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Maximal Subgroups of Infinite Index in Finitely Generated Linear Groups

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INTRODUCTION

This paper arose from the following problem posed by Platonov in [4]: describe all maximal subgroups of $SL_n(\mathbf{Z})$, where $n \geqslant 3$; among these are there any with infinite index? A negative answer to the question would have been of great significance to the description of maximal subgroups in $SL_n(\mathbf{Z})$. But we have shown that: (i) the indices of all maximal subgroups in a finitely generated linear group G over a field are finite if and only if G is solvable-by-finite (in particular, for $n \geqslant 2$ there exists a maximal subgroup of $SL_n(\mathbf{Z})$ of infinite index) and moreover (ii) if a finitely generated linear group over a field is not solvable-by-finite, then the set of its maximal subgroups of infinite index is uncountable; (iii) the group $SL_n(\mathbf{Z})$ has for $n \geqslant 4$ a maximal subgroup of infinite index, which does not contain any free group of finite index. Our paper is devoted to a demonstration of these three main results. We prove our theorems constructing proper subgroups which intersect every residue class modulo any subgroup of finite index. Although essentially we use Tits' techniques developed in [10], the most important step of which is a reduction to linear groups over a local field, in the general case, when the Zariski closure of the group is not necessarily connected, the situation is considerably more complicated and requires some new ideas and methods.

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0. Notation and Terminology

For fundamental definitions of algebraic group theory, group theory and Lie group theory the reader is referred to [2, 5, 7]. The letters $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}_p$ will as usual denote, respectively, the set of integers, non-negative integers, rationals, reals, complex numbers, p-adic numbers. We consider the dimension dim in the sense of the theory of algebraic varieties. The index of a subgroup H of a group G will be denoted by $|G/H|$. If G is a group and $S \subset G$, then by $N_G(S)$, $C_G(S)$ and by $\langle S \rangle$ we denote the normalizer, the centralizer of S in G and the subgroup of G generated by S. The center of G will be denoted by $C(G)$. We denote as usual by GL_n and SL_n the groups of invertible and unimodular $n \times n$ matrices. An algebraic subgroup of GL_n over a field k is called an algebraic k-group or a k-group [2]. The set of all k-points of an algebraic variety W will be denoted by $W(k)$. We denote the maximal unipotent normal subgroup of a k-group G by $M_u(G)$. For an algebraic group G, G^0 will denote the connected component of G containing the identity. We do not assume semi-simple groups to be connected and we say that G is *simple* as soon as G^0 has no infinite proper normal subgroup. If f: $G \to H$ is a k-rational homomorphism of k-groups and the field l is an extension of k, then the natural homomorphism $f(l)$: $G(l) \to H(l)$ will be denoted by f as well. Let V be a finite-dimensional vector space over a field k, $GL(V)$ and $\mathbf{GL}(V)$ will denote the group of its automorphisms and the corresponding k-group. A subgroup G of $GL(V)$ is called irreducible if there is no proper G-invariant subspace of V. A subgroup G of $GL(V)$ is called absolutely irreducible if it remains irreducible for every algebraic extension of k. According to that we call a representation f: $G \to GL(V)$ irreducible (absolutely irreducible) if the image $f(G)$ is irreducible (absolutely irreducible). A local field will mean in the sequel a non-discrete locally compact field. By the Kovalski–Pontrjagin theorem every local field is isomorphic to one of the following four valued field: $\mathbf{R}$, $\mathbf{C}$, a finite extension of $\mathbf{Q}_p$, a field of formal power series in one variable with coefficients from some finite field. If k is a local field and W is an algebraic k-variety, then $W(k)$ has two natural topologies induced by the topology of k and by the Zariski topology of W. To avoid confusion the second topology will be distinguished by the prefix "k," e.g., k-open, k-dense, etc. The closure and the interior of a subset X of a topological space will be denoted by $\operatorname{cl} X$ and $\operatorname{Int} X$ and the closure in the Zariski topology of a subset S of an algebraic variety will be denoted by $\mathscr{A}(S)$. The characteristic of a field k will be denoted by $\operatorname{char} k$ and the cardinality of a set X by $\operatorname{card} X$.

1. TRANSFORMATIONS WITH POINTS OF ATTRACTION AND REPULSION

In this section we denote by k a local field with the absolute value $|\cdot|$, by V a finite-dimensional vector space over k and by P the projective space based on V, both equipped with topologies induced in the usual way from that of k. Let $\dim V = n$ and $x = (x_1,..., x_n)$ be a system of linear coordinates for V. We define the distance function on V which is consistent with its topology as follows: $d_x(p, q) = \max_{1 \leqslant i \leqslant n} |x_i(p) - x_i(q)|$, $(p, q \in V)$. There exists a distance function d: $P \times P \to R_+$ on P, which is said to be admissible; that is, for every affine system of coordinates x in P [10] and every compact subset B in its domain there exist m and M in R_+ such that $m \cdot d_x|_{B\times B} \leqslant d|_{B\times B} \leqslant M \cdot d_x|_{B\times B}$. As was observed in [10], if a field k' is an extension of k, if $V' = V \otimes_k k'$ and P' is the projective space based on V' while d': $P' \times P' \to R_+$ is an admissible distance function on P', then the restriction $d'|_{P\times P}$ is an admissible distance function on P. So when proving assertions not dependent on the admissible distance function d on P we shall assume that it can be extended to an admissible distance function d' on P'. Let $g \in GL(V)$ and let $f(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$ be the characteristic polynomial of the transformation g. The set $\{\lambda_i: |\lambda_i| = \max_{1\leqslant j \leqslant n} |\lambda_j|\}$ will be denoted by $\Omega(g)$. Let $f_1(\lambda) = \prod_{\lambda_i \in \Omega(g)} (\lambda - \lambda_i)$ and $f_2(\lambda) = \prod_{\lambda_i \notin \Omega(g)} (\lambda - \lambda_i)$. Let us define by $A(g)$ and $A'(g)$ the subspaces of P corresponding to the kernels of the endomorphisms $f_1(g)$ and $f_2(g)$, respectively. Let $\mathrm{Cr}(g) = A'(g) \cup A'(g^{-1})$. For every subset S of $GL(V)$ we consider the following three sets: $\Omega_+(S) = \{g \in S: A(g) \text{ is a singleton}\}$, $\Omega_-(S) = \{g \in S: A(g^{-1}) \text{ is a singleton}\}$ and $\Omega_0(S) = \Omega_+(S) \cap \Omega_-(S)$. The projective transformation of P corresponding to the linear transformation $g \in GL(V)$ will be denoted by $\hat{g}$. The absolute value of a linear transformation is defined naturally. As for the value of a projective transformation $\hat{g}$ it is defined in the following way: if d: $P \times P \to R_+$ is an admissible distance function on P and $X \subset P$, then the norm of $\hat{g}$ on X relatively to d is $\|\hat{g}|_X\|_d = \sup\{d(g(x), g(x'))/d(x, x'): x \neq x', x, x' \in X\}$ and $=0$ if card $X \leqslant 1$. In the last part of Section 1 and in Section 2 an admissible distance function d will be fixed and the norm $\|\cdot\| = \|\cdot\|_d$ will be considered. We require the following two propositions which are proved in [10, Lemmas 3.7, 3.8].

LEMMA 1. *Let V be a vector space over k, let x be a system of linear coordinates of V, $r \in R_+$ and $g \in GL(V)$.*

(i) *If every eigenvalue of the transformation g is less than 1 in absolute value, then there exists an integer N_0 such that $\|g^z\|_{d_x} \leqslant r$ for each $z > N_0$, $z \in \mathbf{Z}$. For a compact set $K \subset V$ and for every neighbourhood U of zero there exists an integer N_1 such that $g^z K \subset U$ for each $z > N_1$, $z \in \mathbf{Z}$.*

(ii) *If there exists a set X linearly generating V such that, for every*

$v \in X$, $\lim_{z\to\infty} g^z v = 0$, *then each eigenvalue of g is less than* 1 *in absolute value.*

LEMMA 2. *Let* $g \in GL(V)$, *K be a compact subset of P*, $r \in \mathbf{R}_+$.

(i) *If* $A(g)$ *is a singleton and* $K \subset P \backslash A'(g)$, *then there exists an integer* N_0 *such that* $\| \hat{g}^z |_K \| < r$ *for each* $z > N_0$, $z \in \mathbf{Z}$ *and for every neighbourhood U of* $A(g)$ *there exists an integer* N_1 *such that* $\hat{g}^z K \subset U$ *for each* $z > N_1$, $z \in \mathbf{Z}$.

(ii) *If for some* $m \in \mathbf{N}$, $\hat{g}^m K \subset \operatorname{Int} K$ *and* $\| \hat{g}^m |_K \| < 1$, *then* $A(g)$ *consists of a single point which belongs to* Int *K*.

The join of two linear subspaces X and Y of P is denoted as usual by $X \vee Y$. If $X \cap Y = \varnothing$ and $X \vee Y = P$, then by $P(X, Y)$ we denote the map π: $P \backslash X \to Y$, where $\pi(p) = (\{p\} \vee X) \cap Y$. The following simple assertion is a generalization of item 1 from Lemma 3.9 in [10].

LEMMA 3. *If* h_i *is a semi-simple element of* $GL(V_i)$, *where* V_i *is a vector space over k*, $i = 1, \ldots, s$, *then there exists an infinite subset* $N \subset \mathbf{N}$ *such that* $\lim_{z\to\infty} (\lambda\mu^{-1})^z = 1$ *for* $z \in N$ *and for every* $\lambda, \mu \in \Omega(h_i)$, $i = 1, \ldots, s$.

Proof. Induction on the number of elements. The base of the induction is item 1 of Lemma 3.9 in [10]. Suppose given a subset $N' \subset \mathbf{N}$ such that $\operatorname{card} N' = \aleph_0$ and $\lim_{z\to\infty} (\lambda\mu^{-1})^z = 1$ for $z \in N'$, $\lambda, \mu \in \Omega(h_i)$, $i = 1, \ldots, s-1$. Let $\lambda, \mu \in \Omega(h_s)$. Since $|\lambda\mu^{-1}| = 1$, we have that $\operatorname{cl}_{k^*}\{(\lambda\mu^{-1})^z \colon z \in N'\}$ is compact and so there exists a sequence $\{z_m\}_{m\in\mathbf{N}}$, $z_m \in N'$ such that $z_m \to \infty$ when $m \to \infty$ and the sequence $\{(\lambda\mu^{-1})^{z_m}\}_{m\in\mathbf{N}}$ is convergent. We construct the sequence $\{z_{m_n}\}_{n\in\mathbf{N}}$ in such a way that with $u_n = z_{m_n} - z_{m_{n-1}}$ we have $u_n \to \infty$ when $n \to \infty$. It is obvious that the set $N = \{u_n \colon n \in \mathbf{N}\}$ satisfies all the conditions of Lemma 3.

For a vector space V we consider the following set $\Delta_V = \{(g, h)$: h is a semi-simple, non-trivial automorphism of V, $g \in GL(V)$ is such that $A(g)$ is a singleton, $A(g) \subset P \backslash A(h)$ and $\pi A(g) \subset P \backslash A'(g)$, where $\pi = P(A'(h), A(h))\}$.

LEMMA 4. *Let* V_i *be a vector space over k*, $g_i, h_i \in GL(V_i)$ *and* $(g_i, h_i) \in \Delta_{V_i}$. *There exists an integer* N_0 *and an infinite subset* $N \subset \mathbf{N}$ *such that* $A(g_i^s h_i^n)$ *is a singleton for each* $s > N_0$, $n \in N$ *and* $i = 1, \ldots, m$.

Proof. Let $\pi_i = P(A'(h_i), A(h_i))$, $i = 1, \ldots, m$. It follows from the definition of Δ_{V_i} that there exist compacta $K_i \subset P$ such that $A(g_i) \subset \operatorname{Int} K_i$, $K_i \subset P \backslash A'(g_i)$ and $\pi_i(K_i) \subset P \backslash A'(g_i)$. We can choose for the elements $h_1, \ldots, h_m$ an infinite subset $M \subset \mathbf{N}$ satisfying the conditions of Lemma 3. The set $F = \{\| \hat{h}_i^z |_{K_i} \| \colon z \in M, i = 1, \ldots, m\}$ is bounded by item 2 of Lemma 3.9 in [10], so let $r = \sup F$. Since for each $i = 1, \ldots, m$, $\pi_i(K_i)$ is compact and

$\pi_i(K_i) \subset P \backslash A'(g_i)$, we can take a neighbourhood U_i of $\pi_i(K_i)$ such that $\mathrm{cl}\, U_i \subset P \backslash A'(g_i)$ and so by (i) of Lemma 2 there exists an integer N_0 such that $\| \hat{g}_i^z |_{U_i} \| < 1/r$ for each $z > N_0$, $1 \leqslant i \leqslant m$. By Lemma 3 and again by item 2 of Lemma 3.9 [10] there exists an infinite subset $N \subset M$ such that $\hat{h}_i^z K_i \subset U_i$ for each $z \in N$ and $1 \leqslant i \leqslant m$. So by the definition of $r \in R_+$ it follows that for every $s > N_0$, $n \in N$ and $i = 1, \ldots, m$ we have that $\| \widehat{g_i^s h_i^n} |_{K_i} \| < 1$ and $\widehat{g_i^s h_i^n} K_i \subset \mathrm{Int}\, K_i$. Hence, by (ii) of Lemma 2, $A(g_i^s h_i^n)$ is a singleton for each $s > N_0$, $n \in N$, $i = 1, \ldots, m$ which completes the proof of Lemma 4.

The following assertion is evident.

LEMMA 5. *Let* $g, h \in GL(V)$, *then* $A(ghg^{-1}) = gA'(h)$, $A'(ghg^{-1}) = gA'(h)$ *and, for all* $z \in \mathbf{N}$, $A(h^z) = A(h)$, $A'(h^z) = A'(h)$.

LEMMA 6. *If, for a subgroup* S *of* $GL(V)$, $\Omega_+(S) \neq \varnothing$ *and* $G = \mathscr{A}(S)$, *then the set* $\Omega_+(S) \cap G^0$ *is k-dense in* $G^0 \cap S$.

Proof. If $g \in \Omega_+(S)$, then, by Lemma 5, $A(g^m)$ is a singleton for each $m \in \mathbf{N}$. So, since $G^0 \cap S$ is a subgroup of finite index in S, $\Omega_+(G^0 \cap S) \neq \varnothing$. Hence, without loss of generality, we can assume that S is k-connected. Let us choose g and consider the set $U = \{x \in S \colon \hat{x}A(g) \not\subset A'(g)\}$. Obviously U is k-open in S and $U \neq \varnothing$, since $g \in U$. Therefore the k-connectedness of S implies that U is k-dense in S. Since $A(g)$ is a singleton, for $h \in U$ we have $\hat{h}A(g) \subset P \backslash A'(g)$ and hence there exists a compact set $K \subset P$ such that $A(g) \subset \mathrm{Int}\, K$ and $\hat{h}K \subset P \backslash A'(g)$. From [10, Lemma 3.5] it follows that there exists a number $r \in R_+$ such that $\| \hat{h} |_K \| < r$. Let us consider the compact set $L = \hat{h}K$. Since $L \subset P \backslash A'(g)$, by virtue of (i) of Lemma 2, there exists an integer N_0 such that $\| \hat{g}^z |_L \| < 1/r$ and $\hat{g}^z L \subset \mathrm{Int}\, K$ for $z > N_0$. Let $N \subset \mathbf{N}$ be given so that the set $\{g^z \colon z \in N\}$ is k-connected and let $N' = \{m \in N \colon m > N_0\}$. By the definition of $r \in R_+$ we have $\| \widehat{g^z h} |_K \| < 1$ and $\widehat{g^z h} K \subset \mathrm{Int}\, K$ for each $z \in N'$, therefore (ii) of Lemma 2 implies that $A(g^z h)$ is a singleton for each $z \in N'$. Since $N \backslash N'$ is finite and the set $\{g^z \colon z \in N\}$ is k-connected, $g^z h \in \mathscr{A}(\Omega_+(S))$ for each $z \in N$. That is true for every $h \in U$ and, since $\mathscr{A}(U) = G$, we have $\mathscr{A}(\Omega_+(S)) = G$. This completes the proof.

2. FREE SYSTEMS OF A LINEAR GROUP

In this section we shall use the same notation as before and we shall suppose that the subgroup S of $GL(V)$ satisfies the following three conditions:

(1) $\Omega_+(S) \neq \varnothing$,

(2) The Zariski closure G of S is a semi-simple group,

(3) $G^0 \cap S$ is an absolutely irreducible subgroup of $GL(V)$. We recall for convenience the following simple observation [10, Lemma 3.10].

LEMMA 7. *If G is an irreducible subgroup of $GL(V)$, if P_1 is a non-trivial subspace and P_2 is a proper subspace of P, then $M = \{g \in G: \hat{g}P_1 \not\subset P_2\}$ is a k-open non-empty subset of G.*

LEMMA 8. *Let H be a subgroup of S such that $\Omega_+(H) \neq \varnothing$ and $H \cap G^0$ is a k-dense subgroup of $S \cap G^0$. If $g \in S$, then $Hg \cap \Omega_0(S) \neq \varnothing$.*

Proof. Let us consider the subgroup $H_1 = H \cap G^0$ of H. Since H_1 is a k-dense subgroup of $S \cap G^0$, condition (3) on S implies that its k-connected subgroup H_1 does not leave invariant any proper non-trivial subspace of V. Besides, since $|H/H_1| < \infty$ and $\Omega_+(H) \neq \varnothing$, by virtue of Lemma 5 we have $\Omega_+(H_1) \neq \varnothing$ and therefore by virtue of [10, Lemma 3.11] the set $\Omega_0(H_1)$ is not empty. Let us consider for $h \in \Omega_0(H_1)$ the following set $U = \{x \in H_1: \widehat{xg}A(h) \subset P \backslash A'(h) \ \& \ (\widehat{xg})^{-1} A(h^{-1}) \subset P \backslash A'(h^{-1})\}$. Applying Lemma 7 to H_1 we see that $U \neq \varnothing$. Let $x \in U$ and $h_1 = xg$. It is obvious that $h_1 \in Hg$. From the choice of x and h_1 it follows that $\hat{h}_1 A(h) \subset P \backslash A'(h)$ and $\hat{h}_1^{-1} A(h^{-1}) \subset P \backslash A'(h^{-1})$. Hence there exist compacta K and M in P such that $A(h) \subset \operatorname{Int} K$, $A(h^{-1}) \subset M$ and $\hat{h}K \subset P \backslash A'(h)$, $\hat{h}^{-1} M \subset P \backslash A'(h^{-1})$. Let $K_1 = \hat{h}_1 K$ and $M_1 = \hat{k}_1^{-1} M$. By [10, Lemma 3.5] there exists a number $r \in R_+$ such that $\max\{\|\hat{h}_1|_K\|, \|\hat{h}_1^{-1}|_M\|\} < r$. Since $K_1 \subset P \backslash A'(h)$ and $M_1 \subset P \backslash A'(h^{-1})$, from (i) of Lemma 2 it follows that there exists an integer N_1 such that $\|\hat{h}^z|_{K_1}\| < 1/r$ and $\|\hat{h}^{-z}|_{M_1}\| < 1/r$ for each $z > N_1$. Moreover, there exists an integer N_2 such that $\hat{h}^z K_1 \subset \operatorname{Int} K$ and $\hat{h}^{-z} M_1 \subset \operatorname{Int} M$ for each $x > N_2$. By the choice of $r \in R_+$ we have that $\widehat{h^z h_1} K \subset \operatorname{Int} K$, $\|\widehat{h^z h_1}|_K\| < 1$, $\|\widehat{h^{-z} h_1^{-1}}|_M\| < 1$ and $\widehat{h^{-z} h_1^{-1}} M \subset \operatorname{Int} M$ for each $z > \max(N_1, N_2)$. So from (ii) of Lemma 2 it follows that, for each $z > \max(N_1, N_2)$, $A(h^z h_1)$ and $A(h^{-z} h_1^{-1})$ are singletons. Since the elements $h^{-z} h_1^{-1}$ and $h_1^{-1} h^{-z}$ are conjugate, it follows from Lemma 5 that $A(h_1^{-1} h^{-z})$ is a singleton. Thus $h^z h_1 \in \Omega_0(S)$ for each $z > \max(N_1, N_2)$ and since $h \in H_1$, we have $h^z h_1 \in Hg$. This completes the proof of Lemma 8.

LEMMA 9. *If H is a subgroup of finite index in S and $g \in S$, then $Hg \cap \Omega_0(S) \neq \varnothing$.*

Proof. Let $y \in \Omega_+(S)$. Since $|S/H| < \infty$, we have $y^m \in H$ for some integer m and therefore by Lemma 5, $\Omega_+(H) \neq \varnothing$. On the other hand H is k-open and hence $H \cap G^0$ is a k-dense subgroup of $S \cap G^0$. Thus H satisfies all the conditions of Lemma 8 and the proof is complete.

DEFINITION 1. Let G be a subgroup of $GL(V)$ and $g \in \Omega_0(G)$. We say that the set $F = \{g_i \in G: i \in A\}$ is a *free system* for G relative to g (or simply a free system, if it is clear which group and element are considered) if $g_i \in \Omega_0(S)$ for each $i \in A$ and there exist open sets $O_i = O_i(F) \subset P$, $i \in A$ and a compact set $K = K(F) \subset P$ such that:

(1) $A(g_i) \cup A(g_i^{-1}) \subset O_i$ for each $i \in A$,

(2) $\mathrm{cl} \bigcup_{i \in A} O_i \subset P \backslash \mathrm{Cr}(g)$,

(3) $\hat{g}_i^z O_j \subset O_i$ for every $i, j \in A$, $i \neq j$ and $z \in \mathbf{Z}$, $z \neq 0$,

(4) $\hat{g}_i^z K \subset O_i$ for every $i \in A$ and $z \in \mathbf{Z}$, $z \neq 0$,

(5) $K \subset P \backslash \mathrm{cl} \bigcup_{i \in A} O_i$,

(6) $A(g) \subset \mathrm{Int}\, K$.

LEMMA 10. *Let G be a subgroup of $GL(V)$, $g \in \Omega_0(G)$ and let $F = \{g_i : i \in A\}$ be a free system for G relative to g. If $H = \langle F \rangle$, then H is freely generated by F.*

Proof. See [10, Proposition 1.1].

Let us denote the set $\Omega_0(GL(V))$ by Ω_0. We also use the notation $X_V = \{(g_1, g_2) : g_1, g_2 \in \Omega_0, A(g_i) \cup A(g_i^{-1}) \subset P \backslash \mathrm{Cr}(g_j), i \neq j, 1 \leqslant i, j \leqslant 2\}$. It can easily be proved, making use of Lemma 7, that:

LEMMA 11. *If G is a k-connected irreducible subgroup of $GL(V)$ and $g_i \in \Omega_0$, $i = 1, 2$, then the set $U = \{x \in G : (xg_1x^{-1}, g_2) \in X_V\}$ is k-open and k-dense in G.*

PROPOSITION 1. *Let $g \in \Omega_0(S)$ and let $F_m = \{y_i \in S : 1 \leqslant i \leqslant m\}$ be a free system for S relative to g. If H is a subgroup of finite index in S and $y \in S$, then there exists $y_{m+1} \in Hy$ such that the set $F_{m+1} = \{y_{m+1}\} \cup F_m$ is a free system for S relative to g.*

Proof. Since every subgroup of finite index contains a normal subgroup of finite index, we may assume without loss of generality that H is normal. By virtue of Lemma 9, $Hy \cap \Omega_0(S) \neq \varnothing$. If $h \in Hy \cap \Omega_0(S)$, then, by virtue of Lemma 11 and condition (3) on S, the set $U = \{x \in S \cap G^0 : (xhx^{-1}, h) \in X_V\}$ is k-open and k-dense in $S \cap G^0$. On the other hand, since the index $|S/H|$ is finite, we have that $H \cap G^0$ is a k-dense subgroup of $S \cap G^0$ and therefore $U \cap H \neq \varnothing$. Let $x \in H \cap U$ and $h_1 = xhx^{-1}$. Since H is normal, $h_1 \in Hy$. Since $F_m = \{y_1, \dots, y_m\}$ is a free system, there exist open sets $O_i(F_m)$, $i = 1, \dots, m$ and a compact set $K = K(F_m)$ in P satisfying the conditions of Definition 1. It follows from $(h_1, g) \in X_V$ that there exists an open subset $E \subset P$ such that $A(g^{-1}) \subset E$ and $\mathrm{cl}\, E \subset P \backslash \mathrm{Cr}(h_1)$. From (2) of Definition 1 it follows by Lemma 2 that there exists an integer N_1 such that $\hat{g}^{-z}(\bigcup_{i=1}^m \mathrm{cl}\, O_i) \subset E$ for each $z > N_1$ and $z \in \mathbf{Z}$. Since $(h_1, g) \in X_V$ we have by the definition of X_V that $A(h_1) \cup A(h_1^{-1}) \subset P \backslash A'(g)$ and therefore, by virtue of (i) of Lemma 2, there exists an integer N_2 such that $\hat{g}^z(A(h_1) \cup A(h_1^{-1})) \subset \operatorname{Int} K$ for each $z > N_2$, $z \in \mathbf{Z}$. Since $|S/H| < \infty$, there exists an integer $k > \max(N_1, N_2)$ such that $g^k \in H$. Let $h_2 = g^k h_1 g^{-k}$. Since H is a normal subgroup of S, $h_2 \in Hy$. On the other hand, from the choice of N_1 and N_2 and Lemma 5, it follows that (a) $\operatorname{cl} O_i \subset P \backslash \operatorname{Cr}(h_1)$ and (b) $A(h_1) \cup A(h_1^{-1}) \subset \operatorname{Int} K$. Since $(h_1, g) \in X_V$ and $h_2 = g^k h_1 g^{-k}$, we have $(h_2, g) \in X_V$. It follows that $A(h_2) \cup A(h_2^{-1}) \subset P \backslash \operatorname{Cr}(g)$, $A(g) \cup A(g^{-1}) \subset P \backslash \operatorname{Cr}(g)$ and $h_2 \in \Omega_0(S)$. From the second inclusion it follows, since $A(g) \subset \operatorname{Int} K$, that $A(g) \subset W$ and $\operatorname{cl} W \subset (\operatorname{Int} K) \backslash \operatorname{Cr}(h_2)$ for some open subset W of P. Hence, the first inclusion and (b) imply the existence of an open set $O_{m+1} \subset \operatorname{Int} K$ such that $O_{m+1} \subset (\operatorname{Int} K) \backslash \operatorname{cl} W$ and $A(h_2) \cup A(h_2^{-1}) \subset O_{m+1}$. From (a) and the definition of W it follows by (i) of Lemma 2 that there exists an integer N_3 such that for every $z \in \mathbf{Z}$, $z^2 > N_3^2$ we have $\hat{h}_2^z \operatorname{cl} W \subset O_{m+1}$ and (c) $\hat{h}_2^z(\bigcup_{i=1}^m \operatorname{cl} O_i) \subset O_{m+1}$. On the other hand the set $N = \{z \in \mathbf{Z}: h_2^z \in Hy\}$ is infinite, since $|S/H| < \infty$, and therefore there exists an integer $z_0 \in N$ such that $z_0^2 > N_3^2$. Let $y_{m+1} = h_2^{z_0}$. Since $h_2 \in \Omega_0(S)$, we have that $y_{m+1} \in Hy$. From (c) and the choice of O_{m+1} it follows by Lemma 5 that the open sets $O_1(F_{m+1}) = O_1(F_m), \ldots, O_m(F_{m+1}) = O_m(F_m)$, $O_{m+1}(F_{m+1}) = O_{m+1}$ and the compact set $K(F_{m+1}) = \operatorname{cl} W$, where $F_{m+1} = F_m \cup \{y_{m+1}\}$, satisfy all the conditions of definition 1. This completes the proof.

Let G be a subgroup of $GL(V)$, $g \in \Omega_0(G)$ and $F = \{g_i \in G: i \in A\}$ be a free system for G relative to g. We consider for this system the set $X(F) = \{y \in G: y \in \Omega_0(G), (y, g) \in X_V, A(y) \cup A(y^{-1}) \subset \operatorname{Int} K(F), \operatorname{cl} \bigcup_{i \in A} O_i(F) \subset P \backslash \operatorname{Cr}(y)\}$. The following assertion is evident.

LEMMA 12. *The set $F^m = \{g_i^m: g_i \in F, i \in A\}$ is also a free system for G relative to g. Furthermore we can choose the sets $O_i(F^m)$, $i \in A$ and $K(F^m)$ in such a way that $X(F^m) \subset X(F)$.*

Suppose G is in addition k-connected and does not leave invariant any proper non-trivial subspace of V. Then the following two propositions hold.

LEMMA 13. *The set $X(F)$ is non-empty.*

Proof. Let $U = \{x \in G: (xgx^{-1}, g) \in X_V\}$. By Lemma 11, $U \neq \varnothing$. Let $x \in U$ and $h = xgx^{-1}$, then we have that $(h, g) \in X_V$. This means that there exists an open set $E \subset P$ such that $A(g^{-1}) \subset E$ and $\operatorname{cl} E \subset P \backslash \operatorname{Cr}(h)$. Since $\operatorname{cl} \bigcup_{i \in A} O_i \subset P \backslash A'(g_{-1})$ (Definition 1, (3)), by (i) of Lemma 2, it follows that there exists an integer N_1 such that $\hat{g}^{-z}(\operatorname{cl} \bigcup_{i \in A} O_i) \subset E$ for each $z > N_1$. Since $(h, g) \in X_V$, we have $A(h) \cup A(h^{-1}) \subset P \backslash A'(g)$ and therefore by (i) of Lemma 2 there exists an integer N_2 such that $\hat{g}^z(A(h) \cup A(h^{-1})) \subset \operatorname{Int} K$ for each $z > N_2$. Pick an integer $z_0 > \max(N_1, N_2)$ and put $h_1 = g^{z_0} h g^{-z_0}$. If $(h, g) \in X_V$, then $(h_1, g) \in X_V$. By Lemma 5 and the choice of z_0 it follows that $h_1 \in X(F)$ and hence $X(F) \neq \varnothing$.

LEMMA 14. *If H is a k-dense proper subgroup of G, then $X(F)\backslash H \neq \varnothing$.*

Proof. If $X(F) \cap H = \varnothing$, then $X(F) \subset X(F)\backslash H$ and by virtue of Lemma 13, $X(F)\backslash H \neq \varnothing$. So we may assume that $X(F) \cap H \neq \varnothing$. Let $h_0 \in X(F) \cap H$, then $h_0 \in \Omega_0(G)$ and hence, by Lemma 8, $Hy \cap \Omega_0(G) \neq \varnothing$ for every $y \in G$. Let $y \in G\backslash H$ and $h \in Hy \cap \Omega_0(G)$. Then by Lemma 11 the set $U = \{x \in G\colon (xhx^{-1}, h_0) \in X_V\}$ is k-open and k-dense in G and therefore $H \cap U \neq \varnothing$. Let $x \in H \cap U$ and $h_1 = xhx^{-1}$. Then $(h_1, h_0) \in X_V$ and $h_1 \in H$, for $x, h \in H$. Since $h_0 \in X(F)$, we have that $A(h_0) \cup A(h_0^{-1}) \subset \operatorname{Int} K$, where $K = K(F)$, and $A(g) \cup A(g^{-1}) \subset P\backslash \operatorname{Cr}(h_0)$. Hence there exists an open set $W \subset P$ such that $A g) \subset W$, $A(g^{-1}) \subset \operatorname{cl} W$ and $A(h_0) \cup A(h_0^{-1}) \subset P\backslash \operatorname{Cr}(g)$. Therefore there exists an open subset $E_1 \subset P$ such that $\operatorname{cl} E_1 \subset P\backslash \operatorname{Cr}(g)$, $A(h_0) \cup A(h_0^{-1}) \subset E_1$ and $\operatorname{cl} E_1 \subset \operatorname{Int} K\backslash \operatorname{cl} W$. Since $(h_1, h_0) \in X_V$, we may assume that $E_1 \subset P\backslash \operatorname{Cr}(h_1)$. Since $h_0 \in X(F)$, we have $\operatorname{cl} \bigcup_{i \in A} O_i \subset P\backslash A'(h_0^{-1})$, where $O_i = O_i(F)$, $i \in A$ and from the definition of N_3 in the proof of Proposition 1, it follows by (i) of Lemma 2 that there exists an integer N_1 such that

$$\hat{h}_0^{-z}((\operatorname{cl} W) \cup A(g^{-1})) \subset E_1 \qquad \text{for every } z > N_1, z \in \mathbb{N}; \tag{1}$$

and

$$\hat{h}_0^{-z}\left(\operatorname{cl} \bigcup_{i \in A} O_i\right) \subset E_1 \qquad \text{for each } z > N_1, z \in \mathbb{N}. \tag{2}$$

Since $(h_1, h_0) \in X_V$, $A(h_1) \cup A(h_1^{-1}) \subset P\backslash A'(h_0^{-1})$ and hence, by (i) of Lemma 2 there exists an integer N_2 such that for every $z > N_2$, $z \in \mathbb{N}$, we have

$$\hat{h}_0^{z}(A(h_1) \cup A(h_1^{-1})) \subset E_1. \tag{3}$$

If an integer $z_0 > \max(N_1, N_2)$ and $h_3 = h_0^{z_0} h_2 h_0^{-z_0}$, then $h_3 \notin H$, but $h_3 \in X(F)$. Indeed, from (3) and the definition of E_1 it follows by Lemma 5 that $A(h_3) \cup A(h_3^{-1}) \subset E_1 \subset P\backslash \operatorname{Cr}(g)$ and therefore from (1) we have that $A(g) \cup A(g^{-1}) \subset \hat{h}_0^{z_0} E_1 \subset P\backslash \operatorname{Cr}(h_3)$. It follows similarly from (2) that $\operatorname{cl} \bigcup_{i \in A} O_i \subset P\backslash \operatorname{Cr}(h_3)$ and therefore $h_3 \in X(F)$. Since $h_3 \notin H$, we have that $h_3 \in X(F)\backslash H$ and this completes the proof.

PROPOSITION 2. *Let H be a subgroup of S such that $H \cap G^0$ is a proper k-dense subgroup of $S \cap G^0$. If $g \in \Omega_0(S)$ and $F_m = \{y_i \in S\colon 1 \leqslant i \leqslant m\}$ is a free system for S relative to g, then there exists an element $y_{m+1} \notin H$ such that $F_{m+1} = F_m \cup \{y_{m+1}\}$ is a free system for S relative to g.*

Proof. Let $O_i = O_i(F_m)$, $i = 1, \ldots, m$, be open in F_m and let $K = K(F_m)$ be a compact set in F_m. Since $S \cap G^0$ is a subgroup of finite index in S, there exists an integer n such that the set $F = \{y_i^n\colon y_i \in F_m\}$ is contained in

$S \cap G^0$. By Lemma 11, F is a free system and we may assume that $X(F) \subset X(F_m)$. Since the group $S \cap G^0$ is k-connected, by Lemma 14, we have that $X(F)\backslash H \neq \varnothing$ and hence $X(F_m)\backslash H \neq \varnothing$. Let $y \in X(F_m)\backslash H$. From the definition of $X(F_m)$ there exist open subsets U and O_{m+1} of $\operatorname{Int} K$ such that $A(g) \subset U$, $\operatorname{cl} U \subset (\operatorname{Int} K)\backslash \operatorname{Cr}(y)$ and (a) $A(y) \cup A(y^{-1}) \subset O_{m+1}$, $\operatorname{cl} O_{m+1} \subset (\operatorname{Int} K)\backslash((\operatorname{cl} U)\backslash \operatorname{Cr}(y))$. Since $y \in X(F_m)$, we have $A(y) \cup A(y^{-1}) \subset (\operatorname{Int} K)\backslash \operatorname{Cr}(g)$, $A(g) \cup A(g^{-1}) \subset P\backslash \operatorname{Cr}(g)$, $y \in \Omega_0(S)$ and $\bigcup_{i=1}^m \operatorname{cl} O_i \subset P\backslash \operatorname{Cr}(g)$. By (ii) of Lemma 2 there exists a natural number N_0 such that $\hat{y}^z(\bigcup_{i=1}^m \operatorname{cl} O_i) \subset O_{m+1}$ and $\hat{y}^z((\operatorname{cl} U) \cup A(g^{-1})) \subset O_{m+1}$ for every integer z with $|z| > N_0$. On the other hand, since $y \notin H$, the set $N = \{z \in \mathbf{Z}: y^z \notin H\}$ is infinite. Let $z_0 \in N$, $z_0 > N_0$. If $y_{m+1} = y^{z_0}$, then $\hat{y}^z_{m+1}(\bigcup_{i=1}^m (\operatorname{cl} O_i) \cup \operatorname{cl} U) \subset O_{m+1}$ for each $z \in \mathbf{Z}$, $z \neq 0$. From (a) and Lemma 5 it follows that $A(y_{m+1}) \cup A(y_{m+1}^{-1}) \subset O_{m+1}$. Since $O_{m+1} \subset \operatorname{Int} K$ it follows from (4) of Definition 1 that $\hat{y}_i^z O_{i+1} \subset O_i$ for every integer $z \neq 0$ and $i = 1, ..., m$. Since $y_{m+1} \in \Omega_0(S)$ and the sets $O_i(F_{m+1}) = O_i$, $1 \leqslant i \leqslant m$, $K(F_{m+1}) = \operatorname{cl} U$ satisfy all the conditions of Definition 1 for the set $F_{m+1} = F_m \cup \{y_{m+1}\}$, we have that F_{m+1} is a free system for S relative to g and $y_{m+1} = y^{z_0} \notin H$.

In the next proposition, k is an *arbitrary field.*

Proposition 3. *Let G be a finitely generated subgroup of $GL_n(k)$. If H is a subgroup of G such that for every subgroup L of G of finite index $HL = G$, then H is k-dense in G.*

Proof. We may assume that the field k contains the algebraic closure $\bar{k}_0$ of some simple field k_0. Now suppose the proposition false. Let k_0 be a simple subfield of k. Since G is finitely generated, there exists a subring $\mathcal{O}$ of k, finitely generated over k_0, such that $G \subset GL_n(\mathcal{O})$. Since the Zariski closures of G and H are different, there exists a polynomial f with coefficients from $\mathcal{O}$ vanishing on H and not equal to zero on some element $g \in G$. It is well known [6, p. 283, Corollary 2] that there exists a homomorphism $\varphi: \mathcal{O} \to \bar{k}_0$ which is the identity on k_0 such that $\varphi(f(g)) \neq 0$. Extending φ to the natural homomorphism φ^*: $G \to GL_n(\bar{k}_0)$ we see that $\varphi^*(g)$ does not belong to the Zariski closure of $\varphi^*(H)$. Thus we can suppose $G \subset GL_n(\bar{k}_0)$. If the field k_0 is finite, then the finitely generated group G is also finite and therefore $H = G$, a contradiction. Let us now consider the case $k_0 = \mathbf{Q}$. Since G is finitely generated, by [6, Theorem 3, p. 79] there exists for some p a finite algebraic extension E of the field $\mathbf{Q}_p$ such that $G \subset GL_n(J)$, where J is the ring of integers of E. Since the Zariski closures of G and H do not coincide, there exist a polynomial f with coefficients from J and an element $g \in G$ such that $f|_H = 0$ and $f(g) \neq 0$. Let $|\cdot|_p$ be the p-adic absolute value on E. Then the ideal $I = \{x \in E: |x|_p < |f(g)|_p\}$ defines the homomorphism ψ of the ring J onto the finite ring J/I such that $\psi(f(g)\cdot) \neq 0$. Extending ψ to the homomorphism ψ^*: $G \to GL_n(J/I)$ we see that $\psi^*(G) = \psi^*(H)$, but $\psi^*(g) \notin \psi^*(H)$, for $\psi(f(g)) \neq 0$. This contradiction completes the proof.

The following proposition is obvious.

PROPOSITION 4. *If $g \in \Omega_0(S)$, then there exists a free system for S relative to g.*

3. TRANSFORMATIONS WITH POINTS OF ATTRACTION IN GROUPS OF AUTOMORPHISMS OF DIRECT SUMS OF VECTOR SPACES

Let V be a vector space over a local field k. We assume that V is a direct sum of isomorphic subspaces $V_1, \dots, V_n$ and that S is a subgroup of $GL(V)$ satisfying the following four conditions:

(1) the Zariski closure G of S is semi-simple,

(2) for any two subspaces V_i and V_j, $1 \leqslant i, j \leqslant n$ there exists an element $y \in S$ such that $yV_i = V_j$,

(3) if $y \in S \cap G^0$, then $yV_i = V_i$ for each $i = 1, \dots, n$,

(4) the subgroup $S \cap G^0$ does not leave invariant any proper non-trivial subspace of V_i, $i = 1, \dots, n$.

It follows from (3) that for every $i = 1, \dots, n$ there exists a natural k-rational homomorphism $\varepsilon_i \colon G^0 \to GL(V_i)$. Let $S_i = \varepsilon_i(G^0 \cap S)$ and let m be a natural number not exceeding n. Let us consider the set $\mathcal{O}^m(S) = \{y \in S \cap G^0 \colon \varepsilon_i(y) \in \Omega_+(S_i) \text{ for each } i,\ 1 \leqslant i \leqslant m\}$. It is easily seen that $\mathcal{O}^{m_1}(S) \subset \mathcal{O}^{m_2}(S)$, if $m_1 > m_2$.

LEMMA 15. *Every set $\mathcal{O}^m(S)$ is empty or k-dense in $S \cap G^0$.*

Proof. Let $\tilde{V}$ be the tensor product of the spaces $V_1, \dots, V_m$ and let ε be the tensor product of the representations $\varepsilon_1, \dots, \varepsilon_m$. It can easily be verified that $y \in S$ belongs to $\mathcal{O}^m(S)$ if and only if $\varepsilon(y) \in \Omega_+(\varepsilon(S \cap G^0))$. By Lemma 6 the set $\Omega_+(\varepsilon(S \cap G^0))$ is empty or k-dense in $\varepsilon(S \cap G^0)$ and therefore, since ε is k-rational, the set $\mathcal{O}^m(S)$ is empty or is k-dense in $S \cap G^0$.

LEMMA 16. *If $\mathcal{O}^1(S) \neq \varnothing$, then all $\mathcal{O}^m(S) \neq \varnothing$, $1 \leqslant m \leqslant n$.*

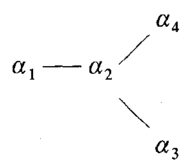
Proof. We shall show that for $m > 1$ if $\mathcal{O}^{m-1}(S) \neq \varnothing$, then $\mathcal{O}^m(S) \neq \varnothing$. Indeed, if $\mathcal{O}^{m-1}(S) \neq \varnothing$, then by Lemma 15 it is k-dense in $S \cap G^0$. Since the set of all semi-simple elements of G^0 is open in the Zariski topology of G^0 [1, p. 193], there exists a semi-simple element $g \in \mathcal{O}^{m-1}(S)$ such that $\varepsilon_i(g) \neq 1$ for each $i = 1, \dots, n$. On the other hand the first condition on S above implies that there exists an element $y \in S$ such that $yV_1 = V_m$. Let

$h = ygy^{-1}$. The image $\varepsilon_i(x)$ of $x \in S \cap G^0$ will be denoted below by x_i. Let π_i be the projection $\mathrm{pr}(A'(h_i), A(h_i))$, $i = 1,..., m-1$ and $\pi_m = \mathrm{pr}(A'(g_m), A(g_m))$. Let us consider the following sets $U_1 = \{x \in S \cap G^0\colon \hat{x}_i^{-1}A(g_i) \not\subset P\backslash A'(h_i)$ for $i = 1, 2,..., m-1\}$ and $U_2 = \{x \in S \cap G^0\colon \pi_i(\hat{x}_i^{-1}A(g_i)) \not\subset P\backslash A'(g_i)$ for every $i = 1,..., m-1\}$. Since $A(g_i)$ is a singleton for $i = 1,..., m-1$, by Lemma 7 and condition (4) on S the sets U_1 and U_2 are k-open and k-dense in $S \cap G^0$. Let $U_3 = \{x \in S \cap G^0\colon \hat{x}_m A(h_m) \subset P\backslash A'(g_m)$ and $\pi_m(\hat{x}_m A(h_m)) \subset P\backslash A'(h_m)\}$. Again as above, since $h = ygy^{-1}$ and $g_1 \in \Omega_+(\varepsilon_1(S \cap G^0))$, by Lemma 7 it follows that U_3 is k-open and k-dense in $S \cap G^0$ and therefore $\bigcap_{i=1}^3 U_i \neq \varnothing$. Let $x \in \bigcap_{i=1}^3 U_i$ and $f = xhx^{-1}$; then by Lemma 5 it follows that for every $i = 1,..., m$ we have $(g_i, f_i) \in \Delta_{V_i}$ and that, in turn, implies by Lemma 4 the existence of an infinite set $N \subset \mathbf{N}$ and an integer N_0 such that $A(g_i^z f_i^r)$ and $A(f_m^z g_m^r)$ are singletons for all $r \in N$, $z \in \mathbf{Z}$, $z > N_0$ and $i = 1,..., m-1$. Since $f_m^r g_m^r$ and $g_m^r f_m^r$ are conjugate, $A(g_i^r f_i^r)$ is a singleton for each $i = 1,..., m$, hence $g^r f^r \in \mathcal{O}^m(S)$ for $r \in N$, $r > N_0$ and this completes the proof.

4. Representations of a Finitely Generated Group over a Local Field

We first introduce some notation that will be used in this section and recall certain well-known results about the group of automorphisms of a semi-simple algebraic group. Let G be a semi-simple connected algebraic group, T be the maximal torus of G and B be a Borel subgroup of G containing T. By Φ we denote the system of roots of G. We introduce the order on Φ such that for the set Φ^+ of all positive roots $B = G_{\Phi^+}$ [2, Theorem 4.5]. By $A(B, T)$ we denote the subgroup of Aut G, that stabilizes T and B. By $\mathrm{Dyn}(\Phi, B)$ we denote the Dynkin diagram of G and by $\mathrm{Aut}(\mathrm{Dyn}(\Phi, B))$ the group of its automorphisms. To every element $a \in A(B, T)$ there corresponds a unique element $a' \in \mathrm{Aut}(\mathrm{Dyn}(\Phi, B))$.

Example I. Let G be a simple, connected group of type D_4. Then its Dynkin diagram is



where α_i, $i = 1,..., 4$ are simple roots of G [3, p. 305]. Thus, the group $\mathrm{Aut}(\mathrm{Dyn}(\Phi, B))$ is in this case the symmetric group of degree 3 and is generated by the two automorphisms σ and τ such that $\sigma^3 = 1$, $\sigma(\alpha_1) = \alpha_3$, $\sigma(\alpha_2) = \alpha_2$, $\tau^2 = 1$, $\tau(\alpha_1) = \alpha_1$, $\tau(\alpha_2) = \alpha_2$, $\tau(\alpha_3) = \alpha_4$. It should be noted that the subgroup generated by σ is normal in $\mathrm{Aut}(\mathrm{Dyn}(\Phi, B))$ and therefore every element of this group has a form $\sigma^i\tau^j$, where $0 \leqslant i \leqslant 2$, $0 \leqslant j \leqslant 1$.

EXAMPLE 2. Let now G be a simple, connected group which is not of type D_4. Then it follows from the well-known classification [3, pp. 301–319] that $\mathrm{Aut}(\mathrm{Dyn}(\Phi, B))$ is a finite cyclic group.

PROPOSITION 5. [1, 14.9] *Let G be a semi-simple, connected group. Then:*

(1) $\mathrm{Aut}\, G = \mathrm{Inn}\, G \cdot A(B, T)$,

(2) $\mathrm{Inn}\, G \cap A(B, T)$ *is the kernel of* $a \mapsto a'$,

(3) *The natural homomorphism* $\mathrm{Aut}\, G/\mathrm{Inn}\, G \to \mathrm{Aut}(\mathrm{Dyn}(\Phi, B))$ *is injective; in particular* $(\mathrm{Aut}\, G)^0 = \mathrm{Inn}\, G$.

LEMMA 17. *Let V be a vector space over a field k, let S be a finitely generated subgroup of GL(V) such that* $G = \mathscr{A}(S)$ *is a simple algebraic group. If X is the set of all torsion elements of S, then the set* $gG^0 \backslash \mathscr{A}(X)$ *is open and dense in the Zariski topology of the variety* gG^0 *for every* $g \in G$.

Proof. Since the variety gG^0 is irreducible, it is sufficient to show that $gG^0 \backslash \mathscr{A}(X) \neq \varnothing$. Suppose on the contrary that $gG^0 \subset \mathscr{A}(X)$. Making use of the conjugacy of Borel subgroups of G^0 and of tori in Borel subgroups, we may assume that the representative g of the coset gG^0 normalizes a torus T and a Borel subgroup B of G^0 containing T. Let $\alpha = \mathrm{ad}_{G^0}\, g$. It is clear that $\alpha \in A(B, T)$. Let us consider $\alpha' \in \mathrm{Aut}(\Phi)$ induced by α. If $\varphi \in \Phi^+$ denotes the dominant root for the order on Φ associated with B, we have $\alpha'(\varphi) = \varphi$. To the root φ there corresponds a connected one-parameter subgroup x_φ of T. If $Y = \{y \in G^0 \colon \alpha(y) = y\}$, then by the choice of φ we have that $x_\varphi \subset Y$. On the other hand all exponents of torsion elements of a finitely generated group S are bounded by a natural number N [10, Lemma 2.3]. Thus, for a large enough n, for instance $n = N!$, $f^n = 1$ for every $f \in \mathscr{A}(X)$. Since $g \in \mathscr{A}(X)$, $gx_\varphi \subset \mathscr{A}(X)$ and $x_\varphi \subset Y$, we have that $x_\varphi \subset \mathscr{A}(x)$. So $x^n = 1$ for each $x \in x_\varphi$. Let $\bar{k}$ be the algebraic closure of k and $\bar{k}^*$ be the multiplicative group of $\bar{k}$. Since over $\bar{k}$ there exists an injective homomorphism $\bar{k}^* \to x_\varphi$, the exponents of all elements of $\bar{k}^*$ are bounded in totality and hence, by [10, Lemma 2.3] the field $\bar{k}$ is finite, a contradiction.

THEOREM 1. *Let G be a simple algebraic group and S a finitely generated subgroup of G dense in the Zariski topology. Then there exist a local field k, a vector space V over k and an absolutely irreducible k-rational* *representation* ρ: $G \to GL(V)$ *such that* $\Omega_+(\rho(S)) \neq \varnothing$ *and the representation* $\rho|_{G^0}$ *is absolutely irreducible.*

To begin with we shall prove the following:

LEMMA 18. *If* G/G^0 *is cyclic, then Theorem* 1 *is valid.*

We shall divide the proof of Lemma 18 into three steps.

Step 1. Pick an element $g \in G$ such that the residue class gG^0 generates G/G^0. Let Y be the set of all semi-simple elements of G and X be the set of all torsion elements of S. Since $(gG^0) \cap Y$ is open and dense in the Zariski topology of the irreducible variety gG^0, we have by Lemma 17 that $(gG^0 \backslash \mathscr{A}(X)) \cap Y \neq \varnothing$. Since S is dense in the Zariski topology of G, there exists an element $y \in S \cap (gG^0 \backslash \mathscr{A}(X)) \cap Y$. It should be noted that G is generated by the set $\{y\} \cup G^0$.

Step 2. Since S is finitely generated and dense in the Zariski topology of G, we may assume that a finitely generated field k_0, a vector space V over k_0 and a k_0-rational representation $\tilde{\rho}$: $G \to GL(V)$ have been chosen in such a way that $\tilde{\rho}(S) \subset \tilde{\rho}(G))(k_0)$, i.e., $\tilde{\rho}(S) \subset GL(V)$, and all eigenvalues of $\tilde{\rho}(y)$ belong to k_0. We identify without loss of generality G and $\tilde{\rho}(G)$, S and $\tilde{\rho}(S)$, y and $\tilde{\rho}(y)$. Since the chosen element y is of infinite exponent ($y \notin \mathscr{A}(X)$), there exists an eigenvalue λ of y such that λ is not a root of unity. By [10, Lemma 4.1] we extend k_0 to a local field with absolute value $|\cdot|$ such that $|\lambda| \neq 1$. Let $V_0 = V \otimes_{k_0} k$ and V_1 be the subspace of V_0 consisting of all eigenvectors of y, corresponding to an eigenvalue λ_0 maximal in absolute value. Let $d = \dim V_1$. Since $\det y = 1$, $d \neq \dim V$. Extending, if necessary, the field k and replacing V_1 by an appropriate quotient of its dth exterior power we get a vector space W over a local field and an absolutely irreducible k-rational representation ρ: $G \to GL(W)$ such that the transformation $\rho(y)$ has a unique eigenvalue that is maximal in absolute value, i.e., $\rho(y) \in \Omega_+(\rho(S))$.

Step 3. It follows from the construction and from the absolute irreducibility of ρ that $\rho|_{G^0}$ is also absolutely irreducible. Indeed, since S is dense in the Zariski topology of G and $S \subset G(k)$, it is sufficient to show that the subgroup $S_1 = S \cap G^0$ leaves invariant no proper subspace of $\tilde{W} = W \otimes_k \bar{k}$. Suppose the contrary. Then there exists a minimal S_1-invariant proper subspace W_0 of $\tilde{W}$. Let $m = |G/G^0|$. It is clear that any S_1-invariant subspace W^* of W either contains W or else $W^* \cap W = \{0\}$ and therefore there exist integers $i_1, \ldots, i_s$ such that $1 \leqslant i_1 < \cdots < i_s < m$ and $\tilde{W} = W_0 \oplus \bigoplus_{k=1}^{s} y^{i_k} W_0$. Let $W_r = y^{i_r} W_0$, $r = 1, \ldots, s$, and v be an eigenvector of y corresponding to the eigenvalue of y which is maximal in absolute value. It is easily seen that v belongs to some W_r or to W_0. Indeed,

$v = v_0 + v_1 + \cdots + v_s$, where $v_i \in W_i$, $i = 0, 1, \ldots, s$, and v is an eigenvector of y^m corresponding to the maximal (in absolute value) eigenvalue. Since $y^m \in S_1$, for $m = |G/G^0|$, and all subspaces W_i are S_1-invariant, so are all the non-zero vectors among $v_0, \ldots, v_s$. On the other hand, by Lemma 5, $A(y^m)$ is a singleton and therefore only one vector from among $v_0, \ldots, v_s$ is non-zero. Suppose, for instance, $v \in W_0$, then $y^{i_1}v \in W_1$ and hence $y^{i_1}v \notin W_0$, but this is impossible because v is an eigenvector of y, a contradiction.

Proof of Theorem 1. It is evident that without loss of generality we may assume that G is a subgroup of the group of automorphisms of a simple, connected group H. Making use of Examples 1 and 2, we see that either H is a simple group of type D_4 and $G = \operatorname{Aut} H$ or G/G^0 is cyclic. In the second case the theorem is valid by Lemma 18, so let $G = \operatorname{Aut} H$, where H is a simple group of type D^4. It follows from Proposition 5 that G^0 is a simple group of type D_4 and the natural homomorphism $G \to \operatorname{Aut} G^0$ is an isomorphism. So we can apply to the group G the results about groups of automorphisms of algebraic groups, mentioned above. Let T and B be a torus and a Borel subgroup of G^0 containing T. Now we shall take advantage of Example 1. Pick an element $g \in G$ such that $g \in N_G(T) \cap N_G(B)$ and $g' = \sigma \in \operatorname{Aut}(\operatorname{Dyn}(\Phi, B))$. Let us consider the set $L = \{x^{-1}gtx \colon x \in G^0,\ t \in T\}$ and let $M = \mathscr{A}(L)$. We define the rational morphism $\alpha\colon T \times G^0 \to gG^0$ by the formula $\alpha(t, x) = x^{-1}gtx$. Let us show that $\dim M = \dim G^0$. Indeed, let t_0 be a regular element of G^0 and $t_0 \in T$. Let us consider the fibre $\alpha^{-1}(gt_0)$ of the morphism α. It is easily seen that if $(t, x) \in \alpha^{-1}(gt_0)$, then $x \in T$ and therefore $\alpha^{-1}(gt_0) = \{(t_0[x, g]^{-1}, x) \colon x \in T\}$. Thus, $\dim \alpha^{-1}(gt_0) = \dim T$. By [1, Theorem 10.1] it follows that $\dim M = \dim G^0 \times T - \dim T$ and hence $\dim M = \dim G^0$. Since $\dim gG^0 = \dim G^0$ and $\alpha(T \times G^0) = L$, we have that $M = gG^0$ and so M contains an open and Zariski dense subset of the variety gG^0. Let us consider the set X of all torsion elements of S and the set Y of all semi-simple elements of G. Since Y is open and dense in the Zariski topology of G [1, p. 193], we have by Lemma 17 that the set $(gG^0 \backslash \mathscr{A}(X)) \cap Y$ is open and dense in the Zariski topology of gG^0. Since L contains an open set of gG^0, we have $S \cap L \cap (gG^0 \backslash \mathscr{A}(X)) \cap Y = D \neq \varnothing$. Let $y \in D^0$. Since $y \in L$, we have $y = x^{-1}gtx$ for some $x \in G^0$ and $t \in T$. Identifying S with xSx^{-1}, we may assume that $y \in (S \cap gT \cap Y) \backslash X$. Thus, y is a semi-simple element of S of infinite exponent and $y \in gT$. Then $y = gt$ for some $t \in T$. Let us observe that $y^3 \in T$ and $gy^3g^{-1} = y^3$. Indeed, since $g' = \sigma$, the element g^3 induces the identical automorphism of the Dynkin diagram $\operatorname{Dyn}(\Phi, B)$ of G^0 and, since $g \in N_G(T) \cap N_G(B)$, we have $g^3 \in T$. Besides, since $gt = y$, we have $y^3 \in T$, so $gy^3g^{-1} = y^3$. Let h be an element of finite exponent of the group $N_G(T) \cap N_G(B)$ such that $h' = \tau \in \operatorname{Aut}(\operatorname{Dyn}(\Phi, B))$. Since $gy^3g^{-1} = y^3$ and $g' = \sigma$, by virtue of Theorem 6 and Lemma 2.8 from [9] we have that

$hy^3h^{-1} = y^3$. Let $\mu = hy^3$, then μ is a semi-simple element of infinite exponent. If $l = |h|$, then $\mu^l = y^{3l}$. The group G is generated by the set $\{y, \mu\} \cup G^0$. Repeating the arguments of step 2 in the proof of Lemma 18, we choose a finitely generated field k_0 and a vector space V over k_0 such that $G \subset GL(V)$ $S \subset \mathbf{G}(k_0)$, $y, \mu \in \mathbf{G}(k_0)$ and all eigenvalues of y and μ belong to k_0. Now we turn to the construction of the desired representation. Since y is a semi-simple element of infinite exponent, at least one of its eigenvalues, call it λ, is not a root of unity. Making use of [10, Lemma 4.1] we extend k_0 to a local field k with absolute value $|\cdot|$ such that $|\lambda| \neq 1$. Repeating the arguments of Step 2, we obtain a vector space W over k and a k-rational, absolutely irreducible representation $\rho: G \to GL(W)$ such that $\rho(y) \in \Omega_+(\rho(S))$. We shall show that $\rho|_{G^0}$ is absolutely irreducible. Let $\bar{k}$ be the algebraic closure of k and $\tilde{W} = W \otimes_k \bar{k}$. We can assume for convenience that $\rho(G) = G$, $\rho(S) = S$, $\rho(y) = y$, $\rho(\mu) = \mu$ and $\rho(G^0) = G^0$. Let us verify that the subgroup $G^0(k)$ of $G(k)$ has no proper invariant subspace in $\tilde{W}$. Since $G^0(k)$ is dense in the Zariski topology of G^0, this will be sufficient to complete the proof. Suppose on the contrary, that $G^0(k)$ has a non-trivial, proper, invariant subspace in $\tilde{W}$. We choose among those subspaces a minimal one W_0. Let W' be any $G^0(k)$-invariant subspace of $\tilde{W}$; then it is clear that either $W_0 \subset W'$ or $W_0 \cap W' = \{0\}$ and therefore, from the properties of y and μ, it follows that there exist s pairs of integers $(i_1, j_1),..., (i_s, j_s)$, $0 \leqslant i_r < 2$, $0 \leqslant j_r < 3$, $i_r^2 + j_r^2 \neq 0$ if $1 \leqslant r \leqslant s$, such that $\tilde{W} = W_0 \oplus \cdots \oplus W_s$, where $W_r = \mu^{i_r} y^{j_r} W_0$. All W_r, $r = 0,..., s$ are invariant by $G^0(k)$. Let v be an eigenvector of y, corresponding to the maximal (in absolute value) eigenvalue of y. Since $y^{3l} = \mu^l$ for some integer l, it follows from Lemma 5 that $A(\mu)$ is a singleton and $A(\mu) = A(y)$. Hence v is an eigenvector of μ, corresponding to an eigenvalue maximal in absolute value. Thus, v belongs to some W_i. Indeed, $v = v_0 + \cdots + v_s$, where $v_i \in W_i$, $i = 0,..., s$, and since $y^3 \in G^0(k)$, each nonvanishing v_i is an eigenvector of y^3, corresponding to the same eigenvalue as v. Since $A(y^3) = A(y)$ is a singleton, by Lemma 5, we have that only one vector among $v_0,..., v_s$ is not zero. Suppose, for instance, $v \in W_0$. Then, since $W_1 = \mu^{i_1} y^{j_1} W_0$, we have that $\mu^{i_1} y^{j_1} v \in W_1$ and on the other hand, since v is an eigenvector of μ and y, we have that $\mu^{i_1} y^{j_1} v \in W_0$. The contradiction completes the proof.

The following theorem allows us to reduce the construction of maximal subgroups in groups over arbitrary fields to the construction of such subgroups in groups over local fields where we can apply the results of Sections 2 and 3.

THEOREM 2. *If G is an algebraic group over a field k_0 such that G^0 is not solvable and S is a finitely generated k_0-dense subgroup of $G(k_0)$, then there exist a vector space V over a local field k and a k-rational, absolutely* *irreducible representation* ρ: $G \to GL(V)$ *such that* $\Omega_+(\rho(S)) \neq \varnothing$, $\rho(G)$ *is a semi-simple group and the restriction* $\rho|_{G^0}$ *is absolutely irreducible.*

Since S is a finitely generated subgroup of G, dense in the Zariski topology, we may assume that the field k_0 is finitely generated

Proof. Without loss of generality we can assume that G is semi-simple and $C_G(G^0)$ is the identity element. Moreover, since the k_0-group G^0 is split over a finite algebraic extension of k_0 [1, p. 258], we may assume that G^0 is split over k^0 and therefore that G^0 is a product of simple algebraic groups over k_0, say $G^0 = H_1 \times \cdots \times H_m$. We may also assume that G^0 does not contain connected subgroups which are normal in G, so the groups $H_1,..., H_m$ are isomorphic and moreover, for every $i = 1,..., m$, there exists an element $g_i \in G$ such that $H_i = g_i H_1 g_i^{-1}$ and every g in G induces a permutation σ_g of the set $\{H_1,..., H_m\}$. Let us consider the subgroup $N_G(H_1)$ of G. Let H be the image of $N_G(H_1)$ under the natural homomorphism $N_G(H_1) \to \operatorname{Aut} H_1$ and let A be the product of m copies of $\operatorname{Aut} H_1$. Let $B = \{(a_1,..., a_m) \in A: a_i \in H, 1 \leqslant i \leqslant m\}$ and let D be the image of G under the homomorphism $g \mapsto \sigma_g$. Let us consider the semi-direct product $A_1 = A \rtimes D$ (the elements of D act by permuting the components of the elements of A) and its subgroup $B_1 = B \rtimes D$. We define the k_0-rational homomorphisms ε_i: $G \to H$ by setting, for $i = 1,..., m$ and $g \in G$, $\varepsilon_i(g) = \operatorname{ad}_{H_1}(g_{\sigma_g(i)}^{-1} g g_i)$ if $i \geqslant 2$ and $\varepsilon_1(g) = \operatorname{ad}_{H_1}(g_{\sigma_g(1)}^{-1} g)$. Let $\varepsilon = (\varepsilon_1,..., \varepsilon_m)$. We define the k_0-rational morphism φ: $G \to A$ as well by the formula $\varphi(g) = \varepsilon(g)\sigma_g$ for $g \in G$. It is easy to verify that φ is a homomorphism and $\varphi(G^0) = A^0 = B^0$. Making use of the inclusions $g_{\sigma_g(i)}^{-1} g g_i \in N_G(H_1)$ for $i \geqslant 2$ and $g_{\sigma_g(1)}^{-1} g \in N_G(H_1)$, we see that $\varphi(G) \subset B_1$. Let $S_1 = \varepsilon_1(G^0 \cap S)$ and $S_2 = \varepsilon_1(N_G(H_1) \cap S)$. Since S is dense in the Zariski topology of G and $G^0 \subset N_G(H_1)$, we have that S_1 and S_2 are dense in the Zariski topology of the groups $\varepsilon_1(G^0)$ and $\varepsilon_1(N_G(H_1))$, respectively, $|S_2/S_1| < \infty$ and S_1 is a normal subgroup of S_2. Let π_r be the projection of A onto the rth component. It is easily seen that $\pi_1(\varphi(S) \cap B) = S_2$ and $\pi_1(\varphi(S) \cap B^0) = S_1$. Since S_2 is a finitely generated, dense subgroup of a simple algebraic group $H = \pi_1(B)$, by Theorem 1 there exist a local field $k \supset k_0$, a vector W over k and an absolutely irreducible k-rational representation ρ_1: $H \to GL(W)$ satisfying all the conditions of Theorem 1. So, since $|S_2/S_1| < \infty$, we have that $\Omega_+(\rho_1(S_1)) \neq \varnothing$. Now we shall extend ρ_1 to the desired representation ρ_2 of G. Since G is k_0-isomorphic to B_1, it is sufficient to extend ρ_1 to a representation ρ_2 of B_1 satisfying the same conditions. The role of a finitely generated subgroup of B_1 will be played by the group $\bar{S} = \varphi(S)$. Let V be the mth tensor power of W. Since B is the product of m copies of H, taking the mth tensor power of ρ_1 as a new representation, we get an absolutely irreducible representation ρ_2 [9, Lemma 68]. Since D acts on B by transpositions of components, this representation is naturally extended to a representation of B_1, which will be denoted by ρ_2 as well. Thus, we have constructed a k-rational, absolutely irreducible representation ρ_2: $B_1 \to \mathbf{GL}(V)$ such that its restriction to $(B_1)^0 = B^0$ is also absolutely irreducible by [9, Lemma 68], for $B^0 = H^0 \times \cdots \times H^0$. It remains to show that $\Omega_+(\rho_2(S)) \neq \varnothing$. Let V^* be the product of m copies of W and ρ^* be the product of m copies of ρ_1. It is clear that ρ^*: $B \to \mathbf{GL}(V^*)$. Making use of the action of D on B, we may extend ρ^* to a representation ρ^*: $B_1 \to \mathbf{GL}(V^*)$. We recall that $\mathscr{O}^i(\rho^*(\tilde{S})) = \{\rho^*(y) \in \rho^*(B^0 \cap \tilde{S})$: $\rho^*(\pi_r(y)) \in \Omega_+(\rho^*(\pi_r(\tilde{S} \cap B^0)))$ for every $r = 1, \ldots, i\}$. Since the group $\rho^*(B_1)$ and its finitely generated subgroup $\rho^*(\tilde{S})$ satisfy all the conditions of Lemma 16, we have that $\mathscr{O}^m(\rho^*(\tilde{S})) \neq \varnothing$. It is evident that the image of $g \in \mathscr{O}^m(\rho^*(\tilde{S}))$ under the natural isomorphism $\rho_2(\rho^*)^{-1}$ belongs to $\Omega_+(\rho_2(S))$ and this completes the proof of Theorem 2.

5. Criterion for Finiteness of the Indices of All Maximal Subgroups

Let G be a group, by $\mathfrak{A}(G)$ we denote the set of all maximal subgroups of G of infinite index in G. If G is a linear group and $\mathbf{G}$ is its closure in the Zariski topology, then we define the set $\mathfrak{A}_n(G) = \{H \in \mathfrak{A}(G)$: $\mathscr{A}(H) = \mathbf{G}\}$. The following simple assertions will be used in the sequel.

Lemma 19. *If M is a maximal subgroup of G and H is a nilpotent, normal subgroup, then* (i) *$M \cap H$ is a normal subgroup of G and* (ii) *$H/M \cap H$ is an Abelian subgroup of $G/M \cap H$ containing no normal, non-trivial, proper subgroup of $G/M \cap H$.*

Proof. It is sufficient to prove the lemma in the case when $H \neq M \cap H$; then $M \cap H$ is a proper subgroup of $N_G(M \cap H)$ [5, Theorem 16.2.2]. On the other hand, since H is normal in G, $M \subset N_G(M \cap H)$. Since $H \cap M \neq N_G(M \cap G)$, therefore $M \neq N_G(M \cap H)$ and so $N_G(M \cap H) = G$. Since $G/M \cap H$ is a semi-direct product of a normal, nilpotent subgroup $H/M \cap H$ with a maximal subgroup $M/M \cap H$, we obtain (ii).

Lemma 20. *Let A and B be arbitrary groups and φ: $A \to B$ be an epimorphism, then $\varphi^{-1}(\mathfrak{A}(B)) \subset \mathfrak{A}(A)$.*

This follows from the fact that the preimage of a maximal subgroup of B is a maximal subgroup of A of the same index.

Theorem 3. *A finitely generated linear group G over a field has maximal subgroups of only finite index if and only if G is solvable-by-finite.*

Proof. We first assume that G has a solvable subgroup of finite index. Then we shall prove that the index of any maximal subgroup of G is finite. It is well known that G has a normal, trigonalizable subgroup A such that $|G/A| < \infty$ [1, Corollary 1 in 10.5]. Let $B = [A, A]$: then B is a normal, nilpotent subgroup of G. By Lemma 19, $H \cap B$, where H is a maximal subgroup of G, is a normal subgroup of G and $B/H \cap B$ is an Abelian subgroup of $G/H \cap B$ containing no proper, non-trivial, normal subgroup of $G/H \cap B$. If $B \neq H \cap B$, then $B^* = B/H \cap B$ is a vector space, not necessarily finite-dimensional a priori, over a simple field k. Indeed, since B^* contains no normal, non-trivial, proper subgroup of $G^* = G/H \cap B$, B^* is either a complete, Abelian torsion free group, i.e., a vector space over $\mathbf{Q}$, or an Abelian group of prime period p, i.e., a vector space over a field of characteristic p. On the other hand B^* is a G^*-module over k and moreover is a G/B-module, for B^* is Abelian. This module is finitely generated, since A is finitely generated and $B = [A, A]$. Since B^* does not contain any proper, non-trivial, normal subgroup of G/B, the module B^* does not contain any proper, non-trivial G/B-module and, since G/B is a finite extension of an Abelian group A/B, we have that $\dim_k B^* < \infty$, see, e.g., [8]. It is easily seen that $k \neq \mathbf{Q}$ and therefore k is finite. But $B^* \neq \{1\}$ and, since $H^* = H/H \cap B$ is a maximal subgroup of G^*, we have that $H^*B^* = G^*$ and $H^* \cap B^*$ is trivial. Hence $|G^*/H^*| = |B^*| < \infty$. In the case when $B \subset H$ the proof is clear.

We prove now the necessity of the condition. Let us suppose on the contrary that G has no solvable subgroups of finite index. By Lemma 20 and Theorem 2 we can reduce the proof to the case of a local field and assume that G satisfies all the conditions of section 2. More precisely, we assume that G is a finitely generated subgroup of $GL(V)$, where V is a vector space over a local field, such that

(i) $\Omega_+(G) \neq \varnothing$;

(ii) If $\mathbf{G}$ is the Zariski closure of G, then $\mathbf{G}$ is semi-simple;

(iii) The subgroup $G \cap \mathbf{G}^0$ does not leave invariant any proper, non-trivial subspace of V.

Since G is finitely generated, the set of its subgroups of finite index is countable and so is the set of residue classes modulo these subgroups. Let $M_1, \ldots, M_n, \ldots$ enumerate all these classes. By Lemma 8, $\Omega_0(G) \neq \varnothing$. Let $g \in \Omega_0(G)$ and $F_0 = \{y_i \in G\colon i = 1, \ldots, m\}$ be a free system for G relative to g. The existence of F_0 follows from Proposition 4. Let us suppose that for some natural number i a free system F_i has already been constructed such that $M_j \cap F_i \neq \varnothing$ for each $j \leqslant i$ and $F_0 \subset F_i$. Applying Proposition 1 to the residue class M_{i+1} and F_i, we can find $y_{i+1} \in M_{i+1}$ such that $F_{i+1} = F_i \cup \{y_{i+1}\}$ is a free system for G relative to g. Let $F = \bigcup_{i=1}^{\infty} F_i$ and $S = \langle F \rangle$; then by the construction of F_i we have that $S \cap M_i \neq \varnothing$ for each $i \in \mathbf{N}$ and, by Lemma 10, S is freely generated by F. Therefore, since G is finitely generated, $S \neq G$ and for every subgroup K of G of finite index $SK = G$. Since G is finitely generated, there exists a maximal proper subgroup H of G such that $S \subset H$. It is evident that H is a maximal subgroup of G. On the other hand, if $|G/H| < \infty$, then $SH = G$, therefore $H \supset SH = G$, but H is a proper subgroup of G. Contradiction. Thus, we have found a maximal subgroup H of G of infinite index and this completes the proof.

COROLLARY 1. *The group $SL_n(\mathbf{Z})$, where $n \geqslant 2$, has a maximal subgroup of infinite index.*

THEOREM 4. *Let G be a finitely generated linear group over a field. If G has no solvable subgroup of finite index, then the set $\mathfrak{A}(G)$ is uncountable.*

Proof. By Lemma 20 we may again assume, making use of Theorem 2, that G satisfies all the conditions of Section 2. By Proposition 3 and Theorem 3, $\mathfrak{A}_n(G) = \mathfrak{A}(G)$ and $\mathfrak{A}_n(G) \neq \varnothing$. Let us assume that $\mathfrak{A}_n(G)$ is countable. Let $M_1,..., M_i,...$ and $H_1,..., H_i,...$ enumerate, respectively, the residue classes modulo the subgroups of G of finite index and all the subgroups of $\mathfrak{A}_n(G)$. By Lemma 8 $\Omega_0(G) \neq \varnothing$. Let $g \in \Omega_0(G)$ and $F_0 = \{y_i \in G\colon i = 1,..., m\}$ be a free system for G relative to g, which exists by Proposition 4. Let us suppose that we have constructed a free system F_i for every natural i such that $F_0 \subset F_i$, $F_i \cap M_r \neq \varnothing$ for every $r \leqslant [(i-1)/2]$, $i \geqslant 1$ and $F_i \cap H_t \neq \varnothing$ for every $t \leqslant [i/2]$, $i \geqslant 2$. If now i is even, then by Proposition 2 we can choose $y \in G$ such that $y \notin H_{i/2}$ and $F_{i+1} = F_i \cup \{y\}$ is a free system. If i is odd, then by Proposition 1 we can choose $y \in G$ such that $y \in M_{(i-1)/2}$ and $F_{i+1} = F_i \cup \{y\}$ is a free system. Let $F = \bigcup_{i=1}^{\infty} F_i$ and $S = \langle F \rangle$. Note that for a subgroup K of G of finite index $SK = G$ and, for $H_i \in \mathfrak{A}_n(G)$, $S \cap (G \backslash H_i) \neq \varnothing$. Just as in the proof of Theorem 3 we choose a maximal subgroup H of G among those containing S. Repeating the arguments of the quoted proof, we conclude that H is a maximal subgroup of G and $|G/H| < \infty$. Then, by proposition 3, $H \in \mathfrak{A}_n(G)$ and on the other hand, since $S \subset H$, $S \cap (G \backslash H_i) \neq \varnothing$ for every $H_i \in \mathfrak{A}_n(G)$; thus the set $S \cap (G \backslash H)$ is not empty, but $G \backslash H \subset G \backslash S$. Contradiction.

COROLLARY 2. *If L is a lattice in a semi-simple Lie group $G_{\mathbf{R}}$, where G is of rank $\geqslant 1$, then the set of maximal subgroups of G of infinite index is uncountable.*

Proof. By Borel's density theorem, the Zariski closures of $\operatorname{Ad} L$ and $\operatorname{Ad} G_{\mathbf{R}}$ coincide. So L has no solvable subgroup of finite index, but L is finitely generated [7, Proposition 13.21]. By Theorem 4 we obtain the conclusion of Corollary 2.

6. The Existence of Maximal Subgroups of Infinite Index without Free Subgroup of Finite Index

Our proof of the existence of maximal subgroups in $SL_n(\mathbf{Z})$ of infinite index stimulated V. P. Platonov and, independently of him, G. Prasad to pose the following question: does there exist in $SL_n(\mathbf{Z})$ a maximal subgroup of infinite index containing no free subgroup of finite index? We answer this question affirmatively.

THEOREM 5. *The group $SL_n(\mathbf{Z})$, where $n \geqslant 4$, has a maximal subgroup of infinite index containing a free Abelian subgroup of rank 2.*

Several lemmas will precede the proof of the theorem. Let a be a semi-simple element of infinite exponent in $SL_2(\mathbf{Z})$ and let us consider the following two elements of $SL_n(\mathbf{Z})$:

$$a_1 = \begin{pmatrix} a & 0 \\ 0 & E_{n-2} \end{pmatrix}, \qquad a_2 = \begin{pmatrix} E_{n-2} & 0 \\ 0 & a \end{pmatrix},$$

where $n \geqslant 4$ and E_{n-2} is the unity $(n-2) \times (n-2)$ matrix. It is obvious that these elements commute. Let $V = \mathbf{R}^n$ and P be the projective space based on V. The natural action of $SL_n(\mathbf{Z})$ on V allows us to consider a_1 and a_2 as elements of $GL(V)$. Let $\pi_{ij} = \mathrm{pr}(A'(a_1^i a_2^j), A(a_1^i a_2^j))$, $i = \pm 1, j = \pm 1$.

LEMMA 21. *If $g = a_1^n a_2^m$, $|n| = |m| \neq 0$, and $x \in P \backslash A'(a_1^i a_2^j)$ for every $i = \pm 1$, $j = \pm 1$, then for every neighbourhood W of x such that $\mathrm{cl}\, W \subset P \backslash A'(a_1^i a_2^j)$, $i = \pm 1$, $j = \pm 1$ and for every neighbourhood U of the set $\bigcup \{\pi_{ij}^{(x)}: i = \pm 1, j = \pm 1\}$ there exists a number $N = N(U, W) \in \mathbf{N}$ such that $g^r \,\mathrm{cl}\, W \subset U$, if $|r| > N$.*

This follows immediately from item 3 of Lemma 3.9 in [10].

LEMMA 22. *If $g = a_1^n a_2^m$ and $|n| \neq |m|$, $x \in P \backslash A'(a_s^i)$ for each $s = 1, 2$, $i = \pm 1$, then for every neighbourhood W of x such that $\mathrm{cl}\, W \subset P \backslash A'(a_s^i)$, $s = 1, 2$, $i = \pm 1$ and for every neighbourhood U of $\bigcup \{A(a_s^i): s = 1, 2, i = \pm 1\}$ there exists $N = N(U, W) \in \mathbf{N}$ such that $\hat{g} W \subset U$, if $\min(|m|, |n|) > N$.*

Proof. It is clear that only four cases are possible for m and n (a) $n \geqslant 0$, $|n| < |m|$, (b) $n \geqslant 0$, $|n| > |m|$, (c) $m \geqslant 0$, $|m| > |n|$, (d) $m \leqslant 0$, $|m| > |n|$. According to them we get: (a) $A(g) = A(a_1)$, (b) $A(g) = A(a_1^{-1})$, (c) $A(g) = A(a_2)$, (d) $A(g) = A(a_2^{-1})$. Thus, all the conditions of (i) in Lemma 2 are satisfied and this completes the proof.

DEFINITION 2. Let $G \subset GL(V)$, $U \subset P$, $W \subset P$, $g \in \Omega_0(G)$ and let $F = \{y_i \in G\colon i \in I\}$ be a free system for G relative to g. We call F compatible with U *and* W if

(1) $K(F) \subset W$ and $O_i(F) \subset W$ for each $i \in I$,

(2) $\hat{y}_i^n \operatorname{cl} U \subset O_i(F)$ for each $n \in \mathbf{Z}$, $n \neq 0$ and $i \in I$,

(3) $\operatorname{cl} U \subset P \backslash A'(g^{-1})$.

PROPOSITION 6. *Let S be a subgroup of $GL(V)$ satisfying all the conditions of Section 2, $U \subset P$, $W \subset P$, $g \in \Omega_0(S)$ and let $F_t = \{y_i \in S\colon i = 1,..., t\}$ be a free system for S relative to g, which is compatible with U and W. If X is a residue class in S modulo a subgroup of finite index, then there exists an element $y_{t+1} \in X$ such that $F_{t+1} = F_t \cup \{y_{t+1}\}$ is a free system for S relative to g compatible with U and W.*

Proof. We may assume (without loss of generality) that X is a residue class modulo a normal subgroup of finite index. Making use of the same argument as in the proof of Proposition 1, we see that there exists $x \in X \cap \Omega_0(S)$ such that $A(x) \cup A(x^{-1}) \subset P \backslash \mathrm{Cr}(g)$ and $A(g) \cup A(g^{-1}) \subset P \backslash \mathrm{Cr}(x)$. By Lemma 1 we can choose $N_1 \in \mathbf{N}$ such that $\hat{g}^r(A(x) \cup A(x^{-1})) \subset \operatorname{Int} K$ for every $r > N_1$, where $K = K(F_t)$. Since $\operatorname{cl} U \subset P \backslash A'(g^{-1})$ and $A(g^{-1}) \subset P \backslash \mathrm{Cr}(x)$, by the same lemma we can find $N_2 \in \mathbf{N}$ such that $\hat{g}^{-r} \operatorname{cl} U \subset P \backslash \mathrm{Cr}(x)$ for every $r > N_2$. Besides, by virtue of (2) in Definition 2 we may assume that the natural number N_2 was chosen in such a way that $\hat{g}^{-r} \operatorname{cl} U \subset P \backslash \mathrm{Cr}(x)$ for every $r > N_2$, $i = 1,..., t$. Let $y = g^{r_0} x g^{-r_0}$, where $r_0 \in \mathbf{N}$ and $r^0 > \max(N_1, N_2)$; then it is clear that $A(y) \cup A(y^{-1}) \subset (\operatorname{Int} K) \backslash A(g)$, $A(y_i) \cup A(y_i^{-1}) \subset P \backslash \mathrm{Cr}(y)$ for each $i = 1,..., t$ and $\operatorname{cl} U \subset P \backslash \mathrm{Cr}(y)$. Therefore there exist an open set O_{t+1} and a compact set $K_1 \subset \operatorname{Int} K$ such that $A(g) \subset \operatorname{Int} K_1$, $\operatorname{cl} O_{t+1} \subset K \backslash \mathrm{Cr}(g)$, $K_1 \subset P \backslash (\operatorname{cl} O_{t+1}) \cup \mathrm{Cr}(y))$. So there exists a natural number N_0 such that $\hat{y}^n \operatorname{cl} O_i \subset O_{t+1}$, $\hat{y}^n K_1 \subset O_{t+1}$, $\hat{y}^n \operatorname{cl} U \subset O_{t+1}$ for every $i = 1,..., t$, $n \in \mathbf{Z}$, $n^2 > N_0^2$. Since the set $X_1 = \{z \in \mathbf{Z}\colon y^z \in X\}$ is infinite, we can choose an integer $z_0 \in X_1$ such that $z_0^2 > N_0^2$. Putting now $y_{t+1} = y^{z_0}$ we complete the proof.

Proof of Theorem 5. Let $G = SL_n(\mathbf{Z})$, $n \geqslant 4$, $V = R^n$, P—the projective space based on V and let a_1, a_2 be the elements of G mentioned above. By Lemma 7 there exists $g \in \Omega_0(G)$ such that $A(g) \subset P \backslash (A'(a_1^i a_2^j) \cup A(a_1^i a_2^j))$ for each $i = 0, \pm 1$ and $j = 0, \pm 1$. Let $T = \bigcup \{\pi_{ij}(A(g))\colon i = \pm 1, j = \pm 1\} \cup A(a_1) \cup A(a_2) \cup A(a_1^{-1}) \cup A(a_2^{-1})$. Since $A(g) \in P \backslash T$, there exist neighbourhoods W and U of $A(g)$ and T, respectively, such that $\operatorname{cl} W \cap \operatorname{cl} U = \varnothing$. Making use of Proposition 6, we construct a free system $F = \{y_i \in G\colon i \in \mathbf{N}\}$ for G relative to g, compatible with U and W such that for every residue class X in G modulo a subgroup of finite index we have that $X \cap F \neq \varnothing$. Let us assume that a natural number $N_1 = N_1(U, W)$

($N_2 = N_2(U, W)$) satisfies the conditions of Lemma 21 (Lemma 22). Let us consider the group $G_1 = \langle a_1^N, a_2^N \rangle$, where $N \in \mathbf{N}$, $N > \max(N_1, N_2)$. By the choice of N, $\hat{g}\,\mathrm{cl}\, W \subset U$ for every $g \in G_1$. From (1) and (2) of Definition 2 it follows that $\hat{y}_i^n\,\mathrm{cl}\, U \subset W$ for each $n \neq 0$, $n \in \mathbf{Z}$. Therefore if $G_2 = \langle F \rangle$, then the group $S = \langle G_1, G_2 \rangle$ is a free product of G_1 and G_2. Thus, S is a proper subgroup of G. On the other hand, since $S \supset G_2$, the index of S in G is infinite. Let M be a maximal proper subgroup of G containing S. Just as in the proof of Theorem 3 we have that M is a maximal subgroup of G and $|G/M| = \infty$; moreover an Abelian free group of rank 2 is contained in M. This completes the proof of Theorem 5.

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