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Mixed type iteration for total asymptotically nonexpansive mappings in hyperbolic spaces

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Abstract

The purpose of this paper is to introduce the concept of *total asymptotically nonexpansive mappings* and to prove some Δ -convergence theorems of the mixed type iteration process to approximating a common fixed point for two *asymptotically nonexpansive mappings* and two *total asymptotically nonexpansive mappings* in *hyperbolic spaces*. The results presented in the paper extend and improve some recent results announced in the current literature.

MSC: 47H09; 47H10

Keywords: asymptotically nonexpansive mapping; total asymptotically nonexpansive mapping; common fixed point; hyperbolic space

1 Introduction and preliminaries

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a 'convex structure'. The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1], defined below, which is more restrictive than the hyperbolic type introduced in [2] and more general than the concept of hyperbolic space in [3].

A hyperbolic space is a metric space (X, d) together with a mapping $W : X^2 \times [0, 1] \rightarrow X$ satisfying

- (i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$;
- (ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$;
- (iii) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$;
- (iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$. A nonempty subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric [4], Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov (see [5]).

A hyperbolic space is uniformly convex [6] if for any $r > 0$ and $\epsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1 - \delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$.

A map $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$, which provides such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X . We call η monotone if it decreases with r (for fixed ϵ), i.e., $\forall \epsilon > 0, \forall r_2 \geq r_1 > 0 (\eta(r_2, \epsilon) \leq \eta(r_1, \epsilon))$.

In the sequel, let (X, d) be a metric space, and let K be a nonempty subset of X . We shall denote the fixed point set of a mapping T by $F(T) = \{x \in K : Tx = x\}$.

A mapping $T : K \rightarrow K$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in K.$$

A mapping $T : K \rightarrow K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \rightarrow 0$ such that

$$d(T^n x, T^n y) \leq (1 + k_n)d(x, y), \quad \forall n \geq 1, x, y \in K.$$

A mapping $T : K \rightarrow K$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq Ld(x, y), \quad \forall n \geq 1, x, y \in K.$$

Definition 1.1 A mapping $T : K \rightarrow K$ is said to be $(\{\mu_n\}, \{\xi_n\}, \rho)$ -total asymptotically nonexpansive if there exist nonnegative sequences $\{\mu_n\}, \{\xi_n\}$ with $\mu_n \rightarrow 0, \xi_n \rightarrow 0$ and a strictly increasing continuous function $\rho : [0, \infty) \rightarrow [0, \infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \mu_n \rho(d(x, y)) + \xi_n, \quad \forall n \geq 1, x, y \in K. \tag{1.1}$$

Remark 1.1 From the definitions, it is to know that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n = 0\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\xi_n\}, \rho)$ -total asymptotically nonexpansive mapping with $\xi_n = 0$, and $\rho(t) = t, t \geq 0$.

The existence of fixed points of various nonlinear mappings has relevant applications in many branches of nonlinear analysis and topology. On the other hand, there are certain situations where it is difficult to derive conditions for the existence of fixed points for certain types of nonlinear mappings. It is worth to mention that fixed point theory for nonexpansive mappings, a limit case of a contraction mapping when the Lipschitz constant is allowed to be 1, requires tools far beyond metric fixed point theory. Iteration schemas are the only main tool for analysis of generalized nonexpansive mappings. Fixed point theory has a computational flavor as one can define effective iteration schemas for the computation of fixed points of various nonlinear mappings. The problem of finding a common

fixed point of some nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics.

The purpose of this paper is to introduce the concepts of total asymptotically nonexpansive mappings and to prove some Δ -convergence theorems of the mixed type iteration process for approximating a common fixed point of two asymptotically nonexpansive mappings and two total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results given in [6–19].

In order to define the concept of Δ -convergence in the general setup of hyperbolic spaces, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X . For $x \in X$, we define a continuous functional $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic center* $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}), \forall y \in K\}. \tag{1.2}$$

This is the set of minimizers of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X , then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that ‘bounded sequences have unique asymptotic centers with respect to closed convex subsets.’ The following lemma is due to Leustean [20] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 [20] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X .*

Recall that a sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$.

A mapping $T : K \rightarrow K$ is semi-compact if every bounded sequence $\{x_n\} \subset K$ satisfying $d(x_n, Tx_n) \rightarrow 0$ has a convergent subsequence.

Lemma 1.2 [9] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1. \tag{1.3}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n \rightarrow \infty} a_n$ exists. If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \rightarrow 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 1.3 [12] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(x_n, x) &\leq c, & \limsup_{n \rightarrow \infty} d(y_n, x) &\leq c, \\ \lim_{n \rightarrow \infty} d(W(x_n, y_n, \alpha_n), x) &= c, \end{aligned}$$

for some $c \geq 0$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 1.4 [12] *Let K be a nonempty closed convex subset of uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{m \rightarrow \infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{m \rightarrow \infty} y_m = y$.*

2 Main results

Theorem 2.1 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η . Let $T_i : K \rightarrow K, i = 1, 2$, be a uniformly L_i -Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$ -total asymptotically nonexpansive mapping with sequence $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n \rightarrow \infty} \mu_n^i = 0, \lim_{n \rightarrow \infty} \xi_n^i = 0$ and a strictly increasing function $\rho^i : [0, \infty) \rightarrow [0, \infty)$ with $\rho^i(0) = 0, i = 1, 2$, let $S_i : K \rightarrow K, i = 1, 2$, be a uniformly \tilde{L}_i -Lipschitzian and asymptotically nonexpansive mapping with sequence $\{k_n^i\}$ satisfying $\lim_{n \rightarrow \infty} k_n^i = 0$. Assume that $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i) \neq \emptyset$, and for arbitrarily chosen $x_1 \in K, \{x_n\}$ is defined as follows:*

$$\begin{cases} x_{n+1} = W(S_1^n x_n, T_1^n y_n, \alpha_n), \\ y_n = W(S_2^n x_n, T_2^n x_n, \beta_n), \end{cases} \tag{2.1}$$

where $\{\mu_n^i\}, \{\xi_n^i\}, \rho^i, k_n^i, i = 1, 2, \{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

- (1) $\sum_{n=1}^{\infty} \mu_n^i < \infty, \sum_{n=1}^{\infty} \xi_n^i < \infty, \sum_{n=1}^{\infty} k_n^i < \infty, i = 1, 2$;
- (2) There exist constants $a, b \in (0, 1)$ with $0 < b(1 - a) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$;
- (3) There exists a constant $M^* > 0$ such that $\rho^i(r) \leq M^* r, r > 0, i = 1, 2$;
- (4) $d(x, T_i y) \leq d(S_i x, T_i y)$ for all $x, y \in K$ and $i = 1, 2$.

Then the sequence $\{x_n\}$ defined by (2.1) Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i)$.

Proof Set $L = \max\{L_i, \tilde{L}_i, i = 1, 2\}, \mu_n = \max\{\mu_n^i, k_n^i, i = 1, 2\}$ and $\xi_n = \max\{\xi_n^i, i = 1, 2\}, \rho = \max\{\rho^i, i = 1, 2\}$. By condition (1), we know that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$. The proof of Theorem 2.1 is divided into four steps.

Step 1. First we prove that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$.

For any given $p \in \mathcal{F}$, since $T_i, i = 1, 2$, is a total asymptotically nonexpansive mapping and $S_i, i = 1, 2$, is an asymptotically nonexpansive mapping, by condition (3) and (2.1), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(S_1^n x_n, T_1^n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n)d(S_1^n x_n, p) + \alpha_n d(T_1^n y_n, p) \end{aligned}$$

$$\begin{aligned} &\leq (1 - \alpha_n)[(1 + \mu_n)d(x_n, p)] \\ &\quad + \alpha_n[d(y_n, p) + \mu_n\rho(d(y_n, p)) + \xi_n] \\ &\leq (1 - \alpha_n)[(1 + \mu_n)d(x_n, p)] + \alpha_n[(1 + \mu_n M^*)d(y_n, p) + \xi_n], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} d(y_n, p) &= d(W(S_2^n x_n, T_2^n x_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(S_2^n x_n, p) + \beta_n d(T_2^n x_n, p) \\ &\leq (1 - \beta_n)[(1 + \mu_n)d(x_n, p)] \\ &\quad + \beta_n[d(x_n, p) + \mu_n\rho(d(x_n, p)) + \xi_n] \\ &\leq (1 - \beta_n)[(1 + \mu_n)d(x_n, p)] + \beta_n[(1 + \mu_n M^*)d(x_n, p) + \xi_n] \\ &= [1 + (1 - \beta_n)\mu_n + \beta_n\mu_n M^*]d(x_n, p) + \beta_n\xi_n \\ &\leq (1 + \mu_n + \beta_n\mu_n M^*)d(x_n, p) + \beta_n\xi_n. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) and simplifying it, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 - \alpha_n)[(1 + \mu_n)d(x_n, p)] \\ &\quad + \alpha_n[(1 + \mu_n M^*)((1 + \mu_n + \beta_n\mu_n M^*)d(x_n, p) + \beta_n\xi_n) + \xi_n] \\ &= \{1 + [1 + \alpha_n M^*(1 + \beta_n + \mu_n + \beta_n\mu_n M^*)]\mu_n\}d(x_n, p) \\ &\quad + (1 + \beta_n + \beta_n\mu_n M^*)\alpha_n\xi_n \\ &= (1 + \delta_n)d(x_n, p) + b_n, \end{aligned} \tag{2.4}$$

where $\delta_n = \mu_n(1 + \alpha_n M^*(1 + \beta_n + \mu_n + \beta_n\mu_n M^*))$, $b_n = (1 + \beta_n + \beta_n\mu_n M^*)\alpha_n\xi_n$. Since $\sum_{n=1}^\infty \mu_n < \infty$, $\sum_{n=1}^\infty \xi_n < \infty$ and condition (2), it follows from Lemma 1.2 that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for $p \in \mathcal{F}$.

Step 2. We show that

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0, \quad \lim_{n \rightarrow \infty} d(x_n, S_i x_n) = 0, \quad i = 1, 2. \tag{2.5}$$

For each $p \in \mathcal{F}$, from the proof of Step 1, we know that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. We may assume that $\lim_{n \rightarrow \infty} d(x_n, p) = c \geq 0$. If $c = 0$, then the conclusion is trivial. Next, we deal with the case $c > 0$. From (2.3), we have

$$d(y_n, p) \leq (1 + \mu_n + \beta_n\mu_n M^*)d(x_n, p) + \beta_n\xi_n. \tag{2.6}$$

Taking lim sup on both sides in (2.6), we have

$$\limsup_{n \rightarrow \infty} d(y_n, p) \leq c. \tag{2.7}$$

In addition, since

$$d(T_1^n y_n, p) \leq d(y_n, p) + \mu_n\rho(d(y_n, p)) + \xi_n \leq (1 + \mu_n M^*)d(y_n, p) + \xi_n,$$

and

$$d(S_1^n x_n, p) \leq (1 + \mu_n)d(x_n, p),$$

then we have

$$\limsup_{n \rightarrow \infty} d(T_1^n y_n, p) \leq c \tag{2.8}$$

and

$$\limsup_{n \rightarrow \infty} d(S_1^n x_n, p) \leq c. \tag{2.9}$$

Since $\lim_{n \rightarrow \infty} d(x_{n+1}, p) = c$, it is easy to prove that

$$\lim_{n \rightarrow \infty} d(W(S_1^n x_n, T_1^n y_n, \alpha_n), p) = c. \tag{2.10}$$

It follows from (2.8)-(2.10) and Lemma 1.3 that

$$\lim_{n \rightarrow \infty} d(S_1^n x_n, T_1^n y_n) = 0. \tag{2.11}$$

By the same method, we can also prove that

$$\lim_{n \rightarrow \infty} d(S_2^n x_n, T_2^n x_n) = 0. \tag{2.12}$$

By virtue of condition (4), it follows from (2.11) and (2.12) that

$$\lim_{n \rightarrow \infty} d(x_n, T_1^n y_n) \leq \lim_{n \rightarrow \infty} d(S_1^n x_n, T_1^n y_n) = 0 \tag{2.13}$$

and

$$\lim_{n \rightarrow \infty} d(x_n, T_2^n x_n) \leq \lim_{n \rightarrow \infty} d(S_2^n x_n, T_2^n x_n) = 0. \tag{2.14}$$

From (2.1) and (2.12) we have

$$\begin{aligned} d(y_n, S_2^n x_n) &= d(W(S_2^n x_n, T_2^n x_n, \beta_n), S_2^n x_n) \\ &\leq \beta_n d(T_2^n x_n, S_2^n x_n) \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned} \tag{2.15}$$

Observe that

$$d(x_n, y_n) = d(x_n, T_2^n x_n) + d(T_2^n x_n, S_2^n x_n) + d(S_2^n x_n, y_n).$$

It follows from (2.14) and (2.15) that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0. \tag{2.16}$$

This together with (2.13) implies that

$$\begin{aligned} d(x_n, T_1^n x_n) &\leq d(x_n, T_1^n y_n) + d(T_1^n y_n, T_1^n x_n) \\ &\leq d(x_n, T_1^n y_n) + Ld(y_n, x_n) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned} \tag{2.17}$$

On the other hand, from (2.11) and (2.16), we have

$$\begin{aligned} d(S_1^n x_n, T_1^n x_n) &\leq d(S_1^n x_n, T_1^n y_n) + d(T_1^n y_n, T_1^n x_n) \\ &\leq d(S_1^n x_n, T_1^n y_n) + Ld(y_n, x_n) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned} \tag{2.18}$$

Hence from (2.17) and (2.18), we have that

$$d(S_1^n x_n, x_n) \leq d(S_1^n x_n, T_1^n x_n) + d(T_1^n x_n, x_n) \rightarrow 0 \quad (n \rightarrow \infty). \tag{2.19}$$

In addition, since

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(S_1^n x_n, T_1^n y_n, \alpha_n), x_n) \\ &\leq (1 - \alpha_n)d(S_1^n x_n, x_n) + \alpha_n d(T_1^n y_n, x_n), \end{aligned}$$

from (2.13) and (2.19), we have

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \tag{2.20}$$

Finally, for all $i = 1, 2$, we have

$$\begin{aligned} d(x_n, T_i x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) \\ &\quad + d(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n) + d(T_i^{n+1} x_n, T_i x_n) \\ &\leq (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) + Ld(T_i^n x_n, x_n). \end{aligned}$$

It follows from (2.14), (2.17) and (2.20) that

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0, \quad i = 1, 2.$$

By virtue of condition (4), $d(S_i x_n, T_i^n x_n) \leq d(S_i^n x_n, T_i^n x_n)$, we have

$$\begin{aligned} d(x_n, S_i x_n) &\leq d(x_n, T_i^n x_n) + d(S_i x_n, T_i^n x_n) \\ &\leq d(x_n, T_i^n x_n) + d(S_i^n x_n, T_i^n x_n), \end{aligned}$$

it follows from (2.12), (2.14), (2.17) and (2.18) that

$$\lim_{n \rightarrow \infty} d(x_n, S_i x_n) = 0, \quad i = 1, 2.$$

Step 3. Now we prove that the sequence $\{x_n\}$ Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i)$.

In fact, for each $p \in F$, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. This implies that the sequence $\{d(x_n, p)\}$ is bounded, so is the sequence $\{x_n\}$. Hence, by virtue of Lemma 1.1, $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$.

Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_K(\{u_n\}) = \{u\}$. It follows from (2.5) that

$$\lim_{n \rightarrow \infty} d(u_n, T_i u_n) = 0. \tag{2.21}$$

Now, we show that $u \in F(T_i)$. For this, we define a sequence $\{z_j\}$ in K by $z_j = T_i^j u$. So we calculate

$$\begin{aligned} d(z_j, u_n) &\leq d(T_i^j u, T_i^j u_n) + d(T_i^j u_n, T_i^{j-1} u_n) + \dots + d(T_i u_n, u_n) \\ &\leq d(u, u_n) + \mu_j \rho(d(u, u_n)) + \xi_j + \sum_{k=1}^j d(T_i^k u_n, T_i^{k-1} u_n) \\ &\leq (1 + \mu_j M^*) d(u, u_n) + \xi_j + \sum_{k=1}^j d(T_i^k u_n, T_i^{k-1} u_n). \end{aligned} \tag{2.22}$$

Since T_i is uniformly L -Lipschitzian, from (2.22) we have

$$d(z_j, u_n) \leq (1 + \mu_j M^*) d(u, u_n) + \xi_j + jLd(T_i u_n, u_n).$$

Taking \limsup on both sides of the above estimate and using (2.21), we have

$$\begin{aligned} r(z_j, \{u_n\}) &= \limsup_{n \rightarrow \infty} d(z_j, u_n) \\ &\leq (1 + \mu_j M^*) \limsup_{n \rightarrow \infty} d(u, u_n) + \xi_j \\ &= (1 + \mu_j M^*) r(u, \{u_n\}) + \xi_j. \end{aligned}$$

And so

$$\limsup_{j \rightarrow \infty} r(z_j, \{u_n\}) \leq r(u, \{u_n\}).$$

Since $A_K(\{u_n\}) = \{u\}$, by the definition of asymptotic center $A_K(\{u_n\})$ of a bounded sequence $\{u_n\}$ with respect to $K \subset X$, we have

$$r(u, \{u_n\}) \leq r(y, \{u_n\}), \quad \forall y \in K.$$

This implies that

$$\liminf_{j \rightarrow \infty} r(z_j, \{u_n\}) \geq r(u, \{u_n\}).$$

Therefore we have

$$\lim_{j \rightarrow \infty} r(z_j, \{u_n\}) = r(u, \{u_n\}).$$

It follows from Lemma 1.4 that $\lim_{j \rightarrow \infty} T_i^j u = u$. As T_i is uniformly continuous, so that $T_i u = T_i(\lim_{j \rightarrow \infty} T_i^j u) = \lim_{j \rightarrow \infty} T_i^{j+1} u = u$. That is, $u \in F(T_i)$. Similarly, we also can show that $u \in F(S_i)$. Hence, u is the common fixed point of T_i and S_i . Reasoning as above by utilizing the uniqueness of asymptotic centers, we get that $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequence $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i)$. This completes the proof. \square

The following theorem can be obtained from Theorem 2.1 immediately.

Theorem 2.2 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η . Let $T_i : K \rightarrow K, i = 1, 2$, be a uniformly L_i -Lipschitzian and asymptotically nonexpansive mapping with sequence $\{t_n^i\} \subset [1, \infty)$ satisfying $\lim_{n \rightarrow \infty} t_n^i = 1$, and $S_i : K \rightarrow K, i = 1, 2$, be a nonexpansive mapping. Assume that $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i)$, for arbitrarily chosen $x_1 \in K, \{x_n\}$ is defined as follows:*

$$\begin{cases} x_{n+1} = W(S_1 x_n, T_1^n y_n, \alpha_n), \\ y_n = W(S_2 x_n, T_2^n x_n, \beta_n), \end{cases} \tag{2.23}$$

where $\{t_n^i\}, i = 1, 2, \{\alpha_n\}$ and β_n satisfy the following conditions:

- (1) $\sum_{n=1}^{\infty} (t_n^i - 1) < \infty, i = 1, 2$;
- (2) There exist constants $a, b \in (0, 1)$ with $0 < b(1 - a) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$;
- (3) $d(x, T_i y) \leq d(S_i x, T_i y)$ for all $x, y \in K$ and $i = 1, 2$.

Then the sequence $\{x_n\}$ defined in (2.23) Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i)$.

Proof Take $\rho^i(t) = t, t \geq 0, \xi_n^i = 0, \mu_n^i = t_n^i - 1, k_n^i = 0, i = 1, 2$, in Theorem 2.1. Since all the conditions in Theorem 2.1 are satisfied, it follows from Theorem 2.1 that the sequence $\{x_n\}$ Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i) \cap F(S_i)$.

This completes the proof of Theorem 2.2. \square

Theorem 2.3 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η . Let $T_i : K \rightarrow K, i = 1, 2$, be a uniformly L_i -Lipschitzian and asymptotically nonexpansive mapping with sequence $\{t_n^i\} \subset [1, \infty)$ satisfying $\lim_{n \rightarrow \infty} t_n^i = 1$. Assume that $\mathcal{F} := \bigcap_{i=1}^2 F(T_i)$, for arbitrarily chosen $x_1 \in K, \{x_n\}$ is defined as follows:*

$$\begin{cases} x_{n+1} = W(x_n, T_1^n y_n, \alpha_n), \\ y_n = W(x_n, T_2^n x_n, \beta_n), \end{cases} \tag{2.24}$$

where $\{t_n^i\}, i = 1, 2, \{\alpha_n\}$ and β_n satisfy the following conditions:

- (1) $\sum_{n=1}^{\infty} (t_n^i - 1) < \infty, i = 1, 2$;
- (2) There exist constants $a, b \in (0, 1)$ with $0 < b(1 - a) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$.

Then the sequence $\{x_n\}$ defined in (2.24) Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i)$.

Proof Take $\rho^i(t) = t$, $t \geq 0$, $\xi_n^i = 0$, $\mu_n^i = t_n^i - 1$, $S_i = I$, $i = 1, 2$, in Theorem 2.1. Since all the conditions in Theorem 2.1 are satisfied, it follows from Theorem 2.1 that the sequence $\{x_n\}$ Δ -converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 F(T_i)$.

This completes the proof of Theorem 2.3. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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