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Formal differential operators, vertex operator algebras and zeta-values, I

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Abstract

We study relationships between spinor representations of certain Lie algebras and Lie superalgebras of differential operators on the circle and values of ζ -functions at the negative integers. By using formal calculus techniques we discuss the appearance of values of ζ -functions at the negative integers underlying the construction. In addition we provide a conceptual explanation of this phenomena through several different notions of normal ordering via vertex operator algebra theory. We also derive a general Jacobi-type identity generalizing our previous construction. At the end we discuss related constructions associated to Dirichlet L -functions.

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1. Introduction

This is the first part in a series of three papers (cf. [34,35]) where we study a relationship between Lie algebras of differential operators on the circle and certain correlation functions associated to vertex operator algebras and superalgebras. This work was motivated by work of Bloch et al. (see [4,5,28–30]). Also, it is closely related to extensive work on the representation theory of Lie algebras of differential operators on the circle (after Kac and Radul [20]). However, our motivation to study these Lie algebras is different.

- (i) This part deals with spinor constructions of classical Lie algebras of differential operators on the circle [1,14,20,23,37,3]. More precisely, we extend [4,28,29] in

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the setting of vertex operator superalgebras (see below for a detailed discussion). In addition, we introduce χ -twisted vertex operators (see Section 6) and study their properties (see also [4]).

- (ii) In Part II [34] we introduce the so-called *n-point correlation functions* (certain graded q -traces) associated to vertex operator algebras and study their elliptic transformation properties (q -difference equations). In a special case, these n -point functions are closely related to generalized characters associated to Lie algebras of differential operators. This part has overlap with some of the results of Zhu [39,40].
- (iii) In Part III [35] we study further modular properties of generalized characters associated to an arbitrary unitary quasi-finite representations [20,14] and modular (resp. elliptic) properties of both g -twisted [11] and χ -twisted correlation functions (resp. n -point functions).

To understand our work we will need several well-known constructions. We start with an infinite-dimensional Heisenberg Lie algebra with a basis consisting of symbols $h(n)$ and 1, for $n \in \mathbb{Z}$, $n \neq 0$, with bracket relations

$$[h(m), h(n)] = m\delta_{m+n,0}1. \quad (1.1)$$

We also add a central element $h(0)$ such that (1.1) still holds. This algebra acts on the “vacuum module” $M(1) := \mathbb{C}[h(-1), h(-2), \dots]$ (a polynomial algebra in infinitely many variables) in a natural way. It is a well-known fact that the Virasoro algebra has a representation on the vacuum module $M(1)$ given by

$$\begin{aligned} c &\mapsto 1 \\ L(n) &\mapsto \frac{1}{2} \sum_{k \in \mathbb{Z}} \bullet h(-k)h(k+n) \bullet, \end{aligned} \quad (1.2)$$

for $n \in \mathbb{Z}$, where $\bullet \bullet$ denotes *normal ordering*, i.e.,

$$\bullet h(n)h(m) \bullet = h(n)h(m)$$

if $n \leq m$ and

$$\bullet h(n)h(m) \bullet = h(m)h(n)$$

if $m < n$. In [4] Bloch realized that a certain infinite-dimensional Lie algebra of differential operators \mathcal{D}^+ (containing the Virasoro algebra as a proper subalgebra) admits a projective representation on $M(1)$ in terms of “quadratic” operators.

Similar constructions have been known in theoretical physics for a while (see [3,37], etc.) under the name \mathcal{W}_∞ and $\mathcal{W}_{1+\infty}$ -algebras. More importantly Bloch found that if we redefine normal ordering procedure in a certain way using the values of the Riemann ζ -function, the central term has an especially simple shape (it is in particular a pure monomial). In the special case of the Virasoro algebra (with $c = 1$) this corresponds to replacing the operator $L(n)$ by the operator

$$\bar{L}(n) = L(n) + \frac{1}{2} \zeta(-1)\delta_{n,0} = L(n) - \frac{1}{24} \delta_{n,0}, \quad (1.3)$$

for which we get

$$[\bar{L}(m), \bar{L}(n)] = (m - n)\bar{L}(m + n) + \frac{m^3}{12} \delta_{m+n,0}c. \tag{1.4}$$

In [28–30] Lepowsky interpreted these phenomena by using formal calculus combined with the theory of vertex operator algebras, as developed in [6,15,13]. More precisely, the appearance of the ζ -function values at negative integers is closely related to a particular (formal) conformal transformation $x \mapsto e^x - 1$ which arises in the geometry underlying vertex operator algebras (see [18,39]). Since the new normal ordering procedure can be defined for an arbitrary vertex operator algebra (cf. [29]), the result of Bloch is a very special case of the general theory. We should mention that in the paper [12] negative integer values of the Riemann zeta function are obtained from a different point of view.

In Section 2 we introduce certain Lie superalgebras of superdifferential operators that we call $\mathcal{S}\mathcal{D}_R^+$ and $\mathcal{S}\mathcal{D}_{NS}^+$ (the subscripts R and NS refer to the Ramond and Neveu–Schwarz sectors) which contain the Lie algebra $\mathcal{D}^+ \oplus \mathcal{D}^-$ (“symplectic” \oplus “orthogonal”). We used several different generating functions and this largely simplified our computations. Also, use of generating functions was very convenient for some results needed in Section 3.

In Section 3 we build up a projective representation of $\mathcal{S}\mathcal{D}_{NS}^+$ in the following way: Let ψ_n , $n \in \mathbb{Z} + \frac{1}{2}$, and 1 be a basis for an infinite dimensional affine superalgebra \mathfrak{f} , with bracket relations

$$[\psi_n, \psi_m] = \delta_{n+m,0}1,$$

and let $F = \bigwedge(\psi_{-1/2}, \psi_{-3/2}, \dots)$ be the exterior algebra, which is an \mathfrak{f} -module. We consider the vector space

$$W := M(1) \otimes F.$$

Then, there is a natural projective representation of $\mathcal{S}\mathcal{D}_{NS}^+$ on W . Here we shall call this representation “orthosymplectic” since it is an extension of the metaplectic representation of the algebra \mathcal{D}^+ and spinor representation of \mathcal{D}^- .

Then we discuss the connection of this construction with the theory of vertex operator superalgebras. Namely, the space W is a $N = 1$ vertex operator superalgebra (see [22]) equipped with a vertex operator map

$$v \mapsto Y(v, x) \in \text{End}(W)[[x, x^{-1}]], \tag{1.5}$$

for $v \in W$. Notice (cf. [28]) that for the construction of $\mathcal{S}\mathcal{D}_{NS}^+$ on W it is more natural to use the slightly modified vertex operators

$$X(v, x) = Y(x^{L(0)}v, x). \tag{1.6}$$

These operator were used, for different purpose, in [15]. Any vertex operator of the form (1.5) (resp. (1.6)) will be called an “ X -vertex operator” (resp. “ Y -vertex operator”) or shorthand an X -operator (resp. Y -operator).

We then define a new normal ordering procedure by using a generating function for negative integer values of the Riemann’s ζ -function and a Hurwitz’s ζ -function $\zeta(s, \frac{1}{2})$

which is essentially the Riemann's ζ -function, since the duplication formula holds. The generating function for $\zeta(1 - n, \frac{1}{2})$, $n \in \mathbb{N}$, is

$$\frac{e^{x/2}}{e^x - 1}. \quad (1.7)$$

Now by using the new normal ordering the central term looks much simpler and it is again a pure monomial. We can illustrate this on the example of the Neveu–Schwarz superalgebra spanned by $L(n)$, $G(m)$, with the bracket relations given by

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m+n, 0} c \quad (1.8)$$

$$[G(m), L(n)] = \left(m - \frac{n}{2}\right) G(m + n) \quad (1.9)$$

$$[G(m), G(n)] = 2L(m + n) + \frac{1}{3} \left(m^2 - \frac{1}{4}\right) \delta_{m+n, 0} c. \quad (1.10)$$

This algebra (with $c = \frac{3}{2}$) has a representation on W , and new generators are given by

$$\tilde{L}(m) = L(m) - \frac{1}{24} c,$$

$$\tilde{G}(n) = G(n),$$

for $m, n \in \mathbb{Z} + \frac{1}{2}$. Therefore we have

$$[\tilde{G}(m), \tilde{G}(n)] = 2\tilde{L}(m + n) + \frac{m^2}{12} \delta_{m+n, 0} c. \quad (1.11)$$

Then by carefully rewriting the Jacobi identity for the vertex operator superalgebra, we establish a new “Jacobi identity”. This is a generalization of the result from [29]. By using the new Jacobi identity for X -vertex operators we can explain most of the calculations obtained in Section 2.2. This new Jacobi identity involves iterates like

$$X(Y[u, y]v, x). \quad (1.12)$$

Even though the operator (1.12) is an iterate of a Y -operator and an X -vertex operator, it should be thought as the X -vertex operator of a *single* “vector” $Y[u, y]v$. For some special u and v , $X(Y[u, y]v, x)$ is essentially the same as $D^{(y, 0)}(x)$, $\bar{D}^{(y, 0)}(x)$ and $G^{(y, 0)}(x)$ (for notation see Section 2). The x variable controls degree, the regular part of y controls filtration of the Lie (super)algebra we are working with, and the singular part of y is related to the ζ -correction term.

When we study $\hat{\mathcal{G}}$ and its super (or sub) generalization, the notion of a graded character (as in [39]) is not the most natural. Specifically, $\text{tr}_{|M} q^{\tilde{L}(0)}$ carries only partial information, ignoring the higher “Hamiltonians” that are present in *deformed* theories. Hence, it is more natural to consider “generalized characters” (cf. [20, 1, 4], etc.):

$$\text{tr}_{|M} \prod_{i \geq 0} q_i^{\tilde{L}^{(i)}(0)}, \quad (1.13)$$

where $q_i = e^{2\pi i \tau_i}$, and the operators $L^{(i)}(0)$ span the “Cartan subalgebra” of $\hat{\mathcal{D}}$. In the above equation τ_i should be thought of—at least for some special models—as a coupling constant in the Hamiltonian theory.

In Section 5 we analyze a generalized character associated to the $\mathcal{S}\mathcal{D}_{\text{NS}}^+$ -module $M(1) \otimes F$ and prove its quasimodularity (in an appropriate sense [5]).

In Section 6 we consider a Lie algebra of (pseudo)differential operators \mathcal{D}_∞ (and its classical subalgebras \mathcal{D}_∞^\pm) and a representation in terms of (new) quadratic operators built up from the expressions of the form

$$X_\chi(u, y) = \sum_{k \in \mathbb{Z}} \chi(-n) u_{n+\text{wt}(u)-1} y^{-n}, \tag{1.14}$$

where χ is a primitive Dirichlet character of conductor N (cf. [17]). When we change the normal ordering, as in the previous cases, the negative integer values of the Dirichlet L -functions $L(s, \chi)$ are related to \mathcal{D}_∞ in the same way as the Riemann ζ -function is related to \mathcal{D}^+ . This is a generalization of the results in [4, Section 6].

At this point we fully do not understand a significance of (1.14) in the theory of vertex operator algebras (or conformal field theory). However, graded q -traces associated to twisted χ -operators have a nice interpretation in terms of automorphic forms [35].

1.1. Notation

Through the whole text we denote by \mathbb{N} the set of positive integers. We work always over the field of complex numbers, \mathbb{C} . We denote by x_i 's y_i 's, x , t , etc. commuting formal variables. $\mathbb{C}(x)$ stands for the field of formal rational functions, $\mathbb{C}((x))$ is the ring of formal Laurent series truncated from below and by $\mathbb{C}[[x, x^{-1}]]$ the vector space of formal series. Also we denote by $\delta(x) = \sum_{n \in \mathbb{Z}} x^n \in \mathbb{C}[[x, x^{-1}]]$ the formal delta function (cf. [13]). Often our generating functions will have coefficients in vector spaces, modules, etc. Most of the results hold if we replace the field \mathbb{C} with some cyclotomic extension of \mathbb{Q} .

2. (Super)differential operators on the circle

2.1. Lie algebra \mathcal{D}

Lie algebras of differential operators on the circle [19] play a prominent rule in conformal field theory (see [14]).

We will use the formal approach that uses formal variables and formal derivatives. Let us denote by \mathcal{D} the Lie algebra of formal differential operators on the circle, i.e., the Lie algebra spanned by $t^k D^n$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, where t is a formal variable and $D = t(d/dt)$. \mathcal{D} has two distinguished bases

$$\left\{ t^k \left(\frac{d}{dt} \right)^l : k \in \mathbb{Z}, l \in \mathbb{N} \right\},$$

and

$$\{t^k D^l : k \in \mathbb{Z}, l \in \mathbb{N}\}.$$

In this work we will mostly use the latter one. By defining $\deg(t^k D^l) = k$, \mathcal{D} becomes a \mathbb{Z} -graded Lie algebra. In what follows we will be interested only in the Lie algebra structure of \mathcal{D} and *not* in the associative algebra structure. We have the following commutation relation:

$$[t^{k_1} D^{l_1}, t^{k_2} D^{l_2}] = t^{k_1+k_2} ((D + k_2)^{l_1} D^{l_2} - (D + k_1)^{l_2} D^{l_1}), \tag{2.1}$$

where $k_1, k_2 \in \mathbb{Z}$ and $l_1, l_2 \in \mathbb{N}$. It is important to notice that \mathcal{D} has a filtration

$$\dots \mathcal{D}_{(i)} \subset \mathcal{D}_{(i+1)} \dots,$$

where $\mathcal{D}_{(i)}$ is spanned by elements of the form $t^l D^k$, $k \leq i$. From (2.1) it follows that $[\mathcal{D}_{(i)}, \mathcal{D}_{(j)}] \subset \mathcal{D}_{(i+j-1)}$. Therefore \mathcal{D} is a *filtered* Lie algebra. For practical reasons one wants to work with a bit smaller Lie algebra $\text{Diff}[t, t^{-1}]$, which is determined with the following exact sequence:

$$0 \rightarrow \text{Diff}[t, t^{-1}] \rightarrow \mathcal{D} \rightarrow \mathbb{C}[t, t^{-1}] \rightarrow 0.$$

In other words, $\text{Diff}[t, t^{-1}]$ is an associative algebra of differential operators $\varphi \in \mathcal{D}$ such that $\varphi(1) = 0$.

The Lie algebra \mathcal{D} has the following 2-cocycle with values in \mathbb{C} (cf. [20])

$$\Psi(t^{k_1} D^{l_1}, t^{k_2} D^{l_2}) = \begin{cases} \sum_{-k_1 \leq i \leq -1} i^{l_1} (i + k_1)^{l_2} & \text{if } k_1 = -k_2 \geq 0, \\ 0 & \text{if } k_1 + k_2 \neq 0. \end{cases} \tag{2.2}$$

Denote by $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}C$ the corresponding central extension. It can be proved (see [10,33]) that $\hat{\mathcal{D}}$ is the unique non-trivial central extension of \mathcal{D} . Therefore $\hat{\mathcal{D}}$ is characterized with the non-split exact sequence:

$$0 \rightarrow \mathbb{C} \rightarrow \hat{\mathcal{D}} \rightarrow \mathcal{D} \rightarrow 0.$$

Representation theory of the Lie algebra $\hat{\mathcal{D}}$ is more interesting than the representation theory of \mathcal{D} itself (see [19–21]).

The Lie algebra \mathcal{D} has many Lie subalgebras. Here we give an example of a large class of subalgebras:

Proposition 2.1. *Let $I \subset \mathbb{C}[x]$ be an ideal and \mathcal{D}_I the vector space of differential operators of the form $t^k f(D)$, $f \in I$, $k \in \mathbb{Z}$.*

Besides Lie subalgebras \mathcal{D}_I there are two distinguished subalgebras of $\text{Diff}[t, t^{-1}]$. We study them in the next sections.

2.2. Lie algebra \mathcal{D}^+

It is known [20] that there is a close connection between \mathcal{D} and a certain Lie algebra of infinite matrices, $gl(\infty)$. By using the same analogy we can construct some

“classical subalgebras” of \mathcal{D} that correspond to classical Lie subalgebras of $gl(\infty)$. In [4] a symplectic subalgebra $\mathcal{D}^+ \subset \text{Diff}[t, t^{-1}]$ is constructed as the θ_1 -stable Lie subalgebra of \mathcal{D} with respect to the involution

$$\theta_1(t^n D^{k+1}) = t^n (-D - n)^k D,$$

where $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Moreover, Bloch showed that \mathcal{D}^+ is the maximal proper Lie subalgebras of \mathcal{D} that contains a Lie algebra of vector fields—Witt algebra—spanned by $t^k D$, $k \in \mathbb{Z}$, that satisfy

$$\text{Witt} \subsetneq \mathcal{D}^+ \subsetneq \text{Diff}[t, t^{-1}].$$

Again, it is reasonable to study the central extension $\hat{\mathcal{D}}^+ \subset \hat{\mathcal{D}}$, induced by (2.2).

Let us recall the definition of the formal delta function $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$. We denote by $e^{xD} \in \mathcal{D}[[x]]$, the formal power series

$$\sum_{n \geq 0} \frac{x^n D^n}{n!}.$$

The following statement is a reformulation of a result from [4, Proposition 1.19], in terms of generating functions. It was also suggested and proved by Lepowsky (see [28,30]).

Proposition 2.2.

$$\begin{aligned} \mathcal{D}^+ = \text{span} \left\{ \text{coeff}_{y_1^k y_2^l x^m} e^{-y_1 D} \delta\left(\frac{t}{x}\right) e^{y_2 D} D \right. \\ \left. + e^{-y_2 D} \delta\left(\frac{t}{x}\right) e^{y_1 D} D, k, l \in \mathbb{N}, m \in \mathbb{Z} \right\}. \end{aligned} \tag{2.3}$$

Moreover, we may assume in (2.3) that $y_2 = 0$.¹ Let

$$\mathcal{D}^{y_1, y_2}(x) = \frac{1}{2} \left(e^{-y_1 D} \delta\left(\frac{t}{x_1}\right) e^{y_2 D} + e^{-y_2 D} \delta\left(\frac{t}{x_1}\right) e^{y_1 D} \right),$$

then

$$\begin{aligned} [\mathcal{D}^{y_1, y_2}(x_1), \mathcal{D}^{y_3, y_4}(x_2)] \\ = \frac{1}{2} \frac{\partial}{\partial y_2} \left(\mathcal{D}^{y_4, y_3 + y_1 - y_2}(x_2) \delta\left(\frac{e^{y_2 - y_3} x_1}{x_2}\right) + \mathcal{D}^{y_3, y_4 + y_1 - y_2}(x_2) \delta\left(\frac{e^{y_2 - y_4} x_1}{x_2}\right) \right) \\ + \frac{1}{2} \frac{\partial}{\partial y_1} \left(\mathcal{D}^{y_3, y_4 + y_2 - y_1}(x_2) \delta\left(\frac{e^{y_1 - y_4} x_1}{x_2}\right) + \mathcal{D}^{y_4, y_3 + y_2 - y_1}(x_2) \delta\left(\frac{e^{y_1 - y_3} x_1}{x_2}\right) \right). \end{aligned} \tag{2.4}$$

¹ Meaning, that \mathcal{D}^+ is spanned by coefficients of $e^{-y_1 D} \delta(t/x) D + \delta(t/x) e^{y_1 D} D$.

Proof. Let us consider the right hand side of (2.3). It is spanned by

$$(-1)^k D^k t^l D^m D + (-1)^m D^m t^l D^k D,$$

for $m, k \in \mathbb{N}$, $l \in \mathbb{Z}$. Since

$$\begin{aligned} & \theta_1((-1)^k D^k t^l D^m D + (-1)^m D^m t^l D^k D) \\ &= \theta_1(t^l((-1)^k (D+l)^k D^m D + (-1)^m (D+l)^m D^k D)) \\ &= t^l((-1)^k (-D)^k (-D-l)^m D + (-1)^m (-D)^m (-D-l)^k D) \\ &= t^l((-1)^m D^k (D+l)^m D + (-1)^k D^m (D+l)^k D) \\ &= (-1)^m D^m t^l D^k D + (-1)^k D^k t^l D^m D, \end{aligned} \tag{2.5}$$

the right hand side of (2.3) is contained in \mathcal{D}^+ . Now suppose that $t^l p(D)D \in \mathcal{D}^+$, where p is some polynomial. Since \mathcal{D}^+ is θ_1 -stable, it follows that $t^l p(-D-l)D = t^l p(D)D$. Hence $p(-D-l) = p(D)$. Now the vector space of all polynomials $\mathcal{P}_l \subset \mathbb{C}[x]$, satisfying $p(x) = p(-x-l)$, is spanned by the set $\{x^k + (-x-l)^k: k \geq 0\}$. Moreover, $\{x^k + (-x-l)^k: k \in 2\mathbb{Z}\}$ is a basis for \mathcal{P}_l . Anyhow, the right hand side of (2.3) is spanned by

$$(-1)^k D^k t^l D + t^l D^k D = t^l((-D-l)^k + D^k)D,$$

$k \in \mathbb{N}$, $l \in \mathbb{Z}$. Thus we may assume in (2.3) that $y_2 = 0$.

We shall prove (2.4) by using the following simple (but crucial) identity

$$e^{-y_1 D} \delta\left(\frac{t}{x}\right) e^{y_2 D} D = \delta\left(\frac{e^{-y_1} t}{x}\right) e^{(y_2 - y_1) D} D.$$

We have

$$\begin{aligned} & [\mathcal{D}^{y_1, y_2}(x_1), \mathcal{D}^{y_3, y_4}(x_2)] \\ &= \frac{1}{4} \left[e^{-y_1 D} \delta\left(\frac{t}{x_1}\right) e^{y_2 D} D + e^{-y_2 D} \delta\left(\frac{t}{x_1}\right) e^{y_1 D} D, \right. \\ & \quad \left. \times e^{-y_3 D} \delta\left(\frac{t}{x_2}\right) e^{y_4 D} D + e^{-y_4 D} \delta\left(\frac{t}{x_2}\right) e^{y_3 D} D \right] \\ &= \frac{1}{4} \left(\frac{\partial}{\partial y_2} \delta\left(\frac{e^{-y_1} t}{x_1}\right) \delta\left(\frac{e^{(y_2 - y_1 - y_3)} t}{x_2}\right) e^{(y_2 - y_1 - y_3 + y_4) D} D \right. \\ & \quad \left. + \frac{\partial}{\partial y_2} \delta\left(\frac{e^{-y_1} t}{x_1}\right) \delta\left(\frac{e^{(y_2 - y_1 - y_4)} t}{x_2}\right) e^{(y_2 - y_1 - y_4 + y_3) D} D \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{\partial}{\partial y_1} \delta \left(\frac{e^{-y_2 t}}{x_1} \right) \delta \left(\frac{e^{(y_1 - y_2 - y_4) t}}{x_2} \right) e^{(y_1 - y_2 - y_4 + y_3) D} D \\
 & + \frac{\partial}{\partial y_1} \delta \left(\frac{e^{-y_2 t}}{x_1} \right) \delta \left(\frac{e^{(y_1 - y_2 - y_3) t}}{x_2} \right) e^{(y_1 - y_2 - y_3 + y_4) D} D \\
 & + \frac{\partial}{\partial y_1} \delta \left(\frac{e^{-y_3 t}}{x_2} \right) \delta \left(\frac{e^{(-y_3 - y_1 + y_4) t}}{x_1} \right) e^{(-y_3 - y_1 + y_4 + y_2) D} D \\
 & + \frac{\partial}{\partial y_2} \delta \left(\frac{e^{-y_3 t}}{x_2} \right) \delta \left(\frac{e^{(-y_3 - y_2 + y_4) t}}{x_1} \right) e^{(-y_3 - y_2 + y_4 + y_1) D} D \\
 & + \frac{\partial}{\partial y_1} \delta \left(\frac{e^{-y_4 t}}{x_2} \right) \delta \left(\frac{e^{(-y_1 - y_4 + y_3) t}}{x_1} \right) e^{(-y_1 - y_4 + y_3 + y_2) D} D \\
 & + \frac{\partial}{\partial y_2} \delta \left(\frac{e^{-y_4 t}}{x_2} \right) \delta \left(\frac{e^{(-y_2 - y_4 + y_3) t}}{x_1} \right) e^{(-y_2 - y_4 + y_3 + y_1) D} D \Big). \tag{2.6}
 \end{aligned}$$

If we apply the delta function substitution property (for more general version see [15])

$$\delta \left(\frac{e^y t}{x_2} \right) e^{-zD} \delta \left(\frac{t}{x_1} \right) = \delta \left(\frac{e^y t}{x_2} \right) \delta \left(\frac{e^{-z} t}{x_1} \right) = \delta \left(\frac{e^y t}{x_2} \right) \delta \left(\frac{e^{y+z} x_1}{x_2} \right),$$

in (2.4) we obtain

$$\begin{aligned}
 & [\mathcal{D}^{y_1, y_2}(x_1), \mathcal{D}^{y_3, y_4}(x_2)] \\
 & = \frac{1}{4} \frac{\partial}{\partial y_2} \left(\delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) \left(\delta \left(\frac{e^{(y_2 - y_1 - y_3) t}}{x_2} \right) e^{(y_2 - y_1 - y_3 + y_4) D} D \right. \right. \\
 & \quad \left. \left. + \delta \left(\frac{e^{-y_4} t}{x_2} \right) e^{(-y_2 - y_4 + y_3 + y_1) D} D \right) \right) \\
 & + \frac{1}{4} \frac{\partial}{\partial y_2} \left(\delta \left(\frac{e^{(y_2 - y_4) x_1}}{x_2} \right) \left(\delta \left(\frac{e^{(y_2 - y_1 - y_4) t}}{x_2} \right) e^{(y_2 - y_1 - y_4 + y_3) D} D \right. \right. \\
 & \quad \left. \left. + \delta \left(\frac{e^{-y_3} t}{x_2} \right) e^{(-y_3 - y_2 + y_4 + y_1) D} D \right) \right) \\
 & + \frac{1}{4} \frac{\partial}{\partial y_1} \left(\delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) \left(\delta \left(\frac{e^{(y_1 - y_2 - y_4) t}}{x_2} \right) e^{(y_1 - y_2 - y_4 + y_3) D} D \right. \right. \\
 & \quad \left. \left. + \delta \left(\frac{e^{-y_3} t}{x_2} \right) e^{(-y_3 - y_1 + y_4 + y_2) D} D \right) \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \frac{\partial}{\partial y_1} \left(\delta \left(\frac{e^{(y_1 - y_3)x_1}}{x_2} \right) \left(\delta \left(\frac{e^{(y_1 - y_2 - y_3)t}}{x_2} \right) e^{(y_1 - y_2 - y_3 + y_4)D} \right. \right. \\
& \left. \left. + \delta \left(\frac{e^{-y_4 t}}{x_2} \right) e^{(-y_1 - y_4 + y_3 + y_2)D} \right) \right) \\
& = \frac{1}{2} \frac{\partial}{\partial y_2} \left(\mathcal{D}^{y_4, y_3 + y_1 - y_2}(x_2) \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) + \mathcal{D}^{y_3, y_4 + y_1 - y_2}(x_2) \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) \right) \\
& + \frac{1}{2} \frac{\partial}{\partial y_1} \left(\mathcal{D}^{y_3, y_4 + y_2 - y_1}(x_2) \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) \right. \\
& \left. + \mathcal{D}^{y_4, y_3 + y_2 - y_1}(x_2) \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) \right). \quad \square \tag{2.7}
\end{aligned}$$

Remark 2.1. In the previous proof we constructed a basis of \mathcal{D}^+ given by

$$\{t^l((D + I)^{2k}D + D^{2k}D): k \geq 0, l \in \mathbb{Z}\}.$$

It is also convenient to work with a basis $\{(-1)^k D^k t^l D^{k+1}: k \geq 0, l \in \mathbb{Z}\}$ as in [4]. This basis is obtained by extracting coefficients $x_1^r x_2^s, r \in \mathbb{N}$, in (2.3).

2.3. Lie algebra \mathcal{D}^-

Another involution of the Lie algebra \mathcal{D} is given by

$$\theta_2(t^k D^l) = -t^k (-D - k)^l.$$

Let \mathcal{D}^- be the θ_2 -stable Lie subalgebra. Notice that this subalgebra is not contained in $\text{Diff}[t, t^{-1}]$. Then we have

Proposition 2.3. (a) \mathcal{D}^- is spanned by the coefficients of the generating function

$$\begin{aligned}
\mathcal{D}^- = \text{span} \left\{ \text{coeff}_{y_1^l y_2^m} e^{-y_1 D} \delta \left(\frac{t}{x} \right) e^{y_2 D} \right. \\
\left. - e^{-y_2 D} \delta \left(\frac{t}{x} \right) e^{y_1 D} k, l \in \mathbb{N}, m \in \mathbb{Z} \right\}. \tag{2.8}
\end{aligned}$$

Moreover, the same statement holds if we assume that $y_2 = 0$.

(b) Let

$$\bar{D}^{y_1, y_2}(x) = e^{-y_1 D} \delta \left(\frac{t}{x} \right) e^{y_2 D} - e^{-y_2 D} \delta \left(\frac{t}{x} \right) e^{y_1 D}. \tag{2.9}$$

Then the following commutation relation holds:

$$\begin{aligned}
 & [\bar{D}^{y_1, y_2}(x_1), \bar{D}^{y_3, y_4}(x_2)] \\
 &= \bar{D}^{y_1, y_4 + y_2 - y_3}(x_1) \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) + \bar{D}^{y_2, y_1 + y_3 - y_4}(x_1) \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) \\
 &\quad - \bar{D}^{y_1, y_2 + y_3 - y_4}(x_1) \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) - \bar{D}^{y_2, y_4 + y_1 - y_3}(x_1) \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right). \quad (2.10)
 \end{aligned}$$

Proof. For every $k, m \in \mathbb{N}$ and $l \in \mathbb{Z}$

$$\begin{aligned}
 & \theta_2((-1)^k D^k t^l D^m + (-1)^{m+1} D^m t^l D^k) \\
 &= \theta_2 t^l ((-1)^k (D + l)^k D^m + (-1)^{m+1} (D + l)^m D^k) \\
 &= -t^l ((-1)^k (-D)^k (-D - l)^m + (-1)^{m+1} (-D)^m (-D - l)^k) \\
 &= -t^l ((-1)^m D^k (D + l)^m + (-1)^{k+1} D^m (D + l)^k) \\
 &= (-1)^{m+1} D^m t^l D^k + (-1)^k D^k t^l D^m. \quad (2.11)
 \end{aligned}$$

Therefore the right hand side of (2.9) is contained in \mathcal{D}^- . Now suppose that $t^l p(D) \in \mathcal{D}^-$. Since \mathcal{D}^- is the θ_2 -stable Lie algebra it follows that $-p(-D - l) = p(D)$. Let us denote the vector space of such polynomials by $\tilde{\mathcal{P}}_l \subset \mathbb{C}[x]$. $-p(-x - l) = p(x)$, so it is easy to see that $\tilde{\mathcal{P}}_l$ is spanned by $x^m + (-1)^{m+1}(x + l)^m$, where $m \in \mathbb{N}$. This corresponds to the case $y_2 = 0$ in (2.9). Actually $\{x^m + (-1)^{m+1}(x + l)^m; m \in 2\mathbb{N} + 1\}$ is a basis for $\tilde{\mathcal{P}}_l$.

The proof of (b) is a straightforward computation so we omit it here. \square

Note that in general $t^k D \notin \mathcal{D}^-$ for every k . Hence, \mathcal{D}^- does not contain the Witt subalgebra spanned by $t^k D$, but rather a subalgebra isomorphic to it. It is possible (compare with [4]) to define a symmetric, nondegenerate bilinear form B on $\mathbb{C}[t, t^{-1}]$ such that for every $\varphi \in \mathcal{D}^-$, $B(\varphi f, g) + B(f, \varphi g) = 0$.

Remark 2.2. It is important to notice that θ_1 and θ_2 are not morphisms of associative algebras, therefore \mathcal{D}^+ and \mathcal{D}^- are not associative algebras.

Notice that $D^{2k+1} \in \mathcal{D}^+$, $k \in \mathbb{N}$. The Lie algebra spanned by these vectors is called *Cartan subalgebra* of \mathcal{D}^+ and we denote it by \mathcal{H}^+ . Similarly, a Lie algebra spanned by $(D + \frac{1}{2})^{2k+1} \in \mathcal{D}^-$, $k \in \mathbb{N}$ is the Cartan subalgebra of \mathcal{D}^- which we denote by \mathcal{H}^- . In the case of $\hat{\mathcal{G}}^\pm$ the Cartan subalgebra is obtained by adding the central subspace $\mathbb{C}C$.

2.4. Lie superalgebras $\mathcal{S}\mathcal{D}_R^+$ and $\mathcal{S}\mathcal{D}_{NS}^+$

Now we switch to *supermathematics* (we will follow [7]). A *supervector* space is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space

$$V = V_0 \oplus V_1.$$

An element $v \in V_0$ (resp. V_1) is said to be even (resp. odd). Its parity is denoted by $p(a)$. A *super algebra* over \mathbb{C} is a super vector space \mathcal{A} , given with a morphisms (product): $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$. By the definition of morphism, the parity of the product of $\mathbb{Z}/2\mathbb{Z}$ -homogeneous elements of \mathcal{A} is the sum of parities of the factors. The super algebra \mathcal{A} is *associative* if $(ab)c = a(bc)$ for every $a, b, c \in \mathcal{A}$. If we define

$$[\cdot, \cdot] : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad (2.12)$$

by

$$[a, b] = ab - (-1)^{p(a)p(b)}ba,$$

for a, b homogeneous, then

$$[a, b] + (-1)^{p(a)p(b)}[b, a] = 0,$$

$$[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0.$$

Therefore $(\mathcal{A}, [\cdot, \cdot])$ is a super Lie algebra.

The aim is to extend results from the previous sections in the Lie superalgebra setting. Denote by

$$\mathbb{C}[t, t^{-1}, \theta] = \mathbb{C}[t, t^{-1}, \theta]_0 \oplus \mathbb{C}[t, t^{-1}, \theta]_1,$$

the associative superalgebra of Laurent polynomials associated to an even, x , and an odd, θ , formal variable. Assume that $\theta^2 = 0$ and $\theta x = x\theta$. A graded element A in the superspace $\text{End}(\mathbb{C}[t, \theta])$ is called a *superderivation* of the sign $p(A) \in \{0, 1\}$ if it satisfies

$$A(uv) = A(u)v + (-1)^{p(A)p(u)}uA(v),$$

for every homogeneous u and v . Denote by $\text{Der}\mathbb{C}[t, \theta]$ a Lie superalgebra of all superderivatives of $\mathbb{C}[t, t^{-1}, \theta]$. Now we construct a superanalogue of $\text{Diff}[t, t^{-1}]$.

Consider the associative algebra $C(1)$ generated by θ , $\partial/\partial\theta$, and 1, modulo the following anticommuting relations

$$\begin{aligned} \left[\theta, \frac{\partial}{\partial\theta} \right] &= \delta_{i,j}, \\ [\theta, \theta] &= \left[\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right] = 0. \end{aligned} \quad (2.13)$$

$C(1)$ is isomorphic to the four-dimensional Clifford algebra, which is isomorphic to $M_2(\mathbb{C})$. We equip $C(1)$ with the \mathbb{Z}_2 -grading such that θ and $\partial/\partial\theta$ span the odd and $\theta\partial/\partial\theta$ and $\partial/\partial\theta\theta$ span the even subspace. Hence $C(1)$ (or $M_2(\mathbb{C})$) is an associative

superalgebra. Then a super associative algebra

$$\mathcal{D} \otimes C(1),$$

can be embedded into the Lie superalgebra $\text{End}(\mathbb{C}[t, t^{-1}, \theta])$ via

$$f(D) \otimes g \mapsto f(D)g.$$

We denote by $\text{Diff}[t, t^{-1}, \theta]$ the corresponding image. Sometimes it is convenient to think of elements $\text{Diff}[t, t^{-1}, \theta]$ in terms of matrices (cf. [1]).

Now

$$\text{Diff}[t, t^{-1}, \theta] = \text{Diff}[t, t^{-1}, \theta]_0 \oplus \text{Diff}[t, t^{-1}, \theta]_1,$$

where

$$\begin{aligned} \text{Diff}[t, t^{-1}, \theta]_0 &= \text{span} \left\{ t^k D^l \frac{\partial}{\partial \theta} \theta, t^k D^l \theta \frac{\partial}{\partial \theta} : k \in \mathbb{Z}, l \in \mathbb{N} \right\}, \\ \text{Diff}[t, t^{-1}, \theta]_1 &= \text{span} \left\{ t^k D^l \theta, t^k D^l \frac{\partial}{\partial \theta} : k \in \mathbb{Z}, l \in \mathbb{N} \right\}. \end{aligned} \tag{2.14}$$

Let

$$\mathcal{SD} = \{ \varphi \in \text{Diff}[t, t^{-1}, \theta] : \varphi(1) = 0 \},$$

where we consider $\mathbb{C}[t, t^{-1}, \theta]$ as a natural $\text{Diff}[t, t^{-1}, \theta]$ -module. Then \mathcal{SD} is a Lie superalgebra with

$$\mathcal{SD} = \mathcal{SD}_0 \oplus \mathcal{SD}_1,$$

where

$$\begin{aligned} \mathcal{SD}_0 &= \text{span} \left\{ t^k D D^l \frac{\partial}{\partial \theta} \theta, t^k D^l \theta \frac{\partial}{\partial \theta} : k \in \mathbb{Z}, l \in \mathbb{N} \right\}, \\ \mathcal{SD}_1 &= \text{span} \left\{ t^k D D^l \theta, t^k D^l \frac{\partial}{\partial \theta} : k \in \mathbb{Z}, l \in \mathbb{N} \right\}. \end{aligned} \tag{2.15}$$

$\text{Diff}[t, t^{-1}, \theta]$ has a \mathbb{Z} -grading induced by $\text{Diff}[t, t^{-1}]$.

Consider a \mathbb{Z}_2 -graded mapping

$$\gamma = \gamma_0 \oplus \gamma_1 : \mathcal{SD} \rightarrow \mathcal{SD},$$

given by

$$\begin{aligned} \gamma_0 \left(t^k F(D) D \frac{\partial}{\partial \theta} \theta \right) &= t^k F(-D - k) D \frac{\partial}{\partial \theta} \theta, \\ \gamma_0 \left(t^k F(D) \theta \frac{\partial}{\partial \theta} \right) &= -t^k F(-D - k) \theta \frac{\partial}{\partial \theta}, \\ \gamma_1 \left(t^k F(D) D \theta \right) &= t^k F(-D - k) \frac{\partial}{\partial \theta}, \\ \gamma_1 \left(t^k F(D) \frac{\partial}{\partial \theta} \right) &= t^k F(-D - k) D \theta, \end{aligned} \tag{2.16}$$

where $F \in \mathbb{C}[x]$.

Proposition 2.4. γ is a Lie superalgebra involution.

Proof. Since θ_1 and θ_2 are involutions of Lie algebras it immediately follows from (2.16) that $\gamma^2 = 1$. Suppose that $v, w \in \mathcal{SD}$, and let $v = v_0 + v_1$ and $w = w_0 + w_1$ are the corresponding graded decompositions. Then

$$\gamma([v_0 + v_1, w_0 + w_1]) = \gamma([v_0, w_0]) + \gamma([v_0, w_1]) + \gamma([v_1, w_0]) + \gamma([v_1, w_1]_+). \quad (2.17)$$

We have to consider several cases. Suppose that

$$\begin{aligned} v_0 &= F_1(D)D \frac{\partial}{\partial \theta} \theta + G_1(D)\theta \frac{\partial}{\partial \theta}, \\ w_0 &= F_2(D)D \frac{\partial}{\partial \theta} \theta + G_2(D)\theta \frac{\partial}{\partial \theta}. \end{aligned}$$

Then

$$\begin{aligned} \gamma([v_0, w_0]) &= \gamma \left([F_1(D)D, F_2(D)D] \frac{\partial}{\partial \theta} \theta + [G_1(D), G_2(D)]\theta \frac{\partial}{\partial \theta} \right) \\ &= [\gamma_0(F_1(D)D), \gamma_0(F_2(D)D)] \frac{\partial}{\partial \theta} \theta + [\gamma_0(G_1(D)), \gamma_0(G_2(D))]\theta \frac{\partial}{\partial \theta} \\ &= [\gamma(v_0), \gamma(w_0)]. \end{aligned} \quad (2.18)$$

In the previous calculations we used the formulas

$$\left(\frac{\partial}{\partial \theta} \theta \right)^2 = \frac{\partial}{\partial \theta} \theta, \quad \left(\theta \frac{\partial}{\partial \theta} \right)^2 = \theta \frac{\partial}{\partial \theta}.$$

In the other three cases: $\gamma([v_0, w_1])$, $\gamma([v_1, w_0])$ and $\gamma([v_1, w_1]_+)$, the proofs are similar. \square

Let us denote by ι_1 and ι_2 the following embeddings:

$$\iota_i : \mathcal{D} \rightarrow \mathcal{SD}_0, \quad i = 1, 2,$$

such that

$$\begin{aligned} \iota_1(F) &= F \frac{\partial}{\partial \theta} \theta, \\ \iota_2(F) &= F\theta \frac{\partial}{\partial \theta}. \end{aligned} \quad (2.19)$$

for every $F \in \mathcal{D}$. Also from now on we will write \mathcal{SD}_R instead of \mathcal{SD} .

Theorem 2.1. Let

$$\mathcal{SD}_R^+ = \mathcal{SD}_{R,0}^+ \oplus \mathcal{SD}_{R,1}^+$$

be the γ -stable Lie subalgebra of \mathcal{SD}_R^+ . Then

$$(a) \quad \mathcal{SD}_{R,0}^+ = \iota_1(\mathcal{D}^+) \oplus \iota_2(\mathcal{D}^-),$$

$$(b) \quad \mathcal{S}\mathcal{D}_{\mathbb{R},1}^+ = \text{span}\{\text{coeff}_{y_1^m y_2^n x^l} G^{y_1, y_2}(x_1): m, n \in \mathbb{N}, l \in \mathbb{Z}\}, \quad (2.20)$$

where

$$G^{y_1, y_2}(x) = e^{-y_1 D} \delta\left(\frac{t}{x}\right) e^{y_2 D} D \theta + e^{-y_2 D} \delta\left(\frac{t}{x}\right) e^{y_1 D} \frac{\partial}{\partial \theta}.$$

Proof. The part (a) follows immediately from the description of the θ_1 and θ_2 -stable subalgebras of $\text{Diff}[t, t^{-1}, \theta]$. The description of the odd part is more complicated.

$$\begin{aligned} & \gamma_1 \left((-1)^k D^k t^l D^m D \theta + (-1)^m D^m t^l D^k \frac{\partial}{\partial \theta} \right) \\ &= t^l \left(D^k (-1)^m (D + l)^m \frac{\partial}{\partial \theta} + (-1)^k (D + l)^k D^m D \theta \right) \\ &= (-1)^m D^m t^l D^k \frac{\partial}{\partial \theta} + (-1)^k D^k t^l D^m D \theta, \end{aligned} \quad (2.21)$$

thus the right hand side in (2.20) is contained in $\mathcal{S}\mathcal{D}_{\mathbb{R},1}^+$. Suppose that

$$t^l F(D) D \theta + t^l G(D) \frac{\partial}{\partial \theta} \in \mathcal{S}\mathcal{D}_{\mathbb{R}}^1.$$

From the definition of γ it follows that $F(-D - l) = G(D)$ and $G(-D - l) = F(D)$. Therefore a homogeneous element of $\mathcal{S}\mathcal{D}_{\mathbb{R},1}^+$ is of the form $t^l F(D) D \theta + t^l F(-D - l) \partial/\partial \theta$, where F is a certain polynomial. Therefore

$$\left\{ t^l D^{k+1} \theta + t^l (-1)^k (D + l)^k \frac{\partial}{\partial \theta}: k \in \mathbb{N}, l \in \mathbb{Z} \right\}, \quad (2.22)$$

is a basis for $\mathcal{S}\mathcal{D}_{\mathbb{R},1}^+$. Since (2.22) are contained in the right hand side of (2.20), we have a proof. \square

It is more convenient to work with a twisted version of $\mathcal{S}\mathcal{D}_{\mathbb{R}}$. Let

$$\mathcal{S}\mathcal{D}_{\text{NS}} := \mathcal{S}\mathcal{D}_{\text{NS},0} \oplus \mathcal{S}\mathcal{D}_{\text{NS},1},$$

where the \mathbb{Z}_2 -graded components are given by

$$\mathcal{S}\mathcal{D}_{\text{NS},0} = \mathcal{S}\mathcal{D}_{\mathbb{R},0}$$

and

$$\mathcal{S}\mathcal{D}_{\text{NS},1} = \{\varphi \in t^{1/2} \text{Diff}[t, t^{-1}, \theta]: \varphi(1) = 0\}.$$

We define an involution γ on $\mathcal{S}\mathcal{D}_{\text{NS}}$ in the same as in (2.16).

In contrast with $\mathcal{S}\mathcal{D}_{\mathbb{R}}$, $\mathcal{S}\mathcal{D}_{\text{NS}}$ is not \mathbb{Z} -graded (it is $\frac{1}{2}\mathbb{Z}$ -graded). The proof of the following result is essentially the same as the proof of Theorem 2.1.

Proposition 2.5. *Let $\mathcal{S}\mathcal{D}_{\text{NS}}^+$ be the γ -fixed Lie superalgebra. Then*

$$\mathcal{S}\mathcal{D}_{\text{NS},0}^+ = \mathcal{S}\mathcal{D}_{\mathbb{R},0}^+$$

$$\mathcal{S}\mathcal{D}_{\text{NS},1}^+ = \text{span}\{\text{coeff}_{y_1^m y_2^n x^l} G^{y_1, y_2}(x): m, n \in \mathbb{N}, l \in \mathbb{Z}\},$$

where

$$G^{y_1, y_2}(x) = e^{-y_1 D} \delta_{1/2} \left(\frac{t}{x} \right) e^{y_2 D} \theta D + e^{-y_2 D} \delta_{1/2} \left(\frac{t}{x} \right) e^{y_1 D} \frac{\partial}{\partial \theta},$$

and $\delta_{1/2}(x) = x^{1/2} \delta(x)$.

Remark 2.3. The centerless Neveu–Schwarz superalgebra has generators L_m and G_n , $m \in \mathbb{Z}$ and $n \in \mathbb{Z} + \frac{1}{2}$, and commutation relations

$$[L_m, L_n] = (m - n)L_{m+n}, \quad (2.23)$$

$$[G_r, L_n] = \left(r - \frac{n}{2} \right) G_{r+n}, \quad (2.24)$$

$$[G_r, G_s] = 2L_{r+s}, \quad (2.25)$$

$n, m \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$. If one considers a Lie superalgebra with the same commutation relations, such that $r, s \in \mathbb{Z}$, then the corresponding algebra is the so-called Ramond Lie superalgebra (with $c = 0$). The mapping

$$\begin{aligned} L_n &\mapsto -t^{n+1} \frac{\partial}{\partial t} - n\theta t^n \frac{\partial}{\partial \theta} \\ G_{n+\frac{1}{2}} &\mapsto -t^{n+1/2} \frac{\partial}{\partial \theta} + t^{n+3/2} \theta \frac{\partial}{\partial t}. \end{aligned} \quad (2.26)$$

is a representation of the Neveu–Schwarz Lie superalgebra. It is easy to see that the Lie superalgebra $\mathcal{S}\mathcal{D}_{\text{NS}}^+$ contains the operators (2.26). A similar property holds for the Ramond Lie superalgebra and the Lie superalgebra $\mathcal{S}\mathcal{D}_{\text{R}}^+$.

Lemma 2.1. *We have the following commutation relations:*

$$\begin{aligned} \text{(a)} \quad & [G^{y_1, y_2}(x_1), G^{y_3, y_4}(x_2)] \\ & = D^{y_2, y_4 + y_1 - y_3}(x_1) \delta_{1/2} \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) - \frac{\partial}{\partial y_4} \bar{D}^{y_1, y_2 + y_3 - y_4}(x_1) \delta_{1/2} \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right), \\ \text{(b)} \quad & [G^{y_1, y_2}(x_1), D^{y_3, y_4}(x_2)] \\ & = \frac{\partial}{\partial y_2} \left(G^{y_1, y_4 + y_2 - y_3}(x_1) \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) + G^{y_1, y_2 - y_4 + y_3}(x_1) \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) \right), \\ \text{(c)} \quad & [G^{y_1, y_2}(x_1), \bar{D}^{y_3, y_4}(x_2)] \\ & = G^{y_2, y_4 + y_1 - y_3}(x_1) \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) - G^{y_2, y_1 - y_4 + y_3}(x_1) \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right), \end{aligned}$$

where \mathcal{D}^+ and \mathcal{D}^- are embedded inside $\mathcal{S}\mathcal{D}_{\text{NS}}^+$ via (2.19).

Proof.

$$\begin{aligned}
 & [G^{y_1, y_2}(x_1), G^{y_3, y_4}(x_2)] \\
 &= \left[e^{-y_1 D} \delta_{1/2} \left(\frac{t}{x_1} \right) e^{y_2 D} \theta D + e^{-y_2 D} \delta_{1/2} \left(\frac{t}{x_1} \right) e^{y_1 D} \frac{\partial}{\partial \theta}, \right. \\
 & \quad \left. \times e^{-y_3 D} \delta_{1/2} \left(\frac{t}{x_2} \right) e^{y_4 D} \theta D + e^{-y_4 D} \delta_{1/2} \left(\frac{t}{x_2} \right) e^{y_3 D} \frac{\partial}{\partial \theta} \right] \\
 &= \frac{\partial}{\partial \theta} \theta \left(\delta_{1/2} \left(\frac{e^{-y_2 t}}{x_1} \right) \delta_{1/2} \left(\frac{e^{y_1 - y_2 - y_3 t}}{x_2} \right) e^{(y_4 + y_1 - y_2 - y_3) D} D \right. \\
 & \quad \left. + \delta_{1/2} \left(\frac{e^{-y_4 t}}{x_2} \right) \delta_{1/2} \left(\frac{e^{y_3 - y_1 - y_4 t}}{x_1} \right) e^{(y_3 + y_2 - y_1 - y_4) D} D \right) \\
 & \quad + \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial y_4} \left(\delta_{1/2} \left(\frac{e^{-y_3 t}}{x_2} \right) \delta_{1/2} \left(\frac{e^{y_4 - y_3 - y_2 t}}{x_2} \right) e^{(y_4 + y_1 - y_2 - y_3) D} \right. \\
 & \quad \left. + \delta_{1/2} \left(\frac{e^{-y_1 t}}{x_1} \right) \delta_{1/2} \left(\frac{e^{y_2 - y_1 - y_4 t}}{x_2} \right) e^{(y_2 + y_3 - y_1 - y_4) D} \right) \\
 &= \frac{\partial}{\partial \theta} \theta \left(\delta \left(\frac{e^{-y_2 t}}{x_1} \right) \delta_{1/2} \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) e^{(y_4 + y_1 - y_2 - y_3) D} D \right. \\
 & \quad \left. + \delta \left(\frac{e^{-y_4 t}}{x_2} \right) \delta_{1/2} \left(\frac{e^{y_3 - y_1} t x_2}{x_1} \right) e^{(y_3 + y_2 - y_1 - y_4) D} D \right) \\
 & \quad + \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial y_4} \left(\delta \left(\frac{e^{-y_3 t}}{x_2} \right) \delta_{1/2} \left(\frac{e^{y_4 - y_2} x_2}{x_1} \right) e^{(y_4 + y_1 - y_2 - y_3) D} \right. \\
 & \quad \left. + \delta \left(\frac{e^{-y_1 t}}{x_1} \right) \delta_{1/2} \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) e^{(y_2 + y_3 - y_1 - y_4) D} \right) \\
 &= \frac{\partial}{\partial \theta} \theta D^{y_2, y_4 + y_1 - y_3}(x_1) \delta_{1/2} \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) \\
 & \quad - \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial y_4} \bar{D}^{y_1, y_2 + y_3 - y_4}(x_1) \delta_{1/2} \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right). \tag{2.27}
 \end{aligned}$$

In the previous formulas we used the following fact:

$$\delta_{1/2} \left(\frac{e^y t}{x_1} \right) \delta_{1/2} \left(\frac{e^x t}{x_2} \right) = \delta \left(\frac{e^y t}{x_1} \right) \delta_{1/2} \left(\frac{e^{x-y} x_1}{x_2} \right).$$

The proofs of (b) and (c) are straightforward. \square

2.5. Lie superalgebras $\mathcal{S}\hat{\mathcal{D}}_{\text{NS}}^+$ and $\mathcal{S}\hat{\mathcal{D}}_{\text{R}}^+$

There is a natural 2-cocycle on $\text{Diff}[t, t^{-1}, \theta]$. Let $F(D), G(D) \in \text{Diff}[t, t^{-1}, \theta]$ such that

$$F(D) = f_1(D) \frac{\partial}{\partial \theta} \theta + f_2(D) \frac{\partial}{\partial \theta} + f_3(D)\theta + f_4(D)\theta \frac{\partial}{\partial \theta},$$

$$G(D) = g_1(D) \frac{\partial}{\partial \theta} \theta + g_2(D) \frac{\partial}{\partial \theta} + g_3(D)\theta + g_4(D)\theta \frac{\partial}{\partial \theta}.$$

Then [1],

$$\begin{aligned} &\Psi^s(t^{k_1}F(D), t^{k_2}G(D)) \\ &= \begin{cases} \sum_{-k_1 \leq i \leq -1} f_1(i)g_1(i+k_1) + f_2(i)g_3(i+k_1) \\ -f_3(i)g_2(i+k_1) - f_4(i)g_4(i+k_1), \text{ for } k_1 = -k_2 \geq 0 \\ 0, \text{ for } k_1 + k_2 \neq 0 \end{cases} \end{aligned} \tag{2.28}$$

defines a 2-cocycle on $\text{Diff}[t, t^{-1}, \theta]$.

We denote Lie superalgebras induced by this 2-cocycle by $\mathcal{S}\hat{\mathcal{D}}_{\text{NS}}^+$ in the Neveu–Schwarz case, and by $\mathcal{S}\hat{\mathcal{D}}_{\text{R}}^+$ in the Ramond case.

3. Representation theory of $\mathcal{S}\hat{\mathcal{D}}_{\text{NS}}^+$ and vertex operator superalgebras

3.1. Highest weight representations for $\hat{\mathcal{G}}^\pm$

Lie algebras $\hat{\mathcal{G}}^+$ and $\hat{\mathcal{G}}^-$ (shorthand $\hat{\mathcal{G}}^\pm$) are \mathbb{Z} -graded, but their homogeneous subspaces are infinite-dimensional. Therefore it is important to distinguish among \mathbb{Z} -graded $\hat{\mathcal{G}}^\pm$ -modules, those representations with the finite-dimensional \mathbb{Z} -graded subspaces. We call such modules *quasifinite* (cf. [20,21]).

We will use the standard triangular decomposition

$$\hat{\mathcal{G}}^\pm = \mathcal{D}_+^\pm \oplus \hat{\mathcal{H}}^\pm \oplus \mathcal{D}_-^\pm,$$

where \mathcal{D}_+^\pm (resp. \mathcal{D}_-^\pm) are elements of strictly positive (resp. negative degree) and $\hat{\mathcal{H}}^\pm$ is the Cartan subalgebra introduced earlier.

For any $\lambda \in (\hat{\mathcal{H}}^\pm)^*$ consider a one-dimensional $\mathcal{D}_+^\pm \oplus \hat{\mathcal{H}}^\pm$ -module \mathbb{C}_λ , with a basis v_λ such that $h \cdot v_\lambda = \lambda(h)v_\lambda$ for $h \in \hat{\mathcal{H}}^\pm$ and $\mathcal{D}_+^\pm|_{\mathbb{C}_\lambda} = 0$.

A Verma module is defined as

$$M(\hat{\mathcal{G}}^\pm, \lambda) = \text{Ind}_{\mathcal{D}_+^\pm \oplus \hat{\mathcal{H}}^\pm}^{\hat{\mathcal{G}}^\pm} \mathbb{C}_\lambda.$$

We denote the corresponding irreducible quotient by $L(\hat{\mathcal{G}}^\pm, \lambda)$.

Besides subalgebras \mathcal{D}_l defined in the first section it is important to consider parabolic subalgebras. A parabolic subalgebra $\mathcal{P} \subset \hat{\mathcal{G}}^\pm$ by definition (cf. [20]) satisfies the properties $\mathcal{P}_j = \hat{\mathcal{G}}_j^\pm$, for every $j \geq 0$, and $\mathcal{P}_j \neq 0$ for some $j < 0$. Similarly we defined a generalized Verma module as

$$M(\hat{\mathcal{G}}^\pm, \mathcal{P}, A) = \text{Ind}_{\mathcal{P}}^{\hat{\mathcal{G}}^\pm} \mathbb{C}A.$$

Notice that the generalized Verma module is well-defined if and only if

$$[\mathcal{P}, \mathcal{P}]|_{\mathbb{C}A} = 0.$$

Parabolic algebras are a very important tool in studying quasifinite representations (cf. [20,23]). We will be interested in a particular parabolic algebra.

The following fact is implicitly contained in [20].

Proposition 3.1. *Subalgebra \mathcal{P}_0 , spanned by $t^k(d/dt)^l$, where $k \geq 0$, is a parabolic subalgebra of $\hat{\mathcal{G}}$. The corresponding generalized Verma module $M(\hat{\mathcal{G}}, \mathcal{P}_0, A)$ is quasifinite for every A . \mathcal{P}_0 is generated by*

$$\mathcal{D}_+ \oplus \mathbb{C}C \oplus \mathbb{C} \frac{d}{dt}.$$

Similarly we consider parabolic subalgebras $\mathcal{P}_0^\pm \subset \hat{\mathcal{G}}^\pm$ spanned by $\mathcal{P}_0 \cap \hat{\mathcal{G}}^\pm$ and construct quasifinite modules $M(\hat{\mathcal{G}}^\pm, \mathcal{P}_0, A)$. In particular if $\mathcal{H}|_{\mathbb{C}A} = 0$ and central element acts as c , then we write shorthand M_c (resp. L_c) for $M(\hat{\mathcal{G}}^\pm, \mathcal{P}_0, A)$ (resp. $L(\hat{\mathcal{G}}^\pm, A)$).

3.2. Highest weight representations for $\hat{\mathcal{S}}\mathcal{D}_{\text{NS}}^+$ and $\hat{\mathcal{S}}\mathcal{D}_{\text{R}}^+$

As in the previous section we consider a triangular decomposition

$$\hat{\mathcal{S}}\mathcal{D}_{\text{NS}}^+ = \mathcal{S}\mathcal{D}_{\text{NS},+}^+ \oplus \hat{\mathcal{S}}\mathcal{H}_{\text{NS}} \oplus \mathcal{S}\mathcal{D}_{\text{NS},+}^+,$$

induced by the $\frac{1}{2} \mathbb{Z}$ -grading and

$$\hat{\mathcal{S}}\mathcal{D}_{\text{R}}^+ = \mathcal{S}\mathcal{D}_{\text{R},+}^+ \oplus \hat{\mathcal{S}}\mathcal{H}_{\text{R}} \oplus \mathcal{S}\mathcal{D}_{\text{R},-}^+,$$

induced by the \mathbb{Z} -grading. Here

$$\hat{\mathcal{S}}\mathcal{H}_{\text{NS}} = \mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathbb{C}C,$$

and

$$\hat{\mathcal{S}}\mathcal{H}_{\text{R}} = \mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathbb{C}C \oplus \bigoplus_{n \geq 0} \mathbb{C}G_n^{(0)},$$

where

$$G_n^{(0)} = D^{n+1}\theta + (-1)^n(D+l)^n \frac{\partial}{\partial \theta}.$$

Notice that the ‘‘Cartan’’ subalgebra $\hat{\mathcal{S}}\mathcal{H}_{\text{R}}$ is not commutative and

$$(G_n^{(0)})^2 = (-1)^n D^{2n+1}. \tag{3.1}$$

Let $\lambda \in (\mathcal{SH}_{NS})^*$ or $\lambda \in (\mathcal{SH}_R)^*$ and \mathbb{C}_λ a one-dimensional module (character) spanned by v_λ . In the Ramond case, in addition, relation (3.1) has to be satisfied in $\text{End}(\lambda)$. The corresponding Verma modules are defined as

$$M(\hat{\mathcal{D}}_R^+, \lambda) = \text{Ind}_{\mathcal{D}_{R,+}^+ \oplus \mathcal{H}_R}^{\mathcal{D}_R^+} \mathbb{C}v_\lambda$$

$$M(\hat{\mathcal{D}}_{NS}^+, \lambda) = \text{Ind}_{\mathcal{D}_{NS,+}^+ \oplus \mathcal{H}_{NS}}^{\mathcal{D}_{NS}^+} \mathbb{C}v_\lambda, \tag{3.2}$$

where

$$\mathcal{D}_{NS,+}^+|_{\mathbb{C}v_\lambda} = \mathcal{D}_{NS,+}^+|_{\mathbb{C}v_\lambda} = 0.$$

3.3. $N = 1$ vertex operator superalgebras

The following definition is from [2,22]:

Definition 3.1. A $N = 1$ vertex operator superalgebra is a quadruple $(V, Y, \mathbf{1}, \tau)$, where $V = V_{(0)} \oplus V_{(1)}$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, equipped with a $\frac{1}{2}\mathbb{Z}$ -grading (that we call degree)

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n,$$

such that

$$V_{(0)} = \bigoplus_{n \in \mathbb{Z}} V_n, \quad V_{(1)} = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} V_n, \tag{3.3}$$

vectors $\tau \in V_{3/2}$ and $\mathbf{1} \in V_0$, and the mapping

$$Y(\cdot, x) \cdot : V \otimes V \rightarrow V[[x, x^{-1}]],$$

satisfying:

1.

$$Y(\mathbf{1}, x) = \text{Id}.$$

2. The *truncation* property: For any $v, w \in V$,

$$Y(v, x)w \in V((x)).$$

3. The *creation* property: For any $v \in V$,

$$Y(v, x)\mathbf{1} \in V[[x]],$$

$$\lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v.$$

4. The *Jacobi identity*: In

$$\text{Hom}(V \otimes V, V)[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]],$$

we have

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) \\ & - \varepsilon_{u,v} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) \end{aligned} \tag{3.4}$$

for any $u, v \in V$ such that u and v homogeneous (here $\varepsilon_{u,v} = (-1)^{p(u)p(v)}$),

5. *Neveu–Schwarz relations*: Let

$$Y(\tau, x) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} G(n) z^{-n-3/2},$$

and

$$\frac{1}{2} Y(G(-1/2)\tau, x) = Y(\omega, x) = \sum_{m \in \mathbb{Z}} L(m) z^{-m-2},$$

then $L(m)$ and $G(n)$ close Neveu–Schwarz superalgebra

$$\begin{aligned} [L(m), L(n)] &= (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m+n,0} c \\ [G(m), L(n)] &= \left(m - \frac{n}{2} \right) G(m + n) \\ [G(m), G(n)] &= 2L(r + s) + \frac{1}{3} \left(m^2 - \frac{1}{4} \right) \delta_{m+n,0} c, \end{aligned} \tag{3.5}$$

such that $L(0) \cdot v = kv$, for every homogeneous $v \in V_k$.

6. *The $L(-1)$ -derivative property*: For any $v \in V$,

$$Y(L(-1)v, x) = \frac{d}{dx} Y(v, x).$$

Remark 3.1. In some literature (cf. [32]) condition (3.3) is omitted. Also, if we drop the Neveu–Schwarz relations (3.5) from the definition then the corresponding structure is called vertex operator superalgebra (cf. [38,32]; see also the introduction in [9])

Let us fix a basis for $\hat{\mathcal{D}}_{NS}^+$:

$$\begin{aligned} l_{m,+}^{(r)} &= \frac{1}{2} t^m D^{2r-1} + \frac{1}{2} t^m (D + m)^{2r-2} D, \\ l_{m,-}^{(r)} &= \frac{1}{2} t^m D^{2r-1} + \frac{1}{2} t^m (D + m)^{2r-1}, \\ g_n^{(r)} &= t^n D^r \theta + t^n (-1)^{r-1} (D + n)^{r-1} \frac{\partial}{\partial \theta}, \end{aligned} \tag{3.6}$$

where $r \geq 1$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z} + \frac{1}{2}$. We always assume that \mathcal{D}^+ and \mathcal{D}^- are embedded inside $\mathcal{S}\mathcal{D}$ by using ι_1 and ι_2 . Notice the relation

$$l_{-1,+}^{(1)} + l_{-1,-}^{(2)} = (g_{-1/2}^{(1)})^2. \quad (3.7)$$

Consider a parabolic algebra generated by $g_{-1/2}^{(1)}$ and $\mathcal{S}\mathcal{D}_{\text{NS},+}^+$ and C . This is a super analogue of the parabolic algebra considered in Proposition 3.1. We recall (see the previous section) definitions of M_c and L_c .

We define fields:

$$\begin{aligned} l_+^{(r)}(x) &= \sum_{m \in \mathbb{Z}} l_{m,+}^{(r)} x^{-m-r-1}, \\ l_-^{(r)}(x) &= \sum_{m \in \mathbb{Z}} l_{m,-}^{(r)} x^{-m-r-1}, \\ g^{(r)}(x) &= \sum_{n \in \mathbb{Z} + 1/2} g_n^{(r)} x^{-n-r-\frac{3}{2}}, \end{aligned} \quad (3.8)$$

acting on some $\widehat{\mathcal{S}\mathcal{D}_{\text{NS},+}^+}$ -module. We will distinguish the following fields:

$$L(x) = -l_+^{(1)}(x) - l_-^{(1)}(x), \quad (3.9)$$

$$G(x) = \sqrt{-1}g^{(1)}(x). \quad (3.10)$$

It is not hard to check that these fields close the Neveu–Schwarz algebra. If we let $L(-1) = \text{Res}_x L(x)$ then:

$$[L(-1), h(x)] = \frac{d}{dx} h(x),$$

where $h(x)$ is any field in (3.8). Hence $l_{\pm}^{(r)}(x), g^{(r)}(x)$ are the so-called *weak* vertex operators [32].

We will show that there is a canonical $N = 1$ vertex operator superalgebra structure on the spaces M_c and L_c .

The following definition is from [32]:

Definition 3.2. Let $M = M(\widehat{\mathcal{S}\mathcal{D}_{\text{NS},+}^+}, A)$. We say that two weak vertex operators $a(x), b(x) \in \text{End}(M)[[x, x^{-1}]]$ are mutually local if there is $n > 0$ such that

$$(x_1 - x_2)^n [a(x_1), b(x_2)] = 0.$$

A weak vertex operator on M which is local with itself is called *vertex operator*. A family of vertex operators $\{a_i(x)\}_{i \in I}$ is *local* if all pairs of operators are mutually local.

Theorem 3.1. (a) *Weak vertex operators $l_+^{(r)}(x), l_-^{(s)}(x)$ and $g^{(t)}(x)$, $r, s, t \geq 1$, are mutually local.*

(b) M_c and L_c have $N = 1$ vertex operator superalgebra structures. For every A , such that central element acts as multiplication with c , $M(\mathcal{S}\hat{\mathcal{D}}_{NS}^+, A)$ is a weak M_c -module.

Proof. We will give the proof for $c = 0$ (in the case of M_c). The proof for general c can be easily deduced by using the same arguments as below and the formula (2.28). Our proof is not the same as Proposition 3.1 in [14] (which deals with an explicit construction). We rather apply general theory of *local systems* developed by Li (cf. [32]). Moreover, some examples in [32] are similar to our construction (especially Section 4.2 in [32]).

Proof of (a). Notice the following relations:

$$\begin{aligned} l_+^{(r)}(x) &= x^{-r-1} \text{Coeff}_{y^r/r!}(\mathcal{D}^{y,0}(x) + \mathcal{D}^{-y,0}(x)), \\ l_-^{(r)}(x) &= x^{-r-1} \text{Coeff}_{y^r/r!}(\bar{\mathcal{D}}^{y,0}(x) + \bar{\mathcal{D}}^{-y,0}(x)), \\ g^{(r)}(x) &= x^{-r-1} \text{Coeff}_{y^r/r!} G^{y,0}(x), \end{aligned} \tag{3.11}$$

where operators $\mathcal{D}^{y,0}(x)$, $\bar{\mathcal{D}}^{y,0}(x)$ and $G^{y,0}(x)$ are defined in Section 2.1. Therefore

$$\sum_{r \geq 1} x^{r+1} \frac{l_+^{(r)}(x) y^n}{n!} = \mathcal{D}^{y,0}(x) + \mathcal{D}^{-y,0}(x), \tag{3.12}$$

and similar formulas hold for $l_+^{(r)}(x)$ and $g^{(r)}(x)$. Hence in order to prove that fields (3.11) are local, it is enough to show that coefficients of $\mathcal{D}^{y,0}(x)$, $\hat{\mathcal{D}}^{y,0}(x)$ and $G^{y,0}(x)$, in the Fourier expansion with respect to y , are local. To prove this fact we will use a well-known result (cf. [15]):

$$(x_1 - x_2)^m \delta^{(n)} \left(\frac{x_1}{x_2} \right) = 0, \tag{3.13}$$

for every $m > n$. By extracting appropriate coefficients in formulas in Propositions 2.2, 2.3 and Lemma 2.1, then multiplying with $(x_1 - x_2)^n$ (n large enough) and applying (3.13) we obtain the result.

(b) Now, we will closely follow general theory obtained in [32] (we skip unnecessary details). Let

$$V = \langle g^{(1)}(x), g^{(2)}(x), \dots, l_{\pm}^{(1)}(x), l_{\pm}^{(2)}(x), \dots \rangle.$$

Notice that $\text{Id}(x)|_M \in V$. Then (cf. Theorem 3.2.10 in [32]), V is a vertex operator superalgebra. Then, because of $g^{(1)}(-1/2)I(x) = 0$ there is a map

$$V \mapsto M_c,$$

such that $I(x)$ is mapped to $\mathbf{1}$. Because of the universal property this is an isomorphism. Also, again by Theorem 3.2.10 in [32], every Verma module $M(\mathcal{S}\hat{\mathcal{D}}_{NS}^+, A)$ is a weak M_0 -module. Similar proof works for arbitrary $c \neq 0$. $L_c \cong M_c/M_c^{(1)}$, where $M_c^{(1)}$ is the maximal ideal. If $c \neq 0$, $\omega \notin M_c^{(1)}$. Therefore (cf. [13]) L_c is a vertex operator (super)algebra. \square

Remark 3.2. It can be shown (by using a result from [31]) that for every A , $M(\hat{\mathcal{S}}_{\mathbb{R}}^+, A)$ is a σ -twisted M_c -module (this notion was formalized in [11,8]; see also [16,26]), where σ is the canonical automorphism of vertex operator superalgebra defined by, for homogeneous u , $\sigma(u) = (-1)^{p(u)}u$.

3.4. Vertex operator superalgebra $W = M(1) \otimes F$

Certain representations of $\hat{\mathcal{S}}_{\mathbb{N}\mathbb{S}}^+$ (or projective representations of $\mathcal{S}_{\mathbb{N}\mathbb{S}}^+$) admit realizations in terms of free fields. In some interesting cases these representations carry a structure of vertex operator superalgebras.

Here we shall not consider the representations of $\hat{\mathcal{S}}_{\mathbb{R}}^+$ in detail, even though they naturally appear. As we mentioned in Remark 3.2 they are related to the so-called σ -twisted modules for vertex operator superalgebras (see also Remark 5.2).

In this section we study a distinguished-simplest-projective representation of $\mathcal{S}_{\mathbb{N}\mathbb{S}}^+$ that admits a realization in terms of free fields.

We already mentioned in the introduction that there is a vertex operator algebra structure on $M(1)$ and a vertex operator superalgebra structure on F . Now we equip $W = M(1) \otimes F$ with a structure of a $N = 1$ vertex operator superalgebra as in [22]. Simply take

$$\omega = \frac{1}{2} h(-1)^2 \mathbf{1} + \frac{1}{2} \varphi(-3/2) \varphi(-1/2) \mathbf{1}$$

and

$$\tau = h(-1) \varphi(-1/2) \mathbf{1}.$$

The central charge of W is equal to $\frac{3}{2}$.

Let

$$X(x) = Y(x^{L(0)} h(-1) \mathbf{1}, x),$$

$$\tilde{X}(x) = Y(x^{L(0)} \varphi(-1/2) \mathbf{1}, x).$$

Also, we denote by $\bullet\bullet$ the normal ordered product defined in [15] (Section 8.4). We study the normal orderings of *quadratic* operators, i.e., expressions of the type $\bullet\bullet X(x_1) X(x_2) \bullet\bullet$. For more detailed discussion concerning normal ordering see Section 3.1. It is convenient to put all derivatives of $X(x_1)$ and $X(x_2)$ into the same generating function. Hence in all our calculations we shall deal with the generating functions, considered in [28–30], of the form

$$\bullet\bullet e^{y_1 D_{x_1}} X(x_1) e^{y_2 D_{x_2}} X(x_2) \bullet\bullet = \bullet\bullet X(e^{y_1} x_1) X(e^{y_2} x_2) \bullet\bullet.$$

In the calculations that follow we will use the following binomial expansion convention: Every formal rational function of the form $1/(x - y)^k$ has to be expanded in the non-negative powers of y by the binomial theorem. The same convention applies if y is not a formal variable but rather a linear combination of several formal variables. For formal expressions of the form, $1/(e^z x - e^y w)^k$ and $1/(1 - e^x y)^k$ we apply the same convention.

On the other hand the formal expression $1/(e^y - 1)^k$ stands for the multiplicative inverse of $(e^y - 1)^k$. Note that more general expressions of the form

$$\frac{1}{(e^{x-y} - 1)^k}, \tag{3.14}$$

where y is a formal variables are ambiguous (since $(e^{x-y} - 1)^k$ has more than one multiplicative inverse inside $\mathbb{C}[[x, x^{-1}, y, y^{-1}]]$). Unless otherwise stated we consider (3.14) as the multiplicative inverse of $(e^{x-y} - 1)^k$ with the non-negative powers in y . We apply the same expansion if y is a linear combination of several formal variables (cf. Theorem 3.2).

The goal is to obtain a projective representation of the Lie superalgebra \mathcal{SQ}_{NS}^+ in terms of the quadratic operators. For these operators we have the following commutation relations (cf. [28–30]).

Theorem 3.2. (a)

$$\begin{aligned} & [\bullet X(e^{y_1}x_1)X(e^{y_2}x_2)\bullet\bullet X(e^{y_3}x_2)X(e^{y_4}x_2)\bullet] \\ &= \frac{\partial}{\partial y_1} \left(\bullet \tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_4+y_1-y_3}x_1)\bullet\delta\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) \right. \\ & \quad \left. + \bullet \tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_3+y_1-y_4}x_1)\bullet\delta\left(\frac{e^{y_1-y_4}x_1}{x_2}\right) \right) \\ & \quad \times \frac{\partial}{\partial y_2} \left(\bullet \tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_4+y_2-y_3}x_1)\bullet\delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right) \right. \\ & \quad \left. + \bullet \tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\bullet\delta\left(\frac{e^{y_2-y_4}x_1}{x_2}\right) \right) \\ & \quad + \frac{\partial}{\partial y_1} \left(\frac{e^{y_3+y_4-y_1-y_2}}{(1 - e^{y_4+y_1-y_3-y_2})^2} \delta\left(\frac{e^{y_1}x_1}{e^{y_3}x_2}\right) \right. \\ & \quad \left. + \frac{e^{y_3+y_4-y_1-y_2}}{(1 - e^{y_3+y_1-y_4-y_2})^2} \delta\left(\frac{e^{y_1}x_1}{e^{y_4}x_2}\right) \right) \\ & \quad + \frac{\partial}{\partial y_2} \left(\frac{e^{y_3+y_4-y_1-y_2}}{(1 - e^{-y_3-y_1+y_2+y_4})^2} \delta\left(\frac{e^{y_2}x_1}{e^{y_3}x_2}\right) \right. \\ & \quad \left. + \frac{e^{y_3+y_4-y_1-y_2}}{(1 - e^{-y_4-y_1+y_2+y_3})^2} \delta\left(\frac{e^{y_2}x_1}{e^{y_4}x_2}\right) \right) \end{aligned} \tag{3.15}$$

(b)

$$\begin{aligned} & [\bullet \tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_2}x_1)\bullet\bullet \tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_4}x_2)\bullet] \\ &= \bullet \tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_4+y_2-y_3}x_1)\bullet\delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right) \end{aligned}$$

$$\begin{aligned}
& + \bullet\tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_3+y_1-y_4}x_1)\bullet\delta\left(\frac{e^{y_1-y_4}x_1}{x_2}\right) \\
& - \bullet\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_1-y_4}x_1)\bullet\delta\left(\frac{e^{y_2-y_4}x_1}{x_2}\right) \\
& - \bullet\tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_4+y_1-y_3}x_1)\bullet\delta\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) \\
& + \frac{e^{(y_1+y_4-y_2-y_3)/2}}{e^{y_1+y_4-y_2-y_3}-1}\delta\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) + \frac{e^{(y_3+y_2-y_1-y_4)/2}}{e^{-y_1-y_4+y_3+y_2}-1}\delta\left(\frac{e^{y_2-y_4}x_1}{x_2}\right) \\
& + \frac{e^{(y_1+y_3-y_2-y_4)/2}}{e^{y_1+y_3-y_2-y_4}-1}\delta\left(\frac{e^{y_1-y_4}x_1}{x_2}\right) + \frac{e^{(y_2+y_4-y_1-y_3)/2}}{e^{-y_1-y_3+y_2+y_4}-1}\delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right),
\end{aligned} \tag{3.16}$$

(c)

$$\begin{aligned}
& [\bullet\tilde{X}(e^{y_1}x_1)X(e^{y_2}x_1)\bullet\bullet\tilde{X}(e^{y_3}x_2)X(e^{y_4}x_2)\bullet] \\
& = \bullet X(e^{y_2}x_1)X(e^{y_4+y_1-y_3}x_1)\bullet\delta_{1/2}\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) \\
& + \frac{\partial}{\partial y_2}\bullet\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\delta_{1/2}\left(\frac{e^{y_2-y_4}x_1}{x_2}\right) \\
& + \frac{\partial}{\partial y_2}\left(\frac{e^{(y_1-y_3+y_4-y_2)/2}}{1-e^{-y_2-y_3+y_1+y_4}}\delta_{1/2}\left(\frac{e^{y_2-y_4}x_1}{x_2}\right)\right. \\
& \left. + \frac{1}{1-e^{y_2+y_3-y_1-y_4}}\delta_{1/2}\left(\frac{e^{y_1-y_3}x_1}{x_2}\right)\right)
\end{aligned} \tag{3.17}$$

(d)

$$\begin{aligned}
& [\bullet\tilde{X}(e^{y_1}x_1)X(e^{y_2}x_1)\bullet\bullet X(e^{y_3}x_2)X(e^{y_4}x_2)\bullet] \\
& = \frac{\partial}{\partial y_2}\left(\bullet\tilde{X}(e^{y_1}x_1)X(e^{y_4+y_2-y_3}x_1)\bullet\delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right)\right. \\
& \left. + \bullet\tilde{X}(e^{y_1}x_1)X(e^{y_2+y_3-y_4}x_1)\bullet\delta\left(\frac{e^{y_2-y_4}x_1}{x_2}\right)\right).
\end{aligned} \tag{3.18}$$

(e)

$$\begin{aligned}
& [\bullet\tilde{X}(e^{y_1}x_1)X(e^{y_2}x_1)\bullet\bullet\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_4}x_2)\bullet] \\
& = \bullet\tilde{X}(e^{y_2}x_1)X(e^{y_4+y_1-y_3}x_1)\bullet\delta\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) \\
& - \bullet\tilde{X}(e^{y_2}x_1)X(e^{y_1+y_3-y_4}x_1)\bullet\delta\left(\frac{e^{y_1-y_4}x_1}{x_2}\right)
\end{aligned} \tag{3.19}$$

Proof. We imitate the proof from [15] (Section 8.7.), as in [28–30], in the case of the Virasoro algebra. (a) Instead of calculating $[\bullet X(e^{y_1}x_1)X(e^{y_2}x_1)\bullet, \bullet X(e^{y_3}x_2)X(e^{y_4}x_2)\bullet]$ we consider

$$\lim_{x_0 \rightarrow x_1, x_3 \rightarrow x_2} [\bullet X(e^{y_1}x_0)X(e^{y_2}x_1)\bullet, \bullet X(e^{y_3}x_2)X(e^{y_4}x_3)\bullet].$$

Since,

$$[X(e^{y_1}x), X(e^{y_2}x_2)] = \frac{\partial}{\partial x_1} \delta\left(\frac{e^{y_1-y_2}x_1}{x_2}\right) = (D\delta)\left(\frac{e^{y_1-y_2}x_1}{x_2}\right),$$

we have

$$\begin{aligned} & \lim_{x_3 \rightarrow x_2} [X(e^{y_1}x), \bullet X(e^{y_3}x_2)X(e^{y_4}x_3)\bullet] \\ &= \frac{\partial}{\partial y_1} \left(X(e^{y_2}x_2) \delta\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) + X(e^{y_3}x_2) \delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right) \right). \end{aligned} \tag{3.20}$$

Now by using (3.20) we obtain

$$\begin{aligned} & \lim_{x_0 \rightarrow x_1} [\bullet X(e^{y_0}x_0)X(e^{y_1}x_1)\bullet, \bullet X(e^{y_2}x_2)X(e^{y_4}x_2)\bullet] \\ &= \lim_{x_0 \rightarrow x_1} \frac{\partial}{\partial y_0} \left(X(e^{y_2}x_2)X(e^{y_1}x_1) \delta\left(\frac{e^{y_0-y_4}x_0}{x_2}\right) + X(e^{y_4}x_2)X(e^{y_1}x_1) \delta\left(\frac{e^{y_0-y_2}x_0}{x_2}\right) \right) \\ & \quad + \frac{\partial}{\partial y_1} \left(X(e^{y_0}x_0)X(e^{y_2}x_2) \delta\left(\frac{e^{y_1-y_4}x_1}{x_2}\right) + X(e^{y_0}x_0)X(e^{y_4}x_2) \delta\left(\frac{e^{y_1-y_2}x_1}{x_2}\right) \right) \\ &= \lim_{x_0 \rightarrow x_1} \frac{\partial}{\partial y_0} \left(\left(\bullet X(e^{y_2}x_2)X(e^{y_1}x_1)\bullet + \frac{e^{y_2+y_1}x_1x_2}{(e^{y_2}x_2 - e^{y_1}x_1)^2} \right) \delta\left(\frac{e^{y_0-y_4}x_0}{x_2}\right) \right) \\ & \quad + \left(\left(\bullet X(e^{y_4}x_2)X(e^{y_1}x_1)\bullet + \frac{e^{y_4+y_1}x_1x_2}{(e^{y_4}x_2 - e^{y_1}x_1)^2} \right) \delta\left(\frac{e^{y_0-y_3}x_0}{x_2}\right) \right) \\ & \quad + \frac{\partial}{\partial y_1} \left(\left(\bullet X(e^{y_0}x_0)X(e^{y_3}x_2)\bullet + \frac{e^{y_1+y_3}x_0x_2}{(e^{y_1}x_0 - e^{y_3}x_2)^2} \right) \delta\left(\frac{e^{y_2-y_4}x_1}{x_2}\right) \right) \\ & \quad + \frac{\partial}{\partial y_2} \left(\left(\bullet X(e^{y_1}x_0)X(e^{y_4}x_2)\bullet + \frac{e^{y_1+y_4}x_0x_2}{(e^{y_1}x_0 - e^{y_4}x_2)^2} \right) \delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right) \right). \end{aligned} \tag{3.21}$$

Let us consider the terms in (3.21) that contain X -operators. Then

$$\begin{aligned}
& \frac{\partial}{\partial y_1} \left(\bullet X(e^{y_3} x_2) X(e^{y_2} x_1) \bullet \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) + \bullet X(e^{y_4} x_2) X(e^{y_2} x_1) \bullet \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) \right) \\
& + \frac{\partial}{\partial y_2} \left(\bullet X(e^{y_1} x_1) X(e^{y_3} x_2) \bullet \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) + \bullet X(e^{y_1} x_1) X(e^{y_4} x_2) \bullet \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) \right) \\
& = \frac{\partial}{\partial y_1} \left(\bullet X(e^{y_3 + y_1 - y_4} x_1) X(e^{y_2} x_1) \bullet \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) + \bullet X(e^{y_4 + y_1 - y_3} x_1) \right. \\
& \quad \left. X(e^{y_2} x_1) \bullet \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) \right) + \frac{\partial}{\partial y_2} \left(\bullet X(e^{y_1} x_1) X(e^{y_3 + y_2 - y_4} x_1) \bullet \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) \right. \\
& \quad \left. + \bullet X(e^{y_1} x_1) X(e^{y_4 + y_2 - y_3} x_1) \bullet \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) \right), \tag{3.22}
\end{aligned}$$

coincides with the first four terms on the right hand side of (3.15). From now on we consider the remaining terms in (3.21), which do not involve X -operators. We have

$$\begin{aligned}
& \lim_{x_0 \rightarrow x_1} \frac{e^{y_3 + y_2} x_1 x_2}{(e^{y_3} x_2 - e^{y_2} x_1)^2} \left(\frac{e^{y_1 + y_4} x_0 x_2}{(e^{y_1} x_0 - e^{y_4} x_2)^2} - \frac{e^{y_1 + y_4} x_0 x_2}{(e^{y_4} x_2 - e^{y_1} x_0)^2} \right) \\
& + \frac{e^{y_4 + y_2} x_1 x_2}{(e^{y_4} x_2 - e^{y_2} x_1)^2} \left(\frac{e^{y_1 + y_3} x_0 x_2}{(e^{y_1} x_0 - e^{y_3} x_2)^2} - \frac{e^{y_1 + y_3} x_0 x_2}{(e^{y_3} x_2 - e^{y_1} x_0)^2} \right) \\
& + \frac{e^{y_1 + y_3} x_0 x_2}{(e^{y_1} x_0 - e^{y_3} x_2)^2} \left(\frac{e^{y_2 + y_4} x_1 x_2}{(e^{y_2} x_1 - e^{y_4} x_2)^2} - \frac{e^{y_2 + y_4} x_1 x_2}{(e^{y_4} x_2 - e^{y_2} x_1)^2} \right) \\
& + \frac{e^{y_1 + y_4} x_0 x_2}{(e^{y_1} x_0 - e^{y_4} x_2)^2} \left(\frac{e^{y_2 + y_3} x_1 x_2}{(e^{y_2} x_1 - e^{y_3} x_2)^2} + \frac{e^{y_2 + y_3} x_1 x_2}{(e^{y_3} x_2 - e^{y_2} x_1)^2} \right). \tag{3.23}
\end{aligned}$$

After cancellations we can substitute x_1 for x_0 in (3.23). Therefore (3.23) is equal to

$$\begin{aligned}
& e^{y_1 + y_2 + y_3 + y_4} x_1^2 x_2^2 \left(\frac{1}{(e^{y_1} x_1 - e^{y_4} x_2)^2 (e^{y_2} x_1 - e^{y_3} x_2)^2} \right. \\
& + \frac{1}{(e^{y_1} x_1 - e^{y_3} x_2)^2 (e^{y_2} x_1 - e^{y_4} x_2)^2} \\
& \left. - \frac{1}{(e^{y_3} x_2 - e^{y_2} x_1)^2 (e^{y_4} x_2 - e^{y_2} x_1)^2} - \frac{1}{(e^{y_4} x_2 - e^{y_2} x_1)^2 (e^{y_3} x_2 - e^{y_1} x_1)^2} \right) \\
& = \frac{\partial^2}{\partial^2 y_1 y_2} \left(\frac{e^{y_3 + y_4} x_2 x_3}{(e^{y_3} x_2 - e^{y_2} x_1)(e^{y_4} x_2 - e^{y_1} x_1)} + \frac{e^{y_3 + y_4} x_2 x_3}{(e^{y_4} x_2 - e^{y_2} x_1)(e^{y_3} x_2 - e^{y_1} x_1)} \right. \\
& \quad \left. - \frac{e^{y_3 + y_4} x_2 x_3}{(e^{y_1} x_1 - e^{y_3} x_2)(e^{y_2} x_1 - e^{y_4} x_2)} - \frac{e^{y_3 + y_4} x_2 x_3}{(e^{y_1} x_1 - e^{y_4} x_2)(e^{y_2} x_1 - e^{y_3} x_2)} \right). \tag{3.24}
\end{aligned}$$

Note that

$$\frac{1}{(1 - e^y z)(1 - e^x z)} = \frac{1}{z} \frac{1}{(e^y - e^x)} \left(\frac{1}{1 - e^y z} - \frac{1}{1 - e^x z} \right),$$

where

$$\frac{1}{e^y - e^x} = \frac{e^{-x}}{e^{y-x} - 1} = \frac{e^{-x}}{(y-x)(1 + (y-x)/2! + \dots)} \in \mathbb{C}[[y, y^{-1}, x]],$$

according to the conventions introduced earlier in this section. Now (3.24) is equal to

$$\begin{aligned} & \frac{\partial^2}{\partial y_1 \partial y_2} \left(\frac{x_2 e^{y_4 - y_1} / x_1}{e^{y_2 - y_3 + y_4 - y_1} - 1} \left(\frac{1}{1 - e^{y_2 - y_3} x_1 / x_2} + \frac{1}{1 - e^{y_3 - y_2} x_2 / x_1} \right) \right. \\ & - \frac{x_2 e^{y_4 - y_1} / x_1}{e^{y_2 - y_3 + y_4 - y_1} - 1} \left(\frac{1}{1 - e^{y_1 - y_4} x_1 / x_2} + \frac{1}{1 - e^{y_4 - y_1} x_2 / x_1} \right) \\ & + \frac{x_2 e^{y_3 - y_1} / x_1}{e^{y_2 - y_4 + y_3 - y_1} - 1} \left(\frac{1}{1 - e^{y_2 - y_4} x_1 / x_2} + \frac{1}{1 - e^{y_4 - y_2} x_2 / x_1} \right) \\ & \left. - \frac{x_2 e^{y_3 - y_1} / x_1}{e^{y_2 - y_4 + y_3 - y_1} - 1} \left(\frac{1}{1 - e^{y_1 - y_3} x_1 / x_2} + \frac{1}{1 - e^{y_3 - y_1} x_2 / x_1} \right) \right) \\ & = \frac{\partial^2}{\partial y_1 \partial y_2} \left(\frac{e^{y_4 - y_1 + y_2 - y_3}}{e^{-y_3 - y_1 + y_4 + y_2} - 1} \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) + \frac{e^{y_3 - y_2 + y_1 - y_4}}{e^{y_3 + y_1 - y_2 - y_4} - 1} \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) \right. \\ & \left. + \frac{e^{y_3 + y_2 - y_4 - y_1}}{e^{-y_4 - y_1 + y_3 + y_2} - 1} \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) + \frac{e^{y_4 + y_1 - y_2 - y_3}}{e^{y_4 + y_1 - y_2 - y_3} - 1} \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) \right). \end{aligned} \tag{3.25}$$

Note that the formal expressions $1/(e^{y_2 - y_3 + y_4 - y_1} - 1)$ and $1/(e^{-y_2 + y_1 + y_3 - y_4} - 1)$ are expanded in the non-negative powers of y_3 , y_4 and y_1 . Finally (3.25) is equal to

$$\begin{aligned} & \frac{\partial}{\partial y_2} \left(\frac{e^{y_4 - y_1 + y_2 - y_3}}{(e^{-y_3 - y_1 + y_2 + y_4} - 1)^2} \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right) \right) \\ & + \frac{\partial}{\partial y_1} \left(\frac{e^{y_3 + y_1 - y_2 - y_4}}{(e^{y_3 + y_1 - y_2 - y_4} - 1)^2} \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) \right) \\ & + \frac{\partial}{\partial y_2} \left(\frac{e^{y_3 - y_1 + y_2 - y_4}}{(e^{-y_4 - y_1 + y_3 + y_2} - 1)^2} \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) \right) \\ & + \frac{\partial}{\partial y_1} \left(\frac{e^{y_4 - y_2 + y_1 - y_3}}{(e^{y_4 + y_1 - y_2 - y_3} - 1)^2} \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) \right). \end{aligned}$$

Proof of (b).

$$\begin{aligned}
& [\bullet\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_2}x_1)\bullet\bullet\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_4}x_2)\bullet] \\
&= \lim_{x_0 \rightarrow x_1} [\bullet\tilde{X}(e^{y_1}x_0)\tilde{X}(e^{y_2}x_1)\bullet\bullet\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_4}x_2)\bullet] \\
&= \lim_{x_0 \rightarrow x_1} \left(\bullet\tilde{X}(e^{y_4}x_2)\tilde{X}(e^{y_2}x_1)\bullet + \frac{e^{(y_4+y_2)/2}x_2^{1/2}x_1^{1/2}}{e^{y_4}x_2 - e^{y_2}x_1} \right) \delta_{1/2} \left(\frac{e^{y_1-y_3}x_0}{x_2} \right) \\
&\quad - \left(\bullet\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_2}x_1)\bullet + \frac{e^{(y_3+y_2)/2}x_2^{1/2}x_1^{1/2}}{e^{y_3}x_2 - e^{y_2}x_1} \right) \delta_{1/2} \left(\frac{e^{y_1-y_4}x_0}{x_2} \right) \\
&\quad + \left(\bullet\tilde{X}(e^{y_1}x_0)\tilde{X}(e^{y_4}x_2)\bullet + \frac{e^{(y_1+y_4)/2}x_0^{1/2}x_2^{1/2}}{e^{y_1}x_0 - e^{y_4}x_2} \right) \delta_{1/2} \left(\frac{e^{y_2-y_3}x_1}{x_2} \right) \\
&\quad - \left(\bullet\tilde{X}(e^{y_1}x_0)\tilde{X}(e^{y_3}x_2)\bullet + \frac{e^{(y_1+y_3)/2}x_0^{1/2}x_2^{1/2}}{e^{y_1}x_0 - e^{y_3}x_2} \right) \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right). \quad (3.26)
\end{aligned}$$

Now

$$\begin{aligned}
& \lim_{x_0 \rightarrow x_1} \bullet\tilde{X}(e^{y_4}x_2)\tilde{X}(e^{y_2}x_1)\bullet\delta_{1/2} \left(\frac{e^{y_1-y_3}x_0}{x_2} \right) - \bullet\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_2}x_1)\bullet\delta_{1/2} \left(\frac{e^{y_1-y_4}x_0}{x_2} \right) \\
&+ \bullet\tilde{X}(e^{y_1}x_0)\tilde{X}(e^{y_4}x_2)\bullet\delta_{1/2} \left(\frac{e^{y_2-y_3}x_1}{x_2} \right) - \bullet\tilde{X}(e^{y_1}x_0)\tilde{X}(e^{y_3}x_2)\bullet\delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right) \\
&= -\bullet\tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_4+y_1-y_3}x_1)\bullet\delta \left(\frac{e^{y_1-y_3}x_0}{x_2} \right) + \bullet\tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\bullet \\
&\quad \delta \left(\frac{e^{y_1-y_4}x_1}{x_2} \right) + \bullet\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_4+y_2-y_3}x_1)\bullet\delta \left(\frac{e^{y_2-y_3}x_1}{x_2} \right) - \bullet\tilde{X}(e^{y_1}x_1) \\
&\quad \tilde{X}(e^{y_3+y_2-y_4}x_1)\bullet\delta \left(\frac{e^{y_2-y_4}x_1}{x_2} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{x_0 \rightarrow x_1} \frac{e^{(y_4+y_2)/2}x_2^{1/2}x_1^{1/2}}{e^{y_4}x_2 - e^{y_2}x_1} \delta_{1/2} \left(\frac{e^{y_1-y_3}x_0}{x_2} \right) - \frac{e^{(y_3+y_2)/2}x_2^{1/2}x_1^{1/2}}{e^{y_3}x_2 - e^{y_2}x_1} \delta_{1/2} \left(\frac{e^{y_1-y_4}x_0}{x_2} \right) \\
&+ \frac{e^{(y_1+y_4)/2}x_0^{1/2}x_2^{1/2}}{e^{y_1}x_0 - e^{y_4}x_2} \delta_{1/2} \left(\frac{e^{y_2-y_3}x_1}{x_2} \right) - \frac{e^{(y_1+y_3)/2}x_0^{1/2}x_2^{1/2}}{e^{y_1}x_0 - e^{y_3}x_2} \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right)
\end{aligned}$$

$$\begin{aligned}
 &= \lim_{x_0 \rightarrow x_1} \frac{e^{(y_1 - y_3 + y_2 - y_4)/2} x_0^{1/2} x_1^{1/2} x_2^{-1/2}}{(1 - e^{y_1 - y_3} \frac{x_0}{x_2})(1 - e^{y_2 - y_3} \frac{x_1}{x_2})} + \frac{e^{(y_3 - y_1 + y_2 - y_4)/2} x_1^{1/2} x_0^{-1/2}}{(1 - e^{y_3 - y_1} \frac{x_2}{x_0})(1 - e^{y_2 - y_4} \frac{x_1}{x_2})} \\
 &\quad - \frac{e^{(y_1 + y_2 - y_3 - y_4)/2} x_0^{1/2} x_1^{1/2} x_2^{-1/2}}{(1 - e^{y_1 - y_4} \frac{x_0}{x_2})(1 - e^{y_2 - y_3} \frac{x_1}{x_2})} - \frac{e^{(y_4 - y_1 + y_2 - y_3)/2} x_1^{1/2} x_0^{-1/2}}{(1 - e^{y_4 - y_1} \frac{x_2}{x_0})(1 - e^{y_2 - y_3} \frac{x_1}{x_2})} \\
 &\quad + \frac{e^{(y_4 + y_2 - y_1 - y_3)/2} x_1^{1/2} x_0^{-1/2}}{(1 - e^{y_2 - y_3} \frac{x_1}{x_2})(1 - e^{y_4 - y_1} \frac{x_2}{x_0})} + \frac{e^{(y_3 - y_2 + y_4 - y_1)/2} x_2 x_0^{-1/2} x_1^{-1/2}}{(1 - e^{y_3 - y_2} \frac{x_2}{x_1})(1 - e^{y_4 - y_1} \frac{x_2}{x_0})} \\
 &\quad - \frac{e^{(y_2 + y_3 - y_1 - y_4)/2} x_1^{1/2} x_0^{-1/2}}{(1 - e^{y_2 - y_4} \frac{x_1}{x_2})(1 - e^{y_3 - y_1} \frac{x_2}{x_0})} - \frac{e^{(y_4 - y_2 + y_3 - y_1)/2} x_2 x_1^{-1/2} x_0^{-1/2}}{(1 - e^{y_4 - y_2} \frac{x_2}{x_1})(1 - e^{y_3 - y_1} \frac{x_2}{x_0})} \\
 &= \frac{e^{(y_1 - y_3 + y_2 - y_4)/2}}{e^{y_1 - y_3} - e^{y_2 - y_4}} \left(\frac{1}{1 - e^{y_1 - y_3} \frac{x_1}{x_2}} + \frac{1}{1 - e^{y_3 - y_1} \frac{x_1}{x_2}} \right) \\
 &\quad - \frac{e^{(y_1 - y_3 + y_2 - y_4)/2}}{e^{y_1 - y_3} - e^{y_2 - y_4}} \left(\frac{1}{1 - e^{y_2 - y_4} \frac{x_1}{x_2}} + \frac{1}{1 - e^{y_4 - y_2} \frac{x_1}{x_2}} \right) \\
 &\quad + \frac{e^{(y_1 - y_4 + y_2 - y_3)/2}}{e^{y_1 - y_4} - e^{y_2 - y_3}} \left(\frac{1}{1 - e^{y_1 - y_4} \frac{x_1}{x_2}} + \frac{1}{1 - e^{y_4 - y_1} \frac{x_1}{x_2}} \right) \\
 &\quad - \frac{e^{(y_1 - y_4 + y_2 - y_3)/2}}{e^{y_1 - y_4} - e^{y_2 - y_3}} \left(\frac{1}{1 - e^{y_2 - y_3} \frac{x_1}{x_2}} + \frac{1}{1 - e^{y_3 - y_2} \frac{x_1}{x_2}} \right) \\
 &= \frac{e^{(y_1 + y_4 - y_2 - y_3)/2}}{e^{y_1 + y_4 - y_2 - y_3} - 1} \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) - \frac{e^{(y_1 + y_4 - y_2 - y_3)/2}}{e^{y_1 + y_4 - y_2 - y_3} - 1} \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) \\
 &\quad + \frac{e^{(y_1 + y_3 - y_2 - y_4)/2}}{e^{y_1 + y_3 - y_2 - y_4} - 1} \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) - \frac{e^{(y_1 + y_3 - y_2 - y_4)/2}}{e^{y_1 + y_3 - y_2 - y_4} - 1} \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right), \tag{3.27}
 \end{aligned}$$

Then (3.27) can be written as

$$\begin{aligned}
 &\frac{e^{(y_1 + y_4 - y_2 - y_3)/2}}{e^{y_1 + y_4 - y_2 - y_3} - 1} \delta \left(\frac{e^{y_1 - y_3} x_1}{x_2} \right) + \frac{e^{(y_3 + y_2 - y_1 - y_4)/2}}{e^{-y_1 - y_4 + y_3 + y_2} - 1} \delta \left(\frac{e^{y_2 - y_4} x_1}{x_2} \right) \\
 &\quad + \frac{e^{(y_1 + y_3 - y_2 - y_4)/2}}{e^{y_1 + y_3 - y_2 - y_4} - 1} \delta \left(\frac{e^{y_1 - y_4} x_1}{x_2} \right) + \frac{e^{(y_2 + y_4 - y_1 - y_3)/2}}{e^{-y_1 - y_3 + y_2 + y_4} - 1} \delta \left(\frac{e^{y_2 - y_3} x_1}{x_2} \right), \tag{3.28}
 \end{aligned}$$

so we have the proof.

Proof of (c).

$$\begin{aligned}
& \lim_{x_0 \rightarrow x_1} [\tilde{X}(e^{y_1}x_0)X(e^{y_2}x_1)\tilde{X}(e^{y_3}x_2)X(e^{y_4}x_2)] \\
&= \lim_{x_0 \rightarrow x_1} \frac{\partial}{\partial y_2} \bullet \tilde{X}(e^{y_1}x_0)\tilde{X}(e^{y_3}x_2)\bullet \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right) \\
&\quad - \bullet X(e^{y_4}x_2)X(e^{y_2}x_1)\bullet \delta_{1/2} \left(\frac{e^{y_1-y_3}x_1}{x_2} \right) \\
&\quad + \frac{\partial}{\partial y_2} \left(- \frac{e^{(y_1+y_3)/2}x_0^{1/2}x_2^{3/2}e^{y_4}}{(e^{y_1}x_0 - e^{y_3}x_2)(e^{y_2}x_1 - e^{y_4}x_2)} - \frac{e^{(y_1+y_3)/2}x_0^{1/2}x_2^{3/2}e^{y_4}}{(e^{y_1}x_0 - e^{y_3}x_2)(e^{y_4}x_2 - e^{y_2}x_1)} \right. \\
&\quad \left. + \frac{e^{(y_1+y_3)/2}x_0^{1/2}x_2^{3/2}e^{y_4}}{(e^{y_4}x_2 - e^{y_2}x_1)(e^{y_1}x_0 - e^{y_3}x_2)} + \frac{e^{(y_1+y_3)/2}x_0^{1/2}x_2^{3/2}e^{y_4}}{(e^{y_4}x_2 - e^{y_2}x_1)(e^{y_3}x_2 - e^{y_1}x_0)} \right) \\
&= \bullet X(e^{y_2}x_1)X(e^{y_4+y_1-y_3}x_1)\bullet \delta_{1/2} \left(\frac{e^{y_1-y_3}x_1}{x_2} \right) \\
&\quad + \frac{\partial}{\partial y_2} \bullet \tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\bullet \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right) \\
&\quad + \frac{\partial}{\partial y_2} \left(\frac{e^{(y_1-y_3)/2}x_1^{-1/2}x_2^{1/2}}{e^{y_2-y_4} - e^{y_1-y_3}} \left(\frac{1}{1 - e^{y_2-y_4}x_1/x_2} - \frac{1}{1 - e^{y_1-y_3}x_1/x_2} \right) \right. \\
&\quad \left. - \frac{e^{(y_3-y_1)/2+y_4-y_2}x_2^{1/2}x_1^{-1/2}}{e^{y_3-y_1} - e^{y_4-y_2}} \left(\frac{1}{1 - e^{y_3-y_1}x_2/x_1} - \frac{1}{1 - e^{y_4-y_2}x_2/x_1} \right) \right) \\
&= \bullet X(e^{y_2}x_1)X(e^{y_4+y_1-y_3}x_1)\bullet \delta_{1/2} \left(\frac{e^{y_1-y_3}x_1}{x_2} \right) \\
&\quad + \frac{\partial}{\partial y_2} \bullet \tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\bullet \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right) \\
&\quad + \frac{\partial}{\partial y_2} \left(\frac{e^{(y_1-y_3+y_4-y_2)/2}}{1 - e^{-y_2-y_3+y_1+y_4}} \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right) \right. \\
&\quad \left. + \frac{1}{1 - e^{y_2+y_3-y_1-y_4}} \delta_{1/2} \left(\frac{e^{y_1-y_3}x_1}{x_2} \right) \right). \tag{3.29}
\end{aligned}$$

Proofs for (d) and (e) are trivial. \square

Remark 3.3. Even though it involves only formal calculus it is interesting that the proof of Theorem 3.2 is very subtle (cf. [28–30]). For instance, while proving (3.16) we encountered series of the form

$$\frac{1}{1 - e^{y_1 - y_4}(x_1/x_2)}, \tag{3.30}$$

which has to be expanded as a geometric series

$$\sum_{n \in \mathbb{N}} \frac{e^{n(y_1 - y_4)} x_1^n}{x_2^n}.$$

In the calculations of the central term we had to multiply such a term with a delta function, i.e.,

$$\frac{1}{1 - e^{y_1 - y_4}(x_1/x_2)} \delta\left(\frac{e^{y_2 - y_3} x_0}{x_2}\right),$$

and then take the limit $x_0 \mapsto x_1$. It is not hard to see that

$$\lim_{x_0 \rightarrow x_1} \frac{1}{1 - e^{y_1 + y_3 - y_2 - y_4}(x_1/x_2)} \delta\left(\frac{e^{y_1 - y_4} x_0}{x_1}\right), \tag{3.31}$$

does not exist. Fortunately, we are taking a limit of four terms of similar form as in (3.31), and then, after some cancellations, we indeed can take the limit $x_0 \mapsto x_1$. After some calculations we get (3.16).

The reader might think that there is a much shorter “proof” of (3.16). First substitute x_1 for x_0 in (3.31) and then apply the delta function substitution so that (3.31) surprisingly becomes

$$\frac{1}{1 - e^{y_1 + y_3 - y_2 - y_4}} \delta\left(\frac{e^{y_1 - y_4} x_0}{x_1}\right),$$

which is a correct central term on the right hand side of (3.16). Of course, these two steps are not rigorous and the “proof” does not work because

$1/(1 - e^{y_1 + y_3 - y_2 - y_4})$ does not make sense as a geometric power series in $e^{y_1 + y_3 - y_2 - y_4}$. This unrigorous procedure has the flavor of Euler’s heuristic interpretation of the formulas $\sum_{n \in \mathbb{N}} n^k$, $k > 0$ mentioned in the introduction.

By combining Theorem 3.2 and Lemma 2.1 we obtain immediately

Theorem 3.3. *The mapping*

$$\Psi : \mathcal{S}\mathcal{D}_{\text{NS}}^+ \rightarrow \text{End}(W),$$

defined in terms of generating functions

$$\begin{aligned} \mathcal{D}^{y_1, y_2}(x) &\mapsto \bullet X(e^{y_1 x}) X(e^{y_2 x}) \bullet \\ \tilde{\mathcal{D}}^{y_1, y_2}(x) &\mapsto \bullet \tilde{X}(e^{y_1 x}) \tilde{X}(e^{y_2 x}) \bullet \\ G^{y_1, y_2}(x) &\mapsto \tilde{X}(e^{y_1 x}) X(e^{y_2 x}) \end{aligned} \quad (3.32)$$

defines a projective representation of the Lie superalgebra $\mathcal{S}\mathcal{D}_{\text{NS}}^+$.

Now we want to make a connection between the quadratic operators considered throughout this section and the subspace $\mathcal{Q} \subset W$ of all *quadratic vectors* in W , i.e., the subspace spanned by the set

$$\begin{aligned} \{h(-i)h(-j)\mathbf{1}, h(-i)\varphi(-j+1/2)\mathbf{1}, \\ \varphi(-j+1/2)\varphi(-i+1/2)\mathbf{1}: i, j \in \mathbb{Z}_{>0}\}. \end{aligned} \quad (3.33)$$

Inside $\text{End}(W)$, the vector space of all Fourier coefficients of vectors from \mathcal{Q} , i.e. the vector space spanned by operators $u(n)$ where $u \in \mathcal{Q}$, $n \in \mathbb{Z}$, coincide with the image of the representation Ψ . To prove this fact first notice that [15]

$$Y(u(-n-1)v(-m-1)\mathbf{1}, x) = \bullet \left\{ \frac{1}{n!} \left(\frac{\partial}{\partial x} \right)^n Y(u, x) \right\} \left\{ \frac{1}{m!} \left(\frac{\partial}{\partial x} \right)^m Y(v, x) \right\} \bullet,$$

for every $m, n \geq 0$, and the formula

$$x^l [D]_k = x^{k+l} \left(\frac{\partial}{\partial x} \right)^k, \quad (3.34)$$

holds for every $k \geq 1$, $l \in \mathbb{Z}$. Then

$$\text{span}\{u(n): u \in \mathcal{Q} \ n \in \mathbb{Z}\}, \quad (3.35)$$

is a subspace of the image of Ψ . Now by inverting the expression (3.34), i.e. by expressing D^l as a linear combination of the operators $x^k (\partial/\partial x)^{l+k}$, $k \in \mathbb{N}$, we get the opposite inclusion.

In fact we do not need all quadratic operators in W to build the representation Ψ . To explain this more precisely, let us first make a small detour.

Definition 3.3. Let V be a vertex operator (super)algebra and U a $L(-1)$ -stable ($L(-1)U \subset U$) graded subspace. We say that a graded subspace $U' \subset U$ is a *field* subspace for U if:

- The Fourier coefficients of elements of U' and U span the same subspace inside $\text{End}(V)$,
- U' is a minimal graded subspace of U (with respect to the inclusion) that satisfy (a).

Lemma 3.1. Let $U \subset \bigoplus_{i \geq 1} V_i$ be as above and U' a graded subspace such that

$$U = U' \oplus \bigoplus_{i \geq 1} L(-1)^i U'. \tag{3.36}$$

Then U' is a field subspace. Moreover,

$$U/L(-1)U \cong U'.$$

Proof. Clearly every subspace U' which satisfies (3.36), also satisfies the property (a). Let us prove the property (b). Suppose on the contrary that there is a proper subspace $U'' \subset U'$ which satisfies (a). Then

$$U = \sum_{i \geq 0} L^i(-1)U''. \tag{3.37}$$

Claim. $L^i(-1)$ is injective on U .

It is enough to prove that $L(-1)$ is injective. Suppose that $L(-1)$ is not injective. Then there exists $u \in U$ such that $(d/dx)Y(u, x) = 0$. But this implies $u \in V_0$, contradiction.

Formula (3.37) and the claim imply that

$$\dim_q(U) \leq \sum_{i \geq 0} q^i \dim_q(U'') = \frac{1}{1-q} \dim_q(U''),$$

where the \leq relation, between two q -series with integer coefficients, means that all coefficients in the q -expansion satisfy the same relation. But (3.36) implies that

$$\dim_q(U) = \frac{1}{1-q} \dim_q(U').$$

Hence $\dim_q(U) \leq \dim_q(U')$, which is a contradiction since U' is a subspace of U . \square

Every vertex operator superalgebra is $\frac{1}{2} \mathbb{Z}$ -graded with finite-dimensional graded subspaces. If the grading is lower bounded then U' can be constructed inductively starting from the lowest weight subspace of U . Also note that U is not unique but the graded dimension

$$\dim_q U' := \text{tr}_{U'} q^{L(0)}$$

does not depend on the choice of U' because

$$\dim_q U' = (1 - q) \dim_q U.$$

Now we shall describe a field subspace for $\mathcal{Q} \subset W$, introduced in (3.33).

Proposition 3.2.

$$U' = \text{Span}\{h(-i)^2 \mathbf{1}, h(-i)\varphi(-1/2)\mathbf{1}, \varphi(-j - 1/2)\varphi(-j + 1/2)\mathbf{1}, \\ i \in \mathbb{N}, j \in \mathbb{N}\}$$

is a field subspace for \mathcal{Q} .

Proof. It is enough to show that

$$U'_1 = \text{Span}\{h(-i)^2 \mathbf{1}, i \in \mathbb{N}\},$$

$$U'_2 = \text{Span}\{\varphi(-k - 1/2)\varphi(-k + 1/2)\mathbf{1}, k \geq 1\}$$

and

$$U'_3 = \text{Span}\{h(-i)\varphi(-1/2)\mathbf{1}, i \in \mathbb{N}\}$$

are field subspaces for

$$\mathcal{Q}_1 = \text{Span}\{h(-i)h(-j)\mathbf{1}: i, j \in \mathbb{N}\},$$

$$\mathcal{Q}_2 = \text{Span}\{\varphi(-j + 1/2)\varphi(-i + 1/2)\mathbf{1}, i, j \in \mathbb{N}\}$$

and

$$\mathcal{Q}_3 = \text{Span}\{h(-i)\varphi(-j + 1/2)\mathbf{1}, i, j \in \mathbb{N}\},$$

respectively. By comparing the graded dimensions of U'_1 , U'_2 and U'_3 with \mathcal{Q}_1 , \mathcal{Q}_2 and \mathcal{Q}_3 respectively we see that it is enough to show, because of Lemma 3.1, that if $j \neq k$ $L^j(-1)U'_i \cap L^k(-1)U'_i = 0$, for $i = 1, 2, 3$. Suppose that for some n

$$\sum_{k=1}^m \alpha_k L(-1)^{n-2i_k} h(-i_k)^2 \mathbf{1} = 0, \quad (3.38)$$

for certain α_k 's (not necessary nonzero). Suppose that n is the minimal one. By using $[L(m), h(n)] = -nh(m+n)$, we have

$$L(1)h(-i)^2 \mathbf{1} = 2ih(-i+1)h(-i)\mathbf{1} = i \frac{L(-1)h(-i+1)^2}{(i-1)} \mathbf{1},$$

for $i \neq 1$. Therefore

$$\begin{aligned} & L(1) \sum_{k=1}^m \alpha_k L(-1)^{n-2i_k} h(-i_k)^2 \mathbf{1} \\ &= \sum_{k=1}^m \alpha_k [L(1), L(-1)^{n-2i_k}] h(-i_k)^2 \mathbf{1} + \sum_k \alpha_k L(-1)^{n-2i_k} L(1)h(-i_k)^2 \mathbf{1} \\ &= \sum_{k=1}^m \beta_k \alpha_k L(-1)^{n-1-2i_k} h(-i_k)^2 \mathbf{1} + \sum_{k=1}^m \gamma_k \alpha_k L(-1)^{n-2i_k+1} h(-i_k+1)^2 \mathbf{1}, \end{aligned}$$

where

$$\beta_k = 2 \sum_{s=2i_k}^{n-1} s \quad \text{and} \quad \gamma_k = \frac{i_k}{i_k - 1}.$$

Now we have a contradiction since n is the minimal integer such that (3.38) holds. For the fermions we obtain

$$L(1)\varphi(-i_k - 1/2)\varphi(-i_k + 1/2)\mathbf{1} = \frac{(i_k + 1/2)L(-1)\varphi(-i_k + 1/2)\varphi(-i_k + 3/2)\mathbf{1}}{i_k - 1/2},$$

for $i_k \neq 1/2$, by using

$$[L(m), \varphi(n + 1/2)] = -(n + 1/2)\varphi(n + m + 1/2).$$

Again suppose that n is the minimal positive integer such that

$$\sum_{k=1}^{m'} \alpha'_k L(-1)^{n-2i_k} \varphi(-i_k - 1/2)\varphi(-i_k + 1/2)\mathbf{1} = 0. \tag{3.39}$$

Then

$$\begin{aligned} L(1) \sum_{k=1}^{m'} \alpha'_k L(-1)^{n-2i_k} \varphi(-i_k + 1/2)\varphi(-i_k - 1/2)\mathbf{1} \\ = \sum_{k=1}^{m'} \alpha'_k [L(1), L(-1)^{n-2i_k}] \varphi(-i_k - 1/2)\varphi(-i_k + 1/2)\mathbf{1} \\ + \sum_{k=1}^{m'} \alpha'_k L(-1)^{n-2i_k} L(1)\psi(-i_k - 1/2)\varphi(-i_k + 1/2)\mathbf{1} \\ = \sum_{k=1}^{m'} \beta'_k \alpha'_k L(-1)^{n-1-2i_k} \varphi(-i_k - 1/2)\varphi(-i_k + 1/2)\mathbf{1} \\ + \sum_k \gamma'_k \alpha'_k L(-1)^{n+1-2i_k} \psi(-i_k + 1/2)\varphi(-i_k + 3/2)\mathbf{1}, \end{aligned} \tag{3.40}$$

where

$$\beta'_k = 2 \sum_{s=2i_k}^{n-1} s \quad \text{and} \quad \gamma'_k = \frac{i_k - 1/2}{i_k + 1/2}.$$

Therefore we have a contradiction. Finally, for \mathcal{Q}_3 the proof is essentially the same as for \mathcal{Q}_1 and \mathcal{Q}_2 by using the relation $L(1)\varphi(-n - 1/2)h(-1)\mathbf{1} = (n + 1/2)\varphi(-n + 1/2)h(-1)\mathbf{1}$. \square

3.5. ζ -function and a central extension of $\mathcal{S}\mathcal{D}_{\text{NS}}^+$

Let us fix a basis for $\mathcal{S}\mathcal{D}_{\text{NS}}^+$ (this basis is different from the one we encountered before, cf. Remark 2.1):

$$\begin{aligned} L_m^{(r)} &= \frac{(r!)^2}{2} \text{coeff}_{y_1^r y_2^s x^{-m}} D^{y_1, y_2}(x), \\ \mathcal{L}_m^{(s)} &= \frac{(s+1)!s!}{2} \text{coeff}_{y_1^{s+1} y_2^s x^{-m}} \bar{D}^{y_1, y_2}(x) \end{aligned}$$

and

$$G_n^{(r)} = r! \operatorname{coeff}_{y_1^r y_2^s x^{-n}} G^{y_1, y_2}(x),$$

where $r \in \mathbb{N}$, $s \in \mathbb{N}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z} + 1/2$.

According to Theorem 3.3 the corresponding operators (acting on W) are given by

$$L^{(r)}(m) = \frac{(r!)^2}{2} \operatorname{coeff}_{y_1^r y_2^s x^{-m}} \bullet X(e^{y_1 x}) X(e^{y_2 x}) \bullet,$$

$$\mathcal{L}^{(s)}(m) = \frac{(s+1)!s!}{2} \operatorname{coeff}_{y_1^{s+1} y_2^s x^{-m}} \bullet \tilde{X}(e^{y_1 x}) \tilde{X}(e^{y_2 x}) \bullet$$

and

$$G^{(r)}(n) = r! \operatorname{coeff}_{y_1^r x^{-n}} \tilde{X}(e^{y_1 x}) X(e^{y_2 x}),$$

where $r \in \mathbb{N}$, $s \in \mathbb{N}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z} + \frac{1}{2}$.

Let us introduce a new normal ordering $\overset{++}{+}$ as in [28–30]:

$$\overset{++}{+} X(e^{y_1 x_1}) X(e^{y_2 x_1}) \overset{++}{+} = \bullet X(e^{y_1 x_1}) X(e^{y_2 x_1}) \bullet - \frac{\partial}{\partial y_1} \frac{e^{y_1 - y_2}}{e^{y_1 - y_2} - 1}, \tag{3.41}$$

$$\overset{++}{+} \tilde{X}(e^{y_1 x_1}) \tilde{X}(e^{y_2 x_1}) \overset{++}{+} = \bullet \tilde{X}(e^{y_1 x_1}) \tilde{X}(e^{y_2 x_1}) \bullet + \frac{e^{(y_1 - y_2)/2}}{e^{y_1 - y_2} - 1} \tag{3.42}$$

and

$$\overset{++}{+} \tilde{X}(e^{y_1 x_1}) X(e^{y_2 x_1}) \overset{++}{+} = \bullet \tilde{X}(e^{y_1 x_1}) X(e^{y_2 x_1}) \bullet. \tag{3.43}$$

The formal expressions on the right hand of formulas (3.41) and (3.42) are ambiguous. We will have a preferable variable for the expansion in applications that follow.

If we rewrite formulas (3.15)–(3.17) by using the new normal ordering, and pick the expansions in (3.41) and (3.42) such that they match with the expansions in (3.2), formulas (3.15)–(3.17) reduce to very simple forms (see also [28]):

$$\begin{aligned} & [\overset{++}{+} X(e^{y_1 x_1}) X(e^{y_2 x_2}) \overset{++}{+}, \overset{++}{+} X(e^{y_3 x_2}) X(e^{y_4 x_2}) \overset{++}{+}] \\ &= \frac{\partial}{\partial y_1} \left(\overset{++}{+} X(e^{y_2 x_1}) X(e^{y_4 + y_1 - y_3 x_1}) \overset{++}{+} \delta \left(\frac{e^{y_1 - y_3 x_1}}{x_2} \right) \right. \\ & \quad \left. + \overset{++}{+} X(e^{y_2 x_1}) X(e^{y_3 + y_1 - y_4 x_1}) \overset{++}{+} \delta \left(\frac{e^{y_1 - y_4 x_1}}{x_2} \right) \right) \\ & \quad + \frac{\partial}{\partial y_2} \left(\overset{++}{+} X(e^{y_1 x_1}) X(e^{y_4 + y_2 - y_3 x_1}) \overset{++}{+} \delta \left(\frac{e^{y_2 - y_3 x_1}}{x_2} \right) \right. \\ & \quad \left. + \overset{++}{+} X(e^{y_1 x_1}) X(e^{y_3 + y_2 - y_4 x_1}) \overset{++}{+} \delta \left(\frac{e^{y_2 - y_4 x_1}}{x_2} \right) \right), \end{aligned} \tag{3.44}$$

$$\begin{aligned}
 & [\overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_2}x_1)\overset{\pm}{\pm}, \overset{\pm}{\pm}\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_4}x_2)\overset{\pm}{\pm}] \\
 &= \overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_4+y_2-y_3}x_1)\overset{\pm}{\pm}\delta\left(\frac{e^{y_2-y_3}x_1}{x_2}\right) \\
 &+ \overset{\pm}{\pm}\tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_3+y_1-y_4}x_1)\overset{\pm}{\pm}\delta\left(\frac{e^{y_1-y_4}x_1}{x_2}\right) \\
 &- \overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_1-y_4}x_1)\overset{\pm}{\pm}\delta\left(\frac{e^{y_2-y_4}x_1}{x_2}\right) \\
 &- \overset{\pm}{\pm}\tilde{X}(e^{y_2}x_1)\tilde{X}(e^{y_4+y_1-y_3}x_1)\overset{\pm}{\pm}\delta\left(\frac{e^{y_1-y_3}x_1}{x_2}\right)
 \end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
 & [\overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)X(e^{y_2}x_1)\overset{\pm}{\pm}, \overset{\pm}{\pm}\tilde{X}(e^{y_3}x_2)X(e^{y_4}x_2)\overset{\pm}{\pm}] \\
 &= \overset{\pm}{\pm}X(e^{y_2}x_1)X(e^{y_4+y_1-y_3}x_1)\overset{\pm}{\pm}\delta_{1/2}\left(\frac{e^{y_1-y_3}x_1}{x_2}\right) \\
 &+ \frac{\partial}{\partial y_2}\left(\overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\overset{\pm}{\pm}\delta_{1/2}\left(\frac{e^{y_2-y_4}x_1}{x_2}\right)\right).
 \end{aligned} \tag{3.46}$$

After extracting the appropriate (normalized) coefficients inside the new normal ordered products we obtain the operators $\bar{L}^{(r)}(m)$ and $\bar{\mathcal{L}}^{(r)}(m)$ given by

$$\begin{aligned}
 \bar{L}^{(r)}(m) &= L^{(r)}(m) + (-1)^r \frac{\zeta(-2r-1)}{2} \delta_{m,0}, \\
 \bar{\mathcal{L}}^{(r)}(m) &= \mathcal{L}^{(r)}(m) + (-1)^{r+1} \frac{\zeta(1-2r, 1/2)}{2} \delta_{m,0}.
 \end{aligned}$$

We used the classical formula

$$\frac{e^{y/2}}{e^y - 1} = \frac{1}{y} \left(\sum_{n \geq 0} \frac{\zeta(1-n, 1/2)y^n}{n!} \right), \tag{3.47}$$

where $\zeta(s, \frac{1}{2})$ is a Hurwitz's ζ -function.

The following formula has been proven in [4] (for another proof see [28,29]):

$$[\bar{L}^{(r)}(m), \bar{L}^{(s)}(-m)] = \sum_j a_j \bar{L}^{(r)}(m) + \frac{(r+s+1)^2}{2(2r+2s+3)!} m^{2r+2s+3}, \tag{3.48}$$

where $a_j \in \mathbb{Q}$ (the structure constants).

Now we derive a similar formula for the Lie algebra \mathcal{D}^- by extracting regular terms in the normal ordering.

$$\begin{aligned}
 \overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_2}x_1)\overset{\pm}{\pm}_{\text{reg}} &= \overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_2}x_1)\overset{\pm}{\pm} + \frac{1}{y_2 - y_1}. \\
 &= [\overset{\pm}{\pm}\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_2}x_1)\overset{\pm}{\pm}, \overset{\pm}{\pm}\tilde{X}(e^{y_3}x_2)\tilde{X}(e^{y_4}x_2)\overset{\pm}{\pm}]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{+}{+} \tilde{X}(e^{y_1} x_1) \tilde{X}(e^{y_4+y_2-y_3} x_1) \frac{+}{\text{reg}} \delta \left(\frac{e^{y_2-y_3} x_1}{x_2} \right) \\
 & + \frac{+}{+} \tilde{X}(e^{y_2} x_1) \tilde{X}(e^{y_3+y_1-y_4} x_1) \frac{+}{\text{reg}} \delta \left(\frac{e^{y_1-y_4} x_1}{x_2} \right) \\
 & - \frac{+}{+} \tilde{X}(e^{y_1} x_1) \tilde{X}(e^{y_3+y_2-y_4} x_1) \frac{+}{\text{reg}} \delta \left(\frac{e^{y_2-y_4} x_1}{x_2} \right) \\
 & - \frac{+}{+} \tilde{X}(e^{y_2} x_1) \tilde{X}(e^{y_4+y_1-y_3} x_1) \frac{+}{\text{reg}} \delta \left(\frac{e^{y_1-y_3} x_1}{x_2} \right) \\
 & + \frac{1}{-y_1 - y_3 + y_2 + y_4} \delta \left(\frac{e^{y_2} x_1}{e^{y_3} x_2} \right) + \frac{1}{y_1 + y_3 - y_2 - y_4} \delta \left(\frac{e^{y_1} x_1}{e^{y_4} x_2} \right) \\
 & - \frac{1}{-y_1 - y_4 + y_2 + y_3} \delta \left(\frac{e^{y_2} x_1}{e^{y_4} x_2} \right) - \frac{1}{y_1 + y_4 - y_2 - y_3} \delta \left(\frac{e^{y_1} x_1}{e^{y_3} x_2} \right), \tag{3.49}
 \end{aligned}$$

where we used the binomial expansion convention introduced earlier. Apparently the choice of the first variable, y_1 , in the expressions

$$\frac{1}{-y_1 - y_3 + y_2 + y_4} \quad \text{and} \quad \frac{1}{y_1 + y_3 - y_2 - y_4},$$

is not as crucial as the fact that both expressions have to be expanded in the positive powers of the *same* triple of variables (y_2 , y_3 and y_4). This also applies to

$$\frac{1}{-y_1 - y_4 + y_2 + y_3} \quad \text{and} \quad \frac{1}{y_1 + y_4 - y_2 - y_3}.$$

Now, let us calculate the central term in the commutator $[\tilde{\mathcal{L}}^{(r)}(m), \tilde{\mathcal{L}}^{(s)}(-m)]$. We are not interested in the explicit calculations of the structural constants; we leave this problem to the reader. By using (3.49) we have

$$\begin{aligned}
 & [\tilde{\mathcal{L}}^{(r)}(m), \tilde{\mathcal{L}}^{(s)}(-m)] \\
 & = \sum_j b_j \tilde{\mathcal{L}}^{(j)}(0) + \text{coeff}_{\frac{y_1^{r+1} y_2^r y_3^{s+1} y_4^s x_1^{-m} x_2^m}{(r+1)!(s+1)!r!s!}} \left(\frac{e^{(y_2-y_3)D} - e^{(y_1-y_4)D}}{(y_2 - y_3) - (y_1 - y_4)} \right. \\
 & \quad \left. - \frac{e^{(y_2-y_4)D} - e^{(y_1-y_3)D}}{(y_2 - y_4) - (y_1 - y_3)} \right) \delta \left(\frac{x_1}{x_2} \right) \\
 & = \sum_j b_j \tilde{\mathcal{L}}^{(j)}(0) - \frac{(r+s+1)!^2}{(2r+2s+3)!} \text{coeff}_{x_1^{-m} x_2^m} D^{2r+2s+3} \delta \left(\frac{x_1}{x_2} \right) \\
 & \quad + \frac{(r+s)!(r+s+2)!}{(2r+2s+3)!} \text{coeff}_{x_1^{-m} x_2^m} D^{2r+2s+3} \delta \left(\frac{x_1}{x_2} \right) \\
 & = \sum_j b_j \tilde{\mathcal{L}}^{(j)}(0) + \frac{(r+s+1)!^2 - (r+s)!(r+s+2)!}{(2r+2s+3)!} m^{2r+2s+3}, \tag{3.50}
 \end{aligned}$$

where $b_j \in \mathbb{Q}$ are the structural constants. Thus only the odd powers of m appear in (3.50).

For the odd generators ($\dagger\dagger = \bullet\bullet$) we have

$$\begin{aligned}
 & [\dagger\tilde{X}(e^{y_1}x_1)X(e^{y_2}x_1)\dagger, \dagger\tilde{X}(e^{y_3}x_2)X(e^{y_4}x_2)\dagger] \\
 &= \dagger X(e^{y_2}x_1)X(e^{y_4+y_1-y_3}x_1)\dagger_{\text{reg}} \delta_{1/2} \left(\frac{e^{y_1-y_3}x_1}{x_2} \right) \\
 &+ \frac{\partial}{\partial y_2} \left(\dagger\tilde{X}(e^{y_1}x_1)\tilde{X}(e^{y_3+y_2-y_4}x_1)\dagger_{\text{reg}} \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right) \right) \\
 &+ \frac{\partial}{\partial y_2} \left(\frac{1}{(-y_2-y_3+y_1+y_4)^2} \delta_{1/2} \left(\frac{e^{y_1-y_3}x_1}{x_2} \right) \right) \\
 &+ \frac{1}{y_2+y_3-y_1-y_4} \delta_{1/2} \left(\frac{e^{y_2-y_4}x_1}{x_2} \right). \tag{3.51}
 \end{aligned}$$

Again we are interested in the commutation relations. For $m \in \mathbb{Z} + \frac{1}{2}$ we have

$$\begin{aligned}
 & [G^{(r)}(m), G^{(s)}(-m)] \\
 &= \sum_j c_j \tilde{L}^{(j)}(0) + d_j \tilde{\mathcal{L}}^{(j)}(0) + \text{coeff}_{\frac{y_1^r y_3^s x_1^{-m-1/2} x_2^{m+1/2}}{r!s!}} \\
 &\quad \times \frac{\partial}{\partial y_2} \left(\frac{e^{(y_1-y_3)D} - e^{(y_2-y_4)D}}{(y_1-y_3) - (y_2-y_4)} \delta \left(\frac{x_1}{x_2} \right) \right) \\
 &= \sum_j c_j \tilde{L}^{(j)}(0) + d_j \tilde{\mathcal{L}}^{(j)}(0) + \text{coeff}_{x_1^{-m} x_2^m} \\
 &\quad \times \frac{(-1)^s (r+s)! D^{r+s+2}}{(r+s+2)!} \delta \left(\frac{x_1}{x_2} \right) \\
 &= \sum_j c_j \tilde{L}^{(j)}(0) + d_j \tilde{\mathcal{L}}^{(j)}(0) + \frac{(-1)^s}{(r+s+1)(r+s+2)} m^{r+s+2}. \tag{3.52}
 \end{aligned}$$

where $c_j, d_j \in \mathbb{Q}$ are structure constants. Therefore only the pure powers of m appear in the central term.

Remark 3.4. This is a generalization of the Neveu–Schwarz case explained in the introduction. Take $V = \mathcal{W}$. The total central charge is $\frac{3}{2}$. The Virasoro element is $\omega^1 + \omega^2$ where ω^1 is the “bosonic” and ω^2 is the “fermionic” Virasoro. In particular $L_0 = L_0^1 + L_0^2$. Now

$$[G_m, G_{-m}] = 2L_0 + 1/3(m^2 - 1/4)c.$$

In terms of the new generators

$$\bar{L}_0^1 = L_0^1 + \frac{1}{2} \zeta(-1) = L_0^1 - \frac{1}{24}$$

and

$$\bar{L}_0^2 = L_0^2 + \frac{1}{2} \zeta(1/2, -1) = L_0^2 - \frac{1}{48},$$

we have

$$[G_m, G_{-m}] = 2\bar{L}_0 + \frac{3m^2}{4}.$$

The previous construction gives us a central extension of $\mathcal{S}\mathcal{D}_{\text{NS}}^+$ isomorphic to the one defined via (2.28). Therefore we constructed a representation of $\hat{\mathcal{S}}\mathcal{D}_{\text{NS}}^+$ with central charge $c = \frac{3}{2}$.

Remark 3.5. As observed in [4], in the formula (3.48) the rational number $(r+s+1)!^2/(2r+2s+3)!$ is a reciprocal of an integer. Notice that the same hold for the corresponding rational numbers in (3.51) and (3.52).

4. New normal ordering and Jacobi identity

4.1. Change of variables in vertex operator superalgebras

Let $(V, Y(x), \mathbf{1}, \omega)$ be a vertex operator algebra. As in [39] for every $u \in V$ we define

$$Y[u, x] := Y(e^{xL(0)}u, e^x - 1) \in \text{End}(V)[[x, x^{-1}]],$$

where we use the previously adopted expansion conventions (cf. Section 3.1).

Let us recall (cf. [13]) that a linear map $f: V_1 \rightarrow V_2$ is a vertex operator algebra isomorphism, where $(V_1, Y_1(x), \mathbf{1}_1, \omega_1)$ and $(V_2, Y_2(x), \mathbf{1}_2, \omega_2)$ are vertex operator algebras, if

$$f(Y_1(a, x)b) = Y_2(f(a), x)f(b),$$

$$f(\omega_1) = \omega_2,$$

$$f(\mathbf{1}_1) = \mathbf{1}_2.$$

The same definition applies for the vertex operator superalgebras. If in addition, the mapping f satisfies

$$f(\tau_1) = \tau_2,$$

then we say that f is an $N = 1$ isomorphism.

It is known (cf. [40, 18, 27]) that $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega})$ is a vertex operator superalgebra isomorphic to $(V, Y(\cdot, x), \mathbf{1}, \omega)$, where $\tilde{\omega} = \omega - (c/24)\mathbf{1}$. It requires more work to show (cf. [18]).

Proposition 4.1. $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega})$ and $(V, Y[(-1)^{p(\cdot)}, \cdot], \mathbf{1}, \tilde{\omega})$ are vertex operator superalgebras isomorphic to $(V, Y(\cdot, x), \omega)$. Moreover, if V is an $N = 1$ vertex operator superalgebra then $(V, Y[(-1)^{p(\cdot)}, \cdot], \mathbf{1}, \tilde{\omega}, -\tau)$ and $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega}, \tau)$ are $N = 1$ vertex operator superalgebras isomorphic to $(V, Y(\cdot, x), \mathbf{1}, \omega, \tau)$.

Proof. First we show that $(V, Y[\cdot, x], \mathbf{1}, \omega)$ is a vertex operator superalgebra isomorphic to $(V, Y(\cdot, x), \mathbf{1}, \tilde{\omega})$. In the case of vertex operator algebras the equivalent result has been proven in [39], under certain conditions, and in [18] unconditionally, where it follows as a corollary of a much stronger theorem. For vertex operator superalgebras the proof is the same. As in [18] (Example 7.4.5) we consider

$$\begin{aligned} \varphi_{e^x-1}: V &\rightarrow V, \\ \varphi_{e^x-1}u &= \exp\left(\sum_{i \geq 1} -a_i L_i\right)u, \end{aligned}$$

where a_i 's are (uniquely) determined with

$$\exp\left(\sum_{i \geq 0} a_i x^{i+2} \frac{d}{dx}\right)x = \log(1+x).$$

Then the same argument as in [18] (cf. formula (7.4.3)) implies that we have a vertex operator superalgebra isomorphism. But in contrast to the vertex operator algebra case, because of the $\frac{1}{2}\mathbb{Z}$ -grading, it is natural to consider

$$\varphi'_{e^x-1}u = \exp\left(\sum_{i \geq 1} -a_i L_i\right)(-1)^{p(u)}u.$$

This reflects multivaluedness of the map φ_f , where f is an arbitrary conformal transformation (see [18, Chapter 7]).

Since $\tilde{\omega} = \varphi_{e^x-1}\omega = \omega - \frac{c}{24}$ and

$$\tilde{\tau} := \varphi_{e^x-1}\tau = \tau,$$

we have to show that τ is the Neveu–Schwarz vector in the new vertex operator superalgebra $(V, Y[\cdot, x], \tilde{\omega})$. Since $L[i] = L(i) + \sum_{j>i} a_j L(j)$ and $G[i + 1/2] = G(i + 1/2) + \sum_{j>i} b_j G(j + 1/2)$ for some $a_j, b_j \in \mathbb{Q}$, it is enough to show that

$$\begin{aligned} L[0]\tau &= \frac{3}{2}\tau, & L[1]\tau &= 0, \\ G[-1/2]\tau &= 2\tilde{\omega}, & G[1/2]\tau &= 0, & G[3/2] &= \frac{2}{3}c\mathbf{1}. \end{aligned} \tag{4.1}$$

A short calculation gives us

$$\begin{aligned} G[-1/2] &= G(-1/2) + 1/2G(1/2) - 1/8G(3/2) + \dots \\ G[1/2] &= G(1/2) + 0 \cdot G(3/2) + \dots \end{aligned}$$

which implies (4.1). Therefore $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega}, \tau)$ is an $N = 1$ vertex operator superalgebra. Clearly, $(V, Y[(-1)^{p(\cdot)}, \cdot], \mathbf{1}, \tilde{\omega}, -\tau)$ is an $N = 1$ vertex operator superalgebra as well. \square

Remark 4.1. As we already mentioned, Proposition 4.1 deals with a particular change of variables (according to [18, Chapter 7]) stemming from the conformal transformation $x \mapsto e^x - 1$. It is possible to generalize this result for the more general conformal transformation. Since the right framework for studying the geometry of $N = 1$ vertex operator superalgebras is by means of the superconformal transformations developed in [2], we do not continue into this direction.

4.2. Jacobi identity and commutator formula

In this part we show how to obtain all the previous results by using some general properties of the vertex operator superalgebras.

It was noticed in [29] that the Jacobi identity in terms of X -operators carries some interesting features. A commutator formula for X -operators was implicitly used in Zhu's thesis as well (cf. [39]). In the case of vertex operator superalgebras we have the following result (see [28–30] for vertex operator algebra setting). $\delta_r(x) = x^r \delta(x)$, for $r \in \mathbb{Q}$. Let us recall $\varepsilon_{u,v} = (-1)^{p(u)p(v)}$.

Proposition 4.2.

$$\begin{aligned} & y_0^{-1} \delta \left(\frac{e^{y_2,1} x_1}{x_0} \right) X(u, x_1) X(v, x_2) - \varepsilon_{u,v} x_0^{-1} \delta \left(\frac{-e^{y_1,2} x_2}{x_0} \right) X(v, x_2) X(u, x_1) \\ &= y_1^{-1} \delta_{\text{rel}(u)} \left(\frac{e^{y_0,1} x_2}{x_1} \right) X(Y[u, y]v, x_2), \end{aligned} \quad (4.2)$$

where

$$y_{2,1} = \log \left(1 - \frac{x_2}{x_1} \right), \quad y_{1,2} = \log \left(1 - \frac{x_1}{x_2} \right), \quad y = \log \left(1 + \frac{x_0}{x_2} \right),$$

and $\text{rel}(u)$ is defined (for homogeneous vectors) as $\deg(u) - [\deg(u)] \in \{0, \frac{1}{2}\}$.²

Moreover, the following commutator formula holds:

$$\begin{aligned} & [X(Y[u_1, x_1]v_1, y_1), X(Y[u_2, x_2]v_2, y_2)] \\ &= \text{Res}_y \delta_{\text{rel}(u_1) + \text{rel}(v_1)} \left(\frac{e^{x_0,1} y_2}{y_1} \right) X(Y[Y[u_1, x_1]v_1, y]Y[u_2, x_2]v_2, y_2). \end{aligned} \quad (4.3)$$

² Here we do not assume that condition (3.3) holds.

Proof. We start from the ordinary Jacobi identity, which involves Y -operators.

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(v_1, x_1) Y(v_2, x_2) - \varepsilon_{u,v} x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) Y(v_2, x_2) Y(v_1, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(v_1, x_0)v_2, y_2). \end{aligned} \tag{4.4}$$

Hence

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) X(u, x_1) X(v, x_2) \\ & - \varepsilon_{u,v} x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) X(v, x_2) X(u, x_1) \\ &= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(x_1^{L(0)}u, x_1) Y(x_2^{L(0)}v, x_2) \\ & - \varepsilon_{u,v} x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) Y(x_2^{L(0)}v, x_2) Y(x_1^{L(0)}u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(x_1^{L(0)}u, x_0)x_2^{L(0)}v, x_2) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) X \left(Y \left(\left(\frac{x_1}{x_2} \right)^{L(0)} u, \frac{x_0}{x_2} \right) v, x_2 \right) \\ &= x_1^{-1} \delta \left(\frac{e^y x_2}{x_1} \right) X \left(Y \left(\left(\frac{x_1}{x_2} \right)^{L(0)} u, \frac{x_0}{x_2} \right) v, x_2 \right) \\ &= x_1^{-1} \delta \left(\frac{e^y x_2}{x_1} \right) X \left(\left(\frac{x_1}{x_2} \right)^{\text{rel}(u)} Y \left(\left(\frac{x_1}{x_2} \right)^{[L(0)]} u, \frac{x_0}{x_2} \right) v, x_2 \right) \\ &= x_1^{-1} \delta \left(\frac{e^y x_2}{x_1} \right) X \left(\left(\frac{x_1}{x_2} \right)^{\text{rel}(u)} Y(e^{y[L(0)]}u, e^y - 1)v, x_2 \right) \\ &= x_1^{-1} \delta \left(\frac{e^y x_2}{x_1} \right) \left(\frac{x_1}{e^y x_2} \right)^{\text{rel}(u)} X(Y[u, y]v, x_2) \\ &= x_1^{-1} \delta_{\text{rel}(u)} \delta \left(\frac{e^y x_2}{x_1} \right) X(Y[u, y]v, x_2), \end{aligned} \tag{4.5}$$

where $y = \log(1 + (x_0/x_2))$ and $\text{rel}(u) = \deg(u) - [\deg(u)]$. At this point, variables $x_{1,2}$ and $x_{2,1}$ are here for cosmetic purpose.

If we take Res_{x_0} of the left hand side of (4.2) we obtain

$$\begin{aligned} [X(u, x_1), X(v, x_2)] &= \text{Res}_{x_0} x_1^{-1} \delta_{\text{rel}(u)} \delta \left(\frac{e^y x_2}{x_1} \right) X(Y[u, y]v, x_2) \\ &= \text{Res}_{x_0} x_2 e^{y-1} \delta_{\text{rel}(u)} \delta \left(\frac{e^y x_2}{x_1} \right) X(Y[u, y]v, x_2) \\ &= \text{Res}_y \delta_{\text{rel}(u)} \delta \left(\frac{e^y x_2}{x_1} \right) X(Y[u, y]v, x_2), \end{aligned} \quad (4.6)$$

where in the last line y is a formal variable and *not* a power series as the above. If we take $u = Y[u_1, y_1]v_1$ and $v = Y[u_2, y_2]v_2$ we obtain formula (4.3). \square

Unfortunately, formula (4.3) is not good for computational purposes—meaning that taking residue of the right hand side of (4.3) might be cumbersome even though the commutators $[Y(u_i, x), Y(v_j, x)]$ might be simple. The following formula (see [29,30], for vertex operator algebra setting) fixes the problem and can be used to calculate expression in a much simpler way.

Theorem 4.1.

$$\begin{aligned} &[X(Y[u_1, x_1]v_1, y_1), X(Y[u_2, x_2]v_1, y_2)] \\ &= \varepsilon_{u_1+v_1, u_2} \text{Res}_y e^{y(\partial/\partial y_1)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) \right. \\ &\quad \times X(Y[u_2, y_2]Y[u_1, y_1]Y[v_1, y]v_2, x_2)) \\ &\quad - \varepsilon_{u_1+v_1, u_2} \varepsilon_{u_1, v_1} \text{Res}_x e^{x(\partial/\partial y_1)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{-y_1} x_2}{x_1} \right) \right. \\ &\quad \times X(Y[u_2, y_2]Y[v_1, -y_1]Y[u_1, x]v_2, x_2)) \\ &\quad + \varepsilon_{u_1+v_1, u_2} \varepsilon_{u_1, u_2} \text{Res}_x e^{x(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) \right. \\ &\quad \times X(Y[Y[u_1, y_1]Y[u_2, -x]v_1, y_2]v_2, x_2)) \\ &\quad - \varepsilon_{u_1+v_1, u_2} \text{Res}_z e^{(-y+z)(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) \right. \\ &\quad \times X(Y[Y[Y[u_2, z]u_1, y_1]v_1, y_2]v_2, x_2)). \end{aligned} \quad (4.7)$$

Proof. Let us recall (cf. Proposition 4.1) that $(V, Y[\], \mathbf{1}, \tilde{\omega})$ has a vertex operator superalgebra structure. Notice that in (4.3) variable y , for which we take a residuum, is on the “wrong” slot—meaning that taking the residuum yields lots of terms that are hard to compute in the closed form. To “shuffle” the terms we apply first the Jacobi

identity for the expression inside the X -operator on the right hand side of (4.3).

$$\begin{aligned}
 & [X(Y[u_1, y_1]v_1, x_1), X(Y[u_2, y_2]v_1, x_2)] \\
 &= \text{Res}_y \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) \\
 & \quad \times X(Y[Y[u_1, y_1]v_1, y]Y[u_2, y_2]v_2, x_2) \\
 &= \text{Res}_y \text{Res}_x \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) y_2^{-1} \delta \left(\frac{y-x}{y_2} \right) X(Y[Y[Y[u_1, y_1]v_1, x]u_2, y]v_2, x_2) \\
 & \quad + \varepsilon_{u_1+v_1, u_2} \text{Res}_y \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) X(Y[u_2, y_2]Y[Y[u_1, y_1]v_1, y]v_2, x_2).
 \end{aligned} \tag{4.8}$$

Let us first work out the second term in (4.8). By the associator formula (cf. [15])

$$\begin{aligned}
 & \varepsilon_{u_1+v_1, u_2} \text{Res}_y \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) X(Y[u_2, y_2]Y[Y[u_1, y_1]v_1, y]v_2, x_2) \\
 &= \varepsilon_{u_1+v_1, u_2} \text{Res}_y \text{Res}_x \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) X(Y[u_2, y_2] \\
 & \quad \times \left(y_1^{-1} \delta \left(\frac{x-y}{y_1} \right) Y[u_1, x]Y[v_1, y] - \varepsilon_{u_1, v_1} y_1^{-1} \delta \right. \\
 & \quad \times \left. \left(\frac{-y+x}{y_1} \right) Y[v_1, y]Y[u_1, x] \right) u_2, x_2) \\
 &= \varepsilon_{u_1+v_1, u_2} \text{Res}_y e^{y(\partial/\partial y_1)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) \right. \\
 & \quad \times X(Y[u_2, y_2]Y[u_1, y_1]Y[v_1, y]v_2, x_2)) \\
 & \quad - \varepsilon_{u_1+v_1, u_2} \varepsilon_{u_1, v_1} \text{Res}_x \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{-y_1-x} x_2}{x_1} \right) \right. \\
 & \quad \times X(Y[u_2, y_2]Y[v_1, -y_1-x]Y[u_1, x]v_2, x_2)) \\
 &= \varepsilon_{u_1+v_1, u_2} \text{Res}_y e^{y(\partial/\partial y_1)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \right. \\
 & \quad \times \left. \left(\frac{e^y x_2}{x_1} \right) X(Y[u_2, y_2]Y[u_1, y_1]Y[v_1, y]v_2, x_2) \right) \\
 & \quad - \varepsilon_{u_1+v_1, u_2} \varepsilon_{u_1, v_1} \text{Res}_x e^{x(\partial/\partial y_1)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{-y_1} x_2}{x_1} \right) \right. \\
 & \quad \times X(Y[u_2, y_2]Y[v_1, -y_1]Y[u_1, x]v_2, x_2)).
 \end{aligned} \tag{4.9}$$

Now we work out the first term in (4.3). We apply the skew-symmetry formula (cf. [13]) for the expression

$$Y[Y[u_1, y_1]v_1, x]u_2$$

which gives us

$$\begin{aligned} & \text{Res}_y \text{Res}_x \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) y_2^{-1} \delta \left(\frac{y-x}{y_2} \right) X(Y[Y[Y[u_1, y_1]v_1, x]u_2, y]v_2, x_2) \\ &= \varepsilon_{u_1+v_1, u_2} \text{Res}_y \text{Res}_x \delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^y x_2}{x_1} \right) y_2^{-1} \delta \left(\frac{y-x}{y_2} \right) \\ & \quad \times X(Y[e^{xL}[-1]Y[u_2, -x]Y[u_1, y_1]v_1, y]v_2, x_2) \\ &= \varepsilon_{u_1+v_1, u_2} \text{Res}_x e^{x(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) \right. \\ & \quad \times X(Y[Y[u_2, -x]Y[u_1, y_1]v_1, y_2]v_2, x_2)) \\ &= \varepsilon_{u_1+v_1, u_2} \varepsilon_{u_1, u_2} \text{Res}_x e^{x(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) \right. \\ & \quad \times X(Y[Y[u_1, y_1]Y[u_2, -x]v_1, y_2]v_2, x_2)) \\ & \quad + \varepsilon_{u_1+v_1, u_2} \text{Res}_x \text{Res}_z e^{x(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) y_1^{-1} \delta \left(\frac{-x-z}{y_1} \right) \right. \\ & \quad \times X(Y[Y[Y[u_2, z]u_1, y_1]v_1, y_2]v_2, x_2)) \\ &= \varepsilon_{u_1+v_1, u_2} \varepsilon_{u_1, u_2} \text{Res}_x e^{x(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) \right. \\ & \quad \times X(Y[Y[u_1, y_1]Y[u_2, -x]v_1, y_2]v_2, x_2)) \\ & \quad - \varepsilon_{u_1+v_1, u_2} \text{Res}_z e^{(-y+z)(\partial/\partial y_2)} \left(\delta_{\text{rel}(u_1)+\text{rel}(v_1)} \left(\frac{e^{y_2} x_2}{x_1} \right) \right. \\ & \quad \times X(Y[Y[Y[u_2, z]u_1, y_1]v_1, y_2]v_2, x_2)) . \end{aligned} \tag{4.10}$$

By combining (4.8), (4.9) and (4.10) we obtain (4.7). \square

4.3. Matching quadratics and iterates

In the previous section we derived a commutator formula for operators of the form $X(Y[u, y]v, x)$. Now, we make a precise link between these iterates and quadratics we encountered in the construction of $\hat{\mathcal{S}}_{\text{NS}}^+$ in terms of boson (see also [28–30]) and fermions.

Proposition 4.3.

$$X(Y[u, x]v, y) = \bullet X(u, e^x y)X(v, y) \bullet + X(e^{xwt(u)} Y^+(u, e^x - 1)v, y), \tag{4.11}$$

where $Y^+(u, x) = \sum_{n \geq 0} u(n)x^{-n-1}$.

Proof.

$$\begin{aligned} X(Y[u, x]v, y) &= Y(y^{L(0)} Y[u, x]v, y) \\ &= e^{xwt(u)} y^{wt(u)+wt(v)} Y(Y(u, y(e^x - 1))v, y) \\ &= e^{xwt(u)} y^{wt(u)+wt(v)} Y(Y^+(u, y(e^x - 1))v, y) \\ &\quad + e^{xwt(u)} y^{wt(u)+wt(v)} Y(Y^-(u, y(e^x - 1))v, y) \\ &= X(e^{xwt(u)} Y^+(u, e^x - 1)v, y) + e^{xwt(u)} y^{wt(u)+wt(v)} Y(Y^-(u, y(e^x - 1))v, y). \end{aligned}$$

Now

$$\begin{aligned} &e^{xwt(u)} y^{wt(u)+wt(v)} Y(Y^-(u, y(e^x - 1))v, y) \\ &= \lim_{y_1 \rightarrow y} e^{xwt(u)} y^{wt(u)+wt(v)} \\ &\quad \times Y(1 + y_1((e^x - 1)L(-1)u)(-1) + y_1^2 \left((e^x - 1)^2 \frac{L(-1)u}{2!}(-1) + \dots v, y \right) \\ &= \lim_{y_1 \rightarrow y} e^{xwt(u)} y^{wt(u)+wt(v)} Y((e^{L(-1)y_1(e^x - 1)}u)(-1)v, y) \\ &= \lim_{y_1 \rightarrow y} e^{xwt(u)} y^{wt(u)+wt(v)} \bullet Y(e^{(e^x - 1)y_1 L(-1)}u, y)Y(v, y) \bullet \\ &= \lim_{y_1 \rightarrow y} e^{xwt(u)} y^{wt(u)+wt(v)} \bullet Y(u, y + y_1(e^x - 1))Y(v, y) \bullet \\ &= e^{xwt(u)} y^{wt(u)+wt(v)} \bullet Y(u, ye^x)Y(v, y) \bullet = \bullet X(u, e^x y)X(v, y) \bullet. \quad \square \end{aligned}$$

Definition 4.1. We say that two homogeneous vectors u and v form a *free pair* if for every $n \geq 0$

$$u(n)v = c_{u,v} \delta_{wt(u)+wt(v)-1, n} \mathbf{1},$$

where $Y(u, v) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1}$, and $c_{u,v} \in \mathbb{C}$. Note that, because of the skew-symmetry (cf. [13]), we do not specify the order.

Corollary 4.1. Suppose that (u, v) is a free pair. Then

$$X(Y[u, x]v, y) = \bullet X(u, e^x y)X(v, y) \bullet + \frac{e^{xwt(u)}}{(e^x - 1)^{wt(u)+wt(v)}}.$$

Proof.

$$\begin{aligned} & y^{\text{wt}(u)+\text{wt}(v)} e^{x\text{wt}(u)} X(Y^+(u, y(e^x - 1))v, y) \\ &= y^{\text{wt}(u)+\text{wt}(v)} e^{x\text{wt}(u)} \frac{1}{y^{\text{wt}(u)+\text{wt}(v)}(e^x - 1)^{\text{wt}(u)+\text{wt}(v)}} = \frac{e^{x\text{wt}(u)}}{(e^x - 1)^{\text{wt}(u)+\text{wt}(v)}}. \quad \square \end{aligned}$$

Corollary 4.1 can be formulated slightly more generally:

$$\begin{aligned} & X(Y[u, x_1 - x_2]v, e^{x_2} y) \\ &= \bullet X(u, e^{x_1} y) X(v, e^{x_2} y) \bullet + \frac{e^{(x_1 - x_2)\text{wt}(u)}}{(e^{x_1 - x_2} - 1)^{\text{wt}(u)+\text{wt}(v)}}. \end{aligned} \quad (4.12)$$

Let us define a new normal ordering

$$\dagger X(u, e^{x_1} y) X(v, e^{x_2} y) \dagger := X(Y[u, x_1 - x_2]v, e^{x_2} y). \quad (4.13)$$

In the case $V = M(1) \otimes F$ and

$$u, v \in \text{Ch}(-1)\mathbf{1} \oplus \mathbb{C}\varphi(-1/2)\mathbf{1},$$

u and v form a free pair. By applying (4.12) we see that (4.13) becomes a tautology with $\dagger\dagger$ defined in Section 3.1 for the special quadratic operators.

Therefore formula (4.8) gives us an effective way of calculating commutators of the form

$$[\dagger X(u_1, e^{x_1} y_1) X(v_1, e^{x_2} y_2) \dagger, \dagger X(u_2, e^{x_3} y_2) X(v_2, e^{x_4} y_2) \dagger],$$

where $u_1, u_2, v_1, v_2 \in \text{Ch}(-1)\mathbf{1} \oplus \mathbb{C}\psi(-1/2)\mathbf{1}$.

Remark 4.2. There has been extensive study of the centrally extended Lie algebra of differential operators, i.e., \mathcal{W}_∞ -algebras, and its subalgebras (see [37,3,20,1] and references therein). In all these approaches the authors obtain many interesting properties of these algebras and corresponding representations. We should stress that their approach resembles classical representation theory. Our approach (also in [28–30]) is different. We use the associativity and the change of variables in vertex operator algebras theory which does not come up from the classical point of view. By doing this the number theoretic counterpart (which is invisible from the classical representation point of view) becomes very natural.

5. Quasi-modularity of generalized characters

Suppose that \mathcal{L} is a Lie algebra which contains an infinite-dimensional abelian subalgebra \mathcal{L}_0 , spanned by L_k , $k = 1, 2, \dots$. Suppose that a representation M of \mathcal{L} is \mathcal{L}_0 -diagonalizable with the finite-dimensional simultaneous eigenspaces. Then we form a formal *generalized character*

$$M(q_1, q_2, \dots) := \text{tr}_M q_1^{L_1} q_2^{L_2} \dots, \quad (5.1)$$

In addition, if M is a projective representation then we may choose a particular lifting for the operators L_i . Let us assume that $q_i^m = e^{2\pi im\tau_i}$, $\tau_i \in \mathbb{H}$ and $M(q_1, q_2, \dots)$ can be expanded in the following way

$$M(q_1, q_2, \dots) = \sum_{(i_2, \dots, i_k)} m_{i_2, \dots, i_k}(\tau_1) \frac{\tau_2^{i_2} \cdots \tau_k^{i_k}}{(i_2 + \cdots + i_k)!}, \tag{5.2}$$

where the coefficients $m_{i_2, \dots, i_k}(\tau_1)$ are analytic functions in a certain domain. It is more convenient to work with multi-indices $I = (i_2, \dots, i_k)$. We will use $|I| = i_2 + \cdots + i_k$. We are interested in the modular properties of $M(q_1, q_2, \dots)$ with respect to some fixed arithmetic subgroup $\Gamma \subset SL(2, \mathbb{Z})$.

We write $\mathcal{QM}(\Gamma)$ for the ring of all quasimodular forms (see [5]) with respect to Γ . Then we have a (graded) ring isomorphism.

$$\mathcal{QM}(\Gamma) \cong \mathcal{M}(\Gamma) \otimes \mathbb{C}[G_2]. \tag{5.3}$$

where $\mathcal{M}(\Gamma)$ is the ring of modular forms with respect to Γ . It is not hard to see that $\mathcal{QM}(\Gamma)$ is stable with respect to $\partial/\partial\tau$. We denote by $\mathcal{QM}_k(\Gamma)$ the graded component of $\mathcal{QM}(\Gamma)$, i.e., the space of quasimodular forms of the weight k . We can also define the notion of quasi-modularity for an arbitrary half-integer weight k (see [24]).

The following definition is from [5].

Definition 5.1. The series (5.2) is a quasimodular form of the weight n if

$$m_I(\tau_1) \in \mathcal{QM}_{n+\text{wt}(I)}(\Gamma),$$

where $\text{wt}(I) = 3a_3 + 5a_5 + \cdots$ (a_i is the multiplicity of i in I).

Let us recall that (normalized) classical Eisenstein series has q -expansion given by

$$G_k(q) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n,$$

for $k = 2, 4, 6, \dots$. It is well known that $G_k(q)$ is a modular form of the weight k for $k \geq 4$, and a *quasi-modular* form of the weight 2 for $k = 2$.

Now we consider a Lie algebra $L = \mathcal{D}^+ \oplus \mathcal{D}^-$ and a L -module $W = M \otimes F$ (cf. Section 3.1). We pick

$$L_{2i-1} = (-1)^r \bar{L}^{(i)}(0) + (-1)^{r+1} \tilde{\mathcal{L}}^{(i)}(0)$$

and

$$L_{2i} = 0.$$

Then

$$1. \quad M(\tau_1, \tau_3, \dots) = q_1^{\zeta(-1)/2} q_3^{\zeta(-3)/2} \cdots \prod_{n=1}^{\infty} (1 - q_1^n q_2^{n^3} \dots)^{-1}. \tag{5.4}$$

$$2. \quad F(\tau_1, \tau_3, \dots) = q_1^{-\zeta(-1, 1/2)/2} q_3^{-\zeta(-3, 1/2)/2} \dots \prod_{n=1}^{\infty} (1 + q_1^{(n-1/2)} q_2^{(n-1/2)^3} \dots). \quad (5.5)$$

$$3. \quad W(\tau_1, \tau_3, \dots) = M(\tau_1, \tau_3, \dots) F(\tau_1, \tau_3, \dots). \quad (5.6)$$

Because of (5.6) it remains to treat the generalized character $F(\tau_1, \tau_3, \dots)$. In [5] the authors proved that $M(\tau_1, \tau_3, \dots)$ is a quasimodular form of the weight $-1/2$. A generalized character for the vertex operator algebra stemming from the pair of charged fermions is also considered. In our case we treat a single free fermion. Again our approach is quite similar to the one in [5].

First notice that if we let $\tau_3 = \tau_5 = \dots = 0$ in (5.5) then we obtain a genuine character of F with respect to $\mathcal{L}(0)$:

$$m_0(q) = q^{-1/48} \prod_{n \geq 1} (1 + q^{(n-1/2)}). \quad (5.7)$$

Since

$$q^{-1/48} \prod_{n \geq 1} (1 - q^{(n-1/2)}) = \frac{q^{1/48} \prod_{n \geq 1} (1 - q^{n/2})}{q^{1/24} \prod_{n \geq 1} (1 - q^n)},$$

we see that (5.7) is equal to

$$\frac{\eta(q)^2}{\eta(q^2)\eta(q^{1/2})},$$

It is well-known that $\eta(\tau)$ is an automorphic form [17,25] of the weight $1/2$ for $SL(2, \mathbb{Z})$ (with respect to some multiplier χ). It is easy to check, by using a relation

$$\gamma \Gamma(2) \gamma^{-1} \in \Gamma(1),$$

where

$$\gamma = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix},$$

that $m(\tau)$ is an automorphic form of the weight zero for $\Gamma(2)$ [25].

Let us proceed with the calculations of the coefficients $m_{i_1, \dots, i_k}(\tau)$. First notice that, since $m_0(\tau)$ is of the weight zero, it is enough to show that

$$\log(F(\tau_1, \tau_2, \dots)),$$

is a quasimodular form of the weight zero which reduces the problem to calculating the expansion coefficients for $\log(F)$. We first consider the case $r = 1$ and the coefficient

$$\frac{1}{2\pi i} \frac{\partial}{\partial \tau_{j-1}} \log(F)|_{\tau_2 = \tau_3 = \dots = 0},$$

for each $j \in \mathbb{N}$. After some calculations

$$\begin{aligned}
 & \frac{1}{2\pi i} \frac{\partial}{\partial \tau_{j-1}} \log(F)|_{\tau_2=\tau_3=\dots=0} \\
 &= -\frac{\zeta(1-2j, \frac{1}{2})}{2} - \left\{ 2 \sum_{n,m=1}^{\infty} \left(\frac{2n-1}{2}\right)^{2j-1} q^{(2n-1)m} \right. \\
 & \quad \left. - \sum_{m,n=1}^{\infty} \left(\frac{2n-1}{2}\right)^{2j-1} q^{(2n-1)m/2} \right\} \\
 &= -\frac{\zeta(1-2j, \frac{1}{2})}{2} - \frac{1}{2^{2j-1}} \left(2 \sum_{l=0}^{\infty} \left(\sum_{d|l, d \text{ odd}} d^{2l-1} \right) q^l \right. \\
 & \quad \left. - \sum_{l=0}^{\infty} \left(\sum_{d|l, d \text{ odd}} d^{2l-1} \right) q^{l/2} \right). \tag{5.8}
 \end{aligned}$$

Let us recall the level two Eisenstein series (introduced in [5,24])

$$F_{2j}^{(2)}(q) := G_{2j}(q^{1/2}) - s^{2j-1} G_{2j}(q).$$

Then

$$\sum_{l=0}^{\infty} \left(\sum_{d|l, d \text{ odd}} d^{2l-1} \right) q^l = F_{2j}^{(2)}(q^2) - \frac{(1-2^{2j-1})}{2} \zeta(1-2j). \tag{5.9}$$

Since,

$$\zeta(1-2j, \frac{1}{2}) = (2^{1-2j} - 1) \zeta(1-2j),$$

from (5.8) and (5.9) we obtain

$$\frac{1}{2\pi i} \frac{\partial}{\partial \tau_{j-1}} \log(F)|_{\tau_2=\tau_3=\dots=0} = \frac{F_{2j}^{(2)}(q)}{2^{2j-1}} - \frac{F_{2j}^{(2)}(q^2)}{2^{2j-2}},$$

which is a quasi-modular form of the weight $2j$ for $\Gamma(4)$.

For $r > 1$

$$\begin{aligned}
 & \frac{1}{(2\pi i)^r} \frac{\partial^r}{\partial \tau_{2j_1-1} \dots \partial \tau_{2j_r-1}} \log(F)|_{\tau_2=\tau_3=\dots=0} \\
 &= -2^{-2(\sum j_k - r)} \sum_{m,n=1}^{\infty} (2n-1)^{2(\sum j_k - r)+1} (m(2n-1))^{r-1} q^{(2n-1)m} \\
 & \quad + \sum_{n,m=1}^{\infty} \left(\frac{2n-1}{2}\right)^{2(\sum j_k - r)+1} \left(\frac{(2n-1)m}{2}\right)^{r-1} q^{(2n-1)m/2}, \tag{5.10}
 \end{aligned}$$

can be expressed as a linear combination of

$$\left(\frac{\partial}{\partial\tau}\right)^{r-1} F_{2(j_k-r+1)}^{(2)}(\tau) \quad \text{and} \quad \left(\frac{\partial}{\partial\tau}\right)^{r-1} F_{2(j_k-r+1)}^{(2)}(2\tau),$$

which are modular forms (since $\partial/\partial\tau$ has degree 2) of the weight $\sum_{k=1}^r j_k$ for $\Gamma(4)$, i.e. an element of $\mathcal{L}\mathcal{M}_{\text{wt}(\Gamma)}(\Gamma(4))$. Thus $\log(F)$ is a quasi-modular form of the weight 0. Since $M(\tau_1, \tau_3, \dots)$ is a modular form of the weight $-1/2$ (see [5]) we obtain

Theorem 5.1. $W(\tau_1, \tau_3, \dots)$ is a quasi-modular form of the weight $-1/2$.

Remark 5.1. It is not very surprising that the vector space spanned with characters of irreducible modules for certain (rational) vertex operator superalgebras is invariant with respect to $\Gamma(2)$ (cf. [35]). The natural question arises: What should be added to the theory such that it is modular invariant (with respect to $\Gamma(1)$)? Because $[\Gamma(1) : \Gamma(2)] = 6$ we expect that the symmetric group S_3 and the σ -twisted modules will play an important role.

Remark 5.2. A similar result holds for the Ramond sector, constructed by using the procedure in [31]. We already discussed (cf. Remark 3.2) that representations of Ramond algebra $\mathcal{S}\mathcal{D}_R^+$ is a twisted M_c -module. In the free field case the construction is much simpler. $M(1) \otimes F$ has a σ -twisted module, which can be constructed—without changing the bosonic part—by using twisted $\mathbb{Z}/2\mathbb{Z}$ -graded vertex operators. We start with a twisted version of the fermionic superalgebra, i.e., the algebra with generators are $\varphi_n, n \in \mathbb{Z}$ and (anti)commuting relations

$$[\varphi(n), \varphi(m)] = \delta_{m+n, 0}.$$

Corresponding Fock space \tilde{F} is spanned by vectors of the form

$$\varphi(-n_k) \cdots \varphi(-n_2)\varphi(-n_1)\mathbf{1},$$

where $n_k > \cdots > n_2 > n_1 > 0$. Also we define the twisted field

$$\tilde{\varphi}(x) = \sum_{n \in \mathbb{Z}} \varphi(n)x^{-n-1/2}.$$

Then

$$[\tilde{\varphi}(x_1), \tilde{\varphi}(x_2)] = x_2^{-1} \delta_{1/2} \left(\frac{x_1}{x_2} \right),$$

on \tilde{F} . Therefore $\langle \varphi(x) \rangle$ is vertex superalgebra isomorphic to F and \tilde{F} is a σ -twisted F -module. Hence, $M(1) \otimes \tilde{F}$ is a σ -twisted $M \otimes F$ -module. The generalized character for \tilde{F} is

$$\tilde{F}(\tau_1, \tau_3, \dots) = q_1^{-\zeta(-1, 1/2)/2} q_3^{-\zeta(-3, 1/2)/2} \cdots \prod_{n=1}^{\infty} (1 + q_1^n q_2^n \cdots). \tag{5.11}$$

It has modular properties similar to $F(\tau_1, \tau_3, \dots)$.

Remark 5.3. The generalized character we consider has $N = 1$ flavor (here N is the number of fermions). Thus it seems plausible to study modular properties of generalized characters for certain $N = 2$ vertex operator superalgebras. The novelty is that in the $N = 2$ case one has to combine $U(1)$ -charge so we expect a substantially different result compared to the $N = 1$ case. From the topological point of view this leads to the so-called *generalized elliptic genus* (cf. [36]).

6. \mathcal{D}_∞^\pm algebras and Dirichlet L -functions

In this section we make a connection between representation theory of an algebra of pseudodifferential operators and vertex operator algebras. This approach, as a special case recovers some results of S. Bloch (cf. [4, Section 6]).

6.1. χ -twisted vertex operators

Fix $N \in \mathbb{N}$. Dirichlet character is a multiplicative homomorphisms

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times.$$

Often, we extend χ to the set of integers by letting $\chi(N + a) = \chi(a)$ and $\chi(a) = 0$ for $(a, N) \neq 1$. If χ can be lifted from a Dirichlet character of $(\mathbb{Z}/M\mathbb{Z})^\times$, for some $M|N$, via the natural homomorphism

$$(\mathbb{Z}/N\mathbb{Z})^\times \rightarrow (\mathbb{Z}/M\mathbb{Z})^\times$$

then we say that χ is imprimitive. A character χ is defined as the minimal value of M . A character χ is called *primitive* modulo N (N is the conductor) if χ is not imprimitive.

For the proof of the following lemma see [17].

Lemma 6.1. *Let χ be a primitive character modulo N . Then*

$$\sum_{a=1}^N \chi(a) e^{2\pi i a k / N} = \bar{\chi}(k) g(\chi), \tag{6.1}$$

where $g(\chi) = \sum_{n=1}^N \chi(n) e^{2\pi i n / N}$ is the Gauss sum of χ and $\bar{\chi}$ is the complex conjugate (character) of χ .

Suppose now that V is a vertex operator algebra and $v \in V$. For every X -operator

$$X(u, x) = Y(x^{L(0)}u, y),$$

we consider a χ -twisted operator³

$$X_\chi(u, x) = \sum_{k \in \mathbb{Z}} \chi(-k) u_{n+\text{wt}(u)-1} x^{-k}. \tag{6.2}$$

³ This operator should not be confused with the twisted operators that appear in the theory of twisted modules [11,8,31,15].

In our applications u is homogeneous of the weight 1 (or $\frac{1}{2}$ in superalgebra case). From Lemma 6.1 it follows (provided that $\chi \neq 1$ is primitive modulo N):

$$X_\chi(u, x) = g(\bar{\chi})^{-1} \sum_{a=1}^N \bar{\chi}(a) X(u, e^{(2\pi ia/N)} x). \tag{6.3}$$

Note that $\chi(a) = \chi(-a)$ implies $\chi(-1) = 1$ and vice versa.

Now if we use (6.3) we obtain the following commutator formula ($\chi \neq 1, \mu \neq 1$)

$$\begin{aligned} & [X_\chi(u, x_1), X_\mu(v, x_2)] \\ &= g(\bar{\chi})^{-1} g(\bar{\mu})^{-1} \operatorname{Res}_x \sum_{a,b=1}^N \bar{\chi}(a) \bar{\mu}(b) \delta\left(\frac{e^{x+(2\pi i(b-a)/N)} x_2}{x_1}\right) \\ & \quad \times X(Y[u, x]v, e^{(2\pi ib/N)} x_2). \end{aligned} \tag{6.4}$$

However we always have (no matter what χ is)

$$\begin{aligned} & [X_\chi(u, x_1), X_\mu(u, x_2)] \\ &= \operatorname{Res}_y \sum_{n,m \in \mathbb{Z}} \left(\frac{e^y x_2}{x_1}\right)^n \mu(n-m) \chi(-n) (Y[u, y]v)_m(x_2)^{-m}. \end{aligned} \tag{6.5}$$

Suppose that $Y[u, y]v$ has the form

$$Y[u, y]v = \frac{1}{y^k} \mathbf{1} + \text{regular terms},$$

for some $k \in \mathbb{N}$. Then formula (6.4) (or (6.5)) become very simple

$$[X_\chi(u, x_1), X_\mu(u, x_2)] = \chi(-1) D^{k-1} \delta_{\chi\mu} \left(\frac{x_2}{x_1}\right), \tag{6.6}$$

where

$$\delta_\tau(x) = \sum_{n \in \mathbb{Z}} \tau(n) x^n.$$

Note that if χ and μ are primitive modulo N then $\mu\chi$ is primitive as well.

From now on we assume, if otherwise stated, that all characters are primitive and nontrivial modulo N .

As in the previous sections we study quadratic operators (now equipped with an appropriate twisting):

$$g(\bar{\chi})^{-1} g(\bar{\mu})^{-1} \sum_{a,b=1}^N \bullet \bar{\chi}(a) \bar{\mu}(b) X(u, e^{y_1+(2\pi ia/N)} x) X(v, e^{y_2+(2\pi ib/N)} x) \bullet. \tag{6.7}$$

Remark 6.1. Notice that the factor $e^{(2\pi ia/N)}$ in formula (6.7) is closely related to the “operators” $e^{(2\pi iaD/N)}$, i.e.,

$$e^{aD} f(x) = f(e^a x).$$

This is well-defined. On the other hand, for $c \neq 0$

$$e^{c(d/dx)} f(x) = f(x + c),$$

does not have rigorous interpretation inside $\mathbb{C}[[x, x^{-1}]]$. Even though the operator on the right hand side can be interpreted as a differential operator of an infinite order, the right hand side might not be defined.

6.2. Dirichlet L-functions and the new normal ordering

Let $L(s, \chi)$ be a Dirichlet L-function associated with a primitive character χ (cf. [17]). It is easy to see that every Dirichlet L-function can be expressed as a linear combination of certain Hurwitz zeta functions $\zeta(s, u)$. The generalized Bernoulli numbers associated with χ are defined by

$$\sum_{a=1}^N \frac{y\chi(a)e^{ay}}{e^{Ny} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{y^n}{n!},$$

where the expression on the right hand side is the Taylor expansion inside $|y| < 2\pi/N$. As in the rest of the paper we will treat the previous series formally (without referring to convergence).

Then it follows that (for the proof see [17]) $L(1 - m, \chi) = -B_{m,\chi}/m$, for $m \geq 1$.

Now let $u, v \in V$ such that $u(1)v = \mathbf{1}$. Consider

$$\begin{aligned} & \bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet \\ &= g(\chi)^{-1} g(\mu)^{-1} \sum_{a,b=1}^N \bullet \chi(a)\mu(b) X(u, e^{y_1+(2\pi ia/N)x}) X(v, e^{y_2+(2\pi ib/N)x}) \bullet \end{aligned} \quad (6.8)$$

As in (3.30) we introduce a new normal ordering for $X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x})$.

$$\begin{aligned} & \dagger X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \dagger \\ &= \bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet \\ & \quad - g(\bar{\mu})^{-1} g(\bar{\chi})^{-1} \sum_{a,b=1}^N \frac{\partial}{\partial y_2} \left(\frac{\bar{\chi}(a)\bar{\mu}(b)e^{y_1-y_2+(2\pi i(a-b)/N)}}{e^{y_1-y_2+(2\pi i(a-b)/N)} - 1} \right). \end{aligned} \quad (6.9)$$

We will use the following key result:

Proposition 6.1. *Suppose that $\chi \neq 1$ is a primitive Dirichlet character modulo N . Then (formally)*

$$Ng(\chi)^{-1} \sum_{a=1}^N \frac{\chi(a)e^{ax}}{e^{Nx} - 1} = \sum_{a=1}^N \frac{\bar{\chi}(a)e^{x-(2\pi ia/N)}}{e^{x-(2\pi ia/N)} - 1}. \quad (6.10)$$

If x is a complex variable the previous formula makes sense for $0 < |x| < \frac{2\pi}{N}$.

Proof. Note first that

$$g(\chi)^{-1} \sum_{a=1}^N \frac{\chi(a)e^{ax}}{e^{Nx} - 1},$$

has a partial fraction decomposition

$$g(\chi)^{-1} \sum_{a=1}^N \frac{b_a}{e^x - e^{(2\pi ia/N)}},$$

where

$$\begin{aligned} b_a &= \operatorname{Res}_{\chi=e^{(2\pi ia/N)}} \sum_{m=1}^N \frac{\chi(m)e^{mx}}{e^{Nx} - 1} = \sum_{m=1}^N \frac{\chi(m)(e^{(2\pi ia/N)})^{m+1}}{N(e^{(2\pi ia/N)})^N} \\ &= \frac{\sum_{m=1}^N \chi(m)(e^{(2\pi ia/N)})^{m+1}}{N} = e^{(2\pi ia/N)} \frac{\bar{\chi}(a)g(\chi)}{N}. \end{aligned} \quad (6.11)$$

Now

$$\begin{aligned} g(\chi)^{-1} \sum_{a=1}^N \frac{b_a}{e^x - e^{(2\pi ia/N)}} &= \frac{e^{(2\pi ia/N)}}{N} \sum_{a=1}^N \frac{\bar{\chi}(a)}{e^x - e^{(2\pi ia/N)}} \\ &= \sum_{a=1}^N \frac{\bar{\chi}(a)N}{e^{x-(2\pi ia/N)} - 1}. \end{aligned} \quad (6.12)$$

Because of

$$\sum_{a=1}^N \bar{\chi}(a) = 0$$

it follows that

$$\sum_{a=1}^N \frac{\bar{\chi}(a)e^{x-(2\pi ia/N)}}{e^{x-(2\pi ia/N)} - 1} = \sum_{a=1}^N \frac{\bar{\chi}(a)}{e^{x-(2\pi ia/N)} - 1},$$

and the statement follows. \square

By using the previous proposition, formula (6.9) can be written in the form

$$\begin{aligned} &\dagger X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \dagger \\ &= \bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet \\ &\quad - \frac{\partial}{\partial y_2} g(\bar{\mu})^{-1} g(\bar{\chi})^{-1} N \sum_{a=1}^N \bar{\chi}(a) g(\mu)^{-1} \sum_{b=1}^N \frac{\mu(b) e^{(b(y_1 - y_2) + (2\pi ab/N))}}{e^{N(y_1 - y_2)} - 1} \\ &= \bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet \end{aligned}$$

$$\begin{aligned}
 & - \frac{\partial}{\partial y_2} N g(\mu)^{-1} g(\bar{\mu})^{-1} g(\bar{\chi})^{-1} \sum_{b=1}^N \frac{\chi(b) g(\bar{\chi}) \mu(b) e^{b(y_1 - y_2)}}{e^{N(y_1 - y_2)} - 1} \\
 & = \bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet \\
 & - \frac{\partial}{\partial y_2} N g(\bar{\mu})^{-1} g(\mu)^{-1} \sum_{b=1}^N \frac{(\mu\chi)(b) e^{b(y_1 - y_2)}}{e^{N(y_1 - y_2)} - 1}. \tag{6.13}
 \end{aligned}$$

Because

$$|g(\mu)|^2 = \mu(-1)g(\mu)g(\bar{\mu}).$$

and $|g(\mu)|^2 = N$, then it follows that (6.13) is equal to

$$\bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet + \mu(-1) \sum_{b=1}^N \frac{(\mu\chi)(b) e^{b(y_1 - y_2)}}{e^{N(y_1 - y_2)} - 1}. \tag{6.14}$$

Notice that the last expression is symmetric under the involution

$$y_1 \leftrightarrow y_2, \chi \leftrightarrow \mu.$$

6.3. Lie algebra \mathcal{D}_∞ and its subalgebras

Let $\mathcal{H} \subset \mathbb{C}[[t]]$ be an algebra of formal every convergent power series. Let $D = t d/dt$ as before. Then for every $A(t) \in \mathcal{H}$, $A(D) \in \bar{\mathcal{D}}$ acts on $V[[t, t^{-1}]]$, where V is an arbitrary complex vector space. For example consider e^{aD} for $a \in \mathbb{C}$. Then $e^{aD} g(t) = g(e^a t)$ for every $g(t) \in \mathbb{C}[[t]]$. Also we have relations:

$$e^{aiD} = \cos(aD) + i \sin(aD)$$

and

$$e^{2\pi i D} = \text{Id},$$

where \cos and \sin are, as usual, defined in terms of power series. Consider a vector space \mathcal{D}_∞ spanned by all operators of the form

$$t^k f(D) e^{aD},$$

where $a \in \mathbb{C}$, $k \in \mathbb{Z}$ and $f \in \mathbb{C}[t]$, i.e. the algebra of quasi-polynomial differential operators. \mathcal{D}_∞ has a structure of \mathbb{Z} -graded associative algebra and where the zero degree subalgebra is spanned by $D^k e^{aD}$. If $A, B \in \mathcal{D}_\infty$ then

$$A(D) t^k B(D) = t^k A(D + k) B(D),$$

where $B(D + k)$ is a well defined element of \mathcal{D}_∞ .

Now we generalize the generating functions $\mathcal{D}^{y_1, y_2}(x)$ considered in Section 2.2. For every $a, b \in \mathbb{C}$ we define

$$\begin{aligned}
 & \mathcal{D}^{y_1, y_2, a, b}(x) \\
 & := e^{-y_1 D} \delta \left(\frac{e^{-a} t}{x} \right) e^{y_2 D} e^{(b-a)D} D + e^{-y_2 D} \delta \left(\frac{e^{-b} t}{x} \right) e^{y_1 D} e^{(a-b)D} D. \tag{6.15}
 \end{aligned}$$

Then the coefficients of $\mathcal{D}^{y_1, y_2, a, b}(x)$ span a Lie subalgebra $\mathcal{D}_\infty^+ \subset \mathcal{D}_\infty$, defined as the θ_1 -stable subalgebra, where θ_1 is given by

$$\theta_1(t^k A(D)D) = t^k A(-D - k)D,$$

for $A(D) \in \mathcal{D}_\infty$.

If we choose a different involution θ_2 then we denote the corresponding θ_2 -fixed Lie algebra by \mathcal{D}_∞^- .

Actually we obtain the same algebra if we assume in (6.15) that a or b (but not both) are equal to zero. The generators are given by

$$L_m^{(r, a, b)} := \frac{r!^2}{2} \text{coeff}_{y_1^r y_2^a x^0} \mathcal{D}^{y_1, y_2, a, b}(x).$$

Then every $L_m^{(r, a, b)}$ is a linear combination of some $L_m^{(r, b)} := L_m^{(r, 0, b)}$. We define in the same way the operators $L^{(r, a, b)}(m)$ acting on M and the corresponding operators $\bar{L}^{(r, a, b)}(m)$.

Suppose that χ and μ are primitive and nontrivial mod N . Let $\mu(-1) = 1$ (μ is even). Then

$$\begin{aligned} \bar{L}^{(r, \chi, \mu)}(0) &:= \sum_{a, b=1}^N g(\bar{\chi})^{-1} g(\bar{\mu})^{-1} \bar{\chi}(a) \bar{\mu}(b) \bar{L}^{(r, 2\pi ia/N, 2\pi ib/N)}(0) \\ &= g(\bar{\chi})^{-1} g(\bar{\mu})^{-1} \sum_{a, b=1}^N \bar{\chi}(a) \bar{\mu}(b) L^{(r, 2\pi ia/N, 2\pi ib/N)}(0) \\ &\quad + (-1)^r \frac{1}{2} L(-2r - 1, \chi\mu). \end{aligned} \quad (6.16)$$

Remark 6.2. It was noticed in [4] that commutators

$$[\bar{L}^{(r_1, \chi_1, \mu_1)}(m), \bar{L}^{(r_2, \chi_2, \mu_2)}(-m)],$$

written in terms of ζ -regularized operators, have the trivial central terms. This is a consequence of the following observation:

The generating function for $\bar{L}^{(r_1, \chi_1, \mu_1)}(m)$'s is given by

$$\bullet X_\chi(u, e^{y_1 x}) X_\mu(v, e^{y_2 x}) \bullet + \mu(-1) \sum_{b=1}^N \frac{(\mu\chi)(b) e^{b(y_1 - y_2)}}{e^{N(y_1 - y_2)} - 1}. \quad (6.17)$$

If $\chi\mu \neq 1$ then it does not involve any negative powers of y_1 and y_2 . This is clear since

$$\sum_{a=1}^N \rho(a) = 0,$$

for every nontrivial character $\rho \neq 1$. On the contrary, in the case of \mathcal{D}^+ , \mathcal{D}^- and $\mathcal{S}\mathcal{D}_{\mathbb{N}\mathbb{S}}^+$ the y -singular terms were “responsible” for appearance of the pure monomials in the commutators written in terms of ζ -regularized operators. Here singular terms are absent.

Remark 6.3. Notice that the whole section can be generalized for Lie superalgebras, by constructing the ∞ -analogue of $\mathcal{SD}_{\text{NS}}^+$ and $\mathcal{SD}_{\text{R}}^+$.

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