Embedding Steiner triple systems in hexagon triple systems

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Abstract

A hexagon triple is the graph consisting of the three triangles (triples) \{a, b, c\}, \{c, d, e\}, and \{e, f, a\}, where a, b, c, d, e, and f are distinct. The triple \{a, c, e\} is called an inside triple. A hexagon triple system of order n is a pair \((X, H)\) where \(H\) is a collection of edge disjoint hexagon triples which partitions the edge set of \(K_n\) with vertex set \(X\). The inside triples form a partial Steiner triple system. We show that any Steiner triple system of order \(n\) can be embedded in the inside triples of a hexagon triple system of order approximately \(3n\).

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1. Introduction

A Steiner triple system of order \(n\) (more simply triple system) is a pair \((S, T)\), where \(T\) is a collection of edge disjoint triangles (or triples) which partition the edge set of \(K_n\) with vertex set \(S\). It is well known [3] that the spectrum for Steiner triple systems is precisely the set of all \(n \equiv 1\) or 3 (mod 6) and that if \((S, T)\) is a triple system, \(|T| = n(n - 1)/6\).

A hexagon triple is the graph consisting of three triangles (triples) \{a, b, c\}, \{c, d, e\}, and \{e, f, a\}, where a, b, c, d, e, and f are distinct.

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We will denote this hexagon triple by any cyclic 2-shift of \([a, b, c, d, e, f]\) or \([a, f, e, d, c, b]\). A \(\lambda\)-fold hexagon triple system of order \(n\) is a pair \((X, H)\) where \(\lambda\) is a collection of edge disjoint hexagon triples which partitions the edge set of \(\lambda K_n\) with vertex set \(X\). The spectrum for three-fold hexagon triple systems is precisely the set of all \(n \equiv 1\) or 3 (mod 6) [4], and if \((X, H)\) is a three-fold hexagon triple system then \(|H| = n(n - 1)/6\).

The shrewd observer will notice that the spectra for triple systems and three-fold hexagon triple systems are precisely the same. Not only that, but the number of hexagon triples in a three-fold hexagon triple system of order \(n\) is the same as the number of triples in a Steiner triple system. The following question is immediate: For which \(n \equiv 1\) or 3 (mod 6) is it possible to construct a three-fold hexagon triple system with the property that the “inside” triples form a Steiner triple system of order \(n\)?

In [4] Selda Küçükçiçi and C. C. Lindner proved that this is always possible for all \(n \equiv 1\) or 3 (mod 6). Subsequently, a much stronger result was obtained by Lucia Gionfriddo [2] who proved that every Steiner triple system is the inside of some three-fold hexagon triple system.

In everything that follows “hexagon triple system” means a one-fold hexagon triple system. It is well known that the spectrum for hexagon triple systems is precisely the set of all \(n \equiv 1\) or 9 (mod 18) and that if \((X, H)\) is a hexagon triple system of order \(n\) then \(|H| = n(n - 1)/18\). Clearly the “inside” triples of a hexagon triple system cannot form a triple system: not enough inside triples. So the following problem is immediate. What is the largest Steiner triple system that can be embedded in the partial triple system consisting of the inside triples of a hexagon triple system? We need to be a bit more specific. Let \((X, H)\) be a hexagon triple system of order \(n\) and denote by \(P\) the collection of inside triples. Then \((X, P)\) is a partial triple system of order \(n\). We will say that the triple system \((S, T)\) is embedded in \((X, H)\) provided \(S \subseteq X\) and \(T \subseteq P\). Of course, we would like \(|S|\) to be as big as possible.

**Example 1.1 (STS(7) Embedded in a HTS(19)).** \(HTS(19) = \{(19, 8, 1, 7, 2, 13), (19, 10, 3, 9, 4, 15), (19, 12, 5, 11, 6, 17), (7, 19, 14, 1, 13, 8), (9, 19, 16, 3, 15, 10), (11, 19, 18, 5, 17, 12), (8, 19, 7, 14, 11, 15), (7, 17, 10, 13, 11, 16), (7, 18, 9, 13, 12, 15), (14, 5, 15, 2, 17, 3), (13, 5, 16, 1, 17, 4), (13, 6, 15, 1, 18, 3), (11, 13, 7, 5, 9), (11, 12, 4, 7, 6, 10), (1, 11, 4, 8, 5, 10), (2, 12, 3, 8, 6, 9), (2, 14, 8, 10, 18, 16), (4, 16, 10, 12, 14, 18), (6, 18, 12, 8, 16, 14)\). Inside triples = \{(19, 1, 2), (19, 3, 4), (19, 5, 6), (7, 14, 13), (9, 16, 15), (11, 18, 17), (8, 9, 11), (7, 10, 11), (7, 9, 12), (14, 15, 17), (13, 16, 17), (13, 15, 18), (1, 3, 5), (1, 4, 6), (2, 4, 5), (2, 3, 6), (2, 8, 18), (4, 10, 14), (6, 12, 16)\}.

**STS(7) = \{(19, 1, 2), (19, 3, 4), (19, 5, 6), (1, 3, 5), (1, 4, 6), (2, 4, 5), (2, 3, 6)\}.

The following observation is important in what follows. Let \((S, T)\) be a triple system embedded in the hexagon triple system \((X, H)\). Let \(3H(T) = \{(a, b, c), (c, d, e), (e, f, a) \mid \{a, b, c, d, e, f\} \in H \text{ and } \{a, c, e\} \in T\}. \) Then \((X, 3H(T))\) is a partial triple system and the set \(S\) is independent in the partial triple system \((X, 3H(T))\); i.e., if \(\{x, y\} \subseteq S\) then \(\{x, y, a\} \in 3H(T)\), where \(a \in X \setminus S\). As a consequence if \((S, T^*)\) is any triple system we can rearrange the triples in \(3H(T)\) to form a collection of hexagon triples \(3H(T^*)\) whose inside triples cover the triples in \(T^*\). Hence \((X, (H \setminus H(T)) \cup H(T^*))\) is a hexagon triple system containing \((S, T^*)\). We have the following lemma.

**Lemma 1.2.** If a Steiner triple system of order \(n\) can be embedded in a hexagon triple system of order \(v\), then every Steiner triple system of order \(n\) can be embedded in a hexagon triple system of order \(v\). □

**Example 1.3 (Any STS(7) can be Embedded in a HTS(19)).** Let \((X, H)\) and \((S, T)\) be the hexagon triple system and Steiner triple system in Example 1.1. Then \(3H(T) = \{(19, 8, 1), (1, 7, 2), (2, 13, 19), (19, 10, 3), (3, 9, 4), (4, 15, 19), (19, 12, 5), (5, 11, 6), (6, 17, 19), (1, 11, 3), (3, 7, 5), (5, 9, 1), (1, 12, 4), (4, 7, 6), (6, 10, 1), (2, 11, 4), (4, 8, 5), (5, 10, 2), (2, 12, 3), (3, 8, 6), (6, 9, 2)\}.

Let \((S, T^*)\) be the triple system given by \(T^* = \{(1, 2, 19), (1, 4, 6), (1, 3, 5), (2, 5, 6), (2, 3, 4), (4, 5, 19), (3, 6, 19)\}\) and rearrange the triples in \(3H(T)\) to form the hexagon triples \(H(T^*) = \{(1, 7, 2, 13, 19, 8), (1, 12, 4, 7, 6, 10), (1, 11, 3, 7, 5, 9), (2, 10, 5, 11, 6, 9), (2, 12, 3, 9, 4, 11), (4, 8, 5, 12, 19, 15), (3, 8, 6, 17, 19, 10)\}\).

Then the hexagon triples in \(H(T^*)\) cover \(T^*\) and \((X, (H \setminus H(T)) \cup H(T^*))\) is a hexagon triple system.

The following two constructions show that any triple system of order \(6k + 3\) can be embedded in a hexagon triple system of order \(18k + 9\) and any triple system of order \(6k + 1\) can be embedded in a hexagon triple system of order \(18k + 1\). Whether or not these embeddings are best possible remains an open and interesting question. (See Section 4.)

The interested reader is referred to [1,6] for related work on embedding Steiner triple systems into \(K_4 \setminus e\) designs and \(S(2, 4, v)\).
2. The $18k + 9$ construction

In view of Lemma 1.2 we need only embed a Kirkman triple system of order $6k + 3$ into a hexagon triple system of order $18k + 9$.

Let $(S, K)$ be a Kirkman triple system of order $6k + 3$ and set $X = S \times \{1, 2, 3\}$. Define a collection $H$ of hexagon triples as follows:

1. Let $\pi$ be a parallel class in $K$. For each triple $\{a, b, c\} \in \pi$ place the four hexagon triples $[(a, 1), (c, 2), (b, 1), (a, 2), (c, 1), (b, 2)], [(a, 1), (a, 3), (a, 2), (b, 2), (c, 3), (b, 3)], [(b, 1), (b, 3), (b, 2), (c, 2), (a, 3), (c, 3)],$ and $[(c, 1), (c, 3), (c, 2), (a, 2), (b, 3), (a, 3)]$ in $H$.

2. For each triple $\{a, b, c\} \not\in \pi$ place the three hexagon triples $[(a, 1), (c, 2), (b, 1), (a, 2), (c, 1), (b, 2)], [(a, 2), (c, 3), (b, 2), (a, 3), (c, 2), (b, 3)]$ and $[(a, 3), (c, 1), (b, 3), (a, 1), (c, 3), (b, 1)]$ in $H$.

Then $(X, H)$ is a hexagon triple system of order $18k + 9$ and each $\{a, b, c\} \in K$ regardless of whether or not $\{a, b, c\} \in \pi$ is covered by the hexagon triple $[(a, 1), (c, 2), (b, 1), (a, 2), (c, 1), (b, 2)]$.

Hence $(S, K)$ is embedded in $(X, H)$ and we have the following result.

**Theorem 2.1.** Any triple system of order $6k + 3$ can be embedded in a hexagon triple system of order $18k + 9$. □

3. The $18k + 1$ construction

Let $\{\infty\} \cup S, T$ be a Steiner triple system of order $6k + 1$ constructed using the Skolem construction [8] and let $T(\infty) = \{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, ..., \{\infty, x_{3k}, y_{3k}\}$ be the $3k$ triples in $T$ containing $\infty$. Then $T \setminus T(\infty)$ contains a collection $\pi$ of $k$ parallel triples which we can consider to cover $\{y_1, y_2, ..., y_{3k}\}$. (See [5].) Set $X = \{\infty\} \cup (S \times \{1, 2, 3\})$ and define a collection $H$ of hexagon triples as follows:

1. For each triple $\{\infty, x_i, y_i\} \in T$ place the two hexagon triples $[\infty, (y_1, 2), (x_i, 1), (x_i, 2), (y_1, 1), (x_i, 3)]$ and $[(x_i, 2), \infty, (y_1, 3), (x_i, 1), (x_i, 3), (y_1, 2)]$ in $H$.

   \[\begin{align*}
   \text{Note that these two hexagon triples contain all of the edges in } \{\infty\} \cup (\{x_i, y_i\} \times \{1, 2, 3\}) \text{ except for the missing} \\
   \text{“vertical” triple } \{(y_1, 1), (y_1, 2), (y_1, 3)\}. \]

2. For each $\{a, b, c\} \in \pi$ place four hexagon triples in $H$ as in (1) in the $18k + 9$ construction. (These triples cover the missing “vertical” triples.)
(3) For each triple in $T \setminus (T(\infty) \cup \pi)$ place three hexagon triples in $H$ as in (2) in the $18k + 9$ construction.

Then $(X, H)$ is a hexagon triple system of order $18k + 1$ and $(\cup S, T)$ is embedded in it. We have the following theorem.

**Theorem 3.1.** Any triple system of order $6k + 1$ can be embedded in a hexagon triple system of order $18k + 1$. □

### 4. Concluding remarks

Let $(X, 3H)$ be a triple system and $S \subseteq X$. The subset $S$ is said to be independent provided it contains no triples of $3H$. In other words, if $\{x, y\} \subseteq S$ then $\{x, y, a\} \in 3H$, where $a \in X \setminus S$. In [7] N. Sauer and J. Schönheim proved that the largest independent set in a triple system of order $v$ is

$$
\begin{cases}
(v - 1)/2, & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}, \text{ and} \\
(v + 1)/2, & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}.
\end{cases}
$$

Now let $(S, T)$ be a triple system of order $n$ embedded in the hexagon triple system $(X, H)$ of order $v$ and let $3H = \{\{a, b, c\}, \{c, d, e\}, \{e, f, a\} \mid \{a, b, c, d, e, f\} \in H\}$. Then $(X, 3H)$ is a Steiner triple system of order $v$ and $S$ is an independent set.

It follows that $v$ must be admissible; i.e., $v \equiv 1$ or $9 \pmod{18}$, and $v \geq 2n + 1$. In particular,

\[
v = \begin{cases}
2n + 1, & \text{if } n \equiv 9 \text{ or } 13 \pmod{18} \\
2n + 3, & \text{if } n \equiv 3 \pmod{18}, \\
2n + 5, & \text{if } n \equiv 7 \pmod{18} \\
2n + 7, & \text{if } n \equiv 1 \text{ or } 15 \pmod{18}.
\end{cases}
\]

Establishing whether or not these bounds can be obtained is a difficult problem.

### References


