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Two-sided two-cosided Hopf modules and Doi–Hopf modules for quasi-Hopf algebras ☆

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Abstract

Let H be a finite-dimensional quasi-Hopf algebra over a field k and $\mathfrak{A}$ a right H-comodule algebra. We introduce the category of two-sided Hopf modules, and prove that it is isomorphic to a module category. We also show that two-sided Hopf modules are coalgebra over a certain comonad. We introduce Doi–Hopf modules, and show that they are comodules over a certain coring. If the underlying H-module coalgebra is finite-dimensional, then Doi–Hopf modules are modules over a certain smash products. A similar result holds for two-sided two-cosided Hopf modules.

Introduction

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [15] in connection with the Knizhnik–Zamolodchikov equations [19]. Let k be a field, H an associative algebra and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ two algebra morphisms. Roughly speaking, H is a quasi-bialgebra if the category ${}_H\mathcal{M}$ of left H-modules, equipped with the tensor product of vector spaces endowed with the diagonal H-module structure given via Δ, and with unit object k viewed as a left H-module via ε, is a monoidal category. The co-

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multiplication Δ is not coassociative but only quasi-coassociative, in the sense that it is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$. Moreover, H is a quasi-Hopf algebra if and only if each finite-dimensional left H-module has a dual H-module. Note that the definition of a quasi-bialgebra is not self-dual.

From an algebraic point of view, quasi-bialgebras and quasi-Hopf algebras appear naturally. They can be obtained by twisting the comultiplication on a bialgebra H by an invertible element $F \in H \otimes H$ satisfying $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$: a new comultiplication Δ_F making H a quasi-bialgebra is given by $\Delta_F(h) = F\Delta(h)F^{-1}$. Another important example is the Dijkgraaf–Pasquier–Roche quasi-Hopf algebra $D^{\omega}(G)$, where G is a finite group and ω a normalized 3-cocycle. The representations of $D^{\omega}(G)$ are important in physics (see [12]). Altschuler and Coste [3] used them to construct invariants for knots, links, and 3-manifolds. In [7], this construction was generalized to finite-dimensional cocommutative Hopf algebras, and an even more general construction is the quantum double $D(H)$ of a finite-dimensional quasi-Hopf algebra, see [16,17, 21]. Albuquerque and Majid [1] showed recently that the octonions are a twisting of the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in the monoidal category of representations of a quasi-Hopf algebra associated to a group 3-cocycle. In particular, they shown that the octonions are quasi-algebras associative up to a 3-cocycle isomorphism. They provide new quasi-associative algebras beyond the octonions and also introduce a suitable quasi-Hopf algebra of "automorphisms" associated to any quasi-algebra of the type presented above. More examples of quasi-algebras, where the non-associativity constraint is induced by a $\mathbb{Z}_n$-grading and a nontrivial 3-cocycle, were given in [2].

Let H be a bialgebra, A and H-comodule algebra, and C an H-module coalgebra. We can consider several types of modules, such as modules, comodules, (relative) Hopf modules, Long dimodules, and Yetter–Drinfeld modules. Doi [14] and Koppinen [20] introduced Doi–Hopf modules, and it turned out that they generalize and unify all the types of modules mentioned above. Basically, we obtain the definition of a Doi–Hopf module, by combining the definitions of a relative (A, H)-module and its dual notion, a relative $[H, C]$-module: a (H, A, C)-module is a k-linear space together with an A-action and a C-coaction satisfying an appropriate compatibility relation. We recover the two types of relative Hopf modules taking respectively $C = H$ and $A = H$. At the end of last century, Takeuchi [28] observed that $A \otimes C$ is in a canonical way an A-coring, and that Doi–Hopf modules are nothing else than comodules over the coring $A \otimes C$. This observation was the reason for a revived interest in corings and comodules (see, for example, [5]); actually, corings were considered already by Sweedler in 1965 [26], but then forgotten by Hopf-algebra theorists.

The aim of this paper is to introduce the quasi-bialgebraic versions of these categories, including interpretations in terms of monoidal categories, and to give duality theorems in the finite-dimensional case. The conceptual problem that arises comes from the fact that the definition of a quasi-bialgebra H is not self-dual: an immediate consequence is that we cannot consider H-comodules, because a quasi-bialgebra is not coassociative. H-module (co)algebras can be introduced as (co)algebras in the monoidal category of H-modules, but we cannot introduce H-comodule algebras as algebras in the category of comodules. A formal definition of H-comodule algebras was given by Hausser and Nill [16]; we propose the following interpretation: if H is a bialgebra, and $\mathfrak{A}$ is a right H-comodule algebra, then $\mathfrak{A} \otimes H$ is an $\mathfrak{A}$-coring, which means that it is a coalgebra in the category of $\mathfrak{A}$-bimodules. The quasi-bialgebra analog of this property is the following: let H be a quasi-bialgebra, and $\mathfrak{A}$ an algebra. Then the category of $(\mathfrak{A} \otimes H, \mathfrak{A})$-bimodules is monoidal. If $\mathfrak{A}$ is a right H-comodule algebra in the sense of [16], then $\mathfrak{A} \otimes H$ is a coalgebra in the category ${}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$. This coalgebra induces a comonad, and the two-sided Hopf modules that are introduced in Section 3.1 are precisely the coalgebras over this comonad. This will be discussed in detail in Section 3.3.

Given a finite-dimensional quasi-bialgebra H and a right H-comodule algebra $\mathfrak{A}$, we can introduce the quasi-smash product $\mathfrak{A} \mathbin{\overline{\#}} H^*$, which reduces to the usual smash product in the situation where H is a bialgebra. $\mathfrak{A} \mathbin{\overline{\#}} H^*$ is then a left H-module algebra, and we can consider the category $\mathcal{M}^{H^*}_{\mathfrak{A}\overline{\#}H^*}$ of relative Hopf modules (see Section 2). In Section 3, we introduce the category ${}_H\mathcal{M}^H_{\mathfrak{A}}$ of two-sided $(H, \mathfrak{A})$-Hopf modules; the main result of Section 3 is Theorem 3.5, stating that these two categories are isomorphic if H is a quasi-Hopf algebra. This generalizes [11, Proposition 2.3]. Applying results from [6], we find that the category $\mathcal{M}^{H^*}_{\mathfrak{A}\overline{\#}H^*}$ is isomorphic to the category of right modules over the smash product algebra (in the sense of [8]) of $\mathfrak{A} \mathbin{\overline{\#}} H^*$ and H. In the case where $\mathfrak{A} = H$, we recover a result of Nill announced in [18] stating that ${}_H\mathcal{M}^H_H$ is isomorphic to the category of right modules over the two-sided crossed product $H \rtimes H^* \ltimes H$. In Section 4, we will prove that the two-sided crossed product constructed in [16] is in fact a generalized smash product. As a consequence, $(H \mathbin{\overline{\#}} H^*) \# H$ is just the two-sided crossed product $H \rtimes H^* \ltimes H$ (as an algebra).

The second part of this paper is devoted to the study of the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$. Here C is a coalgebra in the monoidal category of (H, H)-bimodules ${}_H\mathcal{M}_H$ (i.e. an H-bimodule coalgebra), and $\mathbb{A}$ is an H-bicomodule algebra in the sense of [16]. Roughly speaking, an object in ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is a two-sided $(H, \mathbb{A})$-Hopf module which is also an "almost" left C-comodule such that the left C-coaction is compatible with the other structure maps. In Section 5 we will show that if C and H are finite-dimensional then ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to a category of right modules. To this end we will describe first ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ as a category of Doi–Hopf modules. If $\mathfrak{B}$ is a left H-comodule algebra and C is a right H-module coalgebra then the category of right–left $(H, \mathfrak{B}, C)$-Doi–Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a straightforward generalization of the category of relative Hopf modules ${}^C\mathcal{M}_H$. When C is finite-dimensional, ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right modules over the generalized smash product $C^* \ltimes \mathfrak{B}$. We also have an interpretation in terms of monoidal categories: $\mathfrak{B} \otimes C$ is a coring, and the Doi–Hopf modules are comodules over this coring. Now, returning to the category ${}^C_H\mathcal{M}^H_{\mathbb{A}}$, if H is finite-dimensional then we will show that $(\mathbb{A} \mathbin{\overline{\#}} H^*) \# H$ is a left $H \otimes H^{\mathrm{op}}$-comodule algebra (here "op" means the opposite multiplication on H) so, it makes sense to consider the category of Doi–Hopf modules ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}$. The main result states that ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}$, generalizing [4, Proposition 2.3]. In particular, if C is finite-dimensional, then ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to the category of right modules over the generalized smash product $\mathcal{A} = C^* \ltimes ((\mathbb{A} \mathbin{\overline{\#}} H^*) \# H)$. In the Hopf case, the left-handed version of this result was first obtained by Cibils and Rosso [10]. More precisely, they define an algebra X having the property that the category ${}^{H^*}_{H^*}\mathcal{M}^{H^*}_{H^*}$ is isomorphic to the category of left X-modules. Recently, Panaite [23] introduced two other algebras Y and Z with the same property as X; Y is the two-sided crossed product $H^* \# (H \otimes H^{\mathrm{op}}) \# H^{*\mathrm{op}}$ and Z is the diagonal crossed product (in the sense of [16]) $(H^* \otimes H^{*\mathrm{op}}) \bowtie (H \otimes H^{\mathrm{op}})$.

1. Preliminary results

1.1. Quasi-Hopf algebras

We work over a field k. All algebras, linear spaces, etc., will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [15], a quasi-bialgebra is a four-tuple $(H, \Delta, \varepsilon, \Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

$$(\mathrm{id} \otimes \Delta)\big(\Delta(h)\big) = \Phi(\Delta \otimes \mathrm{id})\big(\Delta(h)\big)\Phi^{-1}, \tag{1.1}$$

$$(\mathrm{id} \otimes \varepsilon)\big(\Delta(h)\big) = h, \qquad (\varepsilon \otimes \mathrm{id})\big(\Delta(h)\big) = h, \tag{1.2}$$

for all $h \in H$, and Φ has to be a normalized 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(\Phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi), \tag{1.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1. \tag{1.4}$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. We use the Sweedler–Heyneman notation $\Delta(h) = \sum h_1 \otimes h_2$. Since Δ is only quasi-coassociative, we will write

$$(\Delta \otimes \mathrm{id})\big(\Delta(h)\big) = \sum h_{(1,1)} \otimes h_{(1,2)} \otimes h_2,$$
$$(\mathrm{id} \otimes \Delta)\big(\Delta(h)\big) = \sum h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely:

$$\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 = \cdots,$$
$$\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3 = \cdots.$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-automorphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{1.5}$$

$$\sum X^1\beta S\big(X^2\big)\alpha X^3 = 1 \quad \text{and} \quad \sum S\big(x^1\big)\alpha x^2\beta S\big(x^3\big) = 1. \tag{1.6}$$

For a quasi-Hopf algebra, the antipode is determined uniquely up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, where $U \in H$ is invertible. The axioms for a quasi-Hopf algebra imply that $\varepsilon \circ S = \varepsilon$ and $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β, we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$. The identities (1.2)–(1.4) also imply that

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\Phi) = 1 \otimes 1. \tag{1.7}$$

Recall that the definition of a quasi-Hopf algebra is "twist coinvariant" in the following sense. An invertible element $F \in H \otimes H$ is called a gauge transformation or twist if $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$. If H is a quasi-Hopf algebra and $F = \sum F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = \sum G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit, and antipode of H and replacing the comultiplication, reassociator, and the elements α and β by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \tag{1.8}$$

$$\Phi_F = (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})\big(F^{-1}\big)\big(F^{-1} \otimes 1\big), \tag{1.9}$$

$$\alpha_F = \sum S\big(G^1\big)\alpha G^2, \qquad \beta_F = \sum F^1 \beta S\big(F^2\big). \tag{1.10}$$

It is well known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H \otimes H$ such that

$$f\Delta\big(S(h)\big)f^{-1} = (S \otimes S)\big(\Delta^{\mathrm{op}}(h)\big), \quad \text{for all } h \in H, \tag{1.11}$$

where $\Delta^{\mathrm{op}}(h) = \sum h_2 \otimes h_1$. f can be computed explicitly. First set

$$\sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = \big(1 \otimes \Phi^{-1}\big)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi), \tag{1.12}$$

$$\sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi)\big(\Phi^{-1} \otimes 1\big) \tag{1.13}$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = \sum S\big(A^2\big)\alpha A^3 \otimes S\big(A^1\big)\alpha A^4 \quad \text{and} \quad \delta = \sum B^1 \beta S\big(B^4\big) \otimes B^2 \beta S\big(B^3\big). \tag{1.14}$$

f and f^{-1} are then given by the formulas

$$f = \sum (S \otimes S)\big(\Delta^{\mathrm{op}}\big(x^1\big)\big)\gamma\Delta\big(x^2\beta S\big(x^3\big)\big), \tag{1.15}$$

$$f^{-1} = \sum \Delta\big(S\big(x^1\big)\alpha x^2\big)\delta(S \otimes S)\big(\Delta^{\mathrm{op}}\big(x^3\big)\big). \tag{1.16}$$

f satisfies the following relations:

$$f\Delta(\alpha) = \gamma, \qquad \Delta(\beta)f^{-1} = \delta. \tag{1.17}$$

Furthermore, the corresponding twisted reassociator (see (1.9)) is given by

$$\Phi_f = \sum (S \otimes S \otimes S)\big(X^3 \otimes X^2 \otimes X^1\big). \tag{1.18}$$

In a Hopf algebra H, we obviously have the identity

$$\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \quad \text{for all } h \in H.$$

We will need the generalization of this formula to the quasi-Hopf algebra setting. Following [16,17], we define:

$$\begin{aligned} p_R &= \sum p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S\big(x^3\big), \\ q_R &= \sum q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}\big(\alpha X^3\big) X^2, \end{aligned} \tag{1.19}$$

$$\begin{aligned} p_L &= \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}\big(X^1 \beta\big) \otimes X^3, \\ q_L &= \sum q_L^1 \otimes q_L^2 = \sum S\big(x^1\big) \alpha x^2 \otimes x^3. \end{aligned} \tag{1.20}$$

For all $h \in H$, we then have:

$$\begin{aligned} \sum \Delta(h_1) p_R [1 \otimes S(h_2)] &= p_R [h \otimes 1], \\ \sum \big[1 \otimes S^{-1}(h_2)\big] q_R \Delta(h_1) &= (h \otimes 1) q_R, \end{aligned} \tag{1.21}$$

$$\begin{aligned} \sum \Delta(h_2) p_L \big[S^{-1}(h_1) \otimes 1\big] &= p_L (1 \otimes h), \\ \sum [S(h_1) \otimes 1] q_L \Delta(h_2) &= (1 \otimes h) q_L, \end{aligned} \tag{1.22}$$

and

$$\sum \Delta\big(q_R^1\big) p_R \big[1 \otimes S\big(q_R^2\big)\big] = 1 \otimes 1, \qquad \sum \big[1 \otimes S^{-1}\big(p_R^2\big)\big] q_R \Delta\big(p_R^1\big) = 1 \otimes 1, \tag{1.23}$$

$$\sum \big[S\big(p_L^1\big) \otimes 1\big] q_L \Delta\big(p_L^2\big) = 1 \otimes 1, \qquad \sum \Delta\big(q_L^2\big) p_L \big[S^{-1}\big(q_L^1\big) \otimes 1\big] = 1 \otimes 1, \tag{1.24}$$

$$\begin{aligned} &(q_R \otimes 1)(\Delta \otimes \mathrm{id})(q_R) \Phi^{-1} \\ &\quad = \sum \big[1 \otimes S^{-1}\big(X^3\big) \otimes S^{-1}\big(X^2\big)\big]\big[1 \otimes S^{-1}\big(f^2\big) \otimes S^{-1}\big(f^1\big)\big](\mathrm{id} \otimes \Delta)\big(q_R \Delta\big(X^1\big)\big), \end{aligned} \tag{1.25}$$

$$\begin{aligned} &\Phi(\Delta \otimes \mathrm{id})(p_R)(p_R \otimes \mathrm{id}) \\ &\quad = \sum (\mathrm{id} \otimes \Delta)\big(\Delta\big(x^1\big) p_R\big)\big(1 \otimes f^{-1}\big)\big(1 \otimes S\big(x^3\big) \otimes S\big(x^2\big)\big), \end{aligned} \tag{1.26}$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (1.15).

1.2. The smash product

Suppose that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. If U, V, W are left (right) H-modules, define $a_{U,V,W}, \mathbf{a}_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by

$$a_{U,V,W}\big((u \otimes v) \otimes w\big) = \Phi \cdot \big(u \otimes (v \otimes w)\big),$$
$$\mathbf{a}_{U,V,W}\big((u \otimes v) \otimes w\big) = \big(u \otimes (v \otimes w)\big) \cdot \Phi^{-1}.$$

Then the category ${}_H\mathcal{M}$ ($\mathcal{M}_H$) of left (right) H-modules becomes a monoidal category (see [19,22] for the terminology) with tensor product $\otimes$ given via Δ, associativity constraints $a_{U,V,W}$ ($\mathbf{a}_{U,V,W}$), unit k as a trivial H-module and the usual left and right unit constraints.

Now, let H be a quasi-bialgebra. We say that a k-vector space A is a left H-module algebra if it is an algebra in the monoidal category ${}_H\mathcal{M}$, that is, A has a multiplication and a usual unit 1_A satisfying the following conditions:

$$(aa')a'' = \sum \big(X^1 \cdot a\big)\big[\big(X^2 \cdot a'\big)\big(X^3 \cdot a''\big)\big], \tag{1.27}$$

$$h \cdot (aa') = \sum (h_1 \cdot a)(h_2 \cdot a'), \tag{1.28}$$

$$h \cdot 1_A = \varepsilon(h)1_A, \tag{1.29}$$

for all $a, a', a'' \in A$ and $h \in H$, where $h \otimes a \mapsto h \cdot a$ is the H-module structure of A. Following [8], we define the smash product $A \# H$ as follows: as a vector space $A \# H$ is $A \otimes H$ ($a \otimes h$ viewed as an element of $A \# H$ will be written $a \# h$) with multiplication given by

$$(a \# h)(a' \# h') = \sum \big(x^1 \cdot a\big)\big(x^2 h_1 \cdot a'\big) \# x^3 h_2 h', \tag{1.30}$$

for all $a, a' \in A$, $h, h' \in H$. $A \# H$ is an associative algebra and it is defined by a universal property (as Heyneman and Sweedler did for Hopf algebras, see [8]). It is easy to see that H is a subalgebra of $A \# H$ via $h \mapsto 1 \# h$, A is a k-subspace of $A \# H$ via $a \mapsto a \# 1$ and the following relations hold:

$$(a \# h)(1 \# h') = a \# hh', \qquad (1 \# h)(a \# h') = \sum h_1 \cdot a \# h_2 h', \tag{1.31}$$

for all $a \in A$, $h, h' \in H$.

We will also need the notion right H-module coalgebra. This is a coalgebra C in the monoidal category of right modules over a quasi-bialgebra H. This means that C is a right H-module together with a comultiplication $\underline{\Delta} : C \to C \otimes C$ and a counit $\underline{\varepsilon} : C \to k$, satisfying the following relations:

$$(\underline{\Delta} \otimes \mathrm{id}_C)\big(\underline{\Delta}(c)\big)\Phi^{-1} = (\mathrm{id}_C \otimes \underline{\Delta})\big(\underline{\Delta}(c)\big) \quad \forall c \in C, \tag{1.32}$$

$$\underline{\Delta}(c \cdot h) = \sum c_{\underline{1}} \cdot h_1 \otimes c_{\underline{2}} \cdot h_2 \quad \forall c \in C,\ h \in H, \tag{1.33}$$

$$\underline{\varepsilon}(c \cdot h) = \underline{\varepsilon}(c)\varepsilon(h) \quad \forall c \in C,\ h \in H, \tag{1.34}$$

where we used the Sweedler-type notation

$$\underline{\Delta}(c) = c_{\underline{1}} \otimes c_{\underline{2}}, \quad (\underline{\Delta} \otimes \mathrm{id}_C)\big(\underline{\Delta}(c)\big) = \sum c_{(\underline{1},\underline{1})} \otimes c_{(\underline{1},\underline{2})} \otimes c_{\underline{2}}, \quad \text{etc.}$$

2. The quasi-smash product

The category of H-modules is monoidal, and an H-module (co)algebra is a (co)algebra in this category. This categorical definition cannot be used to introduce H-comodule algebras, since we do not have H-comodules. Hausser and Nill [16] gave a purely algebraic definition of an H-comodule algebra. We will show in Section 3.3 how their definition can be justified from a categorical point of view.

Definition 2.1 *[16].* Let H be a quasi-bialgebra. A unital associative algebra $\mathfrak{A}$ is called a right H-comodule algebra if there exists an algebra morphism $\rho : \mathfrak{A} \to \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that

$$\Phi_\rho(\rho \otimes \mathrm{id})\big(\rho(\mathfrak{a})\big) = (\mathrm{id} \otimes \Delta)\big(\rho(\mathfrak{a})\big)\Phi_\rho, \quad \text{for all } \mathfrak{a} \in \mathfrak{A}, \tag{2.1}$$

$$(1_{\mathfrak{A}} \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi_\rho)(\rho \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_\rho), \tag{2.2}$$

$$(\mathrm{id} \otimes \varepsilon) \circ \rho = \mathrm{id}, \tag{2.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H. \tag{2.4}$$

Similarly, a unital associative algebra $\mathfrak{B}$ is called a left H-comodule algebra if there exists an algebra morphism $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes \mathfrak{B}$ such that the following relations hold:

$$(\mathrm{id} \otimes \lambda)\big(\lambda(\mathfrak{b})\big)\Phi_\lambda = \Phi_\lambda(\Delta \otimes \mathrm{id})\big(\lambda(\mathfrak{b})\big), \quad \text{for all } \mathfrak{b} \in \mathfrak{B}, \tag{2.5}$$

$$(1_H \otimes \Phi_\lambda)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi_\lambda)(\Phi \otimes 1_{\mathfrak{B}}) = (\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\Phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_\lambda), \tag{2.6}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \lambda = \mathrm{id}, \tag{2.7}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi_\lambda) = 1_H \otimes 1_{\mathfrak{B}}. \tag{2.8}$$

We notice that, when $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra we also have

$$(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H.$$

Similarly, if $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H-comodule algebra then

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_\lambda) = 1_H \otimes 1_{\mathfrak{B}}.$$

When H is a quasi-bialgebra, particular examples of left and right H-comodule algebras are given by $\mathfrak{A} = \mathfrak{B} = H$ and $\rho = \lambda = \Delta$, $\Phi_\rho = \Phi_\lambda = \Phi$.

For a right H-comodule algebra $(\mathfrak{A}, \rho, \Phi_\rho)$, we will denote

$$\rho(\mathfrak{a}) = \sum \mathfrak{a}_{\langle 0\rangle} \otimes \mathfrak{a}_{\langle 1\rangle}, \qquad (\rho \otimes \mathrm{id})\big(\rho(\mathfrak{a})\big) = \sum \mathfrak{a}_{\langle 0,0\rangle} \otimes \mathfrak{a}_{\langle 0,1\rangle} \otimes \mathfrak{a}_{\langle 1\rangle}, \quad \text{etc.},$$

for any $\mathfrak{a} \in \mathfrak{A}$. Similarly, for a left H-comodule algebra $(\mathfrak{B}, \lambda, \Phi_\lambda)$, if $\mathfrak{b} \in \mathfrak{B}$ then we will denote

$$\lambda(\mathfrak{b}) = \sum \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0]}, \qquad (\mathrm{id} \otimes \lambda)\big(\lambda(\mathfrak{b})\big) = \sum \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0,-1]} \otimes \mathfrak{b}_{[0,0]}, \quad \text{etc.}$$

In analogy with the notation for the reassociator Φ of H, we will write

$$\Phi_\rho = \sum \tilde{X}^1_\rho \otimes \tilde{X}^2_\rho \otimes \tilde{X}^3_\rho = \sum \tilde{Y}^1_\rho \otimes \tilde{Y}^2_\rho \otimes \tilde{Y}^3_\rho = \cdots \quad \text{and}$$
$$\Phi_\rho^{-1} = \sum \tilde{x}^1_\rho \otimes \tilde{x}^2_\rho \otimes \tilde{x}^3_\rho = \sum \tilde{y}^1_\rho \otimes \tilde{y}^2_\rho \otimes \tilde{y}^3_\rho = \cdots.$$

A similar notation is used for the element Φ_λ of a left H-comodule algebra $\mathfrak{B}$. If no confusion is possible, we will omit the subscripts ρ or λ in the tensor components of the $\Phi_\rho, \Phi_\lambda, \Phi_\rho^{-1}$ and Φ_λ^{-1}.

Recall that, if H is an algebra, then H^* is an (H, H)-bimodule, with left and right action given by $\langle h \rightharpoonup \varphi \leftharpoonup h', h''\rangle = \langle \varphi, h'h''h\rangle$, for all $h, h', h'' \in H$ and $\varphi \in H^*$. If H is finite-dimensional, then H^* is a coalgebra.

Now let H be a bialgebra and $\mathfrak{A}$ be a right H-comodule algebra. Then we can consider the smash product $\mathfrak{A} \# H^*$, with multiplication

$$(a \# \varphi)(a' \# \psi) = \sum a a'_{\langle 0\rangle} \# (\varphi \leftharpoonup a'_{\langle 1\rangle})\psi.$$

We will now generalize this construction to quasi-bialgebras. In this situation, the convolution product on H^* is not associative, but only quasi-associative, namely

$$[\varphi\psi]\xi = \sum \big(X^1 \rightharpoonup \varphi \leftharpoonup x^1\big)\big[\big(X^2 \rightharpoonup \psi \leftharpoonup x^2\big)\big(X^3 \rightharpoonup \xi \leftharpoonup x^3\big)\big], \quad \text{for all } \varphi, \psi, \xi \in H^*. \tag{2.9}$$

In addition, for all $h \in H$ and $\varphi, \psi \in H^*$ we have that

$$h \rightharpoonup (\varphi\psi) = \sum (h_1 \rightharpoonup \varphi)(h_2 \rightharpoonup \psi) \quad \text{and} \quad (\varphi\psi) \leftharpoonup h = \sum (\varphi \leftharpoonup h_1)(\psi \leftharpoonup h_2). \tag{2.10}$$

In other words, H^* is an algebra in the monoidal category of (H, H)-bimodules ${}_H\mathcal{M}_H$. Let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right H-comodule algebra. We define a multiplication on $\mathfrak{A} \otimes H^*$ by

$$(\mathfrak{a} \,\overline{\#}\, \varphi)(\mathfrak{a}' \,\overline{\#}\, \psi) = \sum \mathfrak{a}\mathfrak{a}'_{\langle 0\rangle}\tilde{x}^1 \,\overline{\#}\, \big(\varphi \leftharpoonup \mathfrak{a}'_{\langle 1\rangle}\tilde{x}^2\big)\big(\psi \leftharpoonup \tilde{x}^3\big) \tag{2.11}$$

for all $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$ and $\varphi, \psi \in H^*$, where we write $\mathfrak{a} \,\overline{\#}\, \varphi$ for $\mathfrak{a} \otimes \varphi$, $\rho(\mathfrak{a}) = \sum \mathfrak{a}_{\langle 0\rangle} \otimes \mathfrak{a}_{\langle 1\rangle}$, and $\Phi_\rho^{-1} = \sum \tilde{x}^1 \otimes \tilde{x}^2 \otimes \tilde{x}^3$. We denote this structure on $\mathfrak{A} \otimes H^*$ by $\mathfrak{A} \,\overline{\#}\, H^*$. In the next proposition, we prove that $\mathfrak{A}\,\overline{\#}\,H^*$ is an algebra in the category of left H-modules, and this is why we call $\mathfrak{A}\,\overline{\#}\,H^*$ the quasi-smash product.

Proposition 2.2. *Let H be a quasi-bialgebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. Then $\mathfrak{A}\,\overline{\#}\,H^*$ is an H-module algebra with unit $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ and with left H-action given by*

$$h\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)=\mathfrak{a}\,\overline{\#}\,h\rightharpoonup\varphi\quad \textit{for all } h\in H,\ \mathfrak{a}\in\mathfrak{A},\ \textit{and } \varphi\in H^*. \tag{2.12}$$

Proof. Since H^* is a left H-module via the action $\rightharpoonup$, it is easy to see that $\mathfrak{A}\,\overline{\#}\,H^*$ is a left H-module via the action (2.12). Now, we will prove that $\mathfrak{A}\,\overline{\#}\,H^*$ is an algebra in ${}_H\mathcal{M}$ with unit $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$. Indeed, for all $\mathfrak{a}, \mathfrak{a}', \mathfrak{a}''\in\mathfrak{A}$ and $\varphi, \psi, \chi\in H^*$

$$\begin{aligned}
&\big[X^1\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)\big]\big\{\big[X^2\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)\big]\big[X^3\cdot(\mathfrak{a}''\,\overline{\#}\,\chi)\big]\big\}\\
&\quad=\sum\big(\mathfrak{a}\,\overline{\#}\,X^1\rightharpoonup\varphi\big)\big[\big(\mathfrak{a}'\,\overline{\#}\,X^2\rightharpoonup\psi\big)\big(\mathfrak{a}''\,\overline{\#}\,X^3\rightharpoonup\chi\big)\big]\\
&\quad=\sum\big(\mathfrak{a}\,\overline{\#}\,X^1\rightharpoonup\varphi\big)\big[\mathfrak{a}'\mathfrak{a}''_{\langle0\rangle}\tilde{x}^1\,\overline{\#}\,\big(X^2\rightharpoonup\psi\leftharpoonup\mathfrak{a}''_{\langle1\rangle}\tilde{x}^2\big)\big(X^3\rightharpoonup\chi\leftharpoonup\tilde{x}^3\big)\big]\\
&\overset{(2.10)}{=}\sum\mathfrak{a}\mathfrak{a}'_{\langle0\rangle}\mathfrak{a}''_{\langle0,0\rangle}\tilde{x}^1_{\langle0\rangle}\tilde{y}^1\,\overline{\#}\,\big(X^1\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{\langle1\rangle}\mathfrak{a}''_{\langle0,1\rangle}\tilde{x}^1_{\langle1\rangle}\tilde{y}^2\big)\\
&\quad\big[\big(X^2\rightharpoonup\psi\leftharpoonup\mathfrak{a}''_{\langle1\rangle}\tilde{x}^2\tilde{y}^3_1\big)\big(X^3\rightharpoonup\chi\leftharpoonup\tilde{x}^3\tilde{y}^3_2\big)\big]\\
&\overset{(2.9),(2.2)}{=}\sum\mathfrak{a}\mathfrak{a}'_{\langle0\rangle}\mathfrak{a}''_{\langle0,0\rangle}\tilde{x}^1\tilde{y}^1\,\overline{\#}\,\big[\big(\varphi\leftharpoonup\mathfrak{a}'_{\langle1\rangle}\mathfrak{a}''_{\langle0,1\rangle}\tilde{x}^2\tilde{y}^2_1\big)\big(\psi\leftharpoonup\mathfrak{a}''_{\langle1\rangle}\tilde{x}^3\tilde{y}^2_2\big)\big]\big(\chi\leftharpoonup\tilde{y}^3\big)\\
&\overset{(2.1),(2.10)}{=}\sum\mathfrak{a}\mathfrak{a}'_{\langle0\rangle}\tilde{x}^1\mathfrak{a}''_{\langle0\rangle}\tilde{y}^1\,\overline{\#}\,\big\{\big[\big(\varphi\leftharpoonup\mathfrak{a}'_{\langle1\rangle}\tilde{x}^2\big)\big(\psi\leftharpoonup\tilde{x}^3\big)\big]\leftharpoonup\mathfrak{a}''_{\langle1\rangle}\tilde{y}^2\big\}\big(\chi\leftharpoonup\tilde{y}^3\big)\\
&\quad=\sum\big[\mathfrak{a}\mathfrak{a}'_{\langle0\rangle}\tilde{x}^1\,\overline{\#}\,x\big(\varphi\leftharpoonup\mathfrak{a}'_{\langle1\rangle}\tilde{x}^2\big)\big(\psi\leftharpoonup\tilde{x}^3\big)\big](\mathfrak{a}''\,\overline{\#}\,\chi)\\
&\quad=\big[(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\#\,\psi)\big](\mathfrak{a}''\,\overline{\#}\,\chi).
\end{aligned}$$

It is not hard to see that $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ is the unit of $\mathfrak{A}\,\overline{\#}\,H^*$ and that $h\cdot(1_{\mathfrak{A}}\,\#\,\varepsilon)=\varepsilon(h)1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ for all $h\in H$. Finally, for all $h\in H$, $\mathfrak{a}, \mathfrak{a}'\in\mathfrak{A}$, and $\varphi, \psi\in H^*$, we calculate:

$$\begin{aligned}
&\sum\big[h_1\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)\big]\big[h_2\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)\big]\\
&\quad=\sum(\mathfrak{a}\,\overline{\#}\,h_1\rightharpoonup\varphi)(\mathfrak{a}'\,\overline{\#}\,h_2\rightharpoonup\psi)\\
&\quad=\sum\mathfrak{a}\mathfrak{a}'_{\langle0\rangle}\tilde{x}^1\,\overline{\#}\,\big(h_1\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{\langle1\rangle}\tilde{x}^2\big)\big(h_2\rightharpoonup\psi\leftharpoonup\tilde{x}^3\big)\\
&\overset{(2.10)}{=}\sum\mathfrak{a}\mathfrak{a}'_{\langle0\rangle}\tilde{x}^1\,\overline{\#}\,h\rightharpoonup\big[\big(\varphi\leftharpoonup\mathfrak{a}'_{\langle1\rangle}\tilde{x}^2\big)\big(\psi\leftharpoonup\tilde{x}^3\big)\big]\\
&\overset{(2.12)}{=}h\cdot\big[(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\overline{\#}\,\psi)\big].\qquad\square
\end{aligned}$$

(H, Δ, Φ) is a right H-comodule algebra, so it makes sense to consider the quasi-smash product $H\,\overline{\#}\,H^*$. In this case where H is a Hopf algebra, $H\,\overline{\#}\,H^*$ is called the Heisenberg double of H, and we will keep the same terminology for quasi-Hopf algebras. $\mathcal{H}(H)=H\,\overline{\#}\,H^*$ is not an associative algebra but it is an algebra in the monoidal category ${}_H\mathcal{M}$. If H is a finite-dimensional Hopf algebra then $\mathcal{H}(H)$ is isomorphic to the algebra $\mathrm{End}_k(H)$. In order to prove a similar result for a finite-dimensional quasi-Hopf algebra, we first have to deform the algebra structure of $\mathrm{End}_k(H)$.

Proposition 2.3. *Let H be a finite-dimensional quasi-Hopf algebra. Define*

$$\mu : H \,\overline{\#}\, H^* \to \mathrm{End}_k(H), \quad \mu(h \,\overline{\#}\, \varphi)(h') = \sum \varphi\big(h'_2 p_L^2\big) h h'_1 p_L^1$$

for all $h, h' \in H$ and $\varphi \in H^*$, where $p_L = \sum p_L^1 \otimes p_L^2$ is the element defined by (1.20)*. Then μ is a bijection, and therefore there exists a unique H-module algebra structure on* $\mathrm{End}_k(H)$ *such that μ becomes an H-module algebra isomorphism. The multiplication, the unit, and the H-module structure of* $\mathrm{End}_k(H)$ *are given by*

$$(u \,\overline{\circ}\, v)(h) = \sum u\big(v\big(hx^3X_2^3\big)S^{-1}\big(S\big(x^1X^2\big)\alpha x^2 X_1^3\big)\big)S^{-1}\big(X^1\big), \tag{2.13}$$

$$\mathbf{1}_{\mathrm{End}_k(H)}(h) = hS^{-1}(\beta), \qquad (h \cdot u)(h') = \sum u(h'h_2)S^{-1}(h_1) \tag{2.14}$$

for all $u, v \in \mathrm{End}_k(H)$ and $h, h' \in H$.

Proof. Let $\{e_i\}_{i=\overline{1,n}}$ be a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*. We claim that the inverse of μ is $\mu^{-1} : \mathrm{End}_k(H) \to H \,\overline{\#}\, H^*$ given by

$$\mu^{-1}(u) = \sum u\big(q_L^2(e_i)_2\big)S^{-1}\big(q_L^1(e_i)_1\big) \,\overline{\#}\, e^i \quad \text{for all } u \in \mathrm{End}_k(H),$$

where $q_L = \sum q_L^1 \otimes q_L^2$ is the element defined by (1.20). Indeed, for any $h \in H$ and $\varphi \in H^*$ we have:

$$\begin{aligned}
\big(\mu^{-1} \circ \mu\big)(h \,\overline{\#}\, \varphi) &= \sum_{i=1}^n \mu(h \,\overline{\#}\, \varphi)\big(q_L^2(e_i)_2\big)S^{-1}\big(q_L^1(e_i)_1\big) \,\overline{\#}\, e^i \\
&= \sum_{i=1}^n \varphi\big(\big(q_L^2\big)_2(e_i)_{(2,2)}p_L^2\big)h\big(q_L^2\big)_1(e_i)_{(2,1)}p_L^1 S^{-1}\big(q_L^1(e_i)_1\big) \,\overline{\#}\, e^i \\
\overset{(1.22)}{=} &\sum_{i=1}^n \varphi\big(\big(q_L^2\big)_2 p_L^2 e_i\big)h\big(q_L^2\big)_1 p_L^1 S^{-1}\big(q_L^1\big) \,\overline{\#}\, e^i \\
\overset{(1.24)}{=} &\sum_{i=1}^n \varphi(e_i)h \,\overline{\#}\, e^i = h \,\overline{\#}\, \varphi
\end{aligned}$$

and, in a similar way, for $u \in \mathrm{End}_k(H)$ and $h \in H$ we have that $(\mu \circ \mu^{-1})(u)(h) = u(h)$. Using the bijection μ, we transport the H-module algebra structure from $H \,\overline{\#}\, H^*$ to $\mathrm{End}_k(H)$. First we compute the transported multiplication $\overline{\circ}$: for all $u, v \in \mathrm{End}_k(H)$, we find

$$
\begin{aligned}
u \mathbin{\bar{\circ}} v &= \sum \mu\bigl(\mu^{-1}(u)\mu^{-1}(v)\bigr) \\
&= \sum_{i,j=1}^{n} \mu\Bigl(\bigl(u\bigl(q_L^2(e_i)_2\bigr)S^{-1}\bigl(q_L^1(e_i)_1\bigr) \mathbin{\overline{\#}} e^i\bigr)\bigl(v\bigl(Q_L^2(e_j)_2\bigr)S^{-1}\bigl(Q_L^1(e_j)_1\bigr) \mathbin{\overline{\#}} e^j\bigr)\Bigr) \\
&\overset{(2.11)}{=} \sum_{i,j=1}^{n} \mu\Bigl(u\bigl(q_L^2(e_i)_2\bigr)S^{-1}\bigl(q_L^1(e_i)_1\bigr)\bigl[v\bigl(Q_L^2(e_j)_2\bigr)S^{-1}\bigl(Q_L^1(e_j)_1\bigr)\bigr]_1 x^1 \\
&\qquad \mathbin{\overline{\#}} \bigl(e^i \leftharpoonup \bigl[v\bigl(Q_L^2(e_j)_2\bigr)S^{-1}\bigl(Q_L^1(e_j)_1\bigr)\bigr]_2 x^2\bigr)\bigl(e^j \leftharpoonup x^3\bigr)\Bigr)
\end{aligned}
$$

where $\sum Q_L^1 \otimes Q_L^2$ is another copy of q_L. Note that (1.3) and (1.20) imply

$$\sum S\bigl(x^1\bigr)q_L^1 x_1^2 \otimes q_L^2 x_2^2 \otimes x^3 = \sum q_L^1 X^1 \otimes \bigl(q_L^2\bigr)_1 X^2 \otimes (q_L^2)_2 X^3. \tag{2.15}$$

Using the above arguments, a long but straightforward computation shows that

$$(u \mathbin{\bar{\circ}} v)(h) = \sum u\bigl(v\bigl(hx^3X_2^3\bigr)S^{-1}\bigl(S\bigl(x^1X^2\bigr)\alpha x^2 X_1^3\bigr)\bigr)S^{-1}\bigl(X^1\bigr),$$

for all $h \in H$. Thus, we have obtained (2.13). Similar computations show that the transported unit and the H-action on $\mathrm{End}_k(H)$ are given by (2.14). □

Remarks 2.4. Let H be a finite-dimensional quasi-Hopf algebra, $\{e_i\}_{i=\overline{1,n}}$ a basis of H, and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*.

(1) The bijection μ defined in Proposition 2.3 induces an associative algebra structure on the k-vector space $H \otimes H^*$: it suffices to transport the composition on $\mathrm{End}_k(H)$ to $H \otimes H^*$.

(2) Let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right H-comodule algebra. As in the Hopf case, it is possible to associate different (quasi)smash products to $\mathfrak{A}$. Observe first that the map $\nu : \mathfrak{A} \mathbin{\overline{\#}} H^* \to \mathrm{Hom}_k(H, \mathfrak{A})$ given by $\nu(\mathfrak{a} \mathbin{\overline{\#}} \varphi)(h) = \varphi(h)\mathfrak{a}$, for all $\mathfrak{a} \in \mathfrak{A}$, $\varphi \in H^*$, and $h \in H$, is a k-linear isomorphism. The inverse of ν is given by the formula

$$\nu^{-1}(w) = \sum_{i=1}^{n} w(e_i) \mathbin{\overline{\#}} e^i$$

for $w \in \mathrm{Hom}_k(H, \mathfrak{A})$. Secondly, by transporting the quasi-smash algebra structure from $\mathfrak{A} \mathbin{\overline{\#}} H^*$ to $\mathrm{Hom}_k(H, \mathfrak{A})$ via the isomorphism ν, we obtain that $\mathrm{Hom}_k(H, \mathfrak{A})$ is an H-module algebra. So, if H is an arbitrary quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra, then we can define the quasi-smash product $\overline{\#}(H, \mathfrak{A})$ as follows: $\overline{\#}(H, \mathfrak{A})$ is the k-vector space $\mathrm{Hom}_k(H, \mathfrak{A})$ with multiplication given by

$$(v * w)(h) = \sum v\bigl(w\bigl(\tilde{x}^3 h_2\bigr)_{\langle 1\rangle}\tilde{x}^2 h_1\bigr)w\bigl(\tilde{x}^3 h_2\bigr)_{\langle 0\rangle}\tilde{x}^1 \tag{2.16}$$

for $v, w \in \overline{\#}(H, \mathfrak{A})$ and $h \in H$. The unit is $1_{\overline{\#}(H,\mathfrak{A})}(h) = \varepsilon(h)1_{\mathfrak{A}}$ and the H-module structure is given by $(h \cdot v)(h') = v(h'h)$, $h, h' \in H$, $v \in \mathrm{Hom}_k(H, \mathfrak{A})$. Of course, if H is finite-dimensional then $\mathfrak{A} \mathbin{\overline{\#}} H^* \simeq \overline{\#}(H, \mathfrak{A})$ as H-module algebras.

3. Two-sided Hopf modules and relative Hopf modules

3.1. Two-sided Hopf modules

The fact that a quasi-bialgebra is not coassociative entails that it makes no sense to consider comodules over quasi-bialgebras. Nevertheless, we can associate monoidal categories to quasi-bialgebras, in which we can consider coalgebras, and comodules over these coalgebras. This point of view has been used in [6,18,24] in order to define relative Hopf modules, quasi-Hopf bimodules, and two-sided two-cosided Hopf modules. In the sequel, we will study all these categories in a more general context. The categorical background will be presented in Section 3.3.

Definition 3.1. Let H be a quasi-bialgebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. A two-sided $(H, \mathfrak{A})$-Hopf module is an $(H, \mathfrak{A})$-bimodule M together with a k-linear map

$$\rho_M : M \to M \otimes H, \quad \rho_M(m) = \sum m_{(0)} \otimes m_{(1)},$$

satisfying the following relations, for all $m \in M$, $h \in H$, and $\mathfrak{a} \in \mathfrak{A}$ (the actions of $h \in H$ and $\mathfrak{a} \in \mathfrak{A}$ on $m \in M$ are denoted by $h \succ m$ and $m \prec \mathfrak{a}$):

$$(\mathrm{id}_M \otimes \varepsilon) \circ \rho_M = \mathrm{id}_M, \tag{3.1}$$

$$\Phi \cdot (\rho_M \otimes \mathrm{id}_H)\big(\rho_M(m)\big) = (\mathrm{id}_M \otimes \Delta)\big(\rho_M(m)\big) \cdot \Phi_\rho, \tag{3.2}$$

$$\rho_M(h \succ m) = \sum h_1 \succ m_{(0)} \otimes h_2 m_{(1)}, \tag{3.3}$$

$$\rho_M(m \prec \mathfrak{a}) = \sum m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle} \otimes m_{(1)} \mathfrak{a}_{\langle 1\rangle}. \tag{3.4}$$

The category of two-sided $(H, \mathfrak{A})$-Hopf modules and left H-linear, right $\mathfrak{A}$-linear, and right H-colinear maps is denoted by ${}_H\mathcal{M}_{\mathfrak{A}}^H$.

Observe that the category of two-sided (H, H)-Hopf bimodules is nothing else then the category of right quasi-Hopf H-bimodules introduced in [18].

We will use the following notation, similar to the notation for the comultiplication on a quasi-bialgebra:

$$(\rho_M \otimes \mathrm{id}_H)\big(\rho_M(m)\big) = \sum m_{(0,0)} \otimes m_{(0,1)} \otimes m_{(1)},$$

$$(\mathrm{id}_M \otimes \Delta_H)\big(\rho_M(m)\big) = \sum m_{(0)} \otimes m_{(1)_1} \otimes m_{(1)_2}.$$

Examples 3.2. Let H be a quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra.

(1) $\mathcal{V} = \mathfrak{A} \otimes H \in {}_H\mathcal{M}_{\mathfrak{A}}^H$. The structure maps are as follows:

$$h \succ (\mathfrak{a} \otimes h') = \mathfrak{a} \otimes hh', \qquad (\mathfrak{a} \otimes h) \prec \mathfrak{a}' = \sum \mathfrak{a}\mathfrak{a}'_{\langle 0\rangle} \otimes h\mathfrak{a}'_{\langle 1\rangle}, \quad \text{and}$$

$$\rho_{\mathcal{V}}(\mathfrak{a} \otimes h) = \sum \mathfrak{a}\widetilde{X}^1 \otimes h_1\widetilde{X}^2 \otimes h_2\widetilde{X}^3$$

for all $h, h' \in H$ and $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$. Verification of the details is left to the reader.

(2) $\mathcal{U} = H \otimes \mathfrak{A} \in {}_H\mathcal{M}_{\mathfrak{A}}^H$. Now the structure maps are given by the following formulas, for all $h, h' \in H$ and $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$:

$$h \succ (h' \otimes \mathfrak{a}) = hh' \otimes \mathfrak{a}, \qquad (h \otimes \mathfrak{a}) \prec \mathfrak{a}' = h \otimes \mathfrak{a}\mathfrak{a}', \quad \text{and}$$
$$\rho_{\mathcal{U}}(h \otimes \mathfrak{a}) = \sum h_1 S^{-1}\big(q_L^2 \widetilde{X}_2^3 g^2\big) \otimes \widetilde{X}^1 \mathfrak{a}_{\langle 0\rangle} \otimes h_2 S^{-1}\big(q_L^1 \widetilde{X}_1^3 g^1\big) \widetilde{X}^2 \mathfrak{a}_{\langle 1\rangle}. \tag{3.5}$$

Here $q_L = \sum q_L^1 \otimes q_L^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are the elements defined by the formulas (1.20) and (1.16).

To this end, consider $\theta : \mathcal{V} \to \mathcal{U}$ given by

$$\theta(\mathfrak{a} \otimes h) = \sum h S^{-1}\big(\mathfrak{a}_{\langle 1\rangle} \tilde{p}_\rho^2\big) \otimes \mathfrak{a}_{\langle 0\rangle} \tilde{p}_\rho^1$$

for all $h \in H$ and $\mathfrak{a} \in \mathfrak{A}$, where we use the notation

$$\tilde{p}_\rho = \sum \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \sum \tilde{x}^1 \otimes \tilde{x}^2 \beta S\big(\tilde{x}^3\big) \in \mathfrak{A} \otimes H. \tag{3.6}$$

We claim that θ is bijective; its inverse $\theta^{-1} : \mathcal{U} \to \mathcal{V}$ is defined as follows:

$$\theta^{-1}(h \otimes \mathfrak{a}) = \sum \tilde{q}_\rho^1 \mathfrak{a}_{\langle 0\rangle} \otimes h \tilde{q}_\rho^2 \mathfrak{a}_{\langle 1\rangle}$$

with the notation

$$\tilde{q}_\rho = \sum \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2 = \sum \widetilde{X}^1 \otimes S^{-1}\big(\alpha \widetilde{X}^3\big) \widetilde{X}^2 \in \mathfrak{A} \otimes H. \tag{3.7}$$

Furthermore, θ is a morphism of two-sided $(H, \mathfrak{A})$-Hopf bimodules, and we conclude that $\mathcal{U} = H \otimes \mathfrak{A}$ and $\mathfrak{A} \otimes H = \mathcal{V}$ are isomorphic in ${}_H\mathcal{M}_{\mathfrak{A}}^H$.

To prove this, we proceed as follows. First, by [16], we have the following relations, for all $\mathfrak{a} \in \mathfrak{A}$:

$$\sum \rho(\mathfrak{a}_{\langle 0\rangle}) \tilde{p}_\rho [1_{\mathfrak{A}} \otimes S(\mathfrak{a}_{\langle 1\rangle})] = \tilde{p}_\rho [\mathfrak{a} \otimes 1_H], \tag{3.8}$$
$$\sum \big[1_{\mathfrak{A}} \otimes S^{-1}(\mathfrak{a}_{\langle 1\rangle})\big] \tilde{q}_\rho \rho(\mathfrak{a}_{\langle 0\rangle}) = [\mathfrak{a} \otimes 1_H] \tilde{q}_\rho, \tag{3.9}$$
$$\sum \rho\big(\tilde{q}_\rho^1\big) \tilde{p}_\rho \big[1_{\mathfrak{A}} \otimes S\big(\tilde{q}_\rho^2\big)\big] = 1_{\mathfrak{A}} \otimes 1_H, \tag{3.10}$$
$$\sum \big[1_{\mathfrak{A}} \otimes S^{-1}\big(\tilde{p}_\rho^2\big)\big] \tilde{q}_\rho \rho\big(\tilde{p}_\rho^1\big) = 1_{\mathfrak{A}} \otimes 1_H, \tag{3.11}$$
$$\Phi_\rho(\rho \otimes \mathrm{id}_H)(\tilde{p}_\rho) \tilde{p}_\rho = \sum (\mathrm{id} \otimes \Delta)\big(\rho\big(\tilde{x}^1\big) \tilde{p}_\rho\big)\big(1_{\mathfrak{A}} \otimes g^1 S\big(\tilde{x}^3\big) \otimes g^2 S\big(\tilde{x}^2\big)\big), \tag{3.12}$$
$$(\tilde{q}_\rho \otimes 1_H)(\rho \otimes \mathrm{id}_H)(\tilde{q}_\rho) \Phi_\rho^{-1}$$
$$= \sum \big[1_{\mathfrak{A}} \otimes S^{-1}\big(f^2 \widetilde{X}^3\big) \otimes S^{-1}\big(f^1 \widetilde{X}^2\big)\big] (\mathrm{id}_{\mathfrak{A}} \otimes \Delta)\big(\tilde{q}_\rho \rho\big(\widetilde{X}^1\big)\big). \tag{3.13}$$

Here $f = \sum f^1 \otimes f^2$ is the element defined in (1.15) and $f^{-1} = \sum g^1 \otimes g^2$. Using (3.8)–(3.11), we can show easily that θ and θ^{-1} are inverses, and that $\mathcal{U}$ is an $(H, \mathfrak{A})$-bimodule via the actions $\succ$ and $\prec$. One can finally compute the right H-coaction on $\mathcal{U}$ transported from the coaction on $\mathcal{V}$ using θ, and then see that it coincides with (3.5). For, observe that (3.6)–(2.2) and (2.4) imply

$$\sum \widetilde{X}^1_{\langle 1\rangle} \tilde{p}^2_\rho S\big(\widetilde{X}^2\big) \otimes \widetilde{X}^1_{\langle 0\rangle} \tilde{p}^1_\rho \otimes \widetilde{X}^3 = \sum \tilde{x}^2 S\big(\tilde{x}^3_1 p^1_L\big) \otimes \tilde{x}^1 \otimes \tilde{x}^3_2 p^2_L, \tag{3.14}$$

where $p_L = \sum p^1_L \otimes p^2_L$ is the element defined in (1.20). We also mention that the computation uses the formula (3.13); the details are left to the reader.

3.2. Two-sided Hopf modules and relative Hopf modules

Our aim is to prove a duality theorem for two-sided Hopf modules: if H is a finite-dimensional quasi-Hopf algebra, then the category ${}_H\mathcal{M}^H_{\mathfrak{A}}$ is isomorphic to a category of relative Hopf modules as introduced in [6]. Recall that a right (H^*, A)-Hopf module M is a k-vector space M which is also a right H^*-comodule and a right A-module in the monoidal category of right H^*-comodules $\mathcal{M}^{H^*}$. In terms of H this means:

- M is a left H-module; denote the action of $h \in H$ on $m \in M$ by $h \bullet m$;
- A acts on M from the right; denote the action of $a \in A$ on $m \in M$ by $m \bullet a$;
- for all $m \in M$, $h \in H$, and $a, a' \in A$, we have

$$m \bullet 1_A = m,$$

$$(m \bullet a) \bullet a' = \sum \big(X^1 \bullet m\big) \bullet \big[\big(X^2 \cdot a\big)\big(X^3 \cdot a'\big)\big], \tag{3.15}$$

$$h \bullet (m \bullet a) = \sum (h_1 \bullet m) \bullet (h_2 \cdot a). \tag{3.16}$$

$\mathcal{M}^{H^*}_A$ will be the category of right (H^*, A)-Hopf modules and A-linear H^*-colinear maps. Before we can establish the claimed isomorphism of categories, we need some lemmas.

Lemma 3.3. *Let H be a finite-dimensional quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. We have a functor*

$$F : {}_H\mathcal{M}^H_{\mathfrak{A}} \to \mathcal{M}^{H^*}_{\mathfrak{A}\overline{\#}H^*}.$$

For $M \in {}_H\mathcal{M}^H_{\mathfrak{A}}$, $F(M) = M$, with structure maps

- *M is a left H-module via $h \bullet m = S^2(h) \succ m$, $m \in M$, $h \in H$;*
- *$\mathfrak{A} \,\overline{\#}\, H^*$ acts on M from the right by*

$$m \bullet (\mathfrak{a} \,\overline{\#}\, \varphi) = \sum \big\langle \varphi, S^{-1}\big(S\big(U^1\big) f^2 m_{(1)} \mathfrak{a}_{\langle 1\rangle} \tilde{p}^2_\rho\big)\big\rangle S\big(U^2\big) f^1 \succ m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle} \tilde{p}^1_\rho, \tag{3.17}$$

$$\textit{where} \quad U = \sum U^1 \otimes U^2 = \sum g^1 S\big(q^2_R\big) \otimes g^2 S\big(q^1_R\big). \tag{3.18}$$

Proof. The most difficult part of the proof is to show that $F(M)$ satisfies the relations (3.15) and (3.16). It is then straightforward to show that a map in ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is also a map in $\mathcal{M}_{\mathfrak{A}\overline{\#}H^*}^{H^*}$ and that F is a functor.

By [18, Lemma 3.13] we have, for all $h \in H$:

$$U[1 \otimes S(h)] = \sum \Delta\bigl(S(h_1)\bigr)U(h_2 \otimes 1), \tag{3.19}$$

$$\Phi^{-1}(\mathrm{id} \otimes \Delta)(U)(1 \otimes U) = \sum(\Delta \otimes \mathrm{id})\bigl(\Delta\bigl(S\bigl(X^1\bigr)\bigr)U\bigr)\bigl(X^2 \otimes X^3 \otimes 1\bigr). \tag{3.20}$$

Write $f = \sum f^1 \otimes f^2 = \sum F^1 \otimes F^2$, $f^{-1} = \sum g^1 \otimes g^2$, $\tilde{p}_\rho = \sum \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \sum \tilde{P}_\rho^1 \otimes \tilde{P}_\rho^1$, and $U = \sum U^1 \otimes U^2 = \sum \mathbf{U}^1 \otimes \mathbf{U}^2$. For all $m \in M$, $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, and $\varphi, \psi \in H^*$, we compute that

$$\begin{aligned}
&\bigl(X^1 \bullet m\bigr) \bullet \bigl\{\bigl[X^2 \cdot (\mathfrak{a} \,\overline{\#}\, \varphi)\bigr]\bigl[X^3 \cdot (\mathfrak{a}' \,\overline{\#}\, \psi)\bigr]\bigr\} \\
&= \sum \Bigl\langle \bigl(X^2 \rightharpoonup \varphi \leftharpoonup \mathfrak{a}'_{\langle 1\rangle}\tilde{x}^2\bigr)\bigl(X^3 \rightharpoonup \psi \leftharpoonup \tilde{x}^3\bigr), \\
&\qquad S^{-1}\bigl(S\bigl(U^1\bigr)f^2S^2\bigl(X^1\bigr)_2 m_{(1)}\bigl(\mathfrak{a}\mathfrak{a}'_{\langle 0\rangle}\tilde{x}^1\bigr)_{\langle 1\rangle}\tilde{p}_\rho^2\bigr)\Bigr\rangle \\
&\qquad S\bigl(U^2\bigr)f^1S^2\bigl(X^1\bigr)_1 \succ m_{(0)} \prec \bigl(\mathfrak{a}\mathfrak{a}'_{\langle 0\rangle}\tilde{x}^1\bigr)_{\langle 0\rangle}\tilde{p}_\rho^1 \\
\overset{(1.11)}&{=} \sum \bigl\langle \varphi, S^{-1}\bigl(F^2S\bigl(U^1\bigr)_2 S\bigl(S\bigl(X^1\bigr)_1\bigr)_2 f_2^2 m_{(1)_2}\mathfrak{a}_{\langle 1\rangle_2}\mathfrak{a}'_{\langle 0,1\rangle_2}\tilde{x}^1_{\langle 1\rangle_2}\bigl(\tilde{p}_\rho^2\bigr)_2 g^2 S\bigl(\mathfrak{a}'_{\langle 1\rangle}\tilde{x}^2\bigr)\bigr)X^2\bigr\rangle \\
&\qquad \bigl\langle \psi, S^{-1}\bigl(F^1S\bigl(U^1\bigr)_1 S\bigl(S\bigl(X^1\bigr)_1\bigr)_1 f_1^2 m_{(1)_1}\mathfrak{a}_{\langle 1\rangle_1}\mathfrak{a}'_{\langle 0,1\rangle_1}\tilde{x}^1_{\langle 1\rangle_1}\bigl(\tilde{p}_\rho^2\bigr)_1 g^1 S\bigl(\tilde{x}^3\bigr)\bigr)X^3\bigr\rangle \\
&\qquad S\bigl(S\bigl(X^1\bigr)_2 U^2\bigr)f^1 \succ m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle}\mathfrak{a}'_{\langle 0,0\rangle}\tilde{x}^1_{\langle 0\rangle}\tilde{p}_\rho^1 \\
\overset{(1.11),(3.13),(2.1)}&{=} \sum \bigl\langle \varphi, S^{-1}\bigl(S\bigl(S\bigl(X^1\bigr)_{(1,1)}U_1^1X^2\bigr)F^2 f_2^2 m_{(1)_2}\mathfrak{a}_{\langle 1\rangle_2}\widetilde{X}^3\mathfrak{a}'_{\langle 0,1\rangle}\tilde{p}_\rho^2 S\bigl(\mathfrak{a}'_{\langle 1\rangle}\bigr)\bigr)\bigr\rangle \\
&\qquad \bigl\langle \psi, S^{-1}\bigl(S\bigl(S\bigl(X^1\bigr)_{(1,2)}U_2^1X^3\bigr)F^1 f_1^2 m_{(1)_1}\mathfrak{a}_{\langle 1\rangle_1}\widetilde{X}^2\bigl(\mathfrak{a}'_{\langle 0,0\rangle}\tilde{p}_\rho^1\bigr)_{\langle 1\rangle}\tilde{P}_\rho^2\bigr)\bigr\rangle \\
&\qquad S\bigl(S\bigl(X^1\bigr)_2 U^2\bigr)f^1 \succ m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle}\widetilde{X}^1\bigl(\mathfrak{a}'_{\langle 0,0\rangle}\tilde{p}_\rho^1\bigr)_{\langle 0\rangle}\tilde{P}_\rho^1 \\
\overset{(3.20),(3.8)}&{=} \sum \bigl\langle \varphi, S^{-1}\bigl(S\bigl(x^1U^1\bigr)F^2 f_2^2 m_{(1)_2}\mathfrak{a}_{\langle 1\rangle_2}\widetilde{X}^3\tilde{p}_\rho^2\bigr)\bigr\rangle \\
&\qquad \bigl\langle \psi, S^{-1}\bigl(S\bigl(x^2U_1^2\mathbf{U}^1\bigr)F^1 f_1^2 m_{(1)_1}\mathfrak{a}_{\langle 1\rangle_1}\widetilde{X}^2\bigl(\tilde{p}_\rho^1\mathfrak{a}'\bigr)_{\langle 1\rangle}\tilde{P}_\rho^2\bigr)\bigr\rangle \\
&\qquad S\bigl(x^3U_2^2\mathbf{U}^2\bigr)f^1 \succ m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle}\widetilde{X}^1\bigl(\tilde{p}_\rho^1\mathfrak{a}'\bigr)_{\langle 0\rangle}\tilde{P}_\rho^2 \\
\overset{(1.9),(1.18),(2.1)}&{=} \sum \bigl\langle \varphi, S^{-1}\bigl(S\bigl(U^1\bigr)F^2 m_{(1)}\mathfrak{a}_{\langle 1\rangle}\tilde{p}_\rho^2\bigr)\bigr\rangle \\
&\qquad \bigl\langle \psi, S^{-1}\bigl(S\bigl(U_1^2\mathbf{U}^1\bigr)f^2F_2^1 m_{(0,1)}\mathfrak{a}_{\langle 0,1\rangle}\bigl(\tilde{p}_\rho^1\mathfrak{a}'\bigr)_{\langle 1\rangle}\tilde{P}_\rho^2\bigr)\bigr\rangle \\
&\qquad S\bigl(U_2^2\mathbf{U}^2\bigr)f^1F_1^1 \succ m_{(0,0)} \prec \mathfrak{a}_{\langle 0,0\rangle}\bigl(\tilde{p}_\rho^1\mathfrak{a}'\bigr)_{\langle 0\rangle}\tilde{P}_\rho^1 \\
\overset{(1.11),(3.17)}&{=} \sum \bigl\langle \varphi, S^{-1}\bigl(S\bigl(U^1\bigr)F^2 m_{(1)}\mathfrak{a}_{\langle 1\rangle}\tilde{p}_\rho^2\bigr)\bigr\rangle \bigl(S\bigl(U^2\bigr)F^1 \succ m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle}\tilde{p}_\rho^1\bigr) \bullet (\mathfrak{a}' \,\overline{\#}\, \psi) \\
\overset{(3.17)}&{=} [m \bullet (\mathfrak{a} \,\overline{\#}\, \varphi)] \bullet (\mathfrak{a}' \,\overline{\#}\, \psi).
\end{aligned}$$

Similar computations show that

$$\sum (h_1 \bullet m) \bullet \big(h_2 \cdot (\mathfrak{a} \mathbin{\overline{\#}} \varphi)\big) = h \bullet \big[m \bullet (\mathfrak{a} \mathbin{\overline{\#}} \varphi)\big],$$

for all $h \in H$, $\mathfrak{a} \in \mathfrak{A}$, and $\varphi \in H^*$, so the proof is complete. □

Let us next discuss the construction in the converse direction.

Lemma 3.4. *Let H be a finite-dimensional quasi-Hopf algebra, $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra, and M a right $(H^*, \mathfrak{A} \mathbin{\overline{\#}} H^*)$-Hopf module. Then we have a functor*

$$G : \mathcal{M}^{H^*}_{\mathfrak{A} \overline{\#} H^*} \to {}_H\mathcal{M}^H_{\mathfrak{A}}.$$

For $M \in \mathcal{M}^{H^*}_{\mathfrak{A} \overline{\#} H^*}$, $G(M) = M$, with structure maps ($h \in H$, $m \in M$, $\mathfrak{a} \in \mathfrak{A}$):

- *$h \succ m = S^{-2}(h) \bullet m$;*
- *$m \prec \mathfrak{a} = m \bullet (\mathfrak{a} \mathbin{\overline{\#}} \varepsilon)$;*
- *$\rho_M : M \to M \otimes H$ given by*

$$\begin{aligned}\rho_M(m) &= \sum m_{\{0\}} \otimes m_{\{1\}} \\ &= \sum_{i=1}^{n} \big[S^{-1}\big(V^2 g^2\big) \bullet m\big] \bullet \big(\tilde{q}^1_\rho \mathbin{\overline{\#}} S^{-1}\big(V^1 g^1\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho\big) \otimes e_i, \end{aligned} \tag{3.21}$$

where $\{e_i\}_{i=\overline{1,n}}$ and $\{e^i\}_{i=\overline{1,n}}$ are dual bases and

$$V = \sum V^1 \otimes V^2 = \sum S^{-1}\big(f^2 p^2_R\big) \otimes S^{-1}\big(f^1 p^1_R\big). \tag{3.22}$$

Proof. As in the previous part, the main thing to show is that $G(M)$ is an object of ${}_H\mathcal{M}^H_{\mathfrak{A}}$. It is then straightforward to show that G behaves well on the level of the morphisms (G is the identity on the morphisms).

From the fact that S^{-2} is an algebra map, it follows that M is a left H-module via the action $h \succ m = S^{-2}(h) \bullet m$. Take the map

$$i : \mathfrak{A} \to \mathfrak{A} \mathbin{\overline{\#}} H^*, \quad i(\mathfrak{a}) = \mathfrak{a} \mathbin{\overline{\#}} \varepsilon,$$

for all $\mathfrak{a} \in \mathfrak{A}$. Then i is injective map, $i(1_{\mathfrak{A}}) = 1_{\mathfrak{A} \overline{\#} H^*}$, and $i(\mathfrak{a}\mathfrak{a}') = i(\mathfrak{a}) i(\mathfrak{a}')$, for all $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$. Therefore, M becomes a right $\mathfrak{A}$-module by setting $m \prec \mathfrak{a} = m \bullet i(\mathfrak{a}) = m \bullet (\mathfrak{a} \mathbin{\overline{\#}} \varepsilon)$, $m \in M$, $\mathfrak{a} \in \mathfrak{A}$. Moreover, it is not hard to see that, with this structure, M is an $(H, \mathfrak{A})$-bimodule. In order to check the relations (3.1)–(3.3), we need some formulas due to Hausser and Nill [16, Lemma 3.13], namely:

$$\big[1 \otimes S^{-1}(h)\big]V = \sum (h_2 \otimes 1)V\Delta\big(S^{-1}(h_1)\big), \tag{3.23}$$

$$(\Delta \otimes \mathrm{id})(V)\Phi^{-1} = \sum \big(X^2 \otimes X^3 \otimes 1\big)(1 \otimes V)(\mathrm{id} \otimes \Delta)\big(V\Delta\big(S^{-1}\big(X^1\big)\big)\big). \tag{3.24}$$

Also, it is clear that

$$(\varphi \leftharpoonup h)S = S^{-1}(h) \rightharpoonup \varphi S, \qquad (h \rightharpoonup \varphi)S = \varphi S \leftharpoonup S^{-1}(h) \tag{3.25}$$

for all $h \in H$ and $\varphi \in H^*$. Using (1.11), it follows that

$$(\varphi S)(\psi S) = \sum \big[\big(g^1 \rightharpoonup \psi \leftharpoonup f^1\big)\big(g^2 \rightharpoonup \varphi \leftharpoonup f^2\big)\big]S \tag{3.26}$$

for all $\varphi, \psi \in H^*$. Now, for any $h \in H$ and $m \in M$, we compute that

$$\begin{aligned}
&\sum h_1 \succ m_{\{0\}} \otimes h_2 m_{\{1\}} \\
&= \sum_{i=1}^{n} S^{-2}(h_1) \bullet \big[\big(S^{-1}\big(V^2 g^2\big) \bullet m\big) \bullet \big(\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}\big(V^1 g^1\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho\big)\big] \otimes h_2 e_i \\
\overset{(3.16)}{=}& \sum_{i=1}^{n} \big[S^{-2}(h_1)_1 S^{-1}\big(V^2 g^2\big) \bullet m\big] \\
&\quad \bullet \big(\tilde{q}^1_\rho \,\#\, S^{-2}(h_1)_2 S^{-1}\big(V^1 g^1\big) \rightharpoonup (e^i \leftharpoonup h_2) S \leftharpoonup \tilde{q}^2_\rho\big) \otimes e_i \\
\overset{(1.11),(3.25)}{=}& \sum_{i=1}^{n} \big[S^{-1}\big(V^2 S^{-1}(h_1)_2 g^2\big) \bullet m\big] \\
&\quad \bullet \big(\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}\big(h_2 V^1 S^{-1}(h_1)_1 g^1\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho\big) \otimes e_i \\
\overset{(3.23)}{=}& \sum_{i=1}^{n} \big[S^{-1}\big(V^2 g^2\big) S^{-2}(h) \bullet m\big] \bullet \big(\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}\big(V^1 g^1\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho\big) \otimes e_i \\
&= \rho_M\big(S^{-2}(h) \bullet m\big) = \rho_M(h \succ m),
\end{aligned}$$

and similarly, for any $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$ one can show that

$$\sum m_{\{0\}} \prec \mathfrak{a}_{\langle 0\rangle} \otimes m_{\{1\}}\mathfrak{a}_{\langle 1\rangle} = \rho_M(m \prec \mathfrak{a}),$$

so the relations (3.3) hold. (3.1) is obviously satisfied, thus remain to check (3.2) for our structures. This fact is left to the reader since it is a similar computation as above. □

We are now able to prove the main result of this section, generalizing [11, Proposition 2.3].

Theorem 3.5. *Let H be a finite-dimensional quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. Then the category of two-sided $(H, \mathfrak{A})$-Hopf modules ${}_H\mathcal{M}^H_{\mathfrak{A}}$ is isomorphic to the category of right $(H^*, \mathfrak{A} \,\overline{\#}\, H^*)$-Hopf modules $\mathcal{M}^{H^*}_{\mathfrak{A}\overline{\#}H^*}$.*

Proof. It suffices to show that the functors F and G from Lemmas 3.3 and 3.4 are inverses.

First, let $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$. The structures on $G(F(M))$ (using first Lemma 3.3 and then Lemma 3.4) are denoted by $\succ'$, $\prec'$, and ρ'_M. For any $m \in M$, $h \in H$, and $\mathfrak{a} \in \mathfrak{A}$, we have that

$$h \succ' m = S^{-2}(h) \bullet m = S^2\bigl(S^{-2}(h)\bigr) \succ m = h \succ m,$$
$$m \prec' \mathfrak{a} = m \bullet (\mathfrak{a} \,\overline{\#}\, \varepsilon) = m \prec \mathfrak{a}$$

because $\sum \varepsilon(U^1)U^2 = \sum \varepsilon(f^2)f^1 = 1$ and $\sum \varepsilon(m_{(1)})m_{(0)} = m$, $\sum \varepsilon(\mathfrak{a}_{\langle 1\rangle})\mathfrak{a}_{\langle 0\rangle} = \mathfrak{a}$. In order to prove that $\rho'_M = \rho_M$, observe first that

$$\sum g^1 S\bigl(g^2\alpha\bigr) = \beta, \tag{3.27}$$

where we write $f^{-1} = \sum g^1 \otimes g^2$. The proof of (3.27) can be found in [6, Lemma 2.6(i)] (in the equivalent form $\sum g^2\alpha S^{-1}(g^1) = S^{-1}(\beta)$). (3.27) together with (3.18), (1.9), and (1.18) implies

$$\sum g_2^2 U^2 \otimes g^1 S\bigl(g_1^2 U^1\bigr) = \sum p_L^2 \otimes S\bigl(p_L^1\bigr) \tag{3.28}$$

where $p_L = \sum p_L^1 \otimes p_L^2$ is the element defined by (1.20). Secondly, by $\sum S^{-1}(f^2)\beta f^1 = S^{-1}(\alpha)$, (1.9), and (1.18), we have that

$$\sum S\bigl(p_L^2\bigr) f^1 F_1^1 \otimes S^{-1}\bigl(F^2\bigr) S\bigl(p_L^1\bigr) f^2 F_2^1 = q_R \tag{3.29}$$

where $\sum F^1 \otimes F^2$ is another copy of f, and q_R is the element defined by (1.19). Finally, from (3.28), (3.29), and (1.23), it follows that

$$\sum S\bigl(g_2^2 U^2\bigr) f^1 F_1^1 \bigl(p_R^1\bigr)_1 \otimes S^{-1}\bigl(F^2 p_R^2\bigr) g^1 S\bigl(g_1^2 U^1\bigr) f^2 F_2^1 \bigl(p_R^1\bigr)_2 = 1 \otimes 1. \tag{3.30}$$

We now compute for $m \in M$ that

$$\begin{aligned}
\rho'_M(m) &= \sum_{i=1}^n \bigl[S^{-1}\bigl(V^2 g^2\bigr) \bullet m\bigr] \bullet \bigl(\tilde{q}_\rho^1 \,\#\, S^{-1}\bigl(V^1 g^1\bigr) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2\bigr) \otimes e_i \\
&= \sum_{i=1}^n \bigl[S\bigl(V^2 g^2\bigr) \succ m\bigr] \bullet \bigl(\tilde{q}_\rho^1 \,\overline{\#}\, S^{-1}\bigl(V^1 g^1\bigr) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2\bigr) \otimes e_i \\
\overset{(3.17)}{=}& \sum_{i=1}^n \bigl\langle S^{-1}\bigl(V^1 g^1\bigr) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2, S^{-1}\bigl(S\bigl(U^1\bigr) f^2 S\bigl(V^2 g^2\bigr)_2 m_{(1)} \bigl(\tilde{q}_\rho^1\bigr)_{\langle 1\rangle} \tilde{p}_\rho^2\bigr)\bigr\rangle \\
&\quad S\bigl(U^2\bigr) f^1 S\bigl(V^2 g^2\bigr)_1 \succ m_{(0)} \prec \bigl(\tilde{q}_\rho^1\bigr)_{\langle 0\rangle} \tilde{p}_\rho^1 \otimes e_i
\end{aligned}$$

$$\overset{(1.11)}{=} \sum S\big(V_2^2 g_2^2 U^2\big) f^1 \succ m_{(0)} \prec \big(\tilde{q}_\rho^1\big)_{\langle 0\rangle} \tilde{p}_\rho^1 \otimes V^1 g^1 S\big(V_1^2 g_1^2 U^1\big) f^2 m_{(1)} \big(\tilde{q}_\rho^1\big)_{\langle 1\rangle} \tilde{p}_\rho^2 S\big(\tilde{q}_\rho^2\big)$$

$$\overset{(3.10)}{=} \sum S\big(V_2^2 g_2^2 U^2\big) f^1 \succ m_{(0)} \otimes V^1 g^1 S\big(V_1^2 g_1^2 U^1\big) f^2 m_{(1)}$$

$$\overset{(3.22)}{\underset{(1.11)}{=}} \sum S\big(g_2^2 U^2\big) f^1 F_1^1 \big(p_R^1\big)_1 \succ m_{(0)} \otimes S^{-1}\big(F^2 p_R^2\big) g^1 S\big(g_1^2 U^1\big) f^2 F_2^1 \big(p_R^1\big)_2 m_{(1)}$$

$$\overset{(3.30)}{=} \sum m_{(0)} \otimes m_{(1)} = \rho_M(m),$$

and this finishes the proof of the fact that $G(F(M)) = M$.

Conversely, take $M \in \mathcal{M}^{H^*}_{\mathfrak{A}\overline{\#}H^*}$. We want to show that $F(G(M)) = M$. Denote the left H-action and the right $\mathfrak{A}\,\overline{\#}\,H^*$-action on $F(G(M))$ by $\bullet'$. Using Lemmas 3.3 and 3.4, we find, for all $h \in H$ and $m \in M$:

$$h \bullet' m = S^2(h) \succ m = S^{-2}\big(S^2(h)\big) \bullet m = h \bullet m.$$

The proof of the fact that the right $\mathfrak{A}\,\overline{\#}\,H^*$-actions $\bullet$ and $\bullet'$ on M coincide is somewhat more complicated. Since $\sum f^2 S^{-1}(f^1\beta) = \alpha$, (1.9) and (1.18) imply

$$\sum F^1 f_1^1 p_R^1 \otimes f^2 S^{-1}\big(F^2 f_2^1 p_R^2\big) = \sum S\big(q_L^2\big) \otimes q_L^1 \tag{3.31}$$

where $q_L = \sum q_L^1 \otimes q_L^2$ is the element defined by (1.20). Also, by (1.9), (1.18), and using $\sum S(g^1)\alpha g^2 = S(\beta)$, we can prove the following relation:

$$\sum S\big(G^1\big) q_L^1 G_1^2 g^1 \otimes q_L^2 G_2^2 g^2 = \sum S\big(p_R^2\big) \otimes S\big(p_R^1\big) \tag{3.32}$$

where $\sum G^1 \otimes G^2$ is another copy of f^{-1}. Now, from (3.18), (1.11), (3.31), (3.32), and (1.23) it follows that

$$\sum S^{-1}\big(F^1 f_1^1 p_R^1\big) U_2^2 g^2 \otimes S\big(U^1\big) f^2 S^{-1}\big(F^2 f_2^1 p_R^2\big) U_1^2 g^1 = 1 \otimes 1. \tag{3.33}$$

Therefore, for all $m \in M$, $\mathfrak{a} \in \mathfrak{A}$, and $\varphi \in H^*$, we have that

$$m \bullet' (\mathfrak{a}\,\overline{\#}\,\varphi)$$

$$\overset{(3.17)}{=} \sum \big\langle \varphi, S^{-1}\big(S\big(U^1\big) f^2 m_{\{1\}} \mathfrak{a}_{\langle 1\rangle} \tilde{p}_\rho^2\big)\big\rangle S\big(U^2\big) f^1 \succ m_{\{0\}} \prec \mathfrak{a}_{\langle 0\rangle} \tilde{p}_\rho^1$$

$$\overset{(3.21),(3.15),(2.11)}{=} \sum_{i=1}^n \big\langle \varphi, S^{-1}\big(S\big(U^1\big) f^2 e_i \mathfrak{a}_{\langle 1\rangle} \tilde{p}_\rho^2\big)\big\rangle S^{-2}\big(S\big(U^2\big) f^1\big) \bullet \Big\{\big[S^{-1}\big(V^2 g^2\big) \bullet m\big] \bullet \big[\tilde{q}_\rho^1 \mathfrak{a}_{\langle 0,0\rangle} \big(\tilde{p}_\rho^1\big)_{\langle 0\rangle}\,\overline{\#}\, S^{-1}\big(V^1 g^1\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2 \mathfrak{a}_{\langle 0,1\rangle} \big(\tilde{p}_\rho^1\big)_{\langle 1\rangle}\big]\Big\}$$

$$= \sum_{i=1}^n \varphi(e_i) S^{-2}\big(S\big(U^2\big) f^1\big) \bullet \Big\{\big[S^{-1}\big(V^2 g^2\big) \bullet m\big] \bullet \Big[\tilde{q}_\rho^1 \mathfrak{a}_{\langle 0,0\rangle} \big(\tilde{p}_\rho^1\big)_{\langle 0\rangle}\,\overline{\#}\, S^{-1}\big(V^1 g^1\big) \rightharpoonup \big(\mathfrak{a}_{\langle 1\rangle} \tilde{p}_\rho^2 \rightharpoonup e^i S^{-1} \leftharpoonup S\big(U^1\big) f^2\big) S \leftharpoonup \tilde{q}_\rho^2 \mathfrak{a}_{\langle 0,1\rangle} \big(\tilde{p}_\rho^1\big)_{\langle 1\rangle}\Big]\Big\}$$

$$\overset{(3.25)}{\underset{(3.10)}{\overset{(3.8)}{=}}} \sum S^{-2}\big(S(U^2)f^1\big) \bullet \big\{\big[S^{-1}\big(V^2g^2\big) \bullet m\big] \bullet \big[\mathfrak{a} \,\overline{\#}\, S^{-1}\big(S(U^1)f^2V^1g^1\big) \rightharpoonup \varphi\big]\big\}$$

$$\overset{(3.16)}{\underset{(1.11)}{=}} \sum \big[S^{-1}\big(V^2S^{-1}\big(S(U^2)f^1\big)_2 g^2\big) \bullet m\big] \bullet \big[\mathfrak{a} \,\overline{\#}\, S^{-1}\big(S(U^1)f^2V^1S^{-1}\big(S(U^2)f^1\big)_1 g^1\big) \rightharpoonup \varphi\big]$$

$$\overset{(3.22)}{\underset{(1.11)}{=}} \sum \big[S^{-1}\big(S^{-1}\big(F^1f_1^1p_R^1\big)U_2^2g^2\big) \bullet m\big] \bullet \big[\mathfrak{a} \,\overline{\#}\, S^{-1}\big(S(U^1)f^2S^{-1}\big(F^2f_2^1p_R^2\big)U_1^2g^1\big) \rightharpoonup \varphi\big]$$

$$\overset{(3.33)}{=} m \bullet (\mathfrak{a} \,\overline{\#}\, \varphi),$$

and this finishes our proof. □

If H is a finite-dimensional quasi-Hopf algebra and A is a left H-module algebra then the category $\mathcal{M}_A^{H^*}$ is isomorphic to the category of right modules over the smash product $A \# H$ [6, Proposition 2.7]. Let M be a right $A \# H$-module, and denote the right action of $a \# h \in A \# H$ on $m \in M$ by $m \leftarrow (a \# h)$. Following [6], M is a right (H^*, A)-Hopf module, with structure maps

$$h \bullet m = m \leftarrow \big(1 \# S(h)\big), \qquad m \bullet a = \sum m \leftarrow \big[g^1S\big(q_R^2\big) \cdot a \# g^2S\big(q_R^1\big)\big] \tag{3.34}$$

for all $m \in M$, $a \in A$, and $h \in H$. Conversely, if M is a right (H^*, A)-Hopf module then M is a right $A \# H$-module, with $A \# H$-action

$$m \leftarrow (a \# h) = \sum S^{-1}(h) \bullet \big[\big(S^{-1}\big(q_L^2g^2\big) \bullet m\big) \bullet \big(S^{-1}\big(q_L^1g^1\big) \cdot a\big)\big]. \tag{3.35}$$

Here $q_R = \sum q_R^1 \otimes q_R^2$, $q_L = \sum q_L^1 \otimes q_L^2$, and $f^{-1} = \sum g^1 \otimes g^2$ are the elements defined by (1.19), (1.20), and (1.16). Combining this with Theorem 3.5, we obtain the following result.

Corollary 3.6. *Let H be a finite-dimensional quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. Then the category ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of right $(\mathfrak{A} \,\overline{\#}\, H^*) \# H$-modules, $\mathcal{M}_{(\mathfrak{A}\overline{\#}H^*)\#H}$.*

For later use, we describe the isomorphism of Corollary 3.6 explicitly, leaving verification of the details to the reader.

First take $M \in \mathcal{M}_{(\mathfrak{A}\overline{\#}H^*)\#H}$. The following structure maps make $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$:

$$h \succ m = m \leftarrow \big((1_{\mathfrak{A}} \,\overline{\#}\, \varepsilon) \# S^{-1}(h)\big), \tag{3.36}$$

$$m \prec \mathfrak{a} = m \leftarrow \big((\mathfrak{a} \,\overline{\#}\, \varepsilon) \# 1\big), \tag{3.37}$$

$$\rho_M(m) = \sum_{i=1}^n m \leftarrow \big[\big(\tilde{q}_\rho^1 \,\overline{\#}\, S^{-1}\big(g^2\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2\big) \# S^{-1}\big(g^1\big)\big] \otimes e_i \tag{3.38}$$

for all $m \in M$, $h \in H$, and $\mathfrak{a} \in \mathfrak{A}$; $\tilde{q}_\rho = \sum \tilde{q}^1_\rho \otimes \tilde{q}^2_\rho$ is the element defined in (3.7), $\{e_i\}$ is a basis of H and $\{e^i\}$ is the corresponding dual basis of H^*.

Now take $M \in {}_H\mathcal{M}^H_{\mathfrak{A}}$. Then M is a right $(\mathfrak{A} \,\overline{\#}\, H^*) \# H$-module via the action

$$m \leftarrow [(\mathfrak{a} \,\overline{\#}\, \varphi) \# h] = \sum \big\langle \varphi, S^{-1}\big(f^2 m_{(1)} \mathfrak{a}_{\langle 1\rangle} \tilde{p}^2_\rho\big)\big\rangle S(h) f^1 \succ m_{(0)} \prec \mathfrak{a}_{\langle 0\rangle} \tilde{p}^1_\rho. \tag{3.39}$$

In [18], it is announced that, for a finite-dimensional quasi-Hopf algebra H, the category of right quasi-Hopf H-bimodules ${}_H\mathcal{M}^H_H$ naturally coincides with the category of representations of the two-sided crossed product $H \rtimes H^* \ltimes H$ constructed in [16]. We will show in Section 4 that the algebras $H \rtimes H^* \ltimes H$ and $(H \# H^*) \# H$ are equal.

3.3. Two-sided Hopf modules and coalgebras over comonads

Now, let H be a quasi-bialgebra and $\mathfrak{A}$ a right H-comodule algebra. We will show that the category ${}_H\mathcal{M}^H_{\mathfrak{A}}$ is isomorphic to the category of $\mathbb{U}$-coalgebras, where $\mathbb{U}$ is a suitable comonad. Recall that if $\mathcal{D}$ is a category then a comonad on $\mathcal{D}$ is a three-tuple $\mathbb{U} = (U, \Delta, \varepsilon)$, where $U : \mathcal{D} \to \mathcal{D}$ is a functor, and $\Delta : U \to U \circ U$ and $\varepsilon : U \to 1_{\mathcal{D}}$ are natural transformations, such that

$$U(\Delta_M) \circ \Delta_M = \Delta_{U(M)} \circ \Delta_M, \tag{3.40}$$

$$U(\varepsilon_M) \circ \Delta_M = \varepsilon_{U(M)} \circ \Delta_M = \mathrm{id}_{U(M)} \tag{3.41}$$

for all $M \in \mathcal{D}$. A morphism between two $\mathcal{D}$-comonads $\mathbb{U} = (U, \Delta, \varepsilon)$ and $\mathbb{U}' = (U', \Delta', \varepsilon')$ is a natural transformation $\vartheta : U \to U'$ such that

$$\varepsilon' \circ \vartheta = \varepsilon \quad \text{and} \quad (\vartheta * \vartheta) \circ \Delta = \Delta' \circ \vartheta \tag{3.42}$$

for all $M \in \mathcal{D}$, where $*$ is the Godement product

$$(\vartheta * \vartheta)_M = \vartheta_{U'(M)} \circ U(\vartheta_M).$$

We denote by $\mathsf{Comonad}(\mathcal{D})$ the category of comonads on $\mathcal{D}$.

For $\mathbb{U}$ a comonad on $\mathcal{D}$, a $\mathbb{U}$-coalgebra is a pair (M, ξ) with $M \in \mathcal{D}$, and $\xi : M \to U(M)$ a morphism in $\mathcal{D}$ such that

$$\varepsilon_M \circ \xi = \mathrm{id}_M \quad \text{and} \quad \Delta_M \circ \xi = U(\xi) \circ \xi. \tag{3.43}$$

A morphism between two $\mathbb{U}$-coalgebras (M, ξ) and (M', ξ') consists of a morphism $\upsilon : M \to M'$ in $\mathcal{D}$ such that

$$U(\upsilon) \circ \xi = \xi' \circ \upsilon. \tag{3.44}$$

The category of $\mathbb{U}$-coalgebras is denoted by $\mathcal{D}^{\mathbb{U}}$.

If H is a quasi-bialgebra and $\mathfrak{A}$ an algebra then we define $\mathcal{C} := {}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$. Thus, an object of $\mathcal{C}$ is an $\mathfrak{A}$-bimodule and an $(H,\mathfrak{A})$-bimodule such that $h(\mathfrak{a}m) = \mathfrak{a}(hm)$, for all $\mathfrak{a} \in \mathfrak{A}$, $h \in H$, and $m \in M$. Morphisms are left H-linear maps which are also $\mathfrak{A}$-bimodule maps. We claim that $\mathcal{C}$ is a monoidal category. Indeed, it is not hard to see that $\mathcal{C}$ becomes a monoidal category with tensor product $\otimes_{\mathfrak{A}}$ given via Δ, in the sense that

$$(\mathfrak{a}\otimes h)(m\otimes_{\mathfrak{A}} n)\mathfrak{a}' := \sum \mathfrak{a}h_1m\otimes_{\mathfrak{A}} h_2n\mathfrak{a}'$$

for all $M, N \in \mathcal{C}$, $m \in M$, $n \in N$, $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, and $h \in H$, associativity constraints

$$\underline{a}_{M,N,P} : (M\otimes_{\mathfrak{A}} N)\otimes_{\mathfrak{A}} P \to M\otimes_{\mathfrak{A}} (N\otimes_{\mathfrak{A}} P),$$
$$\underline{a}_{M,N,P}\big((m\otimes_{\mathfrak{A}} n)\otimes_{\mathfrak{A}} p\big) = \sum X^1m\otimes_{\mathfrak{A}} \big(X^2n\otimes_{\mathfrak{A}} X^3p\big),$$

unit $\mathfrak{A}$ as a trivial left H-module, and the usual left and right unit constraints. We denote by $\mathcal{C}$-Coalgebra the category of coalgebras in $\mathcal{C}$. We are able now to prove the claimed isomorphism.

Theorem 3.7. *Let H be a quasi-bialgebra, $\mathfrak{A}$ an algebra, $\mathcal{C} = {}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$, and $\mathcal{D} := {}_H\mathcal{M}_{\mathfrak{A}}$. Then there exists a functor*

$$F : \mathcal{C}\text{-Coalgebra} \to \text{Comonad}(\mathcal{D}).$$

In addition, if $\mathfrak{A}$ is a right H-comodule algebra then $\mathfrak{C} := \mathfrak{A}\otimes H$ is a coalgebra in $\mathcal{C}$ and, in this particular case, we have an isomorphism of categories

$$\mathcal{D}^{F(\mathfrak{C})} \cong {}_H\mathcal{M}_{\mathfrak{A}}^H.$$

Proof. If $\mathfrak{C}$ is a coalgebra in $\mathcal{C}$ then it is an $(H,\mathfrak{A})$-bimodule and an $\mathfrak{A}$-bimodule so, we have a functor $U = (-)\otimes_{\mathfrak{A}} \mathfrak{C} : \mathcal{D} \to \mathcal{D}$ (for any $M \in \mathcal{D}$, the left H-module structure of $U(M)$ is given via Δ and the right $\mathfrak{A}$-action on $U(M)$ is induced by the one on $\mathfrak{C}$). For all $M \in \mathcal{D}$, we define

$$\Delta_M : M\otimes_{\mathfrak{A}} \mathfrak{C} = U(M) \to U\big(U(M)\big) = (M\otimes_{\mathfrak{A}} \mathfrak{C})\otimes_{\mathfrak{A}} \mathfrak{C},$$
$$\Delta_M(m\otimes_{\mathfrak{A}} c) = \sum \big(x^1m\otimes_{\mathfrak{A}} x^2c_{\underline{1}}\big)\otimes_{\mathfrak{A}} x^3c_{\underline{2}},$$
$$\varepsilon_M := \mathrm{id}_M \otimes_{\mathfrak{A}} \underline{\varepsilon}_{\mathfrak{C}} : M\otimes_{\mathfrak{A}} \mathfrak{C} = U(M) \to M \cong M\otimes_{\mathfrak{A}} A$$

for all $m \in M$ and $c \in \mathfrak{C}$, where $\underline{\Delta}_{\mathfrak{C}}(c) := \sum c_{\underline{1}} \otimes c_{\underline{2}}$ is the comultiplication of $\mathfrak{C}$ and $\underline{\varepsilon}_{\mathfrak{C}}$ is the counit of $\mathfrak{C}$. It is not hard to see that $F(\mathfrak{C}) := (U, \Delta_M, \varepsilon_M)$ is a comonad on $\mathcal{D}$. It is also straightforward to check that a morphism κ in $\mathcal{C}$-Coalgebra provides a morphism $U(\kappa)$ in Comonad$(\mathcal{D})$ and that F is a functor.

Suppose now that $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra and let $\mathfrak{C} = \mathfrak{A}\otimes H$. If we define

$$(\mathfrak{a} \otimes h)(\mathfrak{a}' \otimes h')\mathfrak{a}'' := \sum \mathfrak{a}\mathfrak{a}'\mathfrak{a}''_{\langle 0\rangle} \otimes hh'\mathfrak{a}''_{\langle 1\rangle} \tag{3.45}$$

for all $\mathfrak{a}, \mathfrak{a}', \mathfrak{a}'' \in \mathfrak{A}$, and $h, h' \in H$, then one can easily check that with this structure $\mathfrak{C} \in \mathcal{C}$. Moreover, we claim that $\mathfrak{C}$ with the structure given by

$$\underline{\Delta}_{\mathfrak{C}}(\mathfrak{a} \otimes h) := \sum \big(\mathfrak{a}\widetilde{X}^1 \otimes h_1\widetilde{X}^2\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_2\widetilde{X}^3\big), \tag{3.46}$$

$$\underline{\varepsilon}_{\mathfrak{C}}(\mathfrak{a} \otimes h) := \varepsilon(h)\mathfrak{a}, \tag{3.47}$$

for all $\mathfrak{a} \in \mathfrak{A}$ and $h \in H$, becomes a coalgebra in $\mathcal{C}$. Indeed, the fact that $\underline{\Delta}_{\mathfrak{C}}$ and $\underline{\varepsilon}_{\mathfrak{C}}$ are morphisms in $\mathcal{C}$ and that $\underline{\varepsilon}_{\mathfrak{C}}$ is the counit for $\underline{\Delta}_{\mathfrak{C}}$ follow from straightforward computations (all these verifications are left to the reader). We only show that the comultiplication $\underline{\Delta}_{\mathfrak{C}}$ is coassociative up to the associativity constraints of $\mathcal{C}$. Indeed, we compute that

$$\begin{aligned}
&(\underline{\Delta}_{\mathfrak{C}} \otimes_{\mathfrak{A}} \mathrm{id})\big(\underline{\Delta}_{\mathfrak{C}}(\mathfrak{a} \otimes h)\big)\\
&= \sum \underline{\Delta}_{\mathfrak{C}}\big(\mathfrak{a}\widetilde{X}^1 \otimes h_1\widetilde{X}^2\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_2\widetilde{X}^3\big)\\
&= \sum \big(\mathfrak{a}\widetilde{X}^1\widetilde{Y}^1 \otimes h_{(1,1)}\widetilde{X}^2_1\widetilde{Y}^2\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_{(1,2)}\widetilde{X}^2_2\widetilde{Y}^3\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_2\widetilde{X}^3\big)\\
\overset{(2.2)}{}&= \sum \big(\mathfrak{a}\widetilde{X}^1\widetilde{Y}^1_{\langle 0\rangle} \otimes h_{(1,1)}x^1\widetilde{X}^2\widetilde{Y}^1_{\langle 1\rangle}\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_{(1,2)}x^2\widetilde{X}^3_1\widetilde{Y}^2\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_2x^3\widetilde{X}^3_2\widetilde{Y}^3\big)\\
\overset{(1.1)}{}&= \sum x^1\big(\mathfrak{a}\widetilde{X}^1 \otimes h_1\widetilde{X}^2\big)\widetilde{Y}^1 \otimes_{\mathfrak{A}} x^2\big(1_{\mathfrak{A}} \otimes h_{(2,1)}\widetilde{X}^3_1\widetilde{Y}^2\big) \otimes_{\mathfrak{A}} x^3\big(1_{\mathfrak{A}} \otimes h_{(2,2)}\widetilde{X}^3_2\widetilde{Y}^3\big)\\
&= \Phi^{-1}\sum \big(\mathfrak{a}\widetilde{X}^1 \otimes h_1\widetilde{X}^2\big) \otimes_{\mathfrak{A}} \big(\widetilde{Y}^1 \otimes h_{(2,1)}\widetilde{X}^3_1\widetilde{Y}^2\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes h_{(2,2)}\widetilde{X}^3_2\widetilde{Y}^3\big)\\
&= \Phi^{-1}\sum \big(\mathfrak{a}\widetilde{X}^1 \otimes h_1\widetilde{X}^2\big) \otimes_{\mathfrak{A}} \underline{\Delta}_{\mathfrak{C}}\big(1_{\mathfrak{A}} \otimes h_2\widetilde{X}^3\big)\\
&= \Phi^{-1}(\mathrm{id} \otimes_{\mathfrak{A}} \underline{\Delta}_{\mathfrak{C}})\big(\underline{\Delta}_{\mathfrak{C}}(\mathfrak{a} \otimes h)\big),
\end{aligned}$$

for all $\mathfrak{a} \in \mathfrak{A}$ and $h \in H$, as needed.

Consider now the comonad $F(\mathfrak{C}) = (U, \Delta, \varepsilon)$ and $(M, \xi) \in \mathcal{D}^{F(\mathfrak{C})}$. That means that $M \in \mathcal{D} = {}_H\mathcal{M}_{\mathfrak{A}}$ and $\xi : M \to U(M) = M \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes H)$ is a morphism in $\mathcal{D}$ such that $\Delta_M \circ \xi = U(\xi) \circ \xi$ and $\varepsilon_M \circ \xi = \mathrm{id}_M$, for all $M \in \mathcal{D}$. In other words, if we write

$$\xi(m) = \sum m_{(0)} \otimes_{\mathfrak{A}} \big(m_{(1)^{\mathfrak{A}}} \otimes m_{(1)^H}\big) \quad \forall m \in M,$$

then $(M, \xi) \in \mathcal{D}^{F(\mathfrak{C})}$ if and only if the following relations hold:

$$\xi(hm) = \sum h_1m_{(0)} \otimes_{\mathfrak{A}} \big(m_{(1)^{\mathfrak{A}}} \otimes h_2m_{(1)^H}\big), \tag{3.48}$$

$$\xi(m\mathfrak{a}) = \sum m_{(0)} \otimes_{\mathfrak{A}} \big(m_{(1)^{\mathfrak{A}}}\mathfrak{a}_{\langle 0\rangle} \otimes m_{(1)^H}\mathfrak{a}_{\langle 1\rangle}\big), \tag{3.49}$$

$$\begin{aligned}
&\sum x^1m_{(0)} \otimes_{\mathfrak{A}} \big(m_{(1)^{\mathfrak{A}}}\widetilde{X}^1 \otimes x^2m_{(1)^H_1}\widetilde{X}^2\big) \otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}} \otimes x^3m_{(1)^H_2}\widetilde{X}^3\big)\\
&= \sum m_{(0)_{(0)}} \otimes_{\mathfrak{A}} \big(m_{(0)_{(1)^{\mathfrak{A}}}} \otimes m_{(0)_{(1)^H}}\big) \otimes_{\mathfrak{A}} \big(m_{(1)^{\mathfrak{A}}} \otimes m_{(1)^H}\big),
\end{aligned} \tag{3.50}$$

$$\sum \varepsilon\big(m_{(1)^H}\big) m_{(0)} m_{(1)^{\mathfrak{A}}} = m, \tag{3.51}$$

for all $h \in H$, $m \in M$, and $\mathfrak{a} \in \mathfrak{A}$. Applying the canonical isomorphisms, the first three relations are equivalent to

$$\sum (hm)_{(0)} (hm)_{(1)^{\mathfrak{A}}} \otimes (hm)_{(1)^H} = \sum h_1 m_{(0)} m_{(1)^{\mathfrak{A}}} \otimes h_2 m_{(1)^H}, \tag{3.52}$$

$$\sum (m\mathfrak{a})_{(0)} (m\mathfrak{a})_{(1)^{\mathfrak{A}}} \otimes (m\mathfrak{a})_{(1)^H} = \sum m_{(0)} m_{(1)^{\mathfrak{A}}} \mathfrak{a}_{\langle 0\rangle} \otimes m_{(1)^H} \mathfrak{a}_{\langle 1\rangle}, \tag{3.53}$$

$$\begin{aligned} &\sum x^1 m_{(0)} m_{(1)^{\mathfrak{A}}} \widetilde{X}^1 \otimes x^2 m_{(1)^H_1} \widetilde{X}^2 \otimes x^3 m_{(1)^H_2} \widetilde{X}^3 \\ &\quad = \sum m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{A}}}} m_{(1)^{\mathfrak{A}}_{\langle 0\rangle}} \otimes m_{(0)_{(1)^H}} m_{(1)^{\mathfrak{A}}_{\langle 1\rangle}} \otimes m_{(1)^H}, \end{aligned} \tag{3.54}$$

for all $h \in H$, $m \in M$, and $\mathfrak{a} \in \mathfrak{A}$. Now, if define $\rho_M : M \to M \otimes H$,

$$\rho_M(m) = \sum m_{(0)} m_{(1)^{\mathfrak{A}}} \otimes m_{(1)^H} \quad \forall m \in M,$$

then (3.52) implies that $\rho_M(hm) = \Delta(h)\rho_M(m)$ for all $h \in H$ and $m \in M$, and (3.53) implies that $\rho_M(m\mathfrak{a}) = \rho_M(m)\rho(\mathfrak{a})$ for all $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$, respectively. Moreover, for all $m \in M$ we have that

$$\begin{aligned} (\rho_M \otimes \mathrm{id}_H)\big(\rho_M(m)\big) &= \sum \rho_M\big(m_{(0)} m_{(1)^{\mathfrak{A}}}\big) \otimes m_{(1)^H} \\ &= \sum \big(m_{(0)} m_{(1)^{\mathfrak{A}}}\big)_{(0)} \big(m_{(0)} m_{(1)^{\mathfrak{A}}}\big)_{(1)^{\mathfrak{A}}} \otimes \big(m_{(0)} m_{(1)^{\mathfrak{A}}}\big)_{(1)^H} \otimes m_{(1)^H} \\ &\overset{(3.53)}{=} \sum m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{A}}}} m_{(1)^{\mathfrak{A}}_{\langle 0\rangle}} \otimes m_{(0)_{(1)^H}} m_{(1)^{\mathfrak{A}}_{\langle 1\rangle}} \otimes m_{(1)^H} \\ &\overset{(3.54)}{=} \sum x^1 m_{(0)} m_{(1)^{\mathfrak{A}}} \widetilde{X}^1 \otimes x^2 m_{(1)^H_1} \widetilde{X}^2 \otimes x^3 m_{(1)^H_2} \widetilde{X}^3 \\ &= \Phi^{-1} \cdot \Big(\sum m_{(0)} m_{(1)^{\mathfrak{A}}} \otimes \Delta\big(m_{(1)^H}\big)\Big) \cdot \Phi_\rho \\ &= \Phi^{-1} \cdot (\mathrm{id}_M \otimes \Delta)\big(\rho_M(m)\big) \cdot \Phi_\rho. \end{aligned}$$

By (3.51) it follows that $(\mathrm{id}_M \otimes \varepsilon) \circ \rho_M = \mathrm{id}_M$, so we have obtained that $M \in {}_H\mathcal{M}^H_{\mathfrak{A}}$. In this way, we have a functor $\mathbb{F} : \mathcal{D}^{F(\mathfrak{C})} \to {}_H\mathcal{M}^H_{\mathfrak{A}}$ ($\mathbb{F}$ acts as identity on morphisms). We will show that $\mathbb{F}$ provides the desired isomorphism of categories. For, we define the inverse of $\mathbb{F}$ as follows. Let $M \in {}_H\mathcal{M}^H_{\mathfrak{A}}$, and denote by $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ the right coaction of H on M. Then we define

$$\xi : M \to M \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes H), \quad \xi(m) = \sum m_{(0)} \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes m_{(1)}) \quad \forall m \in M.$$

In the same manner as above one can prove that the axioms which define M as a two-sided $(H, \mathfrak{A})$-bimodule imply that ξ satisfies the relations (3.51)–(3.54). Thus $(M, \xi) \in \mathcal{D}^{F(\mathfrak{C})}$ and we have a well-defined functor $\mathbb{G} : {}_H\mathcal{M}^H_{\mathfrak{A}} \to \mathcal{D}^{F(\mathfrak{C})}$ ($\mathbb{G}$ acts as the identity on morphisms). The fact that the functors $\mathbb{F}$ and $\mathbb{G}$ are inverses is obvious, and this finishes our proof. □

Theorem 3.7 enables us to restate the definition of a comodule algebra in terms of monoidal categories.

Proposition 3.8. *Let H be a quasi-bialgebra and $\mathfrak{A}$ an algebra. If $\mathfrak{A}\otimes H$ is viewed in the canonical way as an object in ${}_{\mathfrak{A}\otimes H}\mathcal{M}$ then $\mathfrak{A}\otimes H$ has a coalgebra structure $(\mathfrak{A}\otimes H, \underline{\Delta}, \underline{\varepsilon})$ in the monoidal category $\mathcal{C} = {}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ such that $\underline{\Delta}(1_{\mathfrak{A}}\otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{A}}\otimes 1_H) = 1_{\mathfrak{A}}$ if and only if $\mathfrak{A}$ is a right H-comodule algebra.*

Proof. One implication follows from the proof of Theorem 3.7. Conversely, suppose that $\mathfrak{A}\otimes H$ is an object of $\mathcal{C}$, and that there exists a coalgebra structure $(\mathfrak{A}\otimes H, \underline{\Delta}, \underline{\varepsilon})$ on $\mathfrak{A}\otimes H$ in the monoidal category $\mathcal{C}$ such that $\underline{\Delta}(1_{\mathfrak{A}}\otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{A}}\otimes 1_H) = 1_{\mathfrak{A}}$. Then we define

$$\mathfrak{A}\ni \mathfrak{a}\mapsto \rho(\mathfrak{a}) = \sum \mathfrak{a}_{\langle 0\rangle}\otimes \mathfrak{a}_{\langle 1\rangle} := (1_{\mathfrak{A}}\otimes 1_H)\mathfrak{a}\in \mathfrak{A}\otimes H,$$

and denote

$$\underline{\Delta}(1_{\mathfrak{A}}\otimes 1_H) := \sum \big(\tilde{X}^1\otimes \tilde{X}^2\big)\otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}}\otimes \tilde{X}^3\big).$$

Since $\mathfrak{A}\otimes H$ is a right $\mathfrak{A}$-module, it is follows that ρ is an algebra map. Also, since $\underline{\Delta}(1_{\mathfrak{A}}\otimes 1_H)$ is invertible, we obtain that $\Phi_\rho := \sum \tilde{X}^1\otimes \tilde{X}^2\otimes \tilde{X}^3$ is an invertible element in $\mathfrak{A}\otimes H\otimes H$. Now, using the fact that $\underline{\Delta}$ and $\underline{\varepsilon}$ are morphisms in $\mathcal{C}$, and that $\underline{\varepsilon}(1_{\mathfrak{A}}\otimes 1_H) = 1_{\mathfrak{A}}$, it is not hard to see that

$$\underline{\Delta}(\mathfrak{a}\otimes h) = \sum \big(\mathfrak{a}\tilde{X}^1\otimes h_1\tilde{X}^2\big)\otimes_{\mathfrak{A}} \big(1_{\mathfrak{A}}\otimes h_2\tilde{X}^3\big), \qquad \underline{\varepsilon}(\mathfrak{a}\otimes h) = \varepsilon(h)\mathfrak{a}$$

for all $\mathfrak{a}\in\mathfrak{A}$, $h\in H$. Now, (2.1) and (2.2) follow because of equalities $\underline{\Delta}((1_{\mathfrak{A}}\otimes 1_H)\mathfrak{a}) = \underline{\Delta}(1_{\mathfrak{A}}\otimes 1_H)\mathfrak{a}$ and $\Phi(\underline{\Delta}\otimes \mathrm{id})\underline{\Delta}(\mathfrak{a}\otimes h) = (\mathrm{id}\otimes\underline{\Delta})\underline{\Delta}(\mathfrak{a}\otimes h)$ for all $\mathfrak{a}\in\mathfrak{A}$ and $h\in H$, respectively. Finally, it is easy to see that $\underline{\varepsilon}((1_{\mathfrak{A}}\otimes 1_H)\mathfrak{a}) = \mathfrak{a}$ implies (2.3), and the fact that $\underline{\varepsilon}$ is the counit for $\underline{\Delta}$ implies (2.4), respectively. We leave all these details to the reader. □

4. Two-sided crossed products are generalized smash products

Let H be a finite-dimensional quasi-bialgebra, and $(\mathfrak{A}, \rho, \Phi_\rho)$, $(\mathfrak{B}, \lambda, \Phi_\lambda)$ respectively a right and a left H-comodule algebra. As in the case of a Hopf algebra, the right H-coaction (ρ, Φ_ρ) on $\mathfrak{A}$ induces a left H^*-action $\triangleright : H^*\otimes\mathfrak{A}\to\mathfrak{A}$ given by

$$\varphi\triangleright\mathfrak{a} = \sum \varphi(\mathfrak{a}_{\langle 1\rangle})\mathfrak{a}_{\langle 0\rangle} \tag{4.1}$$

for all $\varphi\in H^*$ and $\mathfrak{a}\in\mathfrak{A}$, and where $\rho(\mathfrak{a}) = \sum a_{\langle 0\rangle}\otimes \mathfrak{a}_{\langle 1\rangle}$ for any $\mathfrak{a}\in\mathfrak{A}$. Similarly, the left H-action (λ, Φ_λ) on $\mathfrak{B}$ provides a right H^*-action $\triangleleft : \mathfrak{B}\otimes H^*\to\mathfrak{B}$ given by

$$\mathfrak{b} \triangleleft \varphi = \sum \varphi(\mathfrak{b}_{[-1]})\mathfrak{b}_{[0]} \tag{4.2}$$

for all $\varphi \in H^*$ and $\mathfrak{b} \in \mathfrak{B}$, where we now denote $\lambda(\mathfrak{b}) = \sum \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0]}$ for $\mathfrak{b} \in \mathfrak{B}$. Following [16, Proposition 11.4(ii)] we can define an algebra structure on the k-vector space $\mathfrak{A} \otimes H^* \otimes \mathfrak{B}$. This algebra is denoted by $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$ and its multiplication is given by

$$\begin{aligned}(&\mathfrak{a} \rtimes \varphi \ltimes \mathfrak{b})(\mathfrak{a}' \rtimes \psi \ltimes \mathfrak{b}') \\ &= \sum \mathfrak{a}(\varphi_1 \triangleright \mathfrak{a}')\tilde{x}^1_\rho \rtimes \big(\tilde{x}^1_\lambda \rightharpoonup \varphi_2 \leftharpoonup \tilde{x}^2_\rho\big)\big(\tilde{x}^2_\lambda \rightharpoonup \psi_1 \leftharpoonup \tilde{x}^3_\rho\big) \ltimes \tilde{x}^3_\lambda(\mathfrak{b} \triangleleft \psi_2)\mathfrak{b}'\end{aligned} \tag{4.3}$$

for all $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$, and $\varphi, \psi \in H^*$, where we write $\mathfrak{a} \rtimes \varphi \ltimes \mathfrak{b}$ for $\mathfrak{a} \otimes \varphi \otimes \mathfrak{b}$ when viewed as an element of $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$. The comultiplication on H^* is denoted by $\Delta(\varphi) = \sum \varphi_1 \otimes \varphi_2$. The unit of the algebra $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$ is $1_\mathfrak{A} \rtimes \varepsilon \ltimes 1_\mathfrak{B}$. Hausser and Nill called this algebra the two-sided crossed product. In this section we will prove that this two-sided crossed product algebra is a generalized smash product between the quasi-smash product $\mathfrak{A} \,\overline{\#}\, H^*$ and $\mathfrak{B}$.

Proposition 4.1. *Let H be a quasi-bialgebra, A a left H-module algebra, and $\mathfrak{B}$ a left H-comodule algebra. Let $A \blacktriangleright\!\!< \mathfrak{B} = A \otimes \mathfrak{B}$ as a k-module, with newly defined multiplication*

$$(a \blacktriangleright\!\!< \mathfrak{b})(a' \blacktriangleright\!\!< \mathfrak{b}') = \sum \big(\tilde{x}^1 \cdot a\big)\big(\tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a'\big) \blacktriangleright\!\!< \tilde{x}^3 \mathfrak{b}_{[0]}\mathfrak{b}' \tag{4.4}$$

for all $a, a' \in A$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. Then $A \blacktriangleright\!\!< \mathfrak{B}$ is an associative algebra with unit $1_A \blacktriangleright\!\!< 1_\mathfrak{B}$.

Proof. For all $a, a', a'' \in A$ and $\mathfrak{b}, \mathfrak{b}', \mathfrak{b}'' \in \mathfrak{B}$, we have:

$$\begin{aligned}
&\big[(a \blacktriangleright\!\!< b)(a' \blacktriangleright\!\!< \mathfrak{b}')\big](a'' \blacktriangleright\!\!< \mathfrak{b}'') \\
&= \sum \big[\big(\tilde{x}^1 \cdot a\big)\big(\tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a'\big) \blacktriangleright\!\!< \tilde{x}^3 \mathfrak{b}_{[0]}\mathfrak{b}'\big](a'' \blacktriangleright\!\!< \mathfrak{b}'') \\
&= \sum \big[\big(\tilde{y}^1_1 \tilde{x}^1 \cdot a\big)\big(\tilde{y}^1_2 \tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a'\big)\big]\big(\tilde{y}^2 \tilde{x}^3_{[-1]} \mathfrak{b}_{[0,-1]} \mathfrak{b}'_{[-1]} \cdot a''\big) \blacktriangleright\!\!< \tilde{y}^3 \tilde{x}^3_{[0]} \mathfrak{b}_{[0,0]} \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
\overset{(1.27)}{=}& \sum \big(X^1 \tilde{y}^1_1 \tilde{x}^1 \cdot a\big)\big[\big(X^2 \tilde{y}^1_2 \tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a'\big)\big(X^3 \tilde{y}^2 \tilde{x}^3_{[-1]} \mathfrak{b}_{[0,-1]} \mathfrak{b}'_{[-1]} \cdot a''\big)\big] \\
&\quad \blacktriangleright\!\!< \tilde{y}^3 \tilde{x}^3_{[0]} \mathfrak{b}_{[0,0]} \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
\overset{(2.6)}{=}& \sum \big(\tilde{x}^1 \cdot a\big)\big[\big(\tilde{x}^2_1 \tilde{y}^1 \mathfrak{b}_{[-1]} \cdot a'\big)\big(\tilde{x}^2_2 \tilde{y}^2 \mathfrak{b}_{[0,-1]} \mathfrak{b}'_{[-1]} \cdot a''\big)\big] \blacktriangleright\!\!< \tilde{x}^3 \tilde{y}^3 \mathfrak{b}_{[0,0]} \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
\overset{(2.5)}{\underset{(1.28)}{=}}& \sum \big(\tilde{x}^1 \cdot a\big)\big\{\big(\tilde{x}^2 \mathfrak{b}_{[-1]} \cdot \big[\big(\tilde{y}^1 \cdot a'\big)\big(\tilde{y}^2 \mathfrak{b}'_{[-1]} \cdot a''\big)\big]\big)\big\} \blacktriangleright\!\!< \tilde{x}^3 \mathfrak{b}_{[0]} \tilde{y}^3 \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
&= \sum (a \blacktriangleright\!\!< \mathfrak{b})\big[\big(\tilde{y}^1 \cdot a'\big)\big(\tilde{y}^2 \mathfrak{b}'_{[-1]} \cdot a''\big) \blacktriangleright\!\!< \tilde{y}^3 \mathfrak{b}'_{[0]} \mathfrak{b}''\big] \\
&= (a \blacktriangleright\!\!< \mathfrak{b})\big[(a' \blacktriangleright\!\!< \mathfrak{b}')(a'' \blacktriangleright\!\!< \mathfrak{b}'')\big].
\end{aligned}$$

It follows from (2.7), (2.8), and (1.29) that $1_A \blacktriangleright\!\!< 1_\mathfrak{B}$ is the unit for $A \blacktriangleright\!\!< \mathfrak{B}$. $\square$

Remark 4.2. Let H be a quasi-bialgebra and A a left H-module algebra. Then H is a left H-comodule algebra so it make sense to consider $A \ltimes H$. It is not hard to see that in this case $A \ltimes H$ is just the smash product $A \# H$. For this reason, we will call the algebra $A \ltimes \mathfrak{B}$ in Proposition 4.1 the generalized smash product of A and $\mathfrak{B}$. In fact, our terminology is in agreement with the terminology used over Hopf algebras, see [9,14].

Let H be a finite-dimensional quasi-bialgebra, $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra. Then the quasi-smash product $\mathfrak{A} \,\overline{\#}\, H^*$ is a left H-module algebra, so it makes sense to consider the generalized smash product $(\mathfrak{A} \,\overline{\#}\, H^*) \ltimes \mathfrak{B}$. The main result of this section is now the following.

Proposition 4.3. *With notation as above, the algebras* $(\mathfrak{A} \,\overline{\#}\, H^*) \ltimes \mathfrak{B}$ *and* $\mathfrak{A} \rtimes {}_\rho H^* \ltimes {}_\lambda \mathfrak{B}$ *coincide.*

Proof. Using (4.4), (2.12), and (2.11), we compute that the multiplication on $(\mathfrak{A} \,\overline{\#}\, H^*) \ltimes \mathfrak{B}$ is given by

$$\begin{aligned}
&\big[(\mathfrak{a} \,\overline{\#}\, \varphi) \ltimes \mathfrak{b}\big]\big[(\mathfrak{a}' \,\overline{\#}\, \psi) \ltimes \mathfrak{b}'\big] \\
&\quad = \sum \big[\tilde{x}_\lambda^1 \cdot (\mathfrak{a} \,\overline{\#}\, \varphi)\big]\big[\tilde{x}_\lambda^2 \mathfrak{b}_{[-1]} \cdot (\mathfrak{a}' \,\overline{\#}\, \psi)\big] \ltimes \tilde{x}_\lambda^3 \mathfrak{b}_{[0]} \mathfrak{b}' \\
&\quad = \sum \big(\mathfrak{a} \,\overline{\#}\, \tilde{x}_\lambda^1 \rightharpoonup \varphi\big)\big(\mathfrak{a}' \,\overline{\#}\, \tilde{x}_\lambda^2 \mathfrak{b}_{[-1]} \rightharpoonup \psi\big) \ltimes \tilde{x}_\lambda^3 \mathfrak{b}_{[0]} \mathfrak{b}' \\
&\quad = \sum \mathfrak{a}\mathfrak{a}'_{\langle 0\rangle} \tilde{x}_\rho^1 \,\overline{\#}\, \big(\tilde{x}_\lambda^1 \rightharpoonup \varphi \leftharpoonup \mathfrak{a}'_{\langle 1\rangle} \tilde{x}_\rho^2\big)\big(\tilde{x}_\lambda^2 \mathfrak{b}_{[-1]} \rightharpoonup \psi \leftharpoonup \tilde{x}_\rho^3\big) \ltimes \tilde{x}_\lambda^3 \mathfrak{b}_{[0]} \mathfrak{b}' \qquad (4.5) \\
&\overset{(4.1)}{\underset{(4.2)}{=}} \sum \mathfrak{a}(\varphi_1 \triangleright \mathfrak{a}') \tilde{x}_\rho^1 \,\overline{\#}\, \big(\tilde{x}_\lambda^1 \rightharpoonup \varphi_2 \leftharpoonup \tilde{x}_\rho^2\big)\big(\tilde{x}_\lambda^2 \rightharpoonup \psi_1 \leftharpoonup \tilde{x}_\rho^3\big) \ltimes \tilde{x}_\lambda^3 (\mathfrak{b} \triangleleft \psi_2) \mathfrak{b}'
\end{aligned}$$

for $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$, and $\varphi, \psi \in H^*$. This is just the multiplication rule on the two-sided crossed product $\mathfrak{A} \rtimes {}_\rho H^* \ltimes {}_\lambda \mathfrak{B}$. $\square$

It follows from (4.5) that the two-sided crossed product can be defined in the situation where H is not finite-dimensional. Take $\mathfrak{B} = H$ in Proposition 4.3. From Remark 4.2, we obtain:

Corollary 4.4. *Let H be a quasi-bialgebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. Then $(\mathfrak{A} \,\overline{\#}\, H^*) \# H = \mathfrak{A} \rtimes {}_\rho H^* \ltimes {}_\Delta H$ as algebras. In particular, $(H \,\overline{\#}\, H^*) \# H = H \rtimes H^* \ltimes H$ as algebras.*

5. The category of Doi–Hopf modules

5.1. Doi–Hopf modules

Let H be a Hopf algebra over a field k, A an H-comodule algebra, and C an H-module coalgebra. A Doi–Hopf module is a k-vector space together with an A-action and a

C-coaction satisfying a certain compatibility relation. They were introduced independently by Doi [14] and Koppinen [20], and it turns out that most types of Hopf modules that had been studied before were special cases: Sweedler's Hopf modules [25], Doi's relative Hopf modules [13], Takeuchi's relative Hopf modules [27], Yetter–Drinfeld modules, graded modules and modules graded by a G-set.

Over a quasi-Hopf algebra, the category of relative Hopf modules has been introduced and studied [6], as well as the category of Hopf H-bimodules (see [18]), and the category of Hopf modules ${}_H^H\mathcal{M}_H^H$ (see [24]). We will introduce Doi–Hopf modules, and we will show that, at least in the case where H is finite-dimensional, all these categories are isomorphic to certain categories of Doi–Hopf modules. We will also prove that Doi–Hopf modules are special cases of comodules over a coring.

First, we recall from [6] the definition of a relative Hopf module. Let H be a quasi-bialgebra and C a right H-module coalgebra. Let N be a k-vector space furnished with the following additional structure:

- N is a right H-module; the right action of $h \in H$ on $n \in N$ is denoted by nh;
- N is a left C-comodule in the monoidal category $\mathcal{M}_H$; we use the following notation for the left C-coaction on N: $\rho_N : N \to C \otimes N$, $\rho_N(n) = \sum n_{[-1]} \otimes n_{[0]}$; this means that the following conditions hold, for all $n \in N$:

$$\sum \underline{\varepsilon}(n_{[-1]})n_{[0]} = n, \qquad (\underline{\Delta} \otimes \mathrm{id}_N)\big(\rho_N(n)\big)\Phi^{-1} = (\mathrm{id}_C \otimes \rho_N)\big(\rho_N(n)\big); \quad (5.1)$$

- we have the following compatibility relation, for all $n \in N$ and $c \in C$:

$$\rho_N(nh) = \sum n_{[-1]} \cdot h_1 \otimes n_{[0]}h_2. \qquad (5.2)$$

Then N is called a left $[C, H]$-Hopf module. ${}^C\mathcal{M}_H$ is the category of left $[C, H]$-Hopf modules; the morphisms are right H-linear maps which are also left C-comodule maps. We will now generalize this definition.

Definition 5.1. Let H be a quasi-bialgebra over a field k, C a right H-module coalgebra, and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra. A right–left $(H, \mathfrak{B}, C)$-Hopf module (or Doi–Hopf module) is a k-module N, with the following additional structure: N is right $\mathfrak{B}$-module (the right action of $\mathfrak{b}$ on n is denoted by $n\mathfrak{b}$), and we have a k-linear map $\rho_N : N \to C \otimes N$, such that the following relations hold, for all $n \in N$ and $\mathfrak{b} \in \mathfrak{B}$:

$$(\underline{\Delta} \otimes \mathrm{id}_N)\big(\rho_N(n)\big) = (\mathrm{id}_C \otimes \rho_N)\big(\rho_N(n)\big)\Phi_\lambda, \qquad (5.3)$$

$$(\underline{\varepsilon} \otimes \mathrm{id}_N)\big(\rho_N(n)\big) = n, \qquad (5.4)$$

$$\rho_N(n\mathfrak{b}) = \sum n_{[-1]} \cdot \mathfrak{b}_{[-1]} \otimes n_{[0]}\mathfrak{b}_{[0]}. \qquad (5.5)$$

As usual, we use the Sweedler-type notation $\rho_N(n) = \sum n_{[-1]} \otimes n_{[0]}$. ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is the category of right–left $(H, \mathfrak{B}, C)$-Hopf modules and right $\mathfrak{B}$-linear, left C-colinear k-linear maps.

Obviously, if $\mathfrak{B} = H$, $\lambda = \Delta$, and $\Phi_\lambda = \Phi$, then ${}^C\mathcal{M}(H)_{\mathfrak{B}} = {}^C\mathcal{M}_H$.

The main aim of Section 6 will be to define the category of two-sided two-cosided Hopf modules over a quasi-bialgebra and to prove that it is isomorphic to a module category in the finite-dimensional case. To this end, we will need our next result, stating that the category of Doi–Hopf modules is a module category in the case where the coalgebra C is finite-dimensional. In fact, for an arbitrary right H-module coalgebra C, the linear dual space of C, C^*, is a left H-module algebra. The multiplication of C^* is the convolution, that is $(c^*d^*)(c) = \sum c^*(c_{\underline{1}})d^*(c_{\underline{2}})$, the unit is $\underline{\varepsilon}$ and the left H-module structure is given by $(h \rightharpoonup c^*)(c) = c^*(c \cdot h)$, for $h \in H$, $c^*, d^* \in C^*$, $c \in C$. Thus C^* is a left H-module algebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H-comodule algebra. By Proposition 4.1, it makes sense to consider the generalized smash product algebra $C^* \ltimes \mathfrak{B}$.

Proposition 5.2. *Let H be a quasi-bialgebra, C a finite-dimensional right H-module coalgebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra. Then the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ of right–left $(H, \mathfrak{B}, C)$-Hopf modules is isomorphic to the category $\mathcal{M}_{C^* \ltimes \mathfrak{B}}$ of right modules over $C^* \ltimes \mathfrak{B}$.*

Proof. We restrict ourselves to defining the functors that demonstrate the isomorphism of categories, leaving all other details to the reader. Let $\{c_i\}_{i=\overline{1,n}}$ and $\{c^i\}_{i=\overline{1,n}}$ be dual bases in C and C^*.

Let N be a right $C^* \ltimes \mathfrak{B}$-module. Since $\mathbf{i} : \mathfrak{B} \to C^* \ltimes \mathfrak{B}$, $\mathbf{i}(\mathfrak{b}) = \underline{\varepsilon} \ltimes \mathfrak{b}$ for $\mathfrak{b} \in \mathfrak{B}$, is an algebra map, it follows that N is a right $\mathfrak{B}$-module via the action $n\mathfrak{b} = n\mathbf{i}(\mathfrak{b}) = n(\underline{\varepsilon} \ltimes \mathfrak{b})$, $n \in N$, $\mathfrak{b} \in \mathfrak{B}$. The map $j : C^* \to C^* \ltimes \mathfrak{B}$, $j(c^*) = c^* \ltimes 1_{\mathfrak{B}}$, $c^* \in C^*$, is not an algebra map (it is not multiplicative) but it can be used to define a left C-coaction on N:

$$\rho_N(n) = \sum n_{[-1]} \otimes n_{[0]} = \sum_{i=1}^{n} c_i \otimes nj\big(c^i\big) = \sum_{i=1}^{n} c_i \otimes n\big(c^i \ltimes 1_{\mathfrak{B}}\big). \tag{5.6}$$

We can easily check that N becomes an object in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$.

Conversely, take $N \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$. Then N is a right $\mathfrak{B}$-module and C^* acts on M from the right as follows: let $nc^* = \sum c^*(n_{[-1]})n_{[0]}$, $n \in N$, $c^* \in C^*$. Now define

$$n(c^* \ltimes \mathfrak{b}) = (nc^*)\mathfrak{b} = \sum c^*(n_{[-1]})n_{[0]}\mathfrak{b}. \tag{5.7}$$

Then N becomes a right $C^* \ltimes \mathfrak{B}$-module. □

5.2. Doi–Hopf modules and comodules over a coring

Now, we will show that the category of right–left Doi–Hopf modules is isomorphic to a category of right comodules over a certain coring. Let us first recall the definition of a coring.

Let R be a ring (with unit). An R-coring $\mathcal{C}$ is an R-bimodule together with two R-bimodule maps

$$\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C} \quad \text{and} \quad \varepsilon_{\mathcal{C}} : \mathcal{C} \to R$$

such that the usual coassociativity and counit properties hold; that means:

$$(\Delta_{\mathcal{C}} \otimes_R \mathrm{id}_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}} \otimes_R \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}},$$
$$(\varepsilon_{\mathcal{C}} \otimes_R \mathrm{id}_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}} \otimes_R \varepsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = \mathrm{id}_C.$$

A right $\mathcal{C}$-comodule is a right R-module M together with a right R-linear map $\rho^r : M \to M \otimes_R \mathcal{C}$ such that

$$(\rho^r \otimes_R \mathrm{id}_{\mathcal{C}}) \circ \rho^r = (\mathrm{id}_M \otimes_R \Delta_{\mathcal{C}}) \circ \rho^r, \tag{5.8}$$
$$(\mathrm{id}_M \otimes_R \varepsilon_{\mathcal{C}}) \circ \rho^r = \mathrm{id}_M. \tag{5.9}$$

A map $\mathfrak{h} : M \to N$ between two right $\mathcal{C}$-comodules is called a $\mathcal{C}$-comodule map if $\mathfrak{h}$ is a right R-module map and $\rho^r \circ \mathfrak{h} = (\mathfrak{h} \otimes_R \mathrm{id}_{\mathcal{C}}) \circ \rho^r$. We denote by $\mathcal{M}^{\mathcal{C}}$ the category of right $\mathcal{C}$-comodules and $\mathcal{C}$-comodule maps. We will use the Sweedler notation for corings and comodules over corings:

$$\Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes_R c_{(2)}, \qquad \rho^r(m) = \sum m_{(0)} \otimes_R m_{(1)}.$$

Lemma 5.3. *Let H be a quasi-bialgebra, $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra, and C a right H-module coalgebra. Then $\mathcal{C} := \mathfrak{B} \otimes C$ is a $\mathfrak{B}$-coring. First, $\mathcal{C}$ is a $\mathfrak{B}$-bimodule via*

$$\mathfrak{b}(\mathfrak{b}' \otimes c) = \mathfrak{b}\mathfrak{b}' \otimes c \quad \text{and} \quad (\mathfrak{b} \otimes c)\mathfrak{b}' = \sum \mathfrak{b}\mathfrak{b}'_{[0]} \otimes c \cdot \mathfrak{b}'_{[-1]} \tag{5.10}$$

for all $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$ and $c \in C$. Secondly, for all $\mathfrak{b} \in \mathfrak{B}$ and $c \in C$, the two $\mathfrak{B}$-bimodule maps are defined by

$$\Delta_{\mathcal{C}}(\mathfrak{b} \otimes c) = \sum \big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\big), \tag{5.11}$$
$$\varepsilon_{\mathcal{C}}(\mathfrak{b} \otimes c) = \underline{\varepsilon}(c)\mathfrak{b}. \tag{5.12}$$

Proof. Since $\mathfrak{B}$ is an associative unital algebra and $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ is an algebra map, it follows that $\mathfrak{B} \otimes C$ is a $\mathfrak{B}$-bimodule via the actions defined in (5.10). Also, it is not hard to see that $\varepsilon_{\mathcal{C}}$ is a $\mathfrak{B}$-bimodule map. The fact that $\Delta_{\mathcal{C}}$ is left $\mathfrak{B}$-linear is straightforward. It is also right $\mathfrak{B}$-linear since

$$\begin{aligned}
\Delta_{\mathcal{C}}\big((\mathfrak{b} \otimes c)\mathfrak{b}'\big) &= \sum \Delta_{\mathcal{C}}\big(\mathfrak{b}\mathfrak{b}'_{[0]} \otimes c \cdot \mathfrak{b}'_{[-1]}\big) \\
&\overset{(1.33)}{=} \sum \big(\mathfrak{b}\mathfrak{b}'_{[0]}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \mathfrak{b}'_{[-1]_2}\tilde{x}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \mathfrak{b}'_{[-1]_1}\tilde{x}^1\big) \\
&\overset{(2.5),(5.10)}{=} \sum \big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big)\mathfrak{b}'_{[0]} \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\mathfrak{b}'_{[-1]}\big) \\
&= \sum \big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big) \otimes_{\mathfrak{B}} \big(\mathfrak{b}'_{[0]} \otimes c_{\underline{1}} \cdot \tilde{x}^1\mathfrak{b}'_{[-1]}\big) \\
&\overset{(5.10)}{=} \sum \big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\big)\mathfrak{b}' = \Delta_{\mathcal{C}}(\mathfrak{b} \otimes c)\mathfrak{b}'
\end{aligned}$$

for all $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$ and $c \in C$. Now, for all $\mathfrak{b} \in \mathfrak{B}$ and $c \in C$, we have that

$$\begin{aligned}
&(\Delta_{\mathcal{C}} \otimes_{\mathfrak{B}} \mathrm{id}_{\mathcal{C}})\big(\Delta_{\mathcal{C}}(\mathfrak{b} \otimes c)\big)\\
&= \sum \Delta_{\mathcal{C}}\big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\big)\\
\stackrel{(1.33)}{=}& \sum \big(\mathfrak{b}\tilde{x}^3\tilde{y}^3 \otimes c_{(\underline{2},\underline{2})} \cdot \tilde{x}^2_2\tilde{y}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{(\underline{2},\underline{1})} \cdot \tilde{x}^2_1\tilde{y}^1\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\big)\\
\stackrel{(1.32)}{=}& \sum \big(\mathfrak{b}\tilde{x}^3\tilde{y}^3 \otimes c_{\underline{2}} \cdot x^3\tilde{x}^2_2\tilde{y}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{2})} \cdot x^2\tilde{x}^2_1\tilde{y}^1\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{1})} \cdot x^1\tilde{x}^1\big)\\
\stackrel{(2.6)}{=}& \sum \big(\mathfrak{b}\tilde{x}^3\tilde{y}^3_{[0]} \otimes c_{\underline{2}} \cdot \tilde{x}^2\tilde{y}^3_{[-1]}\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{2})} \cdot \tilde{x}^1_2\tilde{y}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{1})} \cdot \tilde{x}^1_1\tilde{y}^1\big)\\
\stackrel{(5.10)}{=}& \sum \big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big) \otimes_{\mathfrak{B}} \big(\tilde{y}^3 \otimes c_{(\underline{1},\underline{2})} \cdot \tilde{x}^1_2\tilde{y}^2\big) \otimes_{\mathfrak{B}} \big(1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{1})} \cdot \tilde{x}^1_1\tilde{y}^1\big)\\
\stackrel{(1.33),(5.11)}{=}& \sum \big(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2\big) \otimes_{\mathfrak{B}} \Delta_{\mathcal{C}}\big(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\big) = (\mathrm{id}_{\mathcal{C}} \otimes_{\mathfrak{B}} \Delta_{\mathcal{C}})\big(\Delta_{\mathcal{C}}(\mathfrak{b} \otimes c)\big),
\end{aligned}$$

as needed. It is easy to see that $\varepsilon_{\mathcal{C}}$ is the counit for $\Delta_{\mathcal{C}}$, so the proof is finished. □

We can now prove the following theorem.

Theorem 5.4. *Let H be a quasi-bialgebra, $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra, and C a right H-module coalgebra. If $\mathcal{C} = \mathfrak{B} \otimes C$ is the $\mathfrak{B}$-coring defined in Lemma* 5.3*, then the category of right–left Doi–Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right $\mathcal{C}$-comodules, $\mathcal{M}^{\mathcal{C}}$.*

Proof. If $M \in \mathcal{M}^{\mathcal{C}}$ then we adopt a similar notation as the one used in the proof of Theorem 3.7. Namely, if $M \in \mathcal{M}^{\mathcal{C}}$ with $\rho^r : M \to M \otimes_{\mathfrak{B}} (\mathfrak{B} \otimes C)$, then we set

$$\rho^r(m) = \sum m_{(0)} \otimes_{\mathfrak{B}} \big(m_{(1)^{\mathfrak{B}}} \otimes m_{(1)^C}\big) \quad \forall m \in M.$$

With this notation, the fact that ρ^r is right $\mathfrak{B}$-linear means

$$\sum (m\mathfrak{b})_{(0)} \otimes_{\mathfrak{B}} \big((m\mathfrak{b})_{(0)^{\mathfrak{B}}} \otimes (m\mathfrak{b})_{(1)^C}\big) = \sum m_{(0)} \otimes_{\mathfrak{B}} \big(m_{(1)^{\mathfrak{B}}}\mathfrak{b}_{[0]} \otimes m_{(1)^C} \cdot \mathfrak{b}_{[-1]}\big)$$

for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$, and this is equivalent to

$$\sum (m\mathfrak{b})_{(0)}(m\mathfrak{b})_{(0)^{\mathfrak{B}}} \otimes (m\mathfrak{b})_{(1)^C} = \sum m_{(0)}m_{(1)^{\mathfrak{B}}}\mathfrak{b}_{[0]} \otimes m_{(1)^C} \cdot \mathfrak{b}_{[-1]} \tag{5.13}$$

for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$. Similarly, in this particular case, the relations (5.8) and (5.9) reduce to

$$\begin{aligned}
&\sum m_{(0)_{(0)}}m_{(0)_{(1)^{\mathfrak{B}}}}m_{(1)^{\mathfrak{B}}_{[0]}} \otimes m_{(0)_{(1)^C}} \cdot m_{(1)^{\mathfrak{B}}_{[-1]}} \otimes m_{(1)^C}\\
&\quad = \sum m_{(0)}m_{(1)^{\mathfrak{B}}}\tilde{x}^3 \otimes m_{(1)^C_{\underline{2}}} \cdot \tilde{x}^2 \otimes m_{(1)^C_{\underline{1}}} \cdot \tilde{x}^1,
\end{aligned} \tag{5.14}$$

$$\sum \underline{\varepsilon}\big(m_{(1)^C}\big)m_{(0)}m_{(1)^{\mathfrak{B}}} = m, \tag{5.15}$$

for all $\mathfrak{b} \in \mathfrak{B}$ and $m \in M$. Now, if we define

$$\rho_M : M \to C \otimes M, \quad \rho_M(m) = \sum m_{(1)^C} \otimes m_{(0)} m_{(1)^{\mathfrak{B}}} \quad \forall m \in M,$$

then (5.13) implies that $\rho_M(m\mathfrak{b}) = \rho_M(m)\lambda(\mathfrak{b})$ for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$, and (5.15) implies that $(\underline{\varepsilon} \otimes \mathrm{id}_M) \circ \rho_M = \mathrm{id}_M$, respectively. Thus, $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$ since

$$\begin{aligned}
(\mathrm{id}_C \otimes \rho_M)\big(\rho_M(m)\big) &= \sum m_{(1)^C} \otimes \rho_M\big(m_{(0)} m_{(1)^{\mathfrak{B}}}\big) \\
\overset{(5.13)}{=} &\sum m_{(1)^C} \otimes m_{(0)_{(1)^C}} \cdot m_{(1)^{\mathfrak{B}}_{[-1]}} \otimes m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{B}}}} m_{(1)^{\mathfrak{B}}_{[0]}} \\
\overset{(5.14)}{=} &\sum m_{(1)^C_{\underline{1}}} \cdot \tilde{x}^1 \otimes m_{(1)^C_{\underline{2}}} \cdot \tilde{x}^2 \otimes m_{(0)} m_{(1)^{\mathfrak{B}}} \tilde{x}^3 \\
&= (\underline{\Delta} \otimes \mathrm{id}_M)\big(\rho_M(m)\big)\Phi_\lambda^{-1}
\end{aligned}$$

for all $m \in M$, as needed. Therefore, we have a functor $\mathfrak{F} : \mathcal{M}^{\mathcal{C}} \to {}^C\mathcal{M}(H)_{\mathfrak{B}}$ which acts on objects as above and sends a morphism to itself (the verification of the fact that a morphism in $\mathcal{M}^{\mathcal{C}}$ becomes a morphism in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is left to the reader). Conversely, if $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$ with $\rho_M(m) = \sum m_{[-1]} \otimes m_{[0]}$, $m \in M$, then we define

$$\rho^r : M \to M \otimes_{\mathfrak{B}} (\mathfrak{B} \otimes C), \quad \rho^r(m) = \sum m_{[0]} \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes m_{[-1]}) \quad \forall m \in M.$$

It is not hard to see that in this way the right $\mathfrak{B}$-module M becomes a right $\mathcal{C}$-comodule, i.e. the relations (5.13)–(5.15) hold. So we also have a functor $\mathfrak{G} : {}^C\mathcal{M}(H)_{\mathfrak{B}} \to \mathcal{M}^{\mathcal{C}}$ ($\mathfrak{G}$ sends a morphism to itself). Finally, it is routine to check that $\mathfrak{F}$ and $\mathfrak{G}$ are inverses; we leave the details to the reader. □

6. Two-sided two-cosided Hopf modules

Now we define the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$. If H is finite-dimensional, then this category is isomorphic to a certain category of right–left Doi–Hopf modules, ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}$. As a consequence, if C is also finite-dimensional then this category is isomorphic to the category of right modules over a generalized smash product, by Proposition 5.2.

Definition 6.1 [16, *Definition* 8.2]. Let H be a quasi-bialgebra. An H-bicomodule algebra $\mathbb{A}$ is a quintuple $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$, where λ and ρ are left and right H-coactions on $\mathbb{A}$, and where $\Phi_\lambda \in H \otimes H \otimes \mathbb{A}$, $\Phi_\rho \in \mathbb{A} \otimes H \otimes H$, and $\Phi_{\lambda,\rho} \in H \otimes \mathbb{A} \otimes H$ are invertible elements, such that

– $(\mathbb{A}, \lambda, \Phi_\lambda)$ is a left H-comodule algebra,
– $(\mathbb{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra,

– the following compatibility relations hold, for all $a \in \mathbb{A}$:

$$\Phi_{\lambda,\rho}(\lambda \otimes \mathrm{id})\big(\rho(a)\big) = (\mathrm{id} \otimes \rho)\big(\lambda(a)\big)\Phi_{\lambda,\rho}, \tag{6.1}$$

$$\begin{aligned}&(1_H \otimes \Phi_{\lambda,\rho})(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\Phi_{\lambda,\rho})(\Phi_\lambda \otimes 1_H)\\ &\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\Phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_{\lambda,\rho}),\end{aligned} \tag{6.2}$$

$$\begin{aligned}&(1_H \otimes \Phi_\rho)(\mathrm{id} \otimes \rho \otimes \mathrm{id})(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho} \otimes 1_H)\\ &\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi_{\lambda,\rho})(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_\rho).\end{aligned} \tag{6.3}$$

It was pointed out in [16] that the following additional relations hold in an H-bicomodule algebra $\mathbb{A}$:

$$(\mathrm{id}_H \otimes \mathrm{id}_{\mathbb{A}} \otimes \varepsilon)(\Phi_{\lambda,\rho}) = 1_H \otimes 1_{\mathbb{A}}, \qquad (\varepsilon \otimes \mathrm{id}_{\mathbb{A}} \otimes \mathrm{id}_H)(\Phi_{\lambda,\rho}) = 1_{\mathbb{A}} \otimes 1_H. \tag{6.4}$$

As the first example, take $\mathbb{A} = H$, $\lambda = \rho = \Delta$, and $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = \Phi$. Related to the left and right comodule algebra structures of $\mathbb{A}$, we will keep the notation of the previous sections. We will use the following notation:

$$\Phi_{\lambda,\rho} = \sum \Omega^1 \otimes \Omega^2 \otimes \Omega^3 = \sum \overline{\Omega}^1 \otimes \overline{\Omega}^2 \otimes \overline{\Omega}^3 = \cdots \quad \text{and}$$
$$\Phi_{\lambda,\rho}^{-1} = \sum \omega^1 \otimes \omega^2 \otimes \omega^3 = \sum \overline{\omega}^1 \otimes \overline{\omega}^2 \otimes \overline{\omega}^3 = \cdots.$$

If H is a quasi-bialgebra, then the opposite algebra H^{op} is also a quasi-bialgebra. The reassociator of H^{op} is $\Phi_{\mathrm{op}} = \Phi^{-1}$. $H \otimes H^{\mathrm{op}}$ is also a quasi-bialgebra with reassociator

$$\Phi_{H\otimes H^{\mathrm{op}}} = \sum \big(X^1 \otimes x^1\big) \otimes \big(X^2 \otimes x^2\big) \otimes \big(X^3 \otimes x^3\big). \tag{6.5}$$

If we identify $H \otimes H^{\mathrm{op}}$-modules and (H, H)-bimodules, then the category of (H, H)-bimodules, ${}_H\mathcal{M}_H$, is monoidal. The associativity constraints are given by $\mathbf{a}'_{U,V,W}: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, where

$$\mathbf{a}'_{U,V,W}\big((u \otimes v) \otimes w\big) = \Phi \cdot \big(u \otimes (v \otimes w)\big) \cdot \Phi^{-1} \tag{6.6}$$

for all $U, V, W \in {}_H\mathcal{M}_H$, $u \in U$, $v \in V$, and $w \in W$. A coalgebra in the category of (H, H)-bimodules will be called an H-bimodule coalgebra. More precisely, an H-bimodule coalgebra C is an (H, H)-bimodule (denote the actions by $h \cdot c$ and $c \cdot h$) with a comultiplication $\underline{\Delta} : C \to C \otimes C$ and a counit $\underline{\varepsilon} : C \to k$ satisfying the following relations, for all $c \in C$ and $h \in H$:

$$\Phi \cdot (\underline{\Delta} \otimes \mathrm{id}_C)\big(\underline{\Delta}(c)\big) \cdot \Phi^{-1} = (\mathrm{id}_C \otimes \underline{\Delta})\big(\underline{\Delta}(c)\big), \tag{6.7}$$

$$\underline{\Delta}(h \cdot c) = \sum h_1 \cdot c_{\underline{1}} \otimes h_2 \cdot c_{\underline{2}}, \qquad \underline{\Delta}(c \cdot h) = \sum c_{\underline{1}} \cdot h_1 \otimes c_{\underline{2}} \cdot h_2, \tag{6.8}$$

$$(\underline{\varepsilon} \otimes \mathrm{id}_C) \circ \underline{\Delta} = (\mathrm{id}_C \otimes \underline{\varepsilon}) \circ \underline{\Delta} = \mathrm{id}_C, \tag{6.9}$$

$$\underline{\varepsilon}(h\cdot c)=\varepsilon(h)\underline{\varepsilon}(c),\qquad \underline{\varepsilon}(c\cdot h)=\underline{\varepsilon}(c)\varepsilon(h),\tag{6.10}$$

where we used the same Sweedler-type notation as before. An H-bimodule coalgebra C becomes a right $H\otimes H^{\mathrm{op}}$-module coalgebra via the right $H\otimes H^{\mathrm{op}}$-action

$$c\cdot(h\otimes h')=h'\cdot c\cdot h\tag{6.11}$$

for $c\in C$ and $h,h'\in H$. Our next definition extends the definition of two-sided two-cosided Hopf modules from [24].

Definition 6.2. Let H be a quasi-bialgebra, $(\mathbb{A},\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$ an H-bicomodule algebra, and C an H-bimodule coalgebra. A two-sided two-cosided $(H,\mathbb{A},C)$-Hopf module is a k-vector space with the following additional structure:

- N is an $(H,\mathbb{A})$-two-sided Hopf module, i.e. $N\in {}_H\mathcal{M}_{\mathbb{A}}^H$; we write $\succ$ for the left H-action, $\prec$ for the right $\mathbb{A}$-action, and $\rho_N^H(n)=\sum n_{(0)}\otimes n_{(1)}$ for the right H-coaction on $n\in N$;
- we have k-linear map $\rho_N^C:N\to C\otimes N$, $\rho_N^C(n)=\sum n_{[-1]}\otimes n_{[0]}$, called the left C-coaction on N, such that $\sum\underline{\varepsilon}(n_{[-1]})n_{[0]}=n$ and

$$\Phi(\underline{\Delta}\otimes\mathrm{id}_N)\big(\rho_N^C(n)\big)=\big(\mathrm{id}_C\otimes\rho_N^C\big)\big(\rho_N^C(n)\big)\Phi_\lambda\tag{6.12}$$

  for all $n\in N$;
- N is a (C,H)-“bicomodule,” in the sense that, for all $n\in N$,

$$\Phi\big(\rho_N^C\otimes\mathrm{id}_H\big)\big(\rho_N^H(n)\big)=\big(\mathrm{id}_C\otimes\rho_N^H\big)\big(\rho_N^C(n)\big)\Phi_{\lambda,\rho};\tag{6.13}$$

- the following compatibility relations hold:

$$\rho_N^C(h\succ n)=\sum h_1\cdot n_{[-1]}\otimes h_2\succ n_{[0]},\tag{6.14}$$

$$\rho_N^C(n\prec a)=\sum n_{[-1]}\cdot a_{[-1]}\otimes n_{[0]}\prec a_{[0]}\tag{6.15}$$

  for all $h\in H$, $n\in N$, and $a\in\mathbb{A}$.

${}_H^C\mathcal{M}_{\mathbb{A}}^H$ will be the category of two-sided two-cosided Hopf modules and maps preserving the actions by H and $\mathbb{A}$ and the coactions by H and C.

Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra, and C an H-bimodule coalgebra. If H is finite-dimensional, then the category ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to a certain category of Doi–Hopf modules. In order to prove this, we first need some lemmas.

Lemma 6.3. *Let H be a quasi-Hopf algebra and $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ an H-bicomodule algebra. Consider the map*

$$\wp : (\mathbb{A} \,\overline{\#}\, H^*) \# H \to (H \otimes H^{\mathrm{op}}) \otimes (\mathbb{A} \,\overline{\#}\, H^*) \# H$$

given by

$$\wp\big((a \,\overline{\#}\, \varphi) \# h\big) = \sum a_{[-1]}\omega^1 \otimes S\big(y^3 h_2\big) \otimes \big(a_{[0]}\omega^2 \,\overline{\#}\, y^1 \rightharpoonup \varphi \leftharpoonup \omega^3\big) \# y^2 h_1 \tag{6.16}$$

for any $a \in \mathbb{A}$, $\varphi \in H^*$, and $h \in H$, where $\Phi_{\lambda,\rho}^{-1} = \sum \omega^1 \otimes \omega^2 \otimes \omega^3$. Set

$$\Phi_\wp = \sum \big(\widetilde{X}^1_\lambda \otimes g^1 S\big(x^3\big)\big) \otimes \big(\widetilde{X}^2_\lambda \otimes g^2 S\big(x^2\big)\big) \otimes \big(\widetilde{X}^3_\lambda \,\overline{\#}\, \varepsilon\big) \# x^1 \tag{6.17}$$

where $f^{-1} = \sum g^1 \otimes g^2$ is the element defined in (1.16)*. Then $((\mathbb{A} \,\overline{\#}\, H^*) \# H, \wp, \Phi_\wp)$ is a left $H \otimes H^{\mathrm{op}}$-comodule algebra.*

Proof. We first show that $\wp$ is an algebra map. Using (1.30) and (2.11), we can easily show that the multiplication on $(\mathbb{A} \,\overline{\#}\, H^*) \# H$ is given by

$$\begin{aligned}
&\big((a \,\overline{\#}\, \varphi) \# h\big)\big((a' \,\overline{\#}\, \psi) \# h'\big) \\
&\quad = \sum \big[a a'_{\langle 0\rangle} \tilde{x}^1_\rho \,\overline{\#}\, \big(x^1 \rightharpoonup \varphi \leftharpoonup a'_{\langle 1\rangle} \tilde{x}^2_\rho\big)\big(x^2 h_1 \rightharpoonup \psi \leftharpoonup \tilde{x}^3_\rho\big)\big] \# x^3 h_2 h'
\end{aligned} \tag{6.18}$$

for all $a, a' \in \mathbb{A}$, $\varphi, \psi \in H^*$, and $h, h' \in H$. Therefore

$$\begin{aligned}
&\wp\big(\big((a \,\overline{\#}\, \varphi) \# h\big)\big((a' \,\overline{\#}\, \psi) \# h'\big)\big) \\
&\quad = \sum a_{[-1]} a'_{\langle 0\rangle_{[-1]}} \big(\tilde{x}^1_\rho\big)_{[-1]} \omega^1 \otimes S\big(y^3 x^3_2 h_{(2,2)} h'_2\big) \otimes \Big[a_{[0]} a'_{\langle 0\rangle_{[0]}} \big(\tilde{x}^1_\rho\big)_{[0]} \omega^2 \\
&\qquad\quad \overline{\#}\, \big(y^1_1 x^1 \rightharpoonup \varphi \leftharpoonup a'_{\langle 1\rangle} \tilde{x}^2_\rho \omega^3_1\big)\big(y^1_2 x^2 h_1 \rightharpoonup \psi \leftharpoonup \tilde{x}^3_\rho \omega^3_2\big)\Big] \# y^2 x^3_1 h_{(2,1)} h'_1 \\
&\quad \stackrel{(6.3),(1.3)}{=} \sum a_{[-1]} a'_{\langle 0\rangle_{[-1]}} \varpi^1 \omega^1 \otimes S\big(y^3 x^3 h_{(2,2)} h'_2\big) \otimes \Big[a_{[0]} a'_{\langle 0\rangle_{[0]}} \varpi^2 \omega^2_{\langle 0\rangle} \tilde{x}^1_\rho \\
&\qquad\quad \overline{\#}\, \big(z^1 y^1 \rightharpoonup \varphi \leftharpoonup a'_{\langle 1\rangle} \varpi^3 \omega^2_{\langle 1\rangle} \tilde{x}^2_\rho\big)\big(z^2 y^2_1 x^1 h_1 \rightharpoonup \psi \leftharpoonup \omega^3 \tilde{x}^3_\rho\big)\Big] \# z^3 y^2_2 x^2 h_{(2,1)} h'_1 \\
&\quad \stackrel{(6.1),(1.1)}{=} \sum a_{[-1]} \varpi^1 a'_{[-1]} \omega^1 \otimes S\big(y^3 h_2\big) \cdot_{\mathrm{op}} S\big(x^3 h'_2\big) \otimes \Big[a_{[0]} \varpi^2 \big(a'_{[0]} \omega^2\big)_{\langle 0\rangle} \tilde{x}^1_\rho \\
&\qquad\quad \overline{\#}\, \big(z^1 y^1 \rightharpoonup \varphi \leftharpoonup \varpi^3 \big(a'_{[0]} \omega^2\big)_{\langle 1\rangle} \tilde{x}^2_\rho\big)\big(z^2 y^2_1 h_{(1,1)} x^1 \rightharpoonup \psi \leftharpoonup \omega^3 \tilde{x}^3_\rho\big)\Big] \\
&\qquad\quad \# z^3 y^2_2 h_{(1,2)} x^2 h'_1 \\
&\quad \stackrel{(2.11)}{=} \sum a_{[-1]} \varpi^1 a'_{[-1]} \omega^1 \otimes S\big(y^3 h_2\big) \cdot_{\mathrm{op}} S\big(x^3 h'_2\big) \otimes \Big[\big(a_{[0]} \varpi^2 \,\overline{\#}\, z^1 y^1 \rightharpoonup \varphi \leftharpoonup \varpi^3\big) \\
&\qquad\quad \big(a'_{[0]} \omega^2 \,\overline{\#}\, z^2 y^2_1 h_{(1,1)} x^1 \rightharpoonup \psi \leftharpoonup \omega^3\big)\Big] \# z^3 y^2_2 h_{(1,2)} x^2 h'_1
\end{aligned}$$

$$\overset{(1.30)}{=} \sum a_{[-1]}\overline{\omega}^1 a'_{[-1]}\omega^1 \otimes S\big(y^3h_2\big) \cdot_{\mathrm{op}} S\big(x^3h'_2\big) \otimes \big[\big(a_{[0]}\overline{\omega}^2 \,\overline{\#}\, y^1 \rightharpoonup \varphi \leftharpoonup \overline{\omega}^3\big) \# y^2h_1\big]$$
$$\big[\big(a'_{[0]}\omega^2 \,\overline{\#}\, x^1 \rightharpoonup \psi \leftharpoonup \omega^3\big) \# x^2h'_1\big]$$
$$= \wp\big((a \,\overline{\#}\, \varphi) \# h\big)\wp\big((a' \,\overline{\#}\, \psi) \# h'\big)$$

where $\cdot_{\mathrm{op}}$ is the product in H^{op}. Obviously $\wp$ respects the unit element and (2.7) and (2.8) hold. (2.5) can be proved using similar computations as above and is left to the reader. Using the notation

$$\Phi_\wp = \sum \widetilde{X}^1_\wp \otimes \widetilde{X}^2_\wp \otimes \widetilde{X}^3_\wp = \cdots,$$

we can compute:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \wp)(\Phi_\wp)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_\wp)$$
$$= \sum \big(\widetilde{X}^1_\lambda \otimes g^1 S\big(x^3\big)\big)\big(\big(\widetilde{Y}^1_\lambda\big)_1 \otimes G^1_1 S\big(y^3\big)_1\big) \otimes \big(\widetilde{X}^2_\lambda \otimes g^2 S\big(x^2\big)\big)\big(\big(\widetilde{Y}^1_\lambda\big)_2 \otimes G^1_2 S\big(y^3\big)_2\big)$$
$$\otimes \big(\big(\widetilde{X}^3_\lambda\big)_{[-1]} \otimes S\big(x^1_2\big)\big)\big(\widetilde{Y}^2_\lambda \otimes G^2 S\big(y^2\big)\big) \otimes \big[\big(\big(\widetilde{X}^3_\lambda\big)_{[0]} \,\overline{\#}\, \varepsilon\big) \# x^1_1\big]\big[\big(\widetilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon\big) \# y^1\big]$$
$$\overset{(1.11),(1.3)}{=} \sum \big(\widetilde{X}^1_\lambda\big(\widetilde{Y}^1_\lambda\big)_1 \otimes G^1_1 g^1 S\big(y^3x^3\big)\big) \otimes \big(\widetilde{X}^2_\lambda\big(\widetilde{Y}^1_\lambda\big)_2 \otimes G^1_2 g^2 S\big(z^3y^2_2x^2\big)\big)$$
$$\otimes \big(\big(\widetilde{X}^3_\lambda\big)_{[-1]}\widetilde{Y}^2_\lambda \otimes G^2 S\big(z^2y^2_1x^1\big)\big) \otimes \big[\big(\big(\widetilde{X}^3_\lambda\big)_{[0]}\widetilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon\big) \# z^1y^1\big]$$
$$\overset{(2.6),(1.9),(1.18)}{=} \sum \big(\widetilde{Y}^1_\lambda X^1 \otimes x^1 g^1 S\big(y^3\big)\big) \otimes \big(\widetilde{X}^1_\lambda\big(\widetilde{Y}^2_\lambda\big)_1 X^2 \otimes x^2 g^2_1 G^1 S\big(z^3y^2_2\big)\big)$$
$$\otimes \big(\widetilde{X}^2_\lambda\big(\widetilde{Y}^2_\lambda\big)_2 X^3 \otimes x^3 g^2_2 G^2 S\big(z^2y^2_1\big)\big) \otimes \big[\big(\widetilde{X}^3_\lambda\widetilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon\big) \# z^1y^1\big]$$
$$\overset{(1.11)}{=} \sum \big(\widetilde{Y}^1_\lambda \otimes g^1 S\big(y^3\big)\big)\big(X^1 \otimes x^1\big) \otimes \big(\widetilde{X}^1_\lambda \otimes G^1 S\big(z^3\big)\big)\big(\big(\widetilde{Y}^2_\lambda\big)_1 \otimes g^2_1 S\big(y^2\big)_1\big)\big(X^2 \otimes x^2\big)$$
$$\otimes \big(\widetilde{X}^2_\lambda \otimes G^2 S\big(z^2\big)\big)\big(\big(\widetilde{Y}^2_\lambda\big)_2 \otimes g^2_2 S\big(y^2\big)_2\big)\big(X^3 \otimes x^3\big)$$
$$\otimes \big[\big(\widetilde{X}^3_\lambda \,\overline{\#}\, \varepsilon\big) \# z^1\big]\big[\big(\widetilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon\big) \# y^1\big]$$
$$\overset{(6.5)}{=} (1_H \otimes \Phi_\wp)(\mathrm{id} \otimes \Delta_{H\otimes H^{\mathrm{op}}} \otimes \mathrm{id})(\Phi_\wp)(\Phi_{H\otimes H^{\mathrm{op}}} \otimes \mathbf{1})$$

where $\sum G^1 \otimes G^2$ is another copy of f^{-1} and $\mathbf{1} = (1_{\mathbb{A}} \,\overline{\#}\, \varepsilon) \# 1_H$ is the unit of the algebra $(\mathbb{A} \,\overline{\#}\, H^*) \# H$. □

Let H be a quasi-Hopf algebra, $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ an H-bicomodule algebra, and C an H-bimodule coalgebra. By Lemma 6.3, we can consider the category of Doi–Hopf modules ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}$. We will prove that it is isomorphic to the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$, in the case where H is finite-dimensional.

Lemma 6.4. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra, and C an H-bimodule coalgebra. We have a functor*

$$F : {}^C_H\mathcal{M}^H_{\mathbb{A}} \to {}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}.$$

$F(N)=N$ as a k-module, with structure maps given by the equations

$$n \leftarrow \big((a\,\overline{\#}\,\varphi)\,\#\,h\big)=\sum\big\langle\varphi, S^{-1}\big(f^2 n_{(1)} a_{\langle 1\rangle}\tilde{p}^2_\rho\big)\big\rangle S(h) f^1 \succ n_{(0)} \prec a_{\langle 0\rangle}\tilde{p}^1_\rho, \qquad (6.19)$$

$$\tilde{\rho}^C_N(n)=\sum n_{\{-1\}}\otimes n_{\{0\}}=\sum f^1\cdot n_{[-1]}\otimes f^2\succ n_{[0]} \qquad (6.20)$$

for all $n\in N$, $a\in\mathbb{A}$, $\varphi\in H^*$, and $h\in H$. F sends a morphism to itself.

Proof. Since N is a two-sided $(H,\mathbb{A})$-Hopf module, we know by (3.39) that N is a right $(\mathbb{A}\,\overline{\#}\,H^*)\,\#\,H$-module via the action defined by (6.19). Let $\sum F^1\otimes F^2$ be another copy of f. For any $n\in N$, we have that

$$
\begin{aligned}
&(\underline{\Delta}\otimes \mathrm{id}_N)\big(\tilde{\rho}^C_N(n)\big)\Phi^{-1}_{\wp}\\
&\overset{(6.17)}{=}\sum n_{\{-1\}_{\underline{1}}}\cdot\big(\tilde{x}^1_\lambda\otimes S\big(X^3\big)F^1\big)\otimes n_{\{-1\}_{\underline{2}}}\cdot\big(\tilde{x}^2_\lambda\otimes S\big(X^2\big)F^2\big)\otimes n_{\{0\}}\leftarrow\big[\big(\tilde{x}^3_\lambda\,\overline{\#}\,\varepsilon\big)\,\#\,X^1\big]\\
&\overset{(6.11)}{\underset{(6.20)}{=}}\sum S\big(X^3\big)F^1\cdot\big(f^1\cdot n_{[-1]}\big)_{\underline{1}}\cdot\tilde{x}^1_\lambda\otimes S\big(X^2\big)F^2\cdot\big(f^1\cdot n_{[-1]}\big)_{\underline{2}}\cdot\tilde{x}^2_\lambda\\
&\qquad\otimes S\big(X^1\big)f^2\succ n_{[0]}\prec\tilde{x}^3_\lambda\\
&\overset{(6.8)}{=}\sum S\big(X^3\big)F^1 f^1_1\cdot n_{[-1]_{\underline{1}}}\cdot\tilde{x}^1_\lambda\otimes S\big(X^2\big)F^2 f^1_2\cdot n_{[-1]_{\underline{2}}}\cdot\tilde{x}^2_\lambda\otimes S\big(X^1\big)f^2\succ n_{[0]}\prec\tilde{x}^3_\lambda\\
&\overset{(6.12),(1.9),(1.18)}{=}\sum f^1\cdot n_{[-1]}\otimes F^1 f^2_1\cdot n_{[0,-1]}\otimes F^2 f^2_2\succ n_{[0,0]}\\
&\overset{(6.14)}{=}\sum f^1\cdot n_{[-1]}\otimes F^1\cdot\big(f^2\succ n_{[0]}\big)_{[-1]}\otimes F^2\succ\big(f^2\succ n_{[0]}\big)_{[0]}\\
&\overset{(6.20)}{=}\sum n_{\{-1\}}\otimes F^1\cdot n_{\{0\}_{[-1]}}\otimes F^2\succ n_{\{0\}_{[0]}}\\
&\overset{(6.20)}{=}\big(\mathrm{id}_C\otimes\tilde{\rho}^C_N\big)\big(\tilde{\rho}^C_N(n)\big).
\end{aligned}
$$

We still have to show the compatibility relation (5.5). For, observe that (3.6), (6.3), and (1.5) imply

$$\sum\Omega^1\big(\tilde{p}^1_\rho\big)_{[-1]}\otimes\Omega^2\big(\tilde{p}^1_\rho\big)_{[0]}\otimes\Omega^3\tilde{p}^2_\rho=\sum\omega^1\otimes\omega^2_{\langle 0\rangle}\tilde{p}^1_\rho\otimes\omega^2_{\langle 1\rangle}\tilde{p}^2_\rho S\big(\omega^3\big). \qquad (6.21)$$

Now, for all $n\in N$, $a\in\mathbb{A}$, $\varphi\in H^*$, and $h\in H$ one can show that

$$\tilde{\rho}^C_N\big(n\leftarrow\big((a\,\overline{\#}\,\varphi)\,\#\,h\big)\big)=\tilde{\rho}^C_N(n)\wp\big((a\,\overline{\#}\,\varphi)\,\#\,h\big),$$

completing the proof. □

Lemma 6.5. *Let H be a finite-dimensional quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra, and C an H-bimodule coalgebra. We have a functor*

$$G:{}^C\mathcal{M}(H\otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}\to{}^C_H\mathcal{M}^H_{\mathbb{A}}.$$

$G(N)=N$ as a k-module, with structure maps given by

$$h \succ n = n \leftarrow \big[(1_{\mathbb{A}} \,\overline{\#}\, \varepsilon) \,\#\, S^{-1}(h)\big], \qquad n \prec a = n \leftarrow \big[(a \,\overline{\#}\, \varepsilon) \,\#\, 1_H\big], \tag{6.22}$$

$$\rho_N^H : N \to N \otimes H,$$

$$\rho_N^H(n) = \sum_{i=1}^{n} n \leftarrow \big[\big(\tilde{q}_\rho^1 \,\overline{\#}\, S^{-1}\big(g^2\big) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2\big) \,\#\, S^{-1}\big(g^1\big)\big] \otimes e_i, \tag{6.23}$$

$$\underline{\rho}_N^C : N \to C \otimes N, \quad \underline{\rho}_N^C(n) = \sum g^1 \cdot n_{[-1]} \otimes g^2 \succ n_{[0]} \tag{6.24}$$

for $n \in N$, $a \in \mathbb{A}$, and $h \in H$. Here $\{e_i\}_{i=\overline{1,n}}$ is a basis of H and $\{e^i\}_{i=\overline{1,n}}$ is the corresponding dual basis of H^*. G sends a morphism to itself.

Proof. Since N is a right $(\mathbb{A} \,\overline{\#}\, H^*) \,\#\, H$-module, we already know by (3.36) and (3.38) that H is a two-sided $(H, \mathbb{A})$-Hopf module via (6.22) and (6.23). Thus we only have to check (6.12)–(6.15). First note that $N \in {}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}$ implies

$$\begin{aligned}
&\sum n_{[-1]} \otimes n_{[0,-1]} \otimes n_{[0,0]} \\
&\quad = \sum S\big(X^3\big) f^1 \cdot n_{[-1]_{\underline{1}}} \cdot \tilde{x}_\lambda^1 \otimes S\big(X^2\big) f^2 \cdot n_{[-1]_{\underline{2}}} \cdot \tilde{x}_\lambda^2 \otimes n_{[0]} \leftarrow \big[\big(\tilde{x}_\lambda^3 \,\overline{\#}\, \varepsilon\big) \,\#\, X^1\big],
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
&\sum \big\{n \leftarrow \big[(a \,\overline{\#}\, \varphi) \,\#\, h\big]\big\}_{[-1]} \otimes \big\{n \leftarrow \big[(a \,\overline{\#}\, \varepsilon) \,\#\, h\big]\big\}_{[0]} \\
&\quad = \sum S\big(x^3 h_2\big) \cdot n_{[-1]} \cdot a_{[-1]} \omega^1 \otimes n_{[0]} \leftarrow \big[\big(a_{[0]} \omega^2 \,\overline{\#}\, x^1 \rightharpoonup \varphi \leftharpoonup \omega^3\big) \,\#\, x^2 h_1\big]
\end{aligned} \tag{6.26}$$

for all $n \in N$, $a \in \mathbb{A}$, $\varphi \in H^*$, and $h \in H$. By the above definitions and (6.26), it is immediate that

$$\underline{\rho}_N^C(h \succ n) = \Delta(h) \underline{\rho}_N^C(n) \quad \text{and} \quad \underline{\rho}_N^C(n \prec a) = \underline{\rho}_N^C(n) \rho_\lambda(a) \tag{6.27}$$

for all $h \in H$, $n \in N$, and $a \in \mathbb{A}$ (we leave it to the reader to verify the details). Let $\sum G^1 \otimes G^2$ be another copy of f^{-1}. We compute that

$$\begin{aligned}
&\Phi(\underline{\Delta} \otimes \mathrm{id}_N)\big(\underline{\rho}_N^C(n)\big) \\
&\overset{(6.24)}{=} \sum X^1 \cdot \big(g^1 \cdot n_{[-1]}\big)_{\underline{1}} \otimes X^2 \cdot \big(g^1 \cdot n_{[-1]}\big)_{\underline{2}} \otimes X^3 g^2 \succ n_{[0]} \\
&\overset{(6.22),(6.8)}{=} \sum X^1 g_1^1 \cdot n_{[-1]_{\underline{1}}} \otimes X^2 g_2^1 \cdot n_{[-1]_{\underline{2}}} \otimes n_{[0]} \leftarrow \big[(1_{\mathbb{A}} \,\overline{\#}\, \varepsilon) \,\#\, S^{-1}\big(X^3 g^2\big)\big] \\
&\overset{(6.25),(6.18)}{=} \sum X^1 g_1^1 G^1 S\big(x^3\big) \cdot n_{[-1]} \cdot \widetilde{X}_\lambda^1 \otimes X^2 g_2^1 G^2 S\big(x^2\big) \cdot n_{[0,-1]} \cdot \widetilde{X}_\lambda^2 \\
&\qquad \otimes n_{[0,0]} \leftarrow \big[\big(\widetilde{X}_\lambda^3 \,\overline{\#}\, \varepsilon\big) \,\#\, S^{-1}\big(X^3 g^2 S\big(x^1\big)\big)\big] \\
&\overset{(1.9),(1.18)}{=} \sum g^1 \cdot n_{[-1]} \cdot \widetilde{X}_\lambda^1 \otimes g_1^2 G^1 \cdot n_{[0,-1]} \cdot \widetilde{X}_\lambda^2 \otimes n_{[0,0]} \leftarrow \big[\big(\widetilde{X}_\lambda^3 \,\overline{\#}\, \varepsilon\big) \,\#\, S^{-1}\big(g_2^2 G^2\big)\big] \\
&\overset{(6.22)}{=} \sum g^1 \cdot n_{[-1]} \cdot \widetilde{X}_\lambda^1 \otimes g_1^2 G^1 \cdot n_{[0,-1]} \cdot \widetilde{X}_\lambda^2 \otimes g_2^2 G^2 \succ n_{[0,0]} \prec \widetilde{X}_\lambda^3 \\
&\overset{(6.24),(6.8),(6.27)}{=} \big(\mathrm{id}_C \otimes \underline{\rho}_N^C\big)\big(\underline{\rho}_N^C(n)\big) \Phi_\lambda.
\end{aligned}$$

The verification of (6.13) is based on similar computations, and we leave the details to the reader. □

As a consequence of Lemmas 6.4 and 6.5, we have the following description of ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ as a category of Doi–Hopf modules; this description generalizes [4, Proposition 2.3].

Theorem 6.6. *Let H be a finite-dimensional quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra, and C an H-bimodule coalgebra. Then the categories ${}^C\mathcal{M}(H\otimes H^{\mathrm{op}})_{(\mathbb{A}\overline{\#}H^*)\#H}$ and ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ are isomorphic.*

Proof. We have to verify that the functors F and G defined in Lemmas 6.4 and 6.5 are inverses. For the C-coactions (6.20) and (6.24), this is obvious; for the other structures, it has been already done in Corollary 3.6. □

Propositions 5.2 and 5.4, and Theorem 6.6 immediately imply the following result.

Corollary 6.7. *Let H be a finite-dimensional quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra, and C an H-bimodule coalgebra. Then ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to the category of right comodules over the coring $\mathbf{C}=((\mathbb{A}\,\overline{\#}\,H^*)\,\#\,H)\otimes C$. If C is finite-dimensional, then the category ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to the category of right modules over the generalized smash product $C^*\ltimes((\mathbb{A}\,\overline{\#}\,H^*)\,\#\,H)$.*

Remark 6.8. Let H be a finite-dimensional Hopf algebra. Cibils and Rosso [10] introduced an algebra $X=(H^{\mathrm{op}}\otimes H)\,\underline{\otimes}\,(H^*\otimes H^{*\mathrm{op}})$ having the property that the category of two-sided two-cosided Hopf modules over H^* coincides with the category of left X-modules. Moreover, it was also proved in [10] that X is isomorphic to the direct tensor product of a Heisenberg double and the opposite of a Drinfeld double. Recently, Panaite [23] introduced two other algebras Y and Z with the same property as X. More precisely, Y is the two-sided crossed product $H^*\#(H\otimes H^{\mathrm{op}})\#H^{*\mathrm{op}}$, and Z is the diagonal crossed product in the sense of [16], $(H^*\otimes H^{*\mathrm{op}})\bowtie(H\otimes H^{\mathrm{op}})$. Using different methods, we proved that the category of two-sided two-cosided Hopf modules over a finite-dimensional quasi-Hopf algebra is isomorphic to the category of right (respectively left) modules over the generalized smash product $\mathcal{A}=H^*\ltimes((H\,\overline{\#}\,H^*)\,\#\,H)$ (respectively $\mathcal{A}^{\mathrm{op}}$). Note that, in general, the multiplication on $C^*\ltimes((\mathbb{A}\,\overline{\#}\,H^*)\,\#\,H)$ is given by the formula

$$
\begin{aligned}
&\big[c^*\ltimes\big((a\,\overline{\#}\,\varphi)\,\#\,h\big)\big]\big[d^*\ltimes\big((a'\,\overline{\#}\,\psi)\,\#\,h'\big)\big]\\
&\quad=\sum\big(\tilde{x}_\lambda^1\rightharpoonup c^*\leftharpoonup S\big(X^3\big)f^1\big)\big(\tilde{x}_\lambda^2a_{[-1]}\omega^1\rightharpoonup d^*\leftharpoonup S\big(X^2x^3h_2\big)f^2\big)\\
&\qquad\ltimes\Big\{\big[\tilde{x}_\lambda^3a_{[0]}\omega^2a'_{\langle 0\rangle}\tilde{x}_\rho^1\,\overline{\#}\,\big(X^1_{(1,1)}y^1x^1\rightharpoonup\varphi\leftharpoonup\omega^3a'_{\langle 1\rangle}\tilde{x}_\rho^2\big)\big(X^1_{(1,2)}y^2x_1^2h_{(1,1)}\rightharpoonup\psi\leftharpoonup\tilde{x}_\rho^3\big)\big]\\
&\qquad\quad\#\,X_2^1y^3x_2^2h_{(1,2)}h'\Big\}.
\end{aligned}
$$

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