Improved Sobolev inequalities on groups of Iwasawa type in presence of symmetry

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Abstract

Let \( G \) be a simple Lie group of real rank one and \( N \) be in the Iwasawa decomposition of \( G \). We prove a refined version of the Sobolev inequality on \( N \) in presence of symmetry.

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1. Introduction

Let \( G \) be a simple Lie group of real rank one and \( N \) be in the Iwasawa decomposition of \( G \) with homogeneous dimension \( Q \geq 3 \) (see Section 2 for definitions and properties), and let us consider the following Sobolev inequality for the \( L^2 \)-norm of the gradient (see Folland [8]):

\[
\| \nabla_N f \|_2^2 \geq S_{\min} \| f \|_{2^*}^2, \quad \forall f \in S^1_0(N),
\]

for appropriate Sobolev space \( S^1_0(N) \), where \( 2^* = \frac{2Q}{Q-2} \) be the critical Sobolev exponent, and \( S_{\min} \) be the best embedding constant. We are interested in the following question on \( N \): is there a natural way to bound the quantity

\[
\| \nabla_N f \|_2^2 - S_{\min} \| f \|_{2^*}^2
\]

from below in terms of the distance of \( f \) from the set of Sobolev minimizers?

The question in the case of \( N = \mathbb{R}^n \) is left by Brezis and Lieb [5] and a positive answer was given by Bianchi and Egnell [3]. The minimizers in (1.1) in the case of Heisenberg group have determined by Jerison and Lee in [11]. Loiudice [14] extended the results in [3] to the subelliptic context of Heisenberg group \( H_n \). In this paper, we consider the analogous question to the Kohn’s subelliptic context of \( N \) in presence of symmetry.

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Let
\[ U(\xi) = U(x, z) = k_0 \left( \left( 1 + \frac{|x|^2}{4} \right)^2 + |z|^2 \right)^{-\frac{Q-2}{4}}, \quad \xi = (x, z) \in N, \] (1.2)
where the constant \( k_0 \) is chosen so that \( \| \nabla_N U \|_2^2 = 1 \). Let \( \mathcal{M} \) be the set of functions of the form
\[ \varphi(\xi) = cU_{\lambda, \eta}(\xi) = c\lambda^{\frac{Q-2}{4}}U(\delta_\lambda(\tau_\eta^{-1}(\xi))), \]
where \( c \in \mathbb{R}, \lambda \in \mathbb{R}^+, \delta_\lambda \) is the natural dilations and \( \tau_\eta : N \to N \) is the operator of left-translation \( \tau_\eta(g) = \eta \circ g \). Thus the set \( \mathcal{M} \) is a \((\dim N + 2)\)-dimensional manifold embedded in \( S_0^1(N) \) by means of map
\[ \mathbb{R} \times \mathbb{R}^+ \times N \ni (c, \lambda, \eta) \mapsto cU_{\lambda, \eta} \in S_0^1(N). \]
Let
\[ S = \frac{\| \nabla_N U_{\lambda, \eta} \|_2^2}{\| U_{\lambda, \eta} \|_2^2} = \frac{\| \nabla_N U \|_2^2}{\| U \|_2^2}. \] (1.3)
Recall that for some constant \( c \), \( cU_{\lambda, \eta}(\xi) \) is a solution of the Yamabe-type equation on \( N \) (see [9, Theorem 1.1]). A simple calculation gives us
\[ \Delta_N U_{\lambda, \eta} + S^{\frac{2^*}{2}}U_{\lambda, \eta}^{2^* - 1} = 0. \] (1.4)
Define the distance between \( \mathcal{M} \) and a function \( f \in S_0^1(N) \) as
\[ d(f, \mathcal{M}) = \inf_{u \in \mathcal{M}} \| \nabla_N (f - u) \|_2 = \inf_{c, \lambda, \eta} \| \nabla_N (f - cU_{\lambda, \eta}) \|_2. \]
Observe that
\[ d\left( c\lambda^{\frac{Q-2}{4}} f \circ \delta_\lambda \circ \tau_\eta, \mathcal{M} \right) = |c|d(f, \mathcal{M}). \]
We say that a function \( u : N \to \mathbb{R} \) has partial symmetry (with respect to a point \( \eta_0 \in N \)) if there exists a function \( \tilde{u} : N \to \mathbb{R} \) such that for every \( \eta \in N \), one has \( \tau_\eta u(\eta) = \tilde{u}(|x|, z) \). If \( u \) has partial symmetry and is a positive entire solution to the following Yamabe-type equation:
\[ \Delta_N u + S^{\frac{2^*}{2}}u^{2^* - 1} = 0, \] (1.5)
then we must have \( u = U_{\lambda, \eta} \) for some \( \lambda \in \mathbb{R}^+ \) and \( \eta \in N \) (see [9]).

We denote by \( S_{ps}(N) \) the subset of \( S_0^1(N) \) of the functions having partial symmetry. In this paper we prove the following.

**Theorem 1.1.** There exists a positive constant \( \alpha \), depending only on the dimension \( Q \), such that
\[ \| \nabla_N f \|_2^2 - S\| f \|_2^2 \geq \alpha d(f, \mathcal{M})^2, \quad \forall f \in S_{ps}(N). \]
Furthermore, the result is optimal in the sense that it is false if the remainder term is replaced by \( d(f, \mathcal{M})^\beta \| \nabla_N f \|_2^{2-\beta} \), with \( \beta < 2 \).

A key argument in the proof of the theorem is the study of the eigenvalues of the following problem:
\[ -U_{\lambda, \eta}^{2^* - 2^*} \Delta_N v = \mu v, \quad v \in S_0^1(N), \] (1.6)
which will be proved in Section 4 by using the Cayley transform and the spherical principal series representation of \( G \).

**Remark 1.2.** Garofalo and Vassilev [9] conjecture that the only positive entire solution of the Yamabe-type equation (1.5) is, up to group translations and dilations, \( U \) given by (1.2). If true, we could work with the space \( S_0^1(N) \) instead of \( S_{cyl}^1(N) \) in the proof of Theorem 1.1, which would provide a generalization of the result of [3] and [14] to the setting of groups of Iwasawa type.
2. Notation and preliminaries

We begin by describing the Lie groups and Lie algebras under consideration. A Heisenberg type group $N$ is a Carnot group of step two with the following properties: the Lie algebra $n$ of $N$ is endowed with an inner product $\langle \cdot, \cdot \rangle$ such that, if $\mathfrak{z}$ is the center of $n$, then $[\mathfrak{z}, \mathfrak{z}] = \mathfrak{z}$ and moreover, for every fixed $z \in \mathfrak{z}$, the map $J_z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$J_z(\omega_1), \omega_2) = \{z, [\omega_1, \omega_2]\}, \quad \forall \omega_1, \omega_2 \in \mathfrak{z}^\perp,$$

(2.1)
is an orthogonal map whenever $\langle z, z \rangle = 1$.

We set $m = \dim \mathfrak{z}^\perp$ and $n = \dim \mathfrak{z}$. Since $N$ has step two and since the stratification of the Lie algebra $\mathfrak{g}$ is evidently $\mathfrak{z}^\perp \oplus \mathfrak{z}$, in the sequel we shall fix on $N$ a system of coordinates $(x, z)$ and that the group law has the form

$$(x, z) \circ (x', z') = \left(\sum_{i=1}^{m} x_i + x'_i, \sum_{j=1}^{n} z_j + z'_j + \frac{1}{2} \sum_{i=1}^{m} U^{(j)}(x)x'_i, \quad j = 1, 2, \ldots, n \right)$$

(2.2)

for suitable skew-symmetric matrices $U^{(j)}$, $j = 1, 2, \ldots, n$.

The following theorem can be found in [4].

**Theorem 2.1.** $N$ is a Heisenberg type group if and only if $N$ is (isomorphic to) $\mathbb{R}^{m+n}$ with the group law in (2.2) and the matrices $U^{(1)}, U^{(2)}, \ldots, U^{(n)}$ have the following properties:

(1) $U^{(j)}$ is an $m \times m$ skew symmetric and orthogonal matrix, for every $j = 1, 2, \ldots, n$;

(2) $U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0$ for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

An easy computation shows that the vector field in the algebra $n$ of $N = (\mathbb{R}^{m+n}, \circ)$ that agrees at the origin with $\frac{\partial}{\partial x_j}$ ($j = 1, \ldots, m$) is given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{n} \left( \sum_{i=1}^{m} U^{(k)}_{i,j} x_i \right) \frac{\partial}{\partial z_k},$$

and that $n$ is spanned by the left-invariant vector fields

$$X_1, \ldots, X_m, \quad Z_1 = \frac{\partial}{\partial z_1}, \ldots, Z_n = \frac{\partial}{\partial z_n}.$$

A simple calculation gives us (see e.g. [4])

$$[X_i, X_j] = \sum_{r=1}^{n} U^{(r)}_{i,j} Z_r$$

(2.3)

and hence we obtain, by (2.1) and (2.3),

$$J_z x = \sum_{r=1}^{n} \sum_{i=1}^{m} z_r x_i J_z (X_i) = \sum_{r=1}^{n} \sum_{i=1}^{m} z_r x_i \left( \sum_{j=1}^{m} U^{(r)}_{i,j} X_j \right)$$

$$= \sum_{j=1}^{m} \left( \sum_{r=1}^{n} \sum_{i=1}^{m} z_r x_i U^{(r)}_{i,j} X_j \right),$$

(2.4)

where

$$z = (z_1, \ldots, z_n) = z_1 Z_1 + \cdots + z_n Z_n,$$

$$x = (x_1, \ldots, x_m) = x_1 X_1 + \cdots + x_m X_m.$$
We denote $\nabla_{N} = (X_{1}, \ldots, X_{m})$. The Kohn’s sublaplacian on the Heisenberg type group $N$ is given by (see [4])

$$
\Delta_{N} = \sum_{j=1}^{m} X_{j}^{2} = \sum_{j=1}^{m} \left( \frac{\partial}{\partial x_{j}} + \frac{1}{2} \sum_{k=1}^{n} \left( \sum_{l=1}^{m} U_{j,i}^{(k)} x_{l} \right) \frac{\partial}{\partial z_{k}} \right)^{2}
$$

$$
= \Delta_{x} + \frac{1}{4} |x|^{2} \Delta_{z} + \sum_{k=1}^{n} (U^{(k)} x, \nabla_{x}) \frac{\partial}{\partial z_{k}},
$$

where

$$
\nabla_{x} = \left( \frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{m}} \right), \quad \Delta_{x} = \sum_{j=1}^{m} \left( \frac{\partial}{\partial x_{j}} \right)^{2}, \quad \Delta_{z} = \sum_{k=1}^{n} \left( \frac{\partial}{\partial z_{k}} \right)^{2}.
$$

When $\lambda > 0$ we define the homogeneous dilation $\delta_{\lambda}$ on $N$ by $\delta_{\lambda}(x, t) = (\lambda x, \lambda^{2} t)$. It is easy to check that $\delta_{\lambda}$ is a group automorphism. We write $Q$ for the homogeneous dimension of $N$, i.e., $Q = \text{dim} \mathfrak{z} + 2 \dim \mathfrak{z} = m + 2n$.

It is known that a Heisenberg type group $N$ is the Iwasawa $N$-group of a real rank one simple Lie group if and only if its Lie algebra satisfies the $J^{2}$-condition (see e.g. [7]). We shall henceforth assume that this condition holds.

We write $S$ for the unit sphere in $\mathfrak{z}^{1} \oplus \mathfrak{z} \oplus \mathbb{R}$, i.e.,

$$
S = \{(x', z', t') \in \mathfrak{z}^{1} \oplus \mathfrak{z} \oplus \mathbb{R} : |x'|^{2} + |z'|^{2} + |t'|^{2} = 1\}.
$$

The Cayley transform $C : N \rightarrow S$, introduced in [6], is given by (see e.g. [1])

$$
C(x, z) = \frac{1}{\mathcal{B}(x, z)} \left( \mathcal{A}(x, z) x, 2z, -1 + \frac{|x|^{4}}{16} + |z|^{2} \right)
$$

$$
= \frac{1}{\mathcal{B}(x, z)} \left( x + \frac{|x|^{2}}{4} x - Jz x, 2z, -1 + \frac{|x|^{4}}{16} + |z|^{2} \right), \quad (2.5)
$$

where $\mathcal{A}(x, z)$ and $\mathcal{A}(x, z)$ denote the linear maps $1 + \frac{|x|^{2}}{4} + Jz$ and $1 + \frac{|x|^{2}}{4} - Jz$, respectively, and the real number $\mathcal{B}(x, z)$ is defined by

$$
\mathcal{B}(x, z) = \left( 1 + \frac{|x|^{2}}{4} \right)^{2} + |z|^{2}. \quad (2.6)
$$

Its inverse $C^{-1} : S \setminus \{o\} \rightarrow N$ is given by

$$
C^{-1}(x', z', t') = \frac{1}{(1 - t')^{2} + |z'|^{2}} \left( 2(1 - t') + Jz' \right) x', 2z'.
$$

Here $o$ stands for $(0, 0, 1)$. Note that if $(x', z', t') = C(x, z)$, then

$$
\mathcal{B}(x, z) = \frac{4}{(1 - t')^{2} + |z'|^{2}}.
$$

When the dependence on $(x, z)$ in $N$ is clear, we just write $\mathcal{A}$, $\mathcal{A}$ and $\mathcal{B}$.

The Jacobian determinant $J_{C^{-1}}$ of the map $C^{-1}$ is given by (see e.g. [2])

$$
J_{C^{-1}}(\xi) = \frac{d\mathcal{B}^{-1}(\xi)}{d\sigma(\xi)} = \left( \frac{4}{(1 - t')^{2} + |z'|^{2}} \right)^{Q / 2}
$$

for all $\xi = (x', z', t') \in S \setminus \{o\}$, where $\sigma$ denotes the standard measure on the sphere. Clearly

$$
J_{C} = J_{C^{-1}} \circ C = \mathcal{B}^{-Q / 2} = \left( \left( 1 + \frac{|x|^{2}}{4} \right)^{2} + |z|^{2} \right)^{-Q / 2}. \quad (2.7)
$$

We obtain, by (1.2), (2.6) and (2.7),
\[ U(\xi) = k_0 B^{-\frac{Q}{2}} = k_0 J_{\xi}^{\frac{1}{2} - \frac{1}{2}}, \]
\[ U(\xi)^{1-2^*} = k_0^{-2} B^{-\frac{Q+2}{2}} = k_0^{-2} J_{\xi}^{\frac{1}{2} - \frac{1}{2}}. \] (2.8)

For real \( \beta \), we define the homogeneous Sobolev space \( S_0^\beta(N) \) to be the completion of \( C_c^\infty(N) \), the space of smooth functions with compact support on \( N \), with respect to the norm
\[ \| f \|_{S_0^\beta(N)} = \left( (-\Delta_N)^{\beta} f \right)^{\frac{1}{2}}_N = \left( (-\Delta_N)^{\frac{\beta}{2}} f \right)_2, \quad \forall f \in C_c^\infty(N), \]
where \( \langle \cdot, \cdot \rangle_N \) denotes the usual inner product on \( L^2(N) \).

As in [2], we shall denote by \( W_j \) the vector field
\[ W_j = \frac{\mathcal{C}_* X_j}{|\mathcal{C}_* X_j|} = J_{\xi}^{\frac{1}{2}} (\mathcal{C}_* X_j), \quad j = 1, \ldots, m, \]
and the sublaplacian \( \mathcal{L} \) on \( S \) by
\[ \langle \mathcal{L} f | f \rangle_S^{\frac{1}{2}} = \sum_{j=1}^m (W_j f | W_j f)_S, \quad \forall f \in C_c^\infty(S). \]

In the real case, when \( \dim Z = 0 \), the operator \( \mathcal{L} \) is the Laplace–Beltrami operator on the Euclidean sphere. In the complex case, when \( \dim Z = 1 \), the operator \( \mathcal{L} \) is the laplacian considered by Geller [10] on the complex sphere.

The sublaplacian \( \Delta_N \) and \( \mathcal{L} \) are linked by the equality (see e.g. [1])
\[ J_{\xi}^{\frac{1}{2} - \frac{1}{2}} (-\Delta_N) (J_{\xi}^{\frac{1}{2} - \frac{1}{2}} f \circ \mathcal{C}) = \left( (\mathcal{L} + b) f \right) \circ \mathcal{C}, \quad \forall f \in C_c^\infty(S), \] (2.9)
where
\[ b = \frac{(Q - 2)}{4} \cdot \dim Z^{\frac{1}{2}}. \] (2.10)

Definition 2.2. For real \( \beta \), we define the Sobolev space \( S_0^\beta(S) \) to be the completion of the space of smooth functions on \( S \), with respect to the norm
\[ \| f \|_{S_0^\beta(S)} = \left( (\mathcal{L} + b)^{\beta} f | f \right)^{\frac{1}{2}}_S = \left( (\mathcal{L} + b)^{\frac{\beta}{2}} f \right)_{L^2(S)}, \quad \forall f \in C_c^\infty(S). \]

An important property of these Sobolev spaces is the following (see [1]).

Theorem 2.3. Let \( -\frac{Q}{2} < \beta < \frac{Q}{2} \). The map
\[ T_\beta: f \mapsto J_{\xi}^{\frac{1}{2} - \frac{\beta}{2}} f \circ \mathcal{C} \]
is a bounded invertible operator from \( S_0^\beta(S) \) to \( S_0^\beta(N) \).

In particular, we shall work with the spaces \( S_0^1(S) \), \( S_0^1(N) \) and the space of cylindrically symmetric functions of \( S_0^1(N) \), namely in
\[ S_{cyl}^1(N) = \{ u \in S_0^1(N): u(x, z) = u(|x|, z) \}. \]
Let us also observe that \( S_{cyl}^1(N) \) is a Hilbert space endowed with the scalar product \( (u, v) = \int_N \nabla_N u \cdot \nabla_N v \, dx \, dz \).

3. The Fourier transform on \( K/M \)

Let \( G \) be a simple Lie group of real rank one. We refer to [12,13] for the spherical principal series of \( G \). Given an Iwasawa decomposition \( KAN \) of \( G \), we write \( M \) for the centralizer of \( A \) in \( K \). Then \( MAN \) is a parabolic subgroup and \( S \) may be identified with \( K/M \) (see [1]).
Let \( \hat{K} \) denote the set of equivalence classes of irreducible finite-dimensional representations of \( K \). For an irreducible representation \((\pi_\tau, V_\tau) \in \hat{K}, \) set
\[
V_\tau^M = \{ v \in V_\tau : \pi(m)v = v, \forall m \in M \}.
\]
It is known that
\[
L^2(K/M) = \bigoplus_{(\pi_\tau, V_\tau) \in \hat{K}} V_\tau^M.
\]
In the degenerate case where \( \dim \mathfrak{z} = 0 \), provided that \( \dim \mathfrak{z} \perp \geq 2 \), the subspaces of spherical harmonics of degree \( d \) are \( K \)-invariant and irreducible. For every \( d \in \mathbb{N} = \{0, 1, 2, \ldots\} \), we denote by \( \tau_d \) the corresponding class one representation.

In the other cases, the space of spherical harmonics of a given degree are not always irreducible. One has to restrict attention to so-called bigraded spherical harmonics. Roughly speaking, to any class one representation there corresponds a pair of integers \( (d_1, d_2) \). When \( \dim \mathfrak{z} = 1 \) (the complex case) we consider \( (d_1, d_2) \in \mathbb{N}^2 \). When \( \dim \mathfrak{z} = 3 \) (the quaternionic case) or \( \dim \mathfrak{z} = 7 \) (the octonionic case), we consider only pairs of integers \( (d_1, d_2) \in \mathbb{N}^2 \) where \( d_1 \geq d_2 \geq 0 \).

Define the sublaplacian \( \mathcal{L}^\sharp \) on \( K/M \) by
\[
\mathcal{L}^\sharp f^\sharp = (\mathcal{L} f)^\sharp, \quad f \in C^\infty(S).
\]
We see that (see e.g. [1]), in the real case (where \( \dim \mathfrak{z} = 0 \)),
\[
(\mathcal{L}^\sharp + b)^\wedge (\tau_d) = \left( \frac{Q}{2} - 1 + d \right) \left( \frac{Q}{2} + d \right),
\]
while in the other cases,
\[
(\mathcal{L}^\sharp + b)^\wedge (\tau(d_1, d_2)) = \left( \frac{Q}{2} - 1 + 2d_1 \right) \left( \frac{\dim \mathfrak{z} \perp}{2} + 2d_2 \right).
\]
Since \( \mathcal{L}^\sharp \) is \( M \)-invariant, it acts on \( V_\tau \) by scalar multiples, say
\[
\mathcal{L}^\sharp Y_\tau = \lambda_\tau Y_\tau, \quad Y_\tau \in V_\tau^M,
\]
then
\[
\lambda_\tau = (\mathcal{L}^\sharp + b)^\wedge (\tau) - b.
\]
We obtain, by (2.10), (3.2) and (3.3),
\[
\lambda_{\tau(d_1, d_2)} = \left( \frac{Q - 2}{2} + 2d_1 \right) \left( \frac{\dim \mathfrak{z} \perp}{2} + 2d_2 \right) - \frac{(Q - 2)^2}{4} \cdot \dim \mathfrak{z} \perp.
\]

4. The main results

In the following lemma, we calculate the first and second eigenvalues of (1.5) and the corresponding eigenspaces.

Lemma 4.1. Let \( \mu_i, \ i = 1, 2, \ldots, \) be the eigenvalues of (1.5) given in increasing order. Then

1. \( \mu_1 = S^{2^*} \) is simple with eigenfunction \( U_{\lambda, \eta} \);
2. \( \mu_2 = S^{2^*}(2^* - 1) \) has multiplicity \( (\dim N + 1) \) with corresponding eigenspace spanned by \( \{ \partial_{\eta} U_{\lambda, \eta}, \nabla_{\eta} U_{\lambda, \eta} \} \).

Furthermore, the eigenvalues do not depend on \( \lambda \) and \( \eta \).

Proof. A simple scaling argument shows that the eigenvalues do not depend on \( \lambda \) and \( \eta \). Hence we may assume that \( \lambda = 1, \eta = 0 \) and consequently \( U_{\lambda, \eta} = U \). We want to study the eigenvalue problem
\[
-\Delta_N v = \mu U^{2^*-2} v, \quad v \in S_0^1(N).
\]
Now we consider the linear map \( T : S^1_0(\mathbb{S}) \rightarrow S^1_0(N) \) defined by
\[
T(u)(\xi) = U(\xi)u(c^2(\xi)), \quad u \in S^1_0(\mathbb{S}), \quad \xi \in N.
\]
According to Theorem 2.3, the map \( T \) is an isometry from \( S^1_0(\mathbb{S}) \) to \( S^1_0(N) \). Let \( u \in S^1_0(\mathbb{S}) \) and let \( T(u) \) be a solution of (4.1). Then we obtain, by (2.2), (2.9) and (4.1),
\[
(L + b)u = J_{c^2}^{-1/2}(-\Delta_N)(J_{c^2}^{-1/2}u)
= k_0^{2\sigma-1}U(\xi)^{1-2\sigma}(-\Delta_N)(k_0^{-1}U(\xi)u)
= k_0^{2\sigma-2}U(\xi)^{1-2\sigma}(-\Delta_N)(T(u))
= \mu k_0^{2\sigma-2}U(\xi)^{1-2\sigma}U(\xi)^{2\sigma-2}T(u)
= \mu k_0^{2\sigma-2}u.
\]
Therefore
\[
(L)u = (\mu k_0^{2\sigma-2} - b)u = \tilde{\mu}u,
\]
where
\[
\tilde{\mu} = \mu k_0^{2\sigma-2} - b.
\]
By (3.4), the first eigenvalue \( \tilde{\mu}_1 = 0 \) (in this case we have \( d_1 = 0 \) and \( d_2 = 0 \)) is simple with corresponding eigenfunction the constant function. Hence, by means of the isometry \( T \), we obtain that the first eigenfunction of (4.1) is \( U \), corresponding to the eigenvalue \( S^2 \) according to (4.1).

The second eigenvalue \( \tilde{\mu}_2 = \dim \mathbb{S}^\sigma \) (in this case we have \( d_1 = 1 \) and \( d_2 = 0 \)) is \( (\dim N + 1) \)-dimensional with corresponding eigenspace spanned by the \( K \)-spherical harmonics of degree 1 restricted to \( \mathbb{S} \), i.e., the function \( \{x_1, \ldots, x_m, z_1, \ldots, z_n, t\} \) restricted to \( \mathbb{S} \).

Let \( \eta = (\eta_1, \ldots, \eta_m, w_1, \ldots, w_n) \in N \). We obtain
\[
U_{\lambda, \eta} = \lambda \Omega^{\sigma_2} \frac{1}{U(\partial_{(\tau_{\eta^{-1}})(\xi)})} = k_0 \lambda \Omega^{\sigma_2} \frac{1}{\left(1 + \frac{|x|^2}{4} \right)^{1/2} + |z|^2} \frac{1}{2} \left(1 + \frac{|x|^2}{4} \right) x_j + \sum_{r=1}^n z_r \left(-\sum_{i=1}^m U^{(r)}_{j,i} x_i\right)
\]
Since \( U^{(j)} \) is an \( m \times m \) skew symmetric matrix, for every \( j = 1, 2, \ldots, n \), we obtain, by a simple calculation,
\[
\frac{\partial U_{\lambda, \eta}}{\partial y_j} \bigg|_{\lambda=1, \eta=0} = -\frac{Q-2}{4} U(\xi) \left(1 + \frac{|x|^2}{4}\right)^{1/2} + |z|^2 \left(2 + \frac{|x|^2}{4}\right) x_j - \frac{1}{4} \sum_{r=1}^n z_r \sum_{i=1}^m U^{(r)}_{j,i} x_i
\]
\[
\frac{\partial U_{\lambda, \eta}}{\partial w_r} \bigg|_{\lambda=1, \eta=0} = -\frac{Q-2}{4} U(\xi) \left(1 + \frac{|x|^2}{4}\right)^{1/2} + |z|^2 \left(2 + \frac{|x|^2}{4}\right) (-2z_r)
\]
\[
\frac{\partial U_{\lambda, \eta}}{\partial \lambda} \bigg|_{\lambda=1, \eta=0} = -\frac{Q-2}{4} U(\xi) \left(1 + \frac{|x|^2}{4}\right)^{1/2} + |z|^2 \left(2 + \frac{|x|^2}{4}\right) \left(2 + \frac{|x|^2}{4} + 4|z|^2\right)
\]
\[
= -\frac{Q-2}{4} U(\xi) \left(1 + \frac{|x|^2}{4} + |z|^2\right).
\]
Therefore, by (2.4) and (2.5),
\[
\frac{Q-2}{4} \cdot T(x'_j) = \left. \frac{\partial U_{\lambda, \eta}}{\partial y_j} \right|_{\lambda=1, \eta=0}, \quad j = 1, \ldots, m, \\
\frac{Q-2}{2} \cdot T(z'_r) = \left. \frac{\partial U_{\lambda, \eta}}{\partial w_r} \right|_{\lambda=1, \eta=0}, \quad r = 1, \ldots, n, \\
-\frac{Q-2}{2} \cdot T(t') = \left. \frac{\partial U_{\lambda, \eta}}{\partial \lambda} \right|_{\lambda=1, \eta=0}.
\]

Now observe that, by (1.4),
\[
\Delta_N \frac{\partial U_{\lambda, \eta}}{\partial y_j} + S \frac{u}{2^*} (2^* - 1) U_{\lambda, \eta}^{2^* - 2} \frac{\partial U_{\lambda, \eta}}{\partial y_j} = 0, \quad j = 1, \ldots, m, \\
\Delta_N \frac{\partial U_{\lambda, \eta}}{\partial w_r} + S \frac{u}{2^*} (2^* - 1) U_{\lambda, \eta}^{2^* - 2} \frac{\partial U_{\lambda, \eta}}{\partial w_r} = 0, \quad r = 1, \ldots, n, \\
\Delta_N \frac{\partial U_{\lambda, \eta}}{\partial \lambda} + S \frac{u}{2^*} (2^* - 1) U_{\lambda, \eta}^{2^* - 2} \frac{\partial U_{\lambda, \eta}}{\partial \lambda} = 0.
\]

Hence, by means of the isometry \( T \), we obtain that the second eigenvalue \( \mu_2 = S \frac{u}{2^*} (2^* - 1) \) and the corresponding eigenspace is spanned by \{\( \partial_\lambda U_{\lambda, \eta}, \nabla_\eta U_{\lambda, \eta} \}\). This completes the proof of the lemma. \( \square \)

**Lemma 4.2.** There exists a positive constant \( \alpha \), depending only on the dimension \( Q \), such that
\[
\| \nabla_N f \|^2_2 - S \| f \|^2_{2^*} \geq \alpha d(f, \mathcal{M})^2 + o(d(f, \mathcal{M})^2),
\]
for all \( f \in S^1_0(N) \) with \( d(f, \mathcal{M}) < \| \nabla_N f \|_2 \).

**Proof.** Recalling the results in Lemma 4.1, we can see that the proof is completely analogue to the context of \( \mathbb{R}^n \) [3] and subelliptic context of Heisenberg group \( \mathbb{H}_n \) [14]. \( \square \)

**Proof of Theorem 1.1.** The optimality of the result follows from the last part of the proof of the lemma above (see [3] and [14]). We note that it is enough to prove the following inequality:
\[
\| \nabla_N f \|^2_2 - S \| f \|^2_{2^*} \geq \alpha d(f, \mathcal{M})^2
\]
for all \( f \in S_{cyl}(N) \), the space of cylindrically symmetric functions of \( S^1_0(N) \). The proof is just the same as in [3] and [14]. These complete the proof of Theorem 1.1. \( \square \)

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**References**