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# On the Movement of a Permutation Group

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## 1. THE THEOREM

In [1] the second-named author defined the movement of a permutation group $(G, \Omega)$ by the formula

$$m := \operatorname{move}(G) := \sup\{|\Gamma^g \setminus \Gamma| \mid \Gamma \subseteq \Omega,\ g \in G\}$$

and proved that if $G$ has no fixed points and $m$ is finite then $n \leq 5m - 2$, where $n$ is the degree of $G$, that is, $n := |\Omega|$. In [2] it is shown that equality holds if and only if $n = 3$ and $G$ is transitive. In this note we aim to improve the bound:

THEOREM. (1) *If $G$ has no fixed points and* $\operatorname{move}(G) = m$ *then* $n \leq \frac{1}{2}(9m - 3)$;

(2) *equality holds infinitely often*;

(3) *moreover, if $n = \frac{1}{2}(9m - 3)$ then either $n = 3$ and $G = \mathrm{Sym}(\Omega)$ or $G$ is an elementary abelian 3-group and all its orbits have length 3.*

Part (2) is proved by examples; (1) and (3) will be proved using a sequence of lemmas.



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## 2. THE PROOF

EXAMPLES. Let $d$ be a positive integer, let $G := Z_3^d$, let $t := \frac{1}{2}(3^d - 1)$, and let $H_1, \ldots, H_t$ be an enumeration of the subgroups of index 3 in $G$. Define $\Omega_i$ to be the coset space of $H_i$ in $G$ and $\Omega := \Omega_1 \cup \cdots \cup \Omega_t$. If $g \in G \setminus \{1\}$ then $g$ lies in $\frac{1}{2}(3^{d-1} - 1)$ of the groups $H_i$ and therefore acts on $\Omega$ as a permutation with $\frac{1}{2}(3^d - 3)$ fixed points and $3^{d-1}$ disjoint 3-cycles. Taking one point from each of these 3-cycles to form a set $\Gamma$ we see that $m(G) \geq 3^{d-1}$, and it is not hard to prove that in fact $m(G) = 3^{d-1}$. Thus $n = 3t = \frac{1}{2}(3^{d+1} - 3) = \frac{1}{2}(9m - 3)$. This proves Part (2) of the theorem.

To prove Parts (1) and (3) we introduce the following notation:

$r_3$ := number of $G$-orbits of length 3 on which $G$ acts as Alt(3);

$r_3'$ := number of $G$-orbits of length 3 on which $G$ acts as Sym(3);

and then

$r_2$ := number of $G$-orbits of length 2;

$r_4$ := number of $G$-orbits of length 4;

$s$ := number of $G$-orbits of length $\geq 5$.

The orbits are labelled accordingly: thus $\Omega_1, \ldots, \Omega_{r_3}$ are those of length 3 on which $G$ acts as Alt(3); $\Omega_{r_3+1}, \ldots, \Omega_{r_3+r_3'}$ are those of length 3 on which $G$ acts as Sym(3); $\Omega_{r_3+r_3'+1}, \ldots, \Omega_{r_3+r_3'+r_2}$ are those of length 2, etc. Define $t := r_3 + r_3' + r_2 + r_4 + s$, $t_0 := r_3 + r_3' + r_2$, $\Sigma_4 := \bigcup_{i=t_0+1}^{t_0+r_4} \Omega_i$, and $\Sigma_5 := \bigcup_{i=t_0+r_4+1}^{t} \Omega_i$. For $1 \leq i \leq t_0$ let $K_i$ be the kernel of the action of $G$ on $\Omega_i$ and for $g \in G$ let $k(g)$ be the number of $i$ in that range for which $g \notin K_i$. For $g \in G$ and a $G$-invariant set $\Sigma$ define $\mathrm{fix}_\Sigma(g)$ := the number of fixed points of $g$ in $\Sigma$, $\mathrm{supp}_\Sigma(g)$ := the size of the support of $g$ in $\Sigma$ (so that $\mathrm{fix}_\Sigma(g) + \mathrm{supp}_\Sigma(g) = |\Sigma|$), and $\mathrm{odd}_\Sigma(g)$ := the number of non-trivial cycles of $g$ in $\Sigma$ that have odd length.

LEMMA 1. *With this notation, let* $\Sigma := \bigcup_{i=t_0+1}^{t} \Omega_i$ *and let* $g \in G$. *Then*

$$k(g) + \tfrac{1}{2}\mathrm{supp}_\Sigma(g) - \tfrac{1}{2}\mathrm{odd}_\Sigma(g) \leq m.$$

*Proof.* For each $i$ such that $1 \leq i \leq t_0$ and $g \notin K_i$ choose a point of $\Omega_i$ not fixed by $g$; then let $\Gamma_0$ be the set of chosen points. For each non-trivial cycle $(\alpha_1 \alpha_2 \cdots \alpha_k)$ of $g$ in $\Sigma$ adjoin the points $\alpha_1, \alpha_3, \ldots, \alpha_{k'}$ to $\Gamma_0$, where $k'$ is odd and $k - 2 \leq k' \leq k - 1$. Let $\Gamma$ be the resulting set. It has been constructed so that $\Gamma^g \cap \Gamma = \varnothing$. Therefore $|\Gamma| \leq m$. Since $|\Gamma| = k(g) + \frac{1}{2}\mathrm{supp}_\Sigma(g) - \frac{1}{2}\mathrm{odd}_\Sigma(g)$, we have the stated inequality.

LEMMA 2. *With the notation defined above,* $\sum_{g \in G} k(g) = |G|(\frac{2}{3}r_3 + \frac{5}{6}r'_3 + \frac{1}{2}r_2)$.

*Proof.* First observe that $k(g) = \sum_{i=1}^{t_0} k_i(g)$, where $k_i(g) := 1$ if $g \notin K_i$ and $k_i(g) := 0$ if $g \in K_i$. Now

$$\sum_{g \in G} k_i(g) = \begin{cases} \frac{2}{3}|G| & \text{if } 1 \le i \le r_3, \\ \frac{5}{6}|G| & \text{if } r_3 + 1 \le i \le r_3 + r'_3, \\ \frac{1}{2}|G| & \text{if } r_3 + r'_3 + 1 \le i \le t_0, \end{cases}$$

and so

$$\sum_{g \in G} k(g) = \sum_{g \in G} \sum_{i=1}^{t_0} k_i(g) = \sum_{i=1}^{t_0} \sum_{g \in G} k_i(g) = |G|\left(\frac{2}{3}r_3 + \frac{5}{6}r'_3 + \frac{1}{2}r_2\right),$$

as the lemma states.

LEMMA 3. $n < \frac{9}{2}m - (\frac{3}{4}r'_3 + \frac{1}{4}r_2 + \frac{5}{4}r_4 + \frac{1}{2}(|\Sigma_5| - 3s))$.

*Proof.* We intend to exploit Lemma 1 by averaging over $G$ and in order to do this we examine $\sum_{g \in G}(\mathrm{supp}_{\Sigma_j}(g) - \mathrm{odd}_{\Sigma_j}(g))$ for $j \in \{4, 5\}$, where, recall, $\Sigma_4$ is the union of the orbits of length 4 and $\Sigma_5$ is the union of the orbits of length $\ge 5$. For any orbit $\Omega_i$ of length 4 we find that $\sum_{g \in G} \mathrm{odd}_{\Omega_i}(g) \le \frac{2}{3}|G|$ (the sum is 0 if $G^{\Omega_i}$ is a 2-group, it is $\frac{2}{3}|G|$ if $G^{\Omega_i} = \mathrm{Alt}(\Omega_i)$, and it is $\frac{1}{3}|G|$ if $G^{\Omega_i} = \mathrm{Sym}(\Omega_i)$). Therefore $\sum_{g \in G} \mathrm{odd}_{\Sigma_4}(g) \le \frac{2}{3}|G| \cdot r_4$. Also,

$$\sum_{g \in G} \mathrm{supp}_{\Sigma_4}(g) = \sum_{g \in G} \left(|\Sigma_4| - \mathrm{fix}_{\Sigma_4}(g)\right) = |G|\left(|\Sigma_4| - r_4\right),$$

by Not Burnside's Lemma, and therefore

$$\sum_{g \in G} \left(\mathrm{supp}_{\Sigma_4}(g) - \mathrm{odd}_{\Sigma_4}(g)\right) \ge |G|\left(|\Sigma_4| - r_4 - \frac{2}{3}r_4\right) = |G| \cdot \frac{7}{3}r_4.$$

Similarly, since $\mathrm{odd}_{\Sigma_5}(g) \le \frac{1}{3}\mathrm{supp}_{\Sigma_5}(g)$,

$$\sum_{g \in G} \left(\mathrm{supp}_{\Sigma_5}(g) - \mathrm{odd}_{\Sigma_5}(g)\right)$$
$$\ge \frac{2}{3} \sum_{g \in G} \mathrm{supp}_{\Sigma_5}(g) = \frac{2}{3} \sum_{g \in G} \left(|\Sigma_5| - \mathrm{fix}_{\Sigma_5}(g)\right) = \frac{2}{3}|G|\left(|\Sigma_5| - s\right).$$

Now take the inequality $k(g) + \frac{1}{2}\mathrm{supp}_\Sigma(g) - \frac{1}{2}\mathrm{odd}_\Sigma(g) \le m$ from Lemma 1 and sum over $G$. Using Lemma 2 and the above inequalities we find that

$$\begin{aligned} m|G| &> \sum_{g\in G} k(g) + \sum_{g\in G} \tfrac{1}{2}\big(\mathrm{supp}_{\Sigma_4}(g) - \mathrm{odd}_{\Sigma_4}(g)\big) \\ &\quad + \sum_{g\in G} \tfrac{1}{2}\big(\mathrm{supp}_{\Sigma_5}(g) - \mathrm{odd}_{\Sigma_5}(g)\big) \\ &\ge |G|\big(\tfrac{2}{3}r_3 + \tfrac{5}{6}r_3' + \tfrac{1}{2}r_2 + \tfrac{7}{6}r_4 + \tfrac{1}{3}|\Sigma_5| - \tfrac{1}{3}s\big), \end{aligned}$$

where the strict inequality recognizes the fact that the inequality of Lemma 1 is strict for the identity element of $G$. Now $n = 3r_3 + 3r_3' + 2r_2 + 4r_4 + |\Sigma_5|$ and so

$$\begin{aligned} m &> \tfrac{2}{3}r_3 + \tfrac{5}{6}r_3' + \tfrac{1}{2}r_2 + \tfrac{7}{6}r_4 + \tfrac{1}{3}|\Sigma_5| - \tfrac{1}{3}s \\ &= \tfrac{2}{9}n + \tfrac{1}{6}r_3' + \tfrac{1}{18}r_2 + \tfrac{5}{18}r_4 + \tfrac{1}{9}|\Sigma_5| - \tfrac{1}{3}s. \end{aligned}$$

Multiplication by $\frac{9}{2}$ and rearrangement yields the inequality of the lemma.

Now define $\eta := 9m - 2n$. Clearly $\eta$ is an integer and from Lemma 3 we have that

$$2\eta > 3r_3' + r_2 + 5r_4 + 2\big(|\Sigma_5| - 3s\big) \ge 0.$$

Since $\eta > 0$, in fact $\eta \ge 1$. To prove the theorem we suppose that $\eta \le 3$ and seek to discover what configurations may occur. Then $2\eta \le 6$ and so, since certainly $|\Sigma_5| \ge 5s$ there are only the following possibilities:

(I) $r_3' = r_4 = 0$, $r_2$ is 0 or 1, $s = 1$, and $|\Sigma_5| = 5$;

(II) $r_3' = r_2 = 0$, $r_4 = 1$, $|\Sigma_5| = s = 0$;

(III) $3r_3' + r_2 \le 5$, $r_4 = |\Sigma_5| = s = 0$.

Cases (I) and (II) cannot arise for arithmetical reasons: in these cases we would have $\eta = 3$ (given the assumption that $\eta \le 3$), so that, since $2n = 9m - \eta$, the degree $n$ would be a multiple of 3, in contradiction to the fact that $n$ is $3r_3 + 5$ or $3r_3 + 7$ or $3r_3 + 4$ in these cases. Case (III) falls to the following lemmas.

LEMMA 4. *If* $r_3 = r_4 = s = 0$ *then* $n \le 4m - 1 \le \frac{1}{2}(9m - 3)$. *The equality* $n = \frac{1}{2}(9m - 3)$ *holds only when* $m = 1$, $r_2 = 0$, $r_3' = 1$, *so that* $G = \mathrm{Sym}(3)$.

*Proof.* Suppose that $r_3 = r_4 = s = 0$. It follows from Lemmas 1 and 2 that then $|G|m > |G|(\frac{1}{2}r_2 + \frac{5}{6}r'_3)$. But also $n = 2r_2 + 3r'_3$ and therefore $m > \frac{1}{4}n + \frac{1}{12}r'_3$. Thus $n < 4m$, and so $n \leq 4m - 1$. Since $4m - 1 = \frac{1}{2}(9m - 3) - \frac{1}{2}(m - 1)$, if $n = \frac{1}{2}(9m - 3)$ then $m = 1$, $n = 3$, and $G = \mathrm{Sym}(3)$.

LEMMA 5. *If* $r'_3 = r_2 = r_4 = s = 0$ *then* $n \leq \frac{1}{2}(9m - 3)$.

*Proof.* In this case our inequalities merely say that $2n < 9m$. On the other hand, $n = 3r_3$, so that $n$ is a multiple of 3. Therefore $2n \leq 9m - 3$, as required.

To complete the proof of the theorem we need to deal with Case (III). Suppose therefore that $r_4 = |\Sigma_5| = s = 0$. Suppose also that $r_3 > 0$ and $r'_3 + r_2 > 0$. Define $\Sigma_1 := \bigcup_{i=1}^{r_3} \Omega_i$, the union of the orbits of length 3 on which $G$ acts as Alt(3), and $\Sigma_2 := \bigcup_{i=r_3+1}^{t_0} \Omega_i$, the union of those orbits of length 3 on which $G$ acts as Sym(3) and those of length 2. Then define $K_1$ to be the kernel of the action of $G$ on $\Sigma_1$ and $K_2$ the kernel of its action on $\Sigma_2$. Clearly, $K_1 \cap K_2 = \{1\}$ since $G$ acts faithfully on $\Omega$. Now let $H$ be the subgroup of $G$ generated by its 2-elements. Then $G^{\Sigma_2} = H^{\Sigma_2}$, that is, $G = HK_2$. But $H \leq K_1$ and therefore $G = K_1K_2$, that is, $G = K_1 \times K_2$. It follows easily that if $m_1 := \mathrm{move}(G^{\Sigma_1})$ and $m_2 := \mathrm{move}(G^{\Sigma_2})$ then $m = m_1 + m_2$. Defining $n_1 := |\Sigma_1|$ and $n_2 := |\Sigma_2|$, we have from Lemma 5 that $n_1 \leq \frac{1}{2}(9m_1 - 3)$, and from Lemma 4 that $n_2 \leq (4m_2 - 1)$, and so $n = n_1 + n_2 \leq \frac{1}{2}(9m - m_2 - 5) < \frac{1}{2}(9m - 3)$.

What this has shown is that in Case (III), if $\eta \leq 3$, that is, if $n \geq \frac{1}{2}(9m - 3)$, then either $r_3 = 0$ or $r_2 + r'_3 = 0$. If $r_3 = 0$ then we have the situation of Lemma 4, so that $n \leq \frac{1}{2}(9m - 3)$ with equality if and only if $G = \mathrm{Sym}(3)$. If $r_2 + r'_3 = 0$ then of course $r_2 = r'_3 = 0$ and we have the situation of Lemma 5. In this case $n \leq \frac{1}{2}(9m - 3)$ by that lemma, and, since $G$ is a subdirect product of copies of Alt(3), it is an elementary abelian 3-group. Thus the proof of the theorem is complete.

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