A New 7-Local Subgroup of the Monster

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Communicated by Walter Feit

Received December 18, 1986

1. INTRODUCTION

In treating a specific problem in finite groups recently, we came across the 7-local structure of the sporadic simple group $F_1$, the "Monster." To our surprise, we found a 7-local subgroup of $F_1$ not covered by the list in the "Atlas of Finite Groups" compiled by J. Conway et al. [3]. This subgroup turned out to be a maximal subgroup of $F_1$. The purpose of this paper is to complete the list of the maximal 7-local subgroups of $F_1$, using the notation from the Atlas, as the following

<table>
<thead>
<tr>
<th>Structure</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(7 : 3 \times \text{He}) : 2$</td>
<td>$N(7A)$</td>
</tr>
<tr>
<td>$7_{1+}^4 : (3 \times 2S_7)$</td>
<td>$N(7B)$</td>
</tr>
<tr>
<td>$(7^2 : (3 \times 2A_4) \times L_2(7)) \cdot 2$</td>
<td>$N(7A^2)$</td>
</tr>
<tr>
<td>$7^2 : 7 \cdot 2^2 : GL_2(7)$</td>
<td>$N(7B_1^2)$</td>
</tr>
<tr>
<td>$7^2 : SL_2(7)$</td>
<td>$N(7B_2^2)$</td>
</tr>
</tbody>
</table>

The first four groups consist of the groups in the list of the Atlas. The fifth subgroup is the new 7-local. We use $7B_i^2$ to denote a subgroup of type $7B_i$ of the $i$th kind, for $i = 1, 2$.

We also prove that there is only one conjugacy class of subgroups isomorphic to the sporadic simple group He in $F_1$; and $F_1$ does not contain any subgroup isomorphic to the sporadic simple group $(Fi_{24})'$.

Our notations are standard. They can be found in [1, 2, or 3]. We start off by recording the following general result.

1.1. LEMMA. Let $G$ be a finite group, $p$ be a prime, and $x \in G \setminus \{1\}$. Suppose $O_p(C_G(x)) = Q_x$ is an extraspecial $p$-group of width $w > 1$, and
$N_G\langle x \rangle/Q_x$ does not contain any extraspecial $p$-group. If $y \in x^G$ and $y \in Q_x$, then $x \in Q_y = O_p(C_G(y))$.

Proof. Since $y \in Q_x$, so $C_{Q_x}(y) = \langle y \rangle \times Q_1$, where $Q_1$ is an extraspecial $p$-group of width $w - 1$. This implies that $\langle x \rangle$ is the unique minimal normal subgroup of $Q_1$. Suppose $Q_1 \cap Q_y = 1$. Then $N_{Q_y}\langle y \rangle/Q_y$ contains $Q_1/Q_1$, where $Q_1/Q_1$ is an extraspecial $p$-group. However $y \not\in x^G$ implies that $N_{Q_y}\langle y \rangle/Q_y \cong N_G\langle x \rangle/Q_x$, which does not contain any extraspecial $p$-group. This contradiction proves that $Q_1 \cap Q_y \neq 1$. Since $Q_1 \cap Q_y$ is a normal subgroup of $Q_1$, it contains $x$. Therefore $x \in Q_y$ as desired.

2. A NEW 7-LOCAL SUBGROUP

In this section, let $G \cong F_1$. There are exactly two conjugacy classes of $7$-elements, namely $7A$ and $7B$ in $G$. For $x \in 7B$, set $Q_x = O_7(C_G(x))$ and $V_x = Q_x/\langle x \rangle$. Then $Q_x \cong \mathbb{Z}^3 \times 4$ and $V_x$ is a $4$-dimensional non-degenerate symplectic space over $GF(7)$. Also $N_G\langle x \rangle = Q_x : S_x$, where $S_x \cong 3 \times 2S_7$, and $S_x$ acts irreducibly on $V_x$. For $\langle \sigma \rangle \in \text{Syl}_7(S_x)$, $\sigma$ induces

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
on V_x.
$$

In particular, $|C_{V_x}(\sigma)| = 7$.

2.1. LEMMA. Suppose $x, y \in 7B$. Then $y \in Q_x$ if and only if $\langle x, y \rangle \cong 7B^2$.

Proof. If $y \in Q_x$, then $x \in Q_y$ by 1.1. Since $N_G\langle x, y \rangle$ contains $Q_x$, $Q_y$, and the central $3 \times 2$ part of $S_x$. This shows that $N_G\langle x, y \rangle \cong 7^2 \cdot 7 \cdot 7^2 : GL_2(7)$ and $\langle x, y \rangle \cong 7B^2$.

Suppose $\langle x, y \rangle \cong 7B^2$. Since $C_G\langle x, y \rangle$ contains a subgroup isomorphic to $7^2 \cdot 7 \cdot 7^2$, $|Q_x \cap C_G(y)| \geq 7^4$. If $y \not\in Q_x$, then $|C_{V_x}(y)| = 7$, which implies $|Q_x \cap C_G(y)| \leq 7^2$. This contradiction proves that $y \in Q_x$ and completes the proof of the lemma.

2.2. PROPOSITION. No subgroup of $F_1$ can be isomorphic to $(F_{16})'$.

Proof. Deny this. Let $F \leq F_1$ such that $F \cong (F_{16})'$. Let $\chi$ be the irreducible character of $F_1$ of degree 196883. The only possible irreducible constituents of the restriction of $\chi$ to $F$ (which is also denoted by $\chi$) are the irreducible characters of $(F_{16})'$ of degree not bigger than 196883. Hence $\chi = a\phi_1 + b\phi_2 + c\phi_3$, where $a$, $b$, $c$ are non-negative integers and $\phi_1$, $\phi_2$, $\phi_3$
are irreducible characters of \((\text{Fi}_{24})'\) of degrees 1, 8671, 57477, respectively. Computing the value of \(\chi\) on an element of order 23 yields 3 = a. Computing the value of \(\chi\) on an element of order 27 now yields 2 = 3 + b \cdot 1 + c \cdot 0, which forces \(b = -1\). This contradiction establishes the proposition.

2.3. LEMMA. Let \(H \leq G\) such that \(H \cong He\). Suppose \(T \in \text{Syl}_7(H)\) and \(\langle t \rangle = Z(T)\). Then \(t \in 7B\) and \(T \leq Q_1\).

Proof. From \(\langle t \rangle = T\), we obtain \(t \in 7B\). Suppose \(T \not\leq Q_1\). For \(X \leq N_G\langle t \rangle = Y\), set \(\bar{X} = X_{Q_1}/Q_1\). Thus \(|\bar{T}| = 7\). Since \(N_H(T) \leq Y\), so \(N_H(T) \leq N_G\langle Q_2 \rangle\). Hence \(N_H(T) \leq N_G\langle Q_2 \rangle \cong 6 \times (7 : 6)\). However \(N_H(T) \leq N_H(T)/N_H(T) \cap Q_1\), which has a subgroup isomorphic to \(S_3 \times 3\). This contradiction proves that \(T \leq Q_1\). The proof of the lemma is complete.

We now fix an element \(s \in 7B\) and set \(Q = Q_s\), \(V = V_s\), \(S = S_s\), \(C = C_G(s)\), and \(N = N_G\langle s \rangle\). Let \(I = \{1\text{-dimensional subspaces of } V\}\). Then \(|I| = 400\). From the existence of \(7B^2\), 2.1 implies that there exists \(W = \langle w, s \rangle \cong 7B^2\) and \(W \leq Q\). Thus \(C_G(W)\) contains a subgroup \(P \cong 7^2 \cdot 7 \cdot 7^2\). Since \(W \not\leq Z(Q)\), \(P \not\leq Q\). So \(QP \in \text{Syl}_7(C)\) and \(Q \cap P = C_Q(w) = \langle w \rangle \times Q_1\), where \(Q_1 \cong 7^1 + 2\). Let \(u \in P \cap Q\), and \(v \in Q \cap P\). Set \(r = uv\) and \(R = \langle r, s \rangle\). Then \(r \not\in Q\) and \(R \cong 7^2\). We will show that \(N_G(R)\) is a new 7-local and \(R \cong 7B^2\). For any \(A \leq N\), set \(\bar{A} = AQ/Q\).

2.4. LEMMA. We have \(C_G(r) = \langle s \rangle\) and \(C_G(R) = R\).

Proof. Let \(C_1 = C_G(r)\). First we show \(C_1 = \langle s \rangle\). Since \(v \in Q\), \(u\), and \(r\) induce the same action of \(V\). So \(C_V(r) = C_V(u) = W/\langle s \rangle\). Hence \(C_1 \leq W\). From \([v, w] \in \langle s \rangle\) and \([u, w] = 1\), we obtain \([r, w] = [vu, w] = [v, w]\). Since \(1 \not\in [v, w]\), \(C_1 \not= W\). Therefore \(C_1 = \langle s \rangle\) as desired.

Let \(y \in C_G(R)\). Then \(\bar{y} \in C_G(\bar{r}) = D\langle \bar{r} \rangle\), where \(D\) is the central \(3 \times 2\) part in \(\bar{N}\). Let \(i\) be an involution in \(C\). Then \(i\) is the central involution in \(\bar{N}\). Suppose \(i \in C_G(R)\). Then \(\bar{i}\) normalizes \(C_V(r) = \bar{W}\). Hence \(i\) normalizes \(O_2(N_G(W))\) which is \(P\). This implies \(u' = u_1 u\), where \(u_1 \in P\). Since \(i\) induces \(-I\) on \(V\), \(v' = v^{-1}s_1\), where \(s_1 \in \langle s \rangle\). From \(r = r'\) we obtain \(v^2 = s_1 u_1\). So \(v \in P\). This contradiction proves that \(|C_G(R)| = 1\). Let \(j \in C_G(R)\) with \(j^2 = 1\). Then \(j\) lies in the \(2S_7\) part of \(\bar{N}\) as the central \(3\text{-part of } \bar{N}\) does not centralize \(s\). But then \([j, r] \not= 1\). This contradiction proves that \(|C_G(R)| = 1\). Therefore \(R_1 = R\langle y \rangle\) is an elementary abelian 7-group as the exponent of a Sylow 7-subgroup in \(G\) is \(7\). Since \(|\bar{N}| = 7\) and \(r \not\in Q\), \(R_1 = (Q \cap R_1)\langle r \rangle\). Hence \(Q \cap R_1 \leq C_G(r) = \langle s \rangle\). So \(R_1 = \langle s \rangle \langle r \rangle = R\). In particular \(y \in R\). The proof of the lemma is complete.

2.5. LEMMA. We have \(R \cong 7B^2\) and \(N_G(R) \cong 7^2 : SL_2(7)\).
Proof. First we claim \( r \in 7B \). Deny this. Then \( C_G(r) = \langle r \rangle \times H \), where \( H \cong H_e \). Since a Sylow 7-subgroup of \( H_e \) is isomorphic to \( 7^{1+2} \), \( |C_H(R \cap H)| \geq 7^2 \). From \( C_G(R) \geq \langle r \rangle \times C_H(R \cap H) \), we get \( |C_G(R)| \geq 7^3 \). This contradicts 2.4. So \( r \in 7B \) as claimed.

Next recall \( W = \langle w, s \rangle \leq \langle w, s, r \rangle = W_1 \), and \( 1 \neq [r, w] \in \langle s \rangle \). Hence \( W_1 \geq 7^{1+2} \) and \( R \leq W_1 \). Now \( |\{\langle r \rangle \langle w \rangle\}| = 7 \) and \( r \in 7B \) imply that \( R \geq 7B^2 \).

On \( R \), \( w \) centralizes \( s \) and moves \( r \). By symmetry, there is \( w_1 \in C_G(R) \cap N_G(R) \) such that \( w_1 \) moves \( s \). Hence \( \langle w, w_1 \rangle \) induces \( SL_2(7) \) on \( R \) by Dickson's theorem. Suppose \( i \) is an involution of \( C \). Then \( i \) is the central involution in \( \tilde{N} \). So \( [i, r] = 1 \). If \( i \in N_G(R) \), then \( i \in C_G(R) \). This contradicts 2.4. Hence \( |C \cap N_G(R)| \geq 1 \). A similar argument shows that \( |C \cap N_G(R)| \geq 1 \). Therefore \( N_G(R) \geq 7^2 : SL_2(7) \) by 2.4. The proof of the lemma is complete.

2.6. THEOREM. \( N_G(R) \) is a maximal subgroup of \( G \).

Proof. Let \( B = N_G(R) \). Suppose \( B < M \), a maximal subgroup of \( G \). Let \( B_1 \in \text{Syl}_7(B) \). Then \( B_1 \geq 7^{1+2} \). First we claim that \( M \) does not contain any normal abelian subgroup.

Deny this. Let \( E \cong p^n \) be a minimal normal subgroup of \( M \), where \( p \) is a prime. Suppose \( p \neq 7 \). On any \( B \)-invariant subspace of \( E \), if the kernel of the action is not trivial, then it contains \( R \) as \( R \) is a minimal normal subgroup of \( B \). This is impossible by 2.4. In particular, \( B_1 \) acts faithfully on any irreducible \( B \)-module of \( E \). Therefore \( n \geq (\dim_{GF(p)} F) \cdot 7 \), where \( F \) is a field containing the 7th root of unity. This rules out all \( p \) except possibly \( p = 2 \). Suppose \( p = 2 \). If \( E \) is an irreducible \( B_1 \)-module, then \( Z(B_1) \) induces scalar linear transformations of \( E \). By 2.4, 1 is not an eigenvalue for \( Z(B_1) \). Since all subgroups of order 7 of \( R \) are conjugate in \( B \), none of these subgroups have 1 as an eigenvalue. However, \( R \geq 7^2 \) implies that the kernel of any irreducible \( R \)-module is not trivial. This contradiction proves that \( E \) has at least two non-linear irreducible \( B_1 \)-module. So \( n \geq 2(21) = 42 \). Since \( G \) does not contain any subgroup isomorphic to \( 2^{42} \), the analysis for \( p \neq 7 \) is complete. Suppose \( p = 7 \). Since \( C_E(R) \neq 1 \), 2.5 implies \( R = E \). Hence \( B = M \). This contradiction proves that \( M \) does not contain any normal abelian subgroup as claimed.

Next let \( E \) be a minimal normal subgroup of \( M \). Then \( E = E_1 \times \cdots \times E_t \), where \( E_1, ..., E_t \) are non-abelian simple groups. If \( 7 \nmid |E| \), then \( B_1 \) will normalize a \( p \)-group for some prime \( p \neq 7 \). The argument in the last paragraph yields a contradiction in this case. If \( B \not\leq E_i \) for \( i = 1, ..., t \), then \( t \geq 7 \) as no simple group involved in \( F_1 \) has an outer automorphism of order 7. This implies that \( |E| \geq 7^7 \). This contradiction proves that \( B \leq E_k \) for some \( k \in \{1, ..., t\} \). Let \( L = E_k \). Using \( 7^3 | |L| \), we obtain that \( L \cong H_e \) or \( (F_{24})' \). By 2.2, we may assume \( L \cong H_e \). Then \( B_1 \in \text{Syl}_7(L) \). Since all subgroups of
order 7 of $R$ are conjugate in $B$, we may assume that $\langle s \rangle = Z(B_1)$. By 2.3, $B_1 \leq Q$. In particular $r \in Q$. This contradiction proves that $L \not\cong He$. The proof of the theorem is complete.

3. Maximal 7-Locals

We continue to use the notations in Section 2, especially, $s$, $Q$, $V$, $S$, $C$, $N$, $\Gamma$, $W$, and $G$. For $A \leq N$, $A = AQ/Q$, and $\hat{A} = A\langle s \rangle / \langle s \rangle$.

For $X \leq G$ such that $X \cong 7^2$, the notation $X \cong 7B_2^2$ means that $X$ is conjugate to $R$.

Set $\omega = \langle s \rangle$. From the 7-elements of $S$ and its fixed points on $V$, we obtain the following easily.

3.1. Lemma. We have $\omega^S = \{ \delta \in \Gamma \mid |N_\delta|_7 \neq 1 \} = \{ Y/\langle x \rangle \mid Y \cong 7B_2^2 \text{ and } Y \leq Q \}$. Also $|\omega^S| = 120$.

3.2. Lemma. If $T \cong 7A^2$ and $T \leq N$, then $T \leq Q$.

Proof. Suppose $T \nleq Q$. Then $T = \langle \alpha, \beta \rangle$, where $\alpha \in Q$ and $\beta \in Q$. So $1 \neq \beta$ centralizes $\hat{s}$. By 3.1, $\langle \alpha, s \rangle \in \omega^S$. This contradicts $\alpha \in 7A$. The proof of the lemma is complete.

Using 3.1, 3.2, and a counting argument we obtain the following.

3.3. Lemma. We have $\Gamma = \{ \omega^S \} \cup \{ \lambda^S \}$, where $|\lambda^S| = 280$ and $\lambda^S = \{ \hat{Y} \mid Y \leq Q \text{ and } Y \cong 7A_7B \}$.

3.4. Lemma. If $E \cong 7^3$ is a subgroup of $G$, then $E$ is not an irreducible $N_G(E)$-module.

Proof. Without loss of generality, we may assume that $s$ is in the center of a Sylow 7-subgroup which contains $E$. First we claim that there is an element in $N_G(E)$ inducing a transvection on $E$.

Suppose $E \leq Q$. Then $s \in E$. Let $e \in E \setminus \langle s \rangle$. Then $e \neq 1$ in $\hat{V}$. Let $\langle s \rangle \leq E_1$ such that $\hat{E}_1 = e^\perp$. Then $|E_1| = 7^4$ and $E \leq E_1$. Since $\hat{E} = (\hat{E})^\perp$, $\hat{E} \cong Z(E_1)$. Let $t \in E \setminus E_1$. Then $t$ induces a transvection on $E$.

Now suppose $E \nleq Q$. Let $E_2 = E \cap Q$. So $|E_2| = 7^2$. Since $\hat{E}_2$ is central-
ized by an element of $E \setminus Q$, 3.1 implies that $E_2 \cong 7B_2^2$ and $s \in E_2$. Clearly $C_Q(E) = E_2$. Let $E_3 = C_Q(E_2)$. Then $E \leq N_G(E_3)$ and $|E_3| = 7^4$. Hence $E$ normalizes a chain of normal subgroups of $E_3$: $1 < \langle s \rangle < E_2 < E_4 < E_3$. From the action of $E$ on $V$, we obtain $[E, E_4] \neq 1$. Since $E_2 \leq E \cap E_4$, $|EE_4| = 7^3$. So $E \nleq EE_4$. Therefore an element in $E_4 \setminus E$ will induce a transvection on $E$. This establishes our claim.
Assume that $N_G(E)$ acts irreducibly on $E$. Since $N_G(E)$ contains an element inducing transvection on $E$, the subgroup generated by these transvections is $SL_3(7)$. However, $F_1$ does not involve $L_3(7)$ by the Atlas [3]. This contradiction completes the proof of the lemma.

3.5. PROPOSITION. There is only one conjugacy class of subgroups isomorphic to $He$ in $F_1$.

Proof. Let $H \leq G \cong F_1$, $H \cong He$. There is a subgroup $D = D_1 D_2$ of $H$ such that $D_1 \leq D$, $D_1 \cong 7^2$, and $D_2 \cong SL_2(7)$. Let $J \in Syl_7(D)$. By 2.3 $J \leq Q_j$, where $\langle j \rangle = Z(J)$ and $j \in 7B$. By conjugation if necessary, we may assume $s = j$ without loss of generality. Thus $D_1 \cong 7B^2$ and $s \in D_1$. So $N_G(D_1) \cong 7^2 \cdot 7^2 : GL_2(7)$. Let $D_3 \geq D_1$ such that $N_G(D_1)/D_3 \cong 7^2 : GL_2(7)$ (i.e., $D_3 = D_1 \cdot 7$). Since $N_G(D_1)$ acts irreducibly on $D_1$, $D_1 \leq Z(D_3)$. Hence $D_3 \cong 7^3$. This implies $D_1 D_2 = (D_1 D_2) \cdot \langle y \rangle$. Since $y \in C$ and $[y, J] = 1$, $y \in Q$. Now $N_H(J) = J J_1$, where $J_1 \cong S_3 \times 3$. In $N$, $N_H(J)$ acts on $V$ leaving $\hat{J}$ invariant. So $N_H(J)$ acts on $(\hat{J})^\perp$. This is the same action induced by the normalizer of an ordinary copy of $He$ in $N(7A)$, where $J_1$ acts trivially on $\hat{J}$. Hence $J_1 \leq C_G(y)$. Therefore $\langle D, J_1 \rangle \leq C_G(y)$. Since $D$ is a maximal subgroup of $H$, so $H = \langle D, J_1 \rangle \leq C_G(y)$. This implies that $y \in 7A$ and $H$ is the unique subgroup of $C_G(y)$ isomorphic to $He$. The proof of the proposition is complete.

3.6. LEMMA. If $T \leq G$ and $T \cong 7^3$, then $T \cap 7A \neq \emptyset$.

Proof. First we claim that the following holds:

(1) If $T \leq Q_x$, for some $x \in 7B$, then $T \cap 7A \neq \emptyset$. Deny this. By conjugation if necessary, we may assume that $T \leq Q$. From $T \cap 7A = \emptyset$, we obtain that all 1-dimensional subspaces of the totally isotropic 2-dimensional subspace $\hat{T}$ belong to $\omega^5$. However, for $e \in \omega^5$, any 2-dimensional subspace in $e^\perp$ containing $e$ always contains a 1-dimensional subspace from $\lambda^5 = \{ \hat{Y} \mid Y \leq Q \text{ and } Y \cong 7A, B \}$. This contradiction establishes (1).

In general, let $T \leq P \in Syl_7(G)$. By conjugation if necessary, we may assume $s \in Z(P)$. Let $T_1 = T \cap Q$. Suppose $T \cap 7A = \emptyset$. By (1), $T \leq Q$. So $|T_1| = 7^2$. Let $t \in T \setminus Q$. Then $t \in 7B$. Let $T_2 = T \cap Q$. By (1), $T \leq Q_t$. So $|T_2| = 7^2$. Since $T_1$ and $T_2$ are subgroups of $T$ and $|T| = 7^3$, there is $1 \neq \theta$ such that $\theta \in T_1 \cap T_2$. Thus $\theta \in 7B$. Since $\theta \in T_1 \leq Q$ (resp. $\theta \in T_2 \leq Q_t$), 1.1 implies $s \in Q_\theta$ (resp. $t \in Q_s$). As $T_1 = \langle s, \theta \rangle$ and $t \notin T_1$, so $T = \langle t, \theta, s \rangle$. Thus $T \leq Q_\theta$ which is impossible by (1). This contradiction completes the proof of the lemma.

3.7. THEOREM. The list: $N_G(L)$ where $L \in \{7A, 7B, 7A^2, 7B_1^2, 7B_2^2 \}$ is complete for the maximal 7-locals.
Proof. By 3.4, it suffices to treat $K = N_G(X)$ where $X \cong 7^2$. We divide the proof into the following steps.

I. $X \cong 7B^2$, then $K$ is in the list.

Proof. We may assume that $s \in X$. The action of $N$ on $V$ implies the following. If $X \leq Q$ (resp. $C_G(X) = \langle s \rangle$), then $X \cong 7B_2^2$ (resp. $X \cong 7B_2^2$).

Suppose $C_G(X) = X_1 \cong 7^2$. Let $E = XX_1(\cong 7^2)$. By 3.4, there is $\beta \in E \cap 7A$. Let $H$ be the unique subgroup of $C_G(\beta)$ which is isomorphic to $He$. Let $T \in \text{Syl}_3(H)$ such that $E \cap H \leq T$. If $\langle t \rangle = Z(T)$, then $E \cap H \leq Q_1$. Since $[\beta, T] = 1$, $\beta \in Q_1$. Therefore $E = \langle \beta \rangle \times (E \cap H) \leq Q_1$. So, $X \leq Q_1$. This implies $t \in X$ and $X \cong 7B_2^2$.

II. If $X \cap 7B \neq \phi$, then $K$ lies in a subgroup in the list.

Proof. If $|X \cap 7B|$ or $|X \cap 7A| = 1$, then $K \leq N(7B)$ or $N(7A)$. Hence we may assume that $|X \cap 7B| > 1$ and $|X \cap 7A| > 1$ by (I). By conjugation if necessary, we may assume $s \in X$. If $X \leq Q$, then $X \cong 7A_7B$, which implies $|X \cap 7B| = 1$. Therefore $X \leq Q$. Let $\alpha \in X \cap 7A$, and let $H \leq C_G(\alpha)$ such that $H \cong He$. Let $X \cap H \leq T \in \text{Syl}_3(G)$ and $\langle t \rangle = Z(T)$. Then $X \leq Q_1$. If $t \in X$, then again $X \cong 7A_7B$, which implies $|X \cap 7B| = 1$. Therefore $t \notin X$. Since $s \in Q_1$, 1.1 implies that $t \in Q$. Since $S \leq Q$, $s \notin Q$. Hence $|C_F(x)| = 7$. As $t \in Q$, this implies that $C_G(x) = \langle t, s \rangle$. From $C_G(x) = C_G(s) \cap C_G(\alpha)$, we obtain that $Y = \langle x, t, x \rangle$ is the unique Sylow 7-subgroup in $C_G(\alpha)$. Note that $Y \cong 3^3$. Since $K$ acts on $X \cap 7A$ and $X \cap 7B$ (both have cardinality bigger than 1), so $|N_G(X)/C_G(X)| = 1$. Hence $K = Y \cdot U$, where $|U|_7 = 1$. So $Y = X \oplus Y_1$, where $Y_1$ is a 1-dimensional invariant $K$-subspace by Maschke's theorem. Therefore $K \leq N_G(Y_1)$, a subgroup in the list.

III. There is only one conjugacy class of subgroups of $7A^2$.

Proof. Let $X \cong 7A^2$, and let $\alpha \in X \cap 7A$. Let $H \leq C_G(\alpha)$ such that $H \cong He$. Let $X \cap H \leq T \in \text{Syl}_3(H)$ and $\langle t \rangle = Z(T)$. Also let $H_1 \leq C_G(\alpha)$ such that $H_1 \cong He \cdot 2$. Then $N_{H_1}(T) \cong T : (S_3 \times G)$. On the 8 points of the projective line $T/\langle t \rangle$, $S_3$ has two orbits. One such orbit has size 2 and the other has size 6. One such orbit comes from $7B_2^2$ types and the other orbit comes from $7A_2^2$ types. Let $T = \langle \beta, \gamma \rangle$, where $X \cap H = \langle \beta \rangle$. As $\langle \alpha \rangle \times T \leq Q_1$, $\gamma$ normalizes $\langle \beta, t \rangle$. Hence $\gamma$ normalizes $T_1 = \langle \alpha, \beta, t \rangle \cong 7^3$. Since $[\gamma, \beta] \neq 1$, subgroups containing $\alpha$ of order $7^2$ of $T_1$ are $\langle \alpha, t \rangle$ and $\langle \alpha, \beta, t \rangle$. Since $t \in 7B$, $\langle \alpha, t \rangle \leq 7A_2^2$. Hence subgroups containing $\alpha$ of order $7^2$ of $T_1$ are all conjugate in $T_1$. Since there is only one class of $7A$ and only one $S_3$-orbit in $T/\langle t \rangle$ corresponding to $7A_2^2$, the proof of III is complete.

The proof of the theorem now follows easily from I, II, and III.
ACKNOWLEDGMENTS

I thank Professor J. Thompson for his interest, encouragement, and the ideas he taught me in this problem. Also, I thank Professors R. Solomon and R. Lyons for their helpful conversations.

REFERENCES