On the Characters of Nilpotent Blocks over Small Ground-Fields

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I characterize how the irreducible characters of a nilpotent block over a small ground-field are determined one by one by the irreducible characters of its defect groups. © 2001 Academic Press

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1. THE MAIN RESULTS

1.1

Modifying the Frobenius condition for finite \( p \)-nilpotent groups, Broué and Puig in [1] introduce nilpotent blocks, and in the case of large enough ground-fields, they proved that the characters of such a block are determined one-by-one by the characters of its defect group in the way of the so-called \( * \)-structure. Based on the theory of pointed groups, Puig in [7] shows that the Frobenius condition is equivalent to the local control condition and determined the precise structure of nilpotent blocks; the characters of such blocks are recharacterized in [7].

However, if the ground-field is small enough, as shown in [2], the Frobenius condition is weaker than the local control condition; with the latter the nilpotent blocks were defined (see 1.2 below), and the precise structure of such blocks was determined in [2]. Here we determine the characters of such a block: they are still determined one-by-one by the characters of its defect group but in a more delicate way. For notations, we follow [2, 6, 7].

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1.2

Let $\mathcal{A}$ be a complete discrete valuation field of characteristic zero with valuation ring $\mathcal{O}$ having residual field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic $p$, where $p$ is a prime integer. Let $G$ be a finite group and let $\mathcal{O}G$ denote the group algebra. Let $b$ be an $\mathcal{O}$-block of $G$ and let $\mathcal{O}Gb$ be the block algebra. Recall that a local pointed group $Q_\beta$ on $\mathcal{O}Gb$ means a pair $(Q, \beta)$ of a $p$-subgroup $Q$ of $G$ and a conjugacy class $\beta$ of primitive idempotents on the algebra $(\mathcal{O}Gb)^Q$ of the $Q$-fixed elements of $\mathcal{O}Gb$, such that $\text{Br}_\mathcal{O}'(\beta) \neq [0]$, where $\text{Br}_\mathcal{O}'(\beta): (\mathcal{O}G)^Q \to kC_G(Q)$ is the Brauer homomorphism associated with the coefficient $\mathcal{O}$ and the $p$-subgroup $Q$, and a local pointed group $R_\delta$ is said to be included in $Q_\beta$, denoted by $R_\delta \subset Q_\beta$, if $R \subset Q$ and $ji = j = ij$ for some $j \in \delta$ and $i \in \beta$. The maximal local pointed groups $P_\gamma$ on $\mathcal{O}Gb$ are called the defect pointed groups of the block. It is well known that the defect pointed groups $P_\gamma$ of the block form exactly a $G$-conjugate class. The block $b$ is said to be $\mathcal{O}$-nilpotent if for any local pointed group $Q_\beta \subset P_\gamma$ on $\mathcal{O}Gb$ and any $x \in G$ such that $(Q_\beta)^x \subset P_\gamma$, there are $u \in P$ and $z \in C_G(Q)$ such that $x = zu$.

1.3

From now on, we always assume that $b$ is a nilpotent $\mathcal{O}$-block of $G$ and $P_\gamma$ is a defect pointed group of the block $\mathcal{O}Gb$. As usual $Z(\mathcal{O}Gb)$ denotes the center. It is an easy fact (see [3, 2.4.3 and 2.5.2]) that $k' = Z(\mathcal{O}Gb)/J(Z(\mathcal{O}Gb)) = k(\tau)$ is a cyclic Galois extension of $k$ generated by a primitive root $\tau$ of unity of degree prime to $p$. By $\tau$ we also denote the corresponding primitive $p'$-root of unity over $\mathcal{O}$ for short. Let $\mathcal{A}' = \mathcal{A}(\tau)$, $\mathcal{O}' = \mathcal{O}[\tau]$, and $k' = \mathcal{O}'/J(\mathcal{O}')$, which are cyclic extensions of $\mathcal{A}$, $\mathcal{O}$, and $k$, respectively. By tr$_{\mathcal{A}'/\mathcal{A}}(a')$ we denote the relative trace of $a' \in \mathcal{A}'$ over $\mathcal{A}$. It is easy to see that (cf. 2.1 below)

(1.3.1) There are a nilpotent $\mathcal{O}'G$-block $b'$ and a defect pointed group $P_\gamma'$ of $\mathcal{O}'Gb'$ such that $bb' = b'$ and $i = ib' \in \gamma'$ for any $i \in \gamma$. In the following, we fix such an $\mathcal{O}'G$-block $b'$ and an $i \in \gamma$; hence $P_\gamma'$ and $i' = ib' \in \gamma'$ are also fixed.

For a $p$-element $u \in G$, let $T_G(u, P) = \{g \in G | u^g \in P\}$. If $\langle u \rangle_{b'}$ is a local pointed group on $\mathcal{O}'Gb'$, we also write it as $u_{b'}$ and call it a local pointed element on $\mathcal{O}'G$, and we write $u_{b'} \in P_\gamma'$ if $\langle u \rangle_{b'} \subset P_\gamma'$. Since $b'$ is $\mathcal{O}'$-nilpotent, it is known from [3, 2.3.5] that

(1.3.2) For any $g \in T_G(u, P)$ there is a unique local pointed element $u_{\mathcal{O}'G}$ such that $u_{\mathcal{O}'G}^g \in P_\gamma'$.

Let $\text{Br}_\mathcal{O}'(\mathcal{O}'G)^u : k'C_G(u)$ denote the Brauer homomorphism. Then $\text{Br}_\mathcal{O}'(\mathcal{O}'G)^u$ is a conjugacy class of primitive idempotents of $k'C_G(u)$, so
it gives a simple $k'C_G(u)$-module and hence affords an irreducible Brauer character $\varphi_{\tilde{g}(u,g)}$ of $\sigma'C_G(u)$.

**Theorem 1.4.** Notations as above. Let $\text{Irr}(\mathcal{A}Gb)$ and $\text{Irr}(\mathcal{A}'P)$ denote the sets of all the irreducible characters of $\mathcal{A}Gb$ and $\mathcal{A}'P$, respectively. Then there is a bijection $\text{Irr}(\mathcal{A}Gb) \rightarrow \text{Irr}(\mathcal{A}'P)$, $\chi \mapsto \lambda'$, and a class function $\omega: P \to \{\pm 1\}$ such that

$$\chi(us) = \sum_{g \in C_G(u) \setminus \text{Tr}(u,P)/P} \omega(u^g) \cdot \text{tr}_{\mathcal{A}'/\mathcal{A}}(\lambda'(u^g) \varphi_{\tilde{g}(u,g)}(s))$$

for any $p$-element $u \in G$ and any $p'$-element $s \in C_G(u)$.

**1.5**

Let $\varepsilon$ be a primitive $\exp(P)$-th root of unity where $\exp(P)$ denotes the exponent of $P$. Similar to (1.3.2), we have the Brauer character $\varphi_{\tilde{g}(u,g)}$ of $\sigma'C_G(u)$ determined by the unique local pointed element $u_{\tilde{g}(u,g)}$ on $\sigma'G$ such that $(u_{\tilde{g}(u,g)})^\varepsilon \in P$. 

**Corollary 1.6.** If $\mathcal{A}' \cap \mathcal{A}(\varepsilon) = \mathcal{A}$, then

$$\chi(us) = \sum_{g \in C_G(u) \setminus \text{Tr}(u,P)/P} \omega(u^g) \lambda'(u^g) \varphi_{\tilde{g}(u,g)}(s).$$

**1.7. Remark.** The corollary covers at least the following two cases:

1. $(\mathcal{A}, \sigma) = (\mathcal{H}B, \sigma)$ is split for $Z(\mathcal{H}B)$, i.e., $\sigma' = \sigma$;
2. $\sigma = \mathbb{Z}_p$, the ring of $p$-adic integers.

In fact, (1.7.1) obviously implies the hypothesis of 1.6. For (1.7.2), $\mathbb{Q}_p(\tau)$ is a totally unramified extension over $\mathbb{Q}_p$, while $\mathbb{Q}_p(\varepsilon)$ is a totally ramified extension, where $\mathbb{Q}_p$ is the field of the $p$-adic rationals. Hence $\mathbb{Q}_p(\omega) \cap \mathbb{Q}_p(\varepsilon) = \mathbb{Q}_p$; see [8, Chap. IV, Sect. 4, Remark 2].

**1.8. Remark.** For 1.6, in most cases $\text{Irr}(\mathcal{A}'P)$ can be replaced by $\text{Irr}(\mathcal{A}P)$. In fact, under its hypothesis there is clearly a bijection $\text{Irr}(\mathcal{A}'P) \rightarrow \text{Irr}(\mathcal{A}P)$, $\lambda' \mapsto \lambda$, such that $\lambda' = \lambda/s_g(\lambda')$, where $s_g(\lambda')$ is the **relative Schur index** of $\lambda'$ over $\mathcal{A}$ and $s_g(\lambda') = 1$ unless $p = 2$ and $\sqrt{-1} \not\in \mathbb{A}$ and $4 \mid \exp(P)$ (see [5, 10.14]), and in the exceptional case, $s_g(\lambda') \leq 2$.

**1.9. Remark.** It is clear that in the case (1.7.1) the version of 1.4 has the same form as that over large enough ground-fields; see [7, 1.13] or [9, 52.8]. Otherwise the following example shows that, in general, there is no bijection $\text{Irr}(\mathcal{A}Gb) \rightarrow \text{Irr}(\mathcal{A}P)$ making (1.4.1) hold, and the bijection $\text{Irr}(\mathcal{A}Gb) \rightarrow \text{Irr}(\mathcal{A}'P)$ in 1.4 is not unique, which depends in fact on the choice of $b'$ in (1.3.1); cf. 2.1 below.
1.10. Example. Let \( p = 2 \), let \( \mathcal{A} = \mathbb{Q}_p(\sqrt{3}) \), and let \( \mathcal{O} \) be the valuation ring of \( \mathcal{A} \). Let \( G = \langle s \rangle \times \langle u \rangle \) with orders \( |s| = 3 \) and \( |u| = 4 \); let \( b = 1 - (1 + s + s^2)/3 \), which is the non-principal block of \( \mathcal{O}G \). Then \( \mathcal{A}' = \mathcal{A}(\tau) = \mathcal{A}(e) \) since \( \tau = (-1 + \sqrt{3} e)/2 \), where \( \tau \) is a primitive 3rd root of unity and \( e \) is a primitive 4th root of unity. It is clear that

\[
\mathcal{A}G \cong \mathcal{A}(\tau) \otimes \mathcal{A}P,
\]

where \( \mathcal{A}(\tau) \) corresponds to the irreducible Brauer character \( \varphi: s \mapsto \text{tr}_{\mathcal{A}'/\mathcal{A}}(\tau) \). Take the simple \( \mathcal{A}P \)-module \( \mathcal{A}(e) \) such that \( u \mapsto e \); its character is \( \lambda: u \mapsto \text{tr}_{\mathcal{A}'/\mathcal{A}}(e) \). We have the following decomposition of simple \( \mathcal{A}G \)-modules

\[
\mathcal{A}(\tau) \otimes \mathcal{A}(e) \cong V_1 \oplus V_2,
\]

where \( V_1 \cong V_2 \) as \( \mathcal{A} \)-vector spaces, but the representation of \( V_1 \) maps \( u \) to \( \varepsilon t \), while the representation of \( V_2 \) maps \( u \) to \( -\varepsilon t \). In other words, the first one affords the character \( \chi_1: u \mapsto \text{tr}_{\mathcal{A}'/\mathcal{A}}(\varepsilon t) \), while the second one affords the character \( \chi_2: u \mapsto \text{tr}_{\mathcal{A}'/\mathcal{A}}(-\varepsilon t) \). The two different irreducible characters \( \chi_1 \) and \( \chi_2 \) of \( \mathcal{A}G \) correspond to one and the same pair: the irreducible \( \mathcal{A}P \)-character \( \lambda \) and the irreducible Brauer character \( \varphi \).

2. PROOFS OF THE RESULTS

2.1

The notations and assumptions in 1.3–1.5 are preserved throughout. Further, let \( \Gamma' = \text{Gal}(\mathcal{A}'/\mathcal{A}) \) be the Galois group, let \( \mathcal{A}'' = \mathcal{A}(e) \), and let \( \mathcal{O}'' \) be the corresponding valuation ring and let \( \Gamma = \text{Gal}(\mathcal{A}''/\mathcal{A}') \). Since \( b \) is \( \mathcal{O} \)-nilpotent, by [2, 4.3; 4, 5.2] we have

\[
(2.1.1) \quad b = \sum_{\iota \in \Gamma} b'' \] is a sum of \( \mathcal{O}G \)-block idempotents, and any \( b'' \) is \( \mathcal{O} \)-nilpotent.

As emphasized in (1.3.1), we fix such a \( b' \). Further, for any subgroup \( Q \leq G \) and any primitive idempotent \( j \in (\mathcal{O}G)^0 \), by [2, 3.3, 3.4.1, and 5.2.1], we have

\[
(2.1.2) \quad j = \sum_{\iota \in \Gamma} j'' \] is an orthogonal decomposition and \( j' = jb' \) is an absolutely primitive idempotent on \( (\mathcal{O}G)^0 \).

Conversely, if \( j' \in (\mathcal{O}G)^0 \) is a primitive idempotent then, by the above fact, \( j' \) is absolutely primitive and \( j = \sum_{\iota \in \Gamma} j'' \) is a primitive idempotent
of \((\mathcal{O}Gb)^0\) and \(j^* = jb'.\) We state the latter fact in a refined form:

\[ i = \sum_{r \in \Gamma} i'' \text{ is an } \mathcal{O}G\text{-block, and } j_i = \sum_{r \in \Gamma} j'' \text{ is a primitive idempotent of } (\mathcal{O}Gb)^0, \text{ where } \Gamma' \text{ is the centralizer of } \Gamma_i \text{ in } \Gamma'. \]

We will apply it to the fixed \(i \in \gamma\) and \(i' = ib' \in \gamma'\) (see (1.3.1)) and to the local pointed element \(u_{\delta(u, g)}\) and \(u_{\delta' (u, g)}\) (see (1.3.2) and 1.5. In particular, we have the following two conclusions.

\[ i = \sum_{r \in \Gamma} i'' \text{ is an orthogonal decomposition and } i' \text{ is absolutely primitive.} \]

\[ \varphi_{\delta(u, g)} = \sum_{r \in \Gamma} (\varphi_{\delta' (u, g)})' \text{ and } \varphi_{\delta' (u, g)} \text{ is absolutely irreducible.} \]

\[ 2.2 \]

By [2, 1.3] for the source algebra \(i\mathcal{A}Gi\) of the block \(\mathcal{O}Gb\) we have an interior \(P\)-algebra isomorphism

\[ i\mathcal{A}Gi \cong S \otimes_\mathcal{O}P, \quad \text{where } S \cong \mathcal{M}_n(\mathcal{O}'), \]

but the matrix algebra \(S\) is considered as an interior \(P\)-algebra over \(\mathcal{O}\), though it is also an interior \(P\)-algebra over \(\mathcal{O}'\). Extending \(\mathcal{O}\) to \(\mathcal{A}\), we get a \(\mathcal{A}\)-algebra isomorphism

\[ i\mathcal{A}Gi \cong S_K \otimes_\mathcal{A}P, \quad \text{where } S_K \cong \mathcal{M}_n(\mathcal{A}'), \]

because \(\mathcal{A} \otimes_\mathcal{O} \mathcal{O}' = \mathcal{A}'\). Further, by [6, 3.5; 3, 3.2.2] we have

\[ (2.2.3) \text{ It is a Morita equivalence from the algebra } \mathcal{A}Gb \text{ to the algebra } i\mathcal{A}Gi \text{ to send a } \mathcal{A}Gb\text{-module } M \text{ to the } i\mathcal{A}Gi\text{-module } iM. \text{ In particular, } \mathcal{A}Gb \text{ and } i\mathcal{A}Gi \text{ have the same splitting fields.} \]

It is the same that \(\mathcal{A}'Gb'\) is Morita equivalent to

\[ (2.2.4) \quad i'\mathcal{A}'Gi' \cong S'_K \otimes_{\mathcal{A}'} P, \quad \text{where } S'_K \cong \mathcal{M}_n(\mathcal{A}'). \]

Let \(V'\) be the unique (up to isomorphism) simple \(S'_K\)-module. It is clear that

\[ (2.2.5) \text{ Any simple } i'\mathcal{A}'G'i'\text{-module is isomorphic to } V' \otimes_{\mathcal{A}'} N' \text{ for a unique (up to isomorphism) simple } \mathcal{A}'P\text{-module } N' \text{ and vice versa.} \]

At last, \(\mathcal{A}'\) is a splitting field for \(S'_K\), \(\mathcal{A}''\) is obviously a splitting field for \(\mathcal{A}P\), and

\[ (2.2.6) \quad i'\mathcal{A}''Gi' \equiv \mathcal{A}'' \otimes_{\mathcal{A}'} (i'\mathcal{A}'Gi') \equiv S''_K \otimes_{\mathcal{A}''} P, \quad \text{where } S''_K \equiv \mathcal{M}_n(\mathcal{A}''). \]
Hence, by (2.2.3) we have

(2.2.7) Both \( \mathcal{A}''Gb' \) and \( \mathcal{A}''P \) are split algebras.

2.3

Let \( \chi \in \text{Irr}(\mathcal{A}Gb) \) be as in 1.4; let \( M \) be the \( \mathcal{A}Gb \)-module affording \( \chi \). Extending it to a \( \mathcal{A}'G \)-module, by (2.1.1) we have

\[
(2.3.1) \quad \mathcal{A}' \otimes M = \bigoplus_{i \in \Gamma'} b'' \cdot \left( \mathcal{A}' \otimes M \right) = \bigoplus_{i \in \Gamma'} M''',
\]

where \( M' = b' \cdot \left( \mathcal{A}' \otimes M \right) \).

We claim that

(2.3.2) \( M' \) is a simple \( \mathcal{A}'G \)-module.

To see it, first note that the fixed element set \( (\mathcal{A}' \otimes M) \Gamma = M \). Suppose that \( M' = M_1 \oplus M_2 \) is a \( \mathcal{A}'G \)-decomposition, then \( M_1 = \Sigma_{\gamma \in \Gamma} (M_1') \subset M \) and is \( G \)-stable, and it is the same for \( M_2 = \Sigma_{\gamma \in \Gamma} (M_2') \); that is, \( M = M_1 \oplus M_2 \) is a \( \mathcal{A}G \)-decomposition.

Let \( \chi' \) denote the character of \( M' \). Noting that \( b' \) is fixed (see 1.3.1), we have

(2.3.3) There is a bijection \( \text{Irr}(\mathcal{A}Gb) \to \text{Irr}(\mathcal{A}'Gb') \), \( \chi \to \chi' \), such that \( \chi = \oplus_{\gamma \in \Gamma'} \chi'' \).

Moreover, there are a simple \( \mathcal{A}''Gb' \)-module \( M'' \) and a positive integer \( s_{\mathcal{A}'}(M'') \), i.e., the relative Schur index of \( M'' \) over \( \mathcal{A}' \), such that

\[
(2.3.4) \quad \mathcal{A}'' \otimes M' = s_{\mathcal{A}'}(M'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{M'}} M'' \right),
\]

where \( \Gamma_{M'} = \{ r \in \Gamma \mid M'' \cong M'' \} \) is the stabilizer of \( M'' \) in \( \Gamma \). By \( \chi'' \) we denote the character afforded by \( M'' \); then

\[
(2.3.5) \quad \chi' = s_{\mathcal{A}'}(\chi'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{M'}} \chi'' \right),
\]

where \( s_{\mathcal{A}'}(\chi'') = s_{\mathcal{A}'}(M'') \) and \( \Gamma_{\chi'} = \Gamma_{M'} \). Multiplying \( i' \) to the two sides of (2.3.4), we get an equality of modules over \( i'\mathcal{A}''Gi' \cong S'_{\mathcal{A}'} \otimes \mathcal{A}''P \),

\[
(2.3.6) \quad \mathcal{A}'' \otimes i'M' = s_{\mathcal{A}'}(M'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{M'}} i'M'' \right),
\]

and \( i'M'' \) is a simple \( i'\mathcal{A}''Gi' \)-module as \( \mathcal{A}''Gb' \) is Morita equivalent to \( i'\mathcal{A}''Gi' \); see (2.2.3).
2.4

On the other hand, by the same argument, from (2.3.2) we have that $i'M'$ is a simple $i'G'i$-module. So, by (2.2.5) there is a simple $\mathcal{A}'P$-module $N'$ such that

\[(2.4.1) \quad i'M' \cong V' \otimes N'.\]

Similar to (2.3.4), there are a simple $\mathcal{A}''P$-module $N''$ and an integer $s_{\mathcal{A}'}(N'')$ such that

\[(2.4.2) \quad \mathcal{A}'' \otimes N' = s_{\mathcal{A}'}(N'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{N'}} N''^r \right).\]

Combining (2.3.6), (2.4.1), and (2.4.2), we have

\[
s_{\mathcal{A}'}(M'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{M'}} i'M''^r \right) = \mathcal{A}'' \otimes i'M' = \mathcal{A}'' \otimes \left( V' \otimes N' \right) = (\mathcal{A}'' \otimes V') \otimes (\mathcal{A}'' \otimes N') = V'' \otimes \left( s_{\mathcal{A}'}(N'') \cdot \bigoplus_{r \in \Gamma / \Gamma_{N'}} N''^r \right),
\]

where $V'' = \mathcal{A}'' \otimes_{\mathcal{A}'} V'$ is the unique simple $S_{\mathcal{A}'}$-module. Hence we get

\[(2.4.3) \quad s_{\mathcal{A}'}(M'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{M'}} i'M''^r \right) = s_{\mathcal{A}'}(N'') \cdot \left( \bigoplus_{r \in \Gamma / \Gamma_{N'}} V'' \otimes N''^r \right),\]

where $V'' \otimes_{\mathcal{A}'} N''$ is a simple $i'\mathcal{A}''G'i$-module; cf. (2.2.6). Thus there is an $N''^r$ on the right-hand side, say $N''$ up to a replacement, such that

\[(2.4.4) \quad i'M'' \cong V'' \otimes N'',\]

and by (2.2.5) such $N''$ is uniquely determined by $M''$. Since $V''$ is the unique (up to isomorphism) $S_{\mathcal{A}'}$-module, it follows from (2.4.4) that

\[(2.4.5) \quad \Gamma_{M''} = \Gamma_{i'M''} = \Gamma_{V'' \otimes_{\mathcal{A}'} N''} = \Gamma_{N''}.\]

Comparing the numbers of the simple summands of the two sides of (2.4.3), we see that

\[(2.4.6) \quad s_{\mathcal{A}'}(\chi'') = s_{\mathcal{A}'}(M'') = s_{\mathcal{A}'}(\lambda'') = s_{\mathcal{A}'}(\lambda''),\]

where $\lambda''$ is the character afforded by the $\mathcal{A}''P$-module $N''$. On the other hand, let $\lambda' \in \text{Irr}(\mathcal{A}'P)$ be the character afforded by the $\mathcal{A}'P$-module $N'$;
then (2.4.2) turns out that
\[(2.4.7) \quad \lambda' = s_{\mathcal{A}}'(\lambda'') \cdot \left( \bigoplus_{r \in \Gamma_{\mathcal{A}}/\Gamma_{\mathcal{B}}} \lambda''^r \right),\]
where (the second equality is by (2.4.5))
\[(2.4.8) \quad \Gamma_{\mathcal{A}'} = \Gamma_{\mathcal{M}} = \Gamma_{\mathcal{M}'} = \Gamma_{\mathcal{A}''}.\]

2.5

Now we have enough information to complete a proof of Theorem 1.4. Since both the algebras $\mathcal{A}'' Gb'$ and $\mathcal{A}'' P$ are split (see (2.2.7)), by [7, 1.13.1] (cf. 1.1) there are a bijection determined by (2.4.4) $\text{Irr}(\mathcal{A}'' Gb') \rightarrow \text{Irr}(\mathcal{A}'' P)$, $\chi'' \mapsto \lambda''$, and a class function $\omega: P \rightarrow \{\pm 1\}$ such that (recall from (2.1.5) that $\varphi_{\subseteq (u, g)}$ is absolutely irreducible)
\[\chi''(us) = \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \lambda''(u^g) \varphi_{\subseteq (u, g)}(s).\]
Combining it with (2.3.5) and (2.4.6)--(2.4.8), we have the computation
\[\chi'(us) = s_{\mathcal{A}}'(\chi'') \cdot \sum_{r \in \Gamma_{\mathcal{A}}/\Gamma_{\mathcal{B}}} \chi''(us)^r\]
\[= s_{\mathcal{A}}'(\chi'') \cdot \sum_{r \in \Gamma_{\mathcal{A}}/\Gamma_{\mathcal{B}}} \left( \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \lambda''(u^g) \varphi_{\subseteq (u, g)}(s) \right)^r\]
\[= \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \left( s_{\mathcal{A}}'(\chi'') \cdot \sum_{r \in \Gamma_{\mathcal{A}}/\Gamma_{\mathcal{B}}} (\lambda''(u^g))^r \varphi_{\subseteq (u, g)}(s) \right)\]
\[= \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \lambda''(u^g) \varphi_{\subseteq (u, g)}(s).\]
Hence, by (2.3.3) and the definition of the relative traces, we have
\[\chi(us) = \sum_{t \in \Gamma_{\mathcal{A}}'} (\chi'(us))^t\]
\[= \sum_{t \in \Gamma_{\mathcal{A}}'} \left( \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \lambda'(u^g) \varphi_{\subseteq (u, g)}(s) \right)^t\]
\[= \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \cdot \sum_{t \in \Gamma_{\mathcal{A}}'} (\lambda'(u^g))^t \varphi_{\subseteq (u, g)}(s)^t\]
\[= \sum_{g \in C_0(u) \setminus T_0(u, P)/P} \omega(u^g) \cdot \text{tr}_{\mathcal{A}''/\mathcal{A}}(\lambda'(u^g) \varphi_{\subseteq (u, g)}(s)).\]
Thus the proof of Theorem 1.4 is completed.
Further, for \( \lambda' \in \text{Irr}(\mathcal{H}' P) \), let \( \mathcal{X}_\lambda' \) be the extension of \( \mathcal{X} \) generated by \( \lambda'(u) \) for all \( u \in P \), and let \( \mathcal{O}_{\mathcal{X}_\lambda} \) be the valuation ring of \( \mathcal{X}_\lambda' \). Then \( \mathcal{X}_\lambda' \) is a subfield of \( \mathcal{X}' \) and hence also a cyclic extension over \( \mathcal{X} \) and the centralizer \( \Gamma'_{\mathcal{X}_\lambda} = \text{Gal}(\mathcal{X}' / \mathcal{X}_\lambda) \); by (2.1.3) we have

(2.6.1) There are a unique \( \mathcal{O}_{\mathcal{X}} G \)-block \( b_\lambda \) and a unique defect pointed group \( P_{\gamma_\lambda} \) of \( \mathcal{O}_{\mathcal{X}} G b_\lambda \) such that \( b_\lambda b' = b' \) and \( i_{\gamma} = i_{\gamma' \lambda} \in \gamma_\lambda \) and a unique local pointed element \( u_{b_\lambda(u, g)} \) on \( \mathcal{O}_{\mathcal{X}} G \) such that \((u_{b_\lambda(u, g)})^g \in P_{\gamma_\lambda}\) and the corresponding Brauer character \( \phi_{b_\lambda(u, g)} \) of \( \mathcal{O}_{\mathcal{X}} G(u) \) fulfills that

Thus, we have

\[
\text{tr}_{\mathcal{X}' / \mathcal{X}}(\lambda'(u^g) \phi_{b_\lambda(u, g)}(s)) = \text{tr}_{\mathcal{X}' / \mathcal{X}_\lambda}(\lambda'(u^g) \phi_{b_\lambda(u, g)}(s)) = \text{tr}_{\mathcal{X}' / \mathcal{X}}(\lambda'(u^g) \cdot \text{tr}_{\mathcal{X}' / \mathcal{X}_\lambda}(\phi_{b_\lambda(u, g)}(s))) = \text{tr}_{\mathcal{X}' / \mathcal{X}_\lambda}(\lambda'(u^g) \phi_{b_\lambda(u, g)})
\]

Therefore, we refine the formula (1.4.1) as follows.

2.7. Corollary. Notations are as in 1.4 and 2.6. Then

\[
(2.7.1) \quad \chi(us) = \sum_{g \in \mathcal{C}(u) \setminus \mathcal{T}(u, P)/P} \omega(u^g) \cdot \text{tr}_{\mathcal{X}' / \mathcal{X}}(\lambda'(u^g) \phi_{b_\lambda(u, g)}(s)).
\]

2.8

At last, it is clear that \( \lambda'(u) \in \mathcal{X}' \cap \mathcal{X}(e) \) for all \( u \in P \); i.e., \( \mathcal{X}_\lambda \subset \mathcal{X}' \cap \mathcal{X}(e) \). Then Corollary 1.6 follows immediately from 2.7.

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REFERENCES