JOURNAL OF ALGEBRA 40, 354–363 (1976)

The Modular Characters of the Symmetric Groups

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Communicated by J. A. Green

Received August 20, 1974

1. INTRODUCTION

The ordinary irreducible characters ζ^λ of the symmetric group S_n are labeled by the partitions $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. Similarly, for a given prime number p, the modular irreducible Brauer characters ϕ^μ and the modular principal indecomposable characters η^μ may be labeled by the p-regular partitions μ of n, that is, partitions in which no part is repeated p times or more. This is simply due to the well-known fact (see, e.g., [8, p. 114]) that the number of p-regular conjugacy classes of S_n is equal to the number of p-regular partitions of n. The lack of a good labeling of characters in the modular theory has always contrasted strongly with the very satisfactory state of affairs in the classical ordinary theory.

In this paper we introduce relatively simple combinatorial notions in order to prove the existence of "good" labeling in the modular theory. We use the p-graph of a partition (first introduced by Littlewood [5]) in order to associate with each p-regular partition μ a *set* $I(\mu)$ of partitions of n, by means of a special process of stripping and reassembly of the nodes. Littlewood's p-graph is also employed to introduce a partial order relation $\alpha \rightarrow^E \beta$ on the set of all partitions which is stronger than dictionary order, and which has the key property that for p-regular μ, $\alpha \in I(\mu) \Rightarrow \alpha \rightarrow^E \mu$. In this context we prove that it is possible to label the modular characters by p-regular diagrams in such a way that the decomposition numbers satisfy:

(i) $d_{\alpha,\mu} = \langle \zeta^\alpha, \eta^\mu \rangle = 0$ whenever $\alpha \notin I(\mu)$;

(ii) $d_{\mu,\mu} = \langle \zeta^\mu, \eta^\mu \rangle \neq 0$.

We consider this a "good" labeling because it provides good information on the ordinary characters which appear in a given principal indecomposable character. This information is, in a sense, best possible; for, when p is large, every partition is p-regular and $I(\mu) = \{\mu\}$.

Our results imply, in particular, that the decomposition matrix of S_n may be arranged so as to have zeros above the main diagonal. This fact has long been suspected from numerical work (cf. [8, 10, 11]).

2. Combinatorial Lemmas

Throughout n denotes a fixed positive integer, and p denotes a prime number. We deal with subsets of the set of all ordered pairs of integers (i, j) such that $1 \leqslant i$, $j \leqslant n$. These ordered pairs are to be thought of as *nodes* arranged in n rows and n columns, so that (i, j) appears in the ith row and jth column, as in matrix notation. The *class* (mod p) of the node (i, j) is defined to be $j - i \pmod p$. If $j - i \equiv r \pmod p$ we call (i, j) an *r-node*. For an arbitrary set of such pairs we can talk of its rows and columns. We say that (i, j) is higher (lower) than (i', j') in a given set if $i < i'$ $(i > i')$.

A *diagram of n nodes* is a set of pairs of the form

$$[\lambda] = \{(i, j) \colon 1 \leqslant j \leqslant \lambda_i\,,\, 1 \leqslant i \leqslant n\}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, $\lambda_1 + \cdots + \lambda_n = n$. Littlewood's p-graph of λ (cf. [5]) consists therefore of the above diagram with each node carrying a label equal to its class (mod p). The node (i, j) of the diagram is called *removable* if its removal leaves a diagram (of $n - 1$ nodes). The class of the highest removable node of the diagram $[\lambda]$ is denoted by $r(\lambda)$.

A diagram is *p-regular* if no p rows have the same number of nodes.

(2.1) Lemma. *If $[\mu]$ is a p-regular diagram then it is possible to remove an $r(\mu)$-node from $[\mu]$ so as to leave another p-regular diagram.*

Proof. Let $(i, j), (i', j'), \ldots$ be the removable nodes of $[\mu]$, where $i < i' < \ldots$. Then $j - i \equiv r(\mu) \pmod p$, and we have

$$j = \mu_i > \mu_{i+1} = \mu_{i+2} = \cdots = \mu_{i'} > \mu_{i'+1}\,.$$

Since $[\mu]$ is p-regular we have $i' - i < p$. Hence the removal of (i, j) will leave a p-regular diagram unless $\mu_i = \mu_{i+1} + 1$ and $i' - i = p - 1$, in which case we have $j' - i' = (\mu_i - 1) - (p + i - 1) = j - i - p \equiv r(\mu)$ (mod p). An obvious repetition of this argument finishes the proof.

(2.2) *Remarks.* Let $[\mu]$ be a p-regular diagram of n nodes. The above proof shows that if (h, μ_h) is the highest removable $r(\mu)$-node whose removal yields a p-regular diagram then $h = b(p - 1) + a(1 \leqslant a \leqslant p - 1, 0 \leqslant b)$ and

$$\mu = (\mu_1, (\mu_1 - 1)^{p-1}, (\mu_1 - 2)^{p-1}, \dots, (\mu_1 - b)^{p-1}, \mu_{h+1}, \dots)$$

where the indices are used here to indicate part-repetitions. It is important to observe that all the removable nodes of $[\mu]$ which lie on or above the hth row have class $r(\mu)$, and that $\mu_i - i \not\equiv r(\mu) - 1$ for $1 \leqslant i \leqslant h$.

We shall write $\mu \xrightarrow{0} \beta$ to indicate that $[\beta]$ is the p-regular diagram arising from $[\mu]$ by removal of the $r(\mu)$-node (h, μ_h) described above. By repeated application of this "stripping" procedure we obtain a sequence of p-regular diagrams of $n - 1, n - 2, \dots, 3, 2, 1$ nodes respectively.

Next we shall write $\alpha \to^r \lambda$ to mean that the diagram of λ arises from that of α by the addition of an r-node (so-called r-induction). Note that $\xrightarrow{0}$ is a mapping of the set of all p-regular partitions into itself, while $\to^r$ is a relation on the set of all partitions.

Let μ be a given p-regular partition of n. Define a set of partitions $I(\mu)$ inductively (on n) as follows:

(i) $I(\mu) = \{\mu\}$, if $n = 1$;

(ii) $I(\mu) = \{\lambda : \exists \alpha \in I(\beta), \alpha \to^r \lambda\}$, if $\mu \xrightarrow{0} \beta$ and $r = r(\mu)$.

Equivalently, we may obtain $I(\mu)$ as follows. Suppose that

$$\mu = \mu^{(n)} \xrightarrow{0} \mu^{(n-1)} \xrightarrow{0} \mu^{(n-2)} \xrightarrow{0} \cdots \xrightarrow{0} \mu^{(1)}$$

and let $r(i) = r(\mu^{(i)})$ $(i = 1, 2, \dots, n)$. Then $\lambda \in I(\mu)$ if and only if there is a sequence of the form

$$\lambda = \lambda^{(n)} \xleftarrow{r(n)} \lambda^{(n-1)} \xleftarrow{r(n-1)} \lambda^{(n-2)} \longleftarrow \cdots \xleftarrow{r(2)} \mu^{(1)}.$$

Note for example that if $n < p$ then $I(\mu) = \{\mu\}$.

Let X be a set of nodes, and for $1 \leqslant i \leqslant n$, $0 \leqslant r \leqslant p - 1$, let $E_{ir}(X)$ denote the number of r-nodes (u, v) of X which are on or above the ith row, i.e., for which $u \leqslant i$. Let $E(X)$ be the $n \times p$ matrix whose (i, r)th element is $E_{ir}(X)$. Plainly

$$E(X \cup Y) = E(X) + E(Y) \qquad \text{if} \quad X \cap Y = \phi;$$

and, if X contains the single node (u, v) and $v - u \equiv r \pmod p$ then $E(X)$ has 1 in each of the positions (i, r), $(i \geqslant u)$, and 0 elsewhere.

Write $E(\lambda)$ for $E([\lambda])$ and use $\lambda \to^E \mu$ to mean $E(\lambda) \leqslant E(\mu)$, i.e., $E_{ir}(\lambda) \leqslant E_{ir}(\mu)$ for all i, r. In this way we obtain a partial order relation $\to^E$ on the set of all partitions.

Note that if $\lambda \to^E \mu$ then $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leqslant \mu_1 + \mu_2 + \cdots + \mu_i$ for all i. In particular, it follows that $\to^E$ is stronger than dictionary order.

(2.3) Key Lemma. *Let μ be a p-regular partition of n and let $r = r(\mu)$. Then*

$$\begin{tikzcd} \lambda & \mu \arrow[d, "0"] \\ \alpha \arrow[u, "r"] \arrow[r, "E"] & \beta \end{tikzcd} \quad \textit{implies} \quad \lambda \xrightarrow{E} \mu.$$

Proof. As in the preceding discussion, let (h, μ_h) be the highest removable r-node of $[\mu]$ whose removal leaves the p-regular diagram $[\beta]$. Suppose further that $[\lambda]$ arises from $[\alpha]$ by the addition of the r-node $(s, \alpha_s + 1)$. We have then

$$\mu_h - h \equiv r, \quad \alpha_s - s \equiv r - 1 \qquad \pmod p,$$

and

$$E(\mu) = E(\beta) + E(\{(h, \mu_h)\}); \qquad E(\lambda) = E(\alpha) + E(\{(s, \alpha_s + 1)\}).$$

Let A_{*j} denote the jth column of the matrix A. The above relations show that, for $j \neq r$,

$$E_{*j}(\mu) = E_{*j}(\beta) \geqslant E_{*j}(\alpha) = E_{*j}(\lambda).$$

On the other hand,

$$E_{*r}(\mu) = E_{*r}(\beta) + C_h; \qquad E_{*r}(\lambda) = E_{*r}(\alpha) + C_s,$$

where C_k denotes the column with 0 in the first $k - 1$ positions and 1 in the rest. In case $h \leqslant s$ we have $C_h \geqslant C_s$ and so

$$E_{*r}(\mu) \geqslant E_{*r}(\beta) + C_s \geqslant E_{*r}(\alpha) + C_s = E_{*r}(\lambda),$$

as required. Again, there is nothing to prove when $\lambda = \mu$. The burden of the proof therefore rests in the case $s < h$, $\lambda \neq \mu$. We have

$$\text{(i)} \quad E_{ir}(\mu) - E_{ir}(\lambda) = \begin{cases} E_{ir}(\beta) - E_{ir}(\alpha) & \text{if } i < s \text{ or } i \geqslant h; \\ E_{ir}(\beta) - E_{ir}(\alpha) - 1 & \text{if } s \leqslant i \leqslant h - 1. \end{cases}$$

Consequently, the proof will be complete when we establish

$$\text{(ii)} \qquad E_{ir}(\beta) > E_{ir}(\alpha) \qquad \text{for} \quad s \leqslant i \leqslant h-1.$$

We do so by proving the apparently weaker statement,

$$\text{(iii)} \qquad E_{ir}(\beta) > E_{ir}(\alpha) \text{ whenever } \alpha_i > \beta_i\,, \quad 1 \leqslant i \leqslant h-1,$$

and deducing (ii) from (iii).

Deduction of (ii) *from* (iii). Proceed by induction in i. When $i = s$ we have

$$\alpha_s - s \equiv r - 1; \qquad \beta_s - s \equiv \mu_s - s \not\equiv r - 1 \qquad (\mathrm{mod}\ p)$$

(cf. (2.2)).

Hence $\alpha_s \neq \beta_s$. If $\alpha_s > \beta_s$ then (iii) yields the required conclusion. If $\alpha_s < \beta_s$ then $(s, \alpha_s + 1)$ is an r-node of $[\beta]$ which is not in $[\alpha]$, and so, by (i), we again deduce that $E_{sr}(\beta) > E_{sr}(\alpha)$.

This shows that the induction starts off. To carry out the inductive step, let $s < i \leqslant h-1$ and assume the result for $i-1$, namely, $E_{i-1,r}(\beta) > E_{i-1,r}(\alpha)$. If $\beta_i \geqslant \alpha_i$ then every r-node on the ith row of $[\alpha]$ is also in $[\beta]$, and so $E_{ir}(\beta) > E_{ir}(\alpha)$. If not, then $\beta_i < \alpha_i$ and the same conclusion follows from (iii).

Proof of (iii). We want to prove

$$E_{ir}([\beta]\setminus[\alpha]) > E_{ir}([\alpha]\setminus[\beta]) \text{ whenever } \alpha_i > \beta_i\,, \qquad 1 \leqslant i \leqslant h-1,$$

where $X \setminus Y$ denotes the set of all elements of X not in Y. This is a consequence of

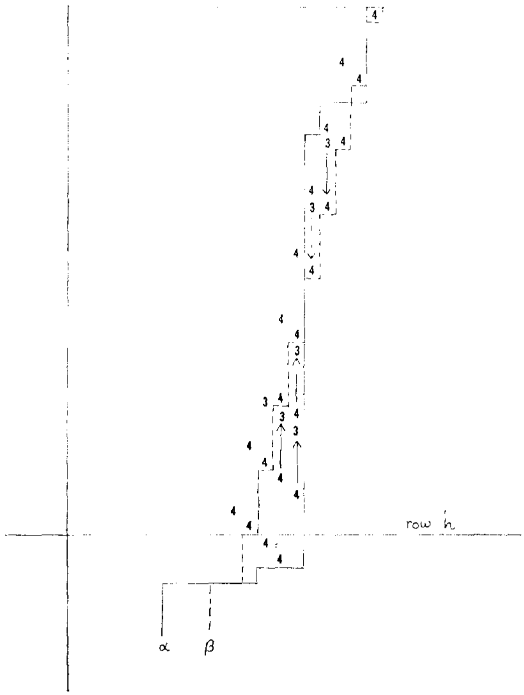
$$\left.\begin{array}{ll} \text{(iv)} & E_{ir}([\beta]\setminus[\alpha]) > E_{i,r-1}([\beta]\setminus[\alpha]) \\ \text{(v)} & E_{i,r-1}([\alpha]\setminus[\beta]) \geqslant E_{ir}([\alpha]\setminus[\beta]) \end{array}\right\} \quad \text{for } \alpha_i > \beta_i; \quad 1 \leqslant i \leqslant h-1,$$

because we are given that $E(\alpha) \leqslant E(\beta)$.

Suppose that (x, y) is an $(r-1)$-node on or above the ith row of $[\beta]\setminus[\alpha]$. Then since $\alpha_i > \beta_i$ we have $x \leqslant i - 1 \leqslant h - 2$, $y > \alpha_i > \beta_i \geqslant \beta_{h-1}$. Recalling our remarks in (2.2) we see that $(x + p - 1, y)$ is an r-node in $[\beta]\setminus[\alpha]$. In this way we obtain an injective mapping $(x, y) \rightarrow (x + p - 1, y)$ of the set of $(r-1)$-nodes in $[\beta]\setminus[\alpha]$ to the set of r nodes. This mapping is not surjective since the highest removable r-node in $[\beta]\setminus[\alpha]$ is not in its image. This proves (iv).

Similarly, let (x, y) be an r-node on or above the ith row of $[\alpha]\setminus[\beta]$. The diagonal through (x, y) will hit the "rim" of $[\beta]$ in an r-node, say $(x - d, y - d)$. Then $(x - d - (p - 1), y - d)$ will be an $(r-1)$-node in $[\beta]$, while the next node down along the same diagonal will be outside $[\beta]$. Hence $(x - (p - 1), y)$ is an $(r-1)$-node in $[\alpha]\setminus[\beta]$ above the ith row. The mapping $(x, y) \rightarrow (x - (p - 1), y)$ therefore establishes (v).

This completes the proof of the key lemma. The diagram below is intended to illustrate parts of the proof.



(2.4) COROLLARY. *Let μ be a p-regular partition of n. Then*

$$\lambda \in I(\mu) \qquad \textit{implies} \quad \lambda \xrightarrow{E} \mu.$$

Proof. The proof is by induction on n. The result is trivial for $n = 1$. Suppose $n > 1$. Let $r = r(\mu)$ and let $\mu \xrightarrow{0} \beta$. Then $\lambda \in I(\mu)$ if and only if $\lambda \leftarrow^{r} \alpha$ for some $\alpha \in I(\beta)$. By the induction hypothesis $\alpha \rightarrow^{E} \beta$. Hence, by (2.3), $\lambda \rightarrow^{E} \mu$.

3. THE MAIN THEOREM

(3.1) THEOREM. *It is possible to associate with each p-regular partition μ of n a principal indecomposable character η^{μ} of S_n in such way that if*

$$\eta^{\mu} = \sum_{\lambda} d_{\lambda,\mu} \zeta^{\lambda}$$

then $d_{\mu,\mu} > 0$ and $d_{\lambda,\mu} = 0$ whenever $\lambda \notin I(\mu)$. The mapping $\mu \rightarrow \eta^{\mu}$ is a *bijection of the set of all p-regular partitions onto the set of all principal indecomposable characters.*

Proof. We proceed by induction on n. If $n < p$ then $n!$ is a p-adic unit and every irreducible character ζ^{μ} is a principal indecomposable character. Also, every partition of n is p-regular, and $I(\mu) = \{\mu\}$. The assertion is therefore true for $n < p$.

Assume that $n > p$ and that the required labeling has been achieved for S_{n-1}. Given a p-regular partition μ of n, let $\mu \to^{0} \beta$. Then β is a p-regular partition of $n - 1$, and we have a principal indecomposable character of S_{n-1}, namely,

$$\eta^{\beta} = \sum_{\alpha \in I(\beta)} d_{\alpha,\beta} \zeta^{\alpha}.$$

Let $r = r(\mu)$. The character η^{β} induces a projective character of S_n. It follows that

$$\sum_{\alpha \in I(\beta)} d_{\alpha,\beta} \sum_{[\lambda] \supset [\alpha]} \zeta^{\lambda}$$

is a sum of principal indecomposable characters of S_n. The sub-sum

$$\sum_{\alpha \in I(\beta)} d_{\alpha,\beta} \sum_{\alpha \xrightarrow{r} \lambda} \zeta^{\lambda} = \sum_{\lambda \in I(\mu)} d'_{\lambda,\mu} \zeta^{\lambda} \quad \text{(say)}$$

contains all the characters ζ^{λ} which are in the same p-block[1] as ζ^{μ}, and is thus itself a sum of principal indecomposable character of S_n. By the induction hypothesis, $d_{\beta,\beta} > 0$. Since $\mu \in I(\mu)$ and $\beta \to \mu$ it follows that $d'_{\mu,\mu} > 0$. Hence, one of these decomposable characters is of the form

$$\sum_{\lambda \in I(\mu)} d_{\lambda,\mu} \zeta^{\lambda} \qquad \text{with} \quad d_{\mu,\mu} > 0, \quad d_{\lambda,\mu} \leqslant d'_{\lambda,\mu}.$$

We choose one such and denote it by η^{μ}.

This establishes the first assertion of the theorem. To complete the proof suppose that μ, μ' are p-regular partitions of n with $\eta^{\mu} = \eta^{\mu'}$. Then both ζ^{μ}, $\zeta^{\mu'}$ appear in this indecomposable character, whence $\mu \in I(\mu')$, $\mu' \in I(\mu)$. The key lemma now implies $\mu \to^{E} \mu'$, $\mu' \to^{E} \mu$, and so $\mu = \mu'$. This shows that the mapping $\mu \to \eta^{\mu}$ is injective, and hence bijective.

[1] This follows from: ζ^{λ} and ζ^{μ} belong to the same p-block if and only if λ and μ have the same p-core, if and only if, for each u $(0 \leqslant u \leqslant p - 1)$, the diagrams $[\lambda]$, $[\mu]$ contain the same number of u-nodes. The first equivalence is Nakayama's criterion conjectured in [2]; [3, 4] constitute the first proof, and [6, 7] constitute a second proof. The second equivalence is discussed in [5, pp. 337, 347]; it can also be derived from [6, Sect. 2]. See also [8].

(3.2) COROLLARY. *In a given p-block of S_n arrange the ordinary characters so that those having p-regular partitions come first, listed according to dictionary order. Then the indecomposable characters may be ordered so that the decomposition matrix has "wedge-shape"*

$$\begin{bmatrix} * & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ & & \ddots & 0 \\ * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \end{bmatrix}$$

This fact has long been suspected (it is conjectured in [11, p. 161]), but the first serious attempt at formulating and proving it was made in [9]. The present paper derives its most crucial techniques from [9].

(3.3) COROLLARY. *In every p-block with $p \neq 2$ of S_n there is at least one ordinary irreducible character which remains irreducible* (mod p).

Proof. Let ζ^μ be the leading ordinary irreducible character in the block, in the sense of (3.2). By (3.2) we deduce that ζ^μ is a *multiple* of a modular irreducible character (mod p). We have to show that it is in fact irreducible (mod p). Let $K[x_1, \ldots, x_n]$ be the polynomial ring in n independent indeterminates over an integral domain K. This is a KS_n-module in an obvious manner. Let the indeterminates $x_1, \ldots, x_n$ be indexed in any way in the form $\{y_{ij}: 1 \leqslant j \leqslant \mu_i, 1 \leqslant i\}$, and write K_μ for the KS_n-submodule of $K[x_1, \ldots, x_n]$ generated by the element

$$\prod_j \prod_{i<i'} (y_{ij} - y_{i'j}).$$

It can be shown easily that $\mathrm{End}(K_\mu) = K$ if K has characteristic not equal to 2. Specht [1] introduced these modules in the case $K = Q$ and showed that they provide a complete set of ordinary irreducible modules. Now when Z_μ is taken modulo p we obtain $(Z/pZ)_\mu$, every endomorphism of which is scalar. It follows that no minimal submodule of $(Z/pZ)_\mu$ can be a homomorphic image as well, unless the module itself is irreducible. Consequently, its character ζ^μ cannot be a multiple of an irreducible character (mod p) without being itself irreducible (mod p).

A discussion of the general properties of the Specht modules can be found in [12].

4. The Decomposition Numbers of Symmetric Groups

We conclude with a few remarks about the decomposition numbers of symmetric groups S_n. These have been calculated for $n \leqslant 10$ at characteristic 2 in [11] and at characteristic 3 in [10]. The method used is the same as that used in the proof of (3.1), namely, r-induction.

Suppose that the p-decomposition numbers of S_{n-1} are known, so that the the principal indecomposable characters of S_{n-1} can be expressed in terms of the ordinary irreducible characters of S_{n-1}. We may induce each of these principal indecomposable characters to S_n and separate the terms into different blocks (by r-induction), thereby obtaining a number of projective characters of S_n expressed as linear combinations of ordinary irreducible characters of S_n in a particular block.

For each p-regular partition μ, at least one of these projective characters has leading term a multiple of ζ^μ, as is shown in the proof of (3.1) using special restriction applied to μ. In some cases, more than one of the projective characters has leading term a multiple of ζ^μ.

The calculations begin by choosing, for each p-regular partition μ in a particular block of S_n, a projective character of S_n of the form

$$\sum_{\lambda \leqslant \mu} d''_{\lambda,\mu} \zeta^\mu.$$

We thus obtain a matrix $R = (d''_{\lambda,\mu})$ whose rows are indexed by the partitions in the block, with the p-regular partitions first in dictionary order, and whose columns are indexed by the principal indecomposable characters of S_n in the block. R has wedge shape. The columns of R represent *sums* of principal indecomposable characters, and the problem is to extract the columns of D from those of R. General properties of decomposition numbers in the block are sometimes useful; sometimes specific information about the decomposition numbers in the block is needed. This information can be obtained from the composition factors of the Specht modules. For full details of the calculations, the reader is referred to [10, pp. 54–74] (characteristic 3) and to [11, pp. 142–161] (characteristic 2). The results of these calculations show that some of the tables given in [8], appendix, are not correct. We do not have sufficient information about the decomposition numbers to guarantee that they could be evaluated for any symmetric group, for any prime p, by this method.

Corollary (3.3) shows that the entry in the first row, first column of the decomposition matrix of a block is 1. In all the examples calculated, the diagonal entries $d_{\mu\mu}$ are all equal to 1; it does not seem possible to strengthen the proof of (3.1) to show that $d_{\mu\mu}$ is always equal to 1.

W. J. Wong has pointed out to the authors that the characters in the principal 3-blocks of the groups $PGL(2, 9)$ and M_{10} cannot be ordered so that the decomposition matrices have wedge shape. Thus (3.2) is not valid for every finite group. Is (3.3) valid for every finite group?

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