right]
\]

\[
= \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=0}^{k_i} p_i(n + s) \frac{x_{n+s-k_i}}{x_n} + \sum_{s=1}^{\infty} p_i(n + k_i + s) \frac{x_{n+s}}{x_n} \right]
\]

\[
\geq \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=0}^{k_i} p_i(n + s) \lambda_+^s - k_i + \sum_{s=1}^{\infty} p_i(n + k_i + s) \mu_+^s \right]
\]

\[
\geq \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \lambda_+^{s-n-k_i} + \sum_{s=1}^{\infty} \liminf_{n \to \infty} p_i(n + k_i + s) \lambda_+^s \right]
\]

\[
\geq \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \lambda_+^{s-n-k_i} + \sum_{s=1}^{\infty} q_i \sum_{s=1}^{\infty} \mu_+^s \right]
\]

\[
= \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \lambda_+^{s-n-k_i} + \sum_{s=1}^{\infty} q_i \frac{\mu_+^{k_i}}{1 - \mu_+} \right]
\]

Thus,

\[
\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{s=n}^{n+k_i} p_i(s) \lambda_+^{s-n-k_i} \leq 1 - \sum_{i=1}^{m} q_i \frac{\mu_+^{k_i}}{1 - \mu_+},
\]

which contradicts (3.6) and so the proof is complete. \(\square\)

**Theorem 3.3.** Assume that (1.5), (2.3) and (2.4) hold, and that

\[
\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{s=n}^{n+k_i} p_i(s) \lambda_+^{s-n-k_i} > 1 - \frac{\mu_+}{1 - \mu_+} \sum_{i=1}^{m} p_i. \quad (3.7)
\]

Then every solution of Eq. (1.2) oscillates.

**Proof.** For the sake of contradiction, assume that Eq. (1.2) has an eventually positive solution \(\{x_n\}\). Then there exists a positive integer \(n_0\) such that

\[
x_{n-k} > 0 \quad \text{and} \quad x_{n+1} - x_n \leq 0, \quad n \geq n_0,
\]

and (3.4) hold. From (3.4) and using Lemmas 2.1 and 2.2, we have

\[
1 \geq \limsup_{n \to \infty} \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_i} p_i(s) \frac{x_{s-k_i}}{x_n} + \sum_{s=1}^{\infty} p_i(n + k_i + s) \frac{x_{n+s}}{x_n} \right]
\]
Thus,
\[
\limsup_{n \to \infty} m \sum_{i=1}^{m} \left[ \sum_{s=0}^{k_{i}} p_{i}(n + s) \frac{x_{n+s-k_{i}}}{x_{n}} + \sum_{s=1}^{\infty} p_{i}(n + k_{i} + s) \liminf_{n \to \infty} \frac{x_{n+s}}{x_{n}} \right]
\geq \limsup_{n \to \infty} m \sum_{i=1}^{m} \left[ \sum_{s=0}^{k_{i}} p_{i}(n + s) \lambda_{s}^{\pm-k_{i}} + p_{i} \sum_{s=1}^{\infty} \mu_{s}^{\pm} \right]
\geq \limsup_{n \to \infty} m \sum_{i=1}^{m} \left[ \sum_{s=n}^{n+k_{i}} p_{i}(s) \lambda_{s}^{\pm-n-k_{i}} + \frac{\mu_{s}^{\pm}}{1 - \nu_{s}^{\pm}} \sum_{i=1}^{m} p_{i} \right].
\]

Thus,
\[
\limsup_{n \to \infty} m \sum_{i=1}^{m} \sum_{s=n}^{n+k_{i}} p_{i}(s) \lambda_{s}^{\pm-n-k_{i}} \leq 1 - \frac{\mu_{s}^{\pm}}{1 - \nu_{s}^{\pm}} \sum_{i=1}^{m} p_{i},
\]
which contradicts (3.7) and so the proof is complete. □

By employing Lemma 2.3 or Corollary 2.1 instead of Lemma 2.2, we can prove the following theorems similarly.

**Theorem 3.4.** Assume that (1.5), (2.3) and (2.4) hold, and that
\[
\limsup_{n \to \infty} m \sum_{i=1}^{m} \sum_{s=n}^{n+k_{i}} p_{i}(s) \lambda_{s}^{\pm-n-k_{i}} > 1 - \sum_{i=1}^{m} q_{i} \frac{k_{i}}{1 - k_{i}}.
\]
Then every solution of Eq. (1.2) oscillates.

**Theorem 3.5.** Assume that (1.5) and (2.4) hold, and that
\[
\limsup_{n \to \infty} m \sum_{i=1}^{m} \sum_{s=n}^{n+k_{i}} p_{i}(s) \lambda_{s}^{\pm-n-k_{i}} > 1 - \frac{1 + d - \sum_{i=1}^{m} p_{i} - \sqrt{(1 + d - \sum_{i=1}^{m} p_{i})^{2} - 4d \sum_{i=1}^{m} p_{i}}}{1 + d - \sum_{i=1}^{m} p_{i} - \sqrt{(1 + d - \sum_{i=1}^{m} p_{i})^{2} - 4d \sum_{i=1}^{m} p_{i}}} \sum_{i=1}^{m} p_{i}.
\]
Then every solution of Eq. (1.2) oscillates.

**Remark 3.2.** Obviously, conditions (3.1), (3.5)–(3.9) improve (3.4). To illustrate the advantage of our results, we give the following example.
Example 3.1. Consider delay difference equation

\[ x_{n+1} - x_n + p_1(n)x_{n-2} + p_2(n)x_{n-3} = 0, \quad n = 0, 1, 2, \ldots, \tag{3.10} \]

where \( k_1 = 2, k_2 = 3, p_1(3n) = a > 0, p_1(3n + 1) = 0, p_1(3n + 2) = 0, p_2(3n) = b > 0, p_2(3n + 1) = 0, p_2(3n + 2) = 0, n = 0, 1, 2, \ldots. \) Observe that

\[ p_1 = \liminf_{n \to \infty} p_1(n) = 0, \quad p_2 = \liminf_{n \to \infty} p_2(n) = 0, \]

\[ q_1 = \liminf_{n \to \infty} \sum_{s=n+1}^{n+2} p_1(s) = 0, \quad q_2 = \liminf_{n \to \infty} \sum_{s=n+1}^{n+3} p_2(s) = b, \]

and

\[ d = \liminf_{n \to \infty} \sum_{i=1}^{2} \sum_{s=n+1}^{n+k_i} p_i(s) b < 1. \]

By a simple calculation, we have

\[ \lambda_s = 1, \quad \nu_s = b, \]

and \( \mu_s \in (b, 1) \) is the smallest root of the equation

\[ \mu(1 - \mu^3) = b. \]

In view of Corollary 3.1, Theorem 3.2 and Theorem 3.4, if \( a + b < 1 \) and one of the following conditions:

\[ a + b + \frac{b}{1 - a - b} > 1, \quad a + 2b > 1 - \frac{b\mu_s^3}{1 - \mu_s^3}, \]

and

\[ a + 2b > 1 - \frac{b^4}{1 - b^3} \]

holds, then every solution of Eq. (3.10) oscillates. Whereas when \( b \leq \frac{81}{256} \) and \( a + 2b \leq 1 \), both conditions (1.3) and (1.4) are not satisfied.

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References


