European Journal of Combinatorics 31 (2010) 694–709

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

# A tiling approach to eight identities of Rogers

David P. Little, James A. Sellers

*Department of Mathematics, Penn State University, University Park, PA 16802, United States*

ARTICLE INFO

*Article history:*
Received 16 February 2009
Received in revised form
11 June 2009
Accepted 22 September 2009
Available online 14 November 2009

ABSTRACT

Beginning in 1893, L.J. Rogers produced a collection of papers in which he considered series expansions of infinite products. Over the years, his identities have been given a variety of partition-theoretic interpretations and proofs. These existing combinatorial techniques, however, do not highlight the similarities and the subtle differences seen in so many of these remarkable identities. It is the goal of this paper to present a new combinatorial approach that unifies numerous $q$-series identities. The eight identities of Rogers that appear in G.E. Andrews' 1986 CBMS monograph on $q$-series will serve as a basis for the collection of identities studied in this paper.



## 1. Introduction

Near the end of the 19th century, L.J. Rogers produced a series of three papers [12–14] in which he considered numerous series expansions of infinite products. His work culminated in the now-celebrated Rogers–Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} \tag{2}$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$. The history behind these works of Rogers, as well as Ramanujan's re-discovery of the Rogers–Ramanujan identities, has been told and re-told on numerous occasions; see, for example, Andrews [1] and Ramanujan's Collected Works [8].

*E-mail addresses:* dlittle@math.psu.edu (D.P. Little), sellersj@math.psu.edu (J.A. Sellers).

0195-6698/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2009.10.005

As noted by Andrews [1], Rogers proved numerous series-product identities in his three-paper series, not just the two identities mentioned above. Many of these series have proven to be invaluable in the field of $q$-series. For example, Baxter [3,4] re-discovered many of Rogers' results on his way to the solution of the now-famous Hard Hexagon model in statistical physics. And while a number of techniques have been employed to prove many of Rogers' identities, both analytic as well as combinatorial, it is the goal of this work to prove a variety of $q$-series identities, most of which appear in Rogers' work (and are re-stated by Andrews [1, pp. 7–8]), from a new, unified combinatorial viewpoint which is described in detail below.

In particular, consider the following identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})(1+q^{2n})} \tag{3}$$

$$\sum_{n=0}^{\infty}\frac{q^{(3n^2-n)/2}}{(q;q)_n(q;q^2)_n}=\prod_{n=1}^{\infty}\frac{(1-q^{10n-4})(1-q^{10n-6})(1-q^{10n})}{(1-q^n)} \tag{4}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}=\prod_{n=1}^{\infty}\frac{(1-q^{20n-8})(1-q^{20n-12})(1-q^{20n})(1+q^{2n-1})}{(1-q^{2n})}. \tag{5}$$

These appear, in one form or another, in Rogers' "trilogy" [12–14] as noted in Andrews [1], and all of these appear in Slater's extensive list of product-series identities [16,17]. (The interested reader may also wish to see the recent survey article of McLaughlin, Sills and Zimmer [11] which provides an expansive, annotated list of Rogers–Ramanujan–Slater type identities.) For our purposes, it will be more convenient to use (1) to rewrite the above identities in the following form:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n} \tag{6}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n}=(q;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2-n)/2}}{(q;q)_n(q;q^2)_n} \tag{7}$$

$$\sum_{n=0}^{\infty}\frac{q^{4n^2}}{(q^4;q^4)_n}=(q;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}. \tag{8}$$

It is in this form that we view such identities from the combinatorial perspective of tilings.

As in recent papers of the authors [7,10], the combinatorial setting of the proofs of the identities studied in this paper (including those mentioned above) will be tilings of a $1\times\infty$ board using some collection of squares and dominoes. The *position* of a tile refers to its location on the board and can be any positive integer, provided that no two tiles cover the same position and that every position on the board is covered by a tile. In the case of a domino, we say that it is in position $i$ if it covers positions $i$ and $i+1$. The parity of a tile refers to the parity of the position of the tile. In other words, we say that a tile is *even* (resp. *odd*) if it is in an even (resp. odd) position.

As we work through the various proofs below, we will vary which types of tiles will be used as well as the weight of the tiles. In every case, white squares will have a weight of 1 and will be used as "filler" to cover positions not covered by a tile with non-trivial weight. In particular, each tiling will contain a finite number of non-white squares and dominoes. Given a tiling $T$, the weight of each tile $t\in T$ will be denoted by $w(t)$ and the weight of a tiling $T$ will be defined as

$$\prod_{t\in T}w(t).$$

Before proceeding, we wish to contrast the use of tilings over the more customary partitions to prove identities of this type. In many instances, partition-theoretic proofs rely on demonstrating a bijection between two *different* collections of partitions. In some cases, the *same* set of partitions is constructed in two different manners in order to prove an identity. For example, the identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n}$$

can be proven by pointing out that the left-hand side generates all partitions according to the size of their Durfee square whereas the right-hand side generates all partitions according to the size of their largest part.

All of our proofs will follow this latter method of proof by constructing the *same* combinatorial objects in two different manners. For example, to prove Eq. (6), Bressoud demonstrated a bijection between partitions where parts differ by at least 2 and partitions with distinct parts where each even part is larger than twice the number of odd parts. In the next section, we will prove the same identity by constructing the *same* set of tilings with squares and dominoes in two different ways. The tiling constructions used to explain different identities are extremely similar, underlining the fact that the identities presented here have much more in common than mere appearance.

We conclude this section by presenting a general outline of the following sections and proofs. Each section will begin with a brief description of the type of tilings to be studied, followed by a definition of the corresponding weight function $w(t)$. Furthermore, each section will contain a unique method for constructing all relevant tilings. Each construction will be based on an operation that moves tiles around on the board. In particular, the term *projection* will be used to refer to any invertible operation on tiles that satisfies the following properties:

P1: Only the position of the tiles are affected. In other words, no tiles are permanently removed from the board and no new tiles are introduced.
P2: The effect on the weight of a tiling is to multiply by $q^k$, where $k$ does not depend on which tile was projected.
P3: The relative position of the projectiles (i.e., tiles that can be projected) cannot change.
P4: When projected, a tile is moved past $r$ white squares, for some fixed value of $r > 0$, while the position of the remaining projectiles does not change.

Given a specific tiling $T$, we can create infinitely many more tilings by systematically projecting all of the projectiles in the following manner. Suppose that $T$ contains projectiles, $t_1, t_2, \ldots, t_n$, where tile $t_{i+1}$ appears to the right of tile $t_i$ for $1 \le i < n$. We begin by projecting tile $t_n$ a total of $p_n \ge 0$ times. Next, project tile $t_{n-1}$ a total of $p_{n-1}$ times where $0 \le p_{n-1} \le p_n$. In general, working in a right-to-left manner, project tile $t_i$ $p_i$ times for $1 \le i \le n$, where $0 \le p_1 \le p_2 \le \cdots \le p_n$. We will refer to this operation as *projecting the tiles*.

Note that this right-to-left manner in which projecting the tiles must take place is essentially dictated by properties P3 and P4. Property P2 and the fact that projection must be invertible leads us to the following lemma.

**Lemma 1.** *Let $T$ be a tiling that contains $n$ projectiles. Then the generating function for all tilings that can be obtained from $T$ by projecting the tiles is given by*

$$\frac{w(T)}{(q^k;q^k)_n}$$

*if each projection increases the weight of a tiling by a factor of $q^k$.*

**Proof.** Note that any tiling that can be obtained from $T$ by projecting the tiles corresponds to a unique sequence, $0 \le p_1 \le \cdots \le p_n$, where $p_i$ represents the number of times the $i$th projectile in $T$ was projected. If each projection increases the weight of the tiling by a factor of $q^k$, then the cumulative effect of projecting the tiles is given by

$$\sum_{0 \le p_1 \le p_2 \le \cdots \le p_n} q^{k(p_1+p_2+p_3+\cdots+p_n)} = \frac{1}{(q^k;q^k)_n}$$

as claimed. □

For example, consider tilings that consist of white squares and gray dominoes where the weight function is given by

$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a gray domino in position } i \\ 1 & \text{if } t \text{ is a white square in position } i. \end{cases}$$

In this case, any domino will be considered a projectile. Since we always project tiles in a right-to-left manner, it is enough to explain how to project a domino that is immediately followed by a white square. To project such a domino, simply increase the position of the domino by one, which increases the weight of the tiling by a factor of $q$.

To construct a tiling with exactly $n$ dominoes, initially place dominoes in positions $1, 3, 5, \ldots, 2n-1$, which accounts for a weight of $z^n q^{n^2}$. It remains to project the tiles (i.e., project the dominoes). Applying Lemma 1 with $k = 1$ shows that

$$\frac{z^n q^{n^2}}{(q;q)_n}$$

is the generating function for tilings with exactly $n$ dominoes.

With the above machinery constructed, we now proceed to prove a variety of $q$-series identities via weighted tilings. The proofs provided are very straightforward and brief in this context, adding to the elegance of this proof approach when dealing with such identities.

## 2. Fibonacci tilings

Consider tilings of a $1 \times \infty$ board using white squares and gray dominoes. We will refer to such tilings as (infinite) Fibonacci tilings since the number of ways to cover a $1 \times n$ board with white squares and gray dominoes is given by the $n$th Fibonacci number, $F_n$, where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 1$ and $F_1 = 1$. The weight of tile $t$ is defined as follows:

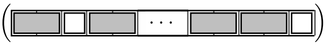
$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a domino in position } i \\ 1 & \text{if } t \text{ is a white square in position } i. \end{cases}$$

Note that Fibonacci tilings together with the above weight function are the same objects used in the example at the end of the previous section. Consequently, the generating function for all Fibonacci tilings with respect to the above weight function is given by

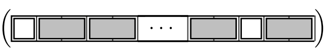
$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n}. \tag{9}$$

However, in this section, we will project a tile in a slightly different manner so that we can count Fibonacci tilings according to the number of odd dominoes. Therefore, in this section, projectiles will refer to odd dominoes only. To project an odd domino that is followed by a white square, simply move it to the beginning of the next collection of odd dominoes. Note that the next collection of odd dominoes could be empty, before this projection is performed.

More formally, suppose that there is an odd domino in position $i$, followed by a white square, followed by $j \geq 0$ even dominoes, followed by another white square. To project the odd domino, rearrange these tiles so that there are white squares in positions $i$ and $i+2j+1$ and dominoes covering the remaining positions. Clearly this operation preserves the number of odd dominoes and increases the weight of the tiling by a factor of $q^2$, as illustrated below.

$w$ (  ) $= zq^i \cdot zq^{i+3} \cdots zq^{i+2j-1} \cdot zq^{i+2j+1}$

$= z^{j+1} q^{(i+j+1)(j+1)-1}$

$w$ (  ) $= zq^{i+1} \cdot zq^{i+3} \cdots zq^{i+2j-1} \cdot zq^{i+2j+2}$

$= z^{j+1} q^{(i+j+1)(j+1)+1}.$

**Theorem 2.**

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}. \tag{10}$$

**Proof.** As mentioned above, the left-hand side is the generating function for all Fibonacci tilings. It remains to show that the right-hand side is also the generating function for all Fibonacci tilings. We will do so by counting tilings according to the number of odd dominoes. To construct a Fibonacci tiling that has exactly $n$ odd dominoes, first place dominoes in positions $1, 3, 5, \ldots, 2n-1$, which accounts for a weight of $z^n q^{n^2}$. Now go through each of the remaining even positions and decide whether or not to place a domino in that position. The factor $(1+zq^{2j})$ represents the choice of whether or not to place a domino in position $2j$ for $j \geq n+1$. This accounts for a weight of

$$\prod_{j \geq n+1} (1+zq^{2j}) = \frac{(-zq^2;q^2)_\infty}{(-zq^2;q^2)_n}.$$

And finally, project the odd dominoes. Applying Lemma 1 with $k = 2$ yields that

$$(-zq^2;q^2)_\infty \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

is the generating function for all Fibonacci tilings that have exactly $n$ odd dominoes. Summing over all values of $n \geq 0$ completes the proof. $\square$

Theorem 2 unifies the following two identities of Rogers which appear as equations (4.11) and (4.7) in Andrews [1].

**Corollary 3.**

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} \tag{11}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = (-q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}}. \tag{12}$$

**Proof.** The above identities follow by setting $z = 1$ and $z = q$ in (10), respectively. $\square$

Eq. (11), after applying (1), appears in Slater's list [17, Equation (20)]. Eq. (12), after applying (2), also appears in Slater's list [17, Equation (17)].

**Theorem 4.**

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+2n}}{(q^2;q^2)_n(-zq^2;q^2)_n}. \tag{13}$$

**Proof.** Note that the left-hand side is obtained by replacing $z$ with $zq$ in (9). Combinatorially, this replacement is equivalent to increasing the position of each domino by one. In other words, we are now counting Fibonacci tilings where position one must be covered with a white square. To complete the proof, we will show that the right-hand side also counts these Fibonacci tilings according to the number of odd dominoes.

As before, we begin by placing $n$ dominoes in positions $1, 3, 5, \ldots, 2n-1$. Next, go through each of the remaining even positions and decide whether or not to place a domino in that position. Now, to make sure that position one is covered by a white square, project each of the odd dominoes exactly once, starting with the domino in position $2n-1$ and working right-to-left. This increases the weight of the tiling by a factor of $q^{2n}$. Now finally, project the odd dominoes. Applying Lemma 1 with $k = 2$ shows that

$$(-zq^2; q^2)_\infty \frac{z^n q^{n^2+2n}}{(q^2; q^2)_n(-zq^2; q^2)_n}$$

is the generating function for Fibonacci tilings that have position one covered with a square and have exactly $n$ odd dominoes. Summing over all values of $n \geq 0$ completes the proof. □

Theorem 4 unifies the following two identities of Rogers which appear as equations (4.8) and (4.12) in Andrews [1].

**Corollary 5.**

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = (-q; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2; q^2)_n(-q; q^2)_n} \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4; q^4)_n}. \tag{15}$$

**Proof.** The above identities follow by setting $z = 1/q$ and $z = 1$ in (13), respectively. □

Eq. (15), after applying (2), also appears in Slater's list [17, Equation (16)].

In a recent article [5], Bowman, McLaughlin and Sills present a collection of Rogers–Ramanujan type identities. Among them are the following identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q; q)_{n+1}}{(q^2; q^2)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-1})(1 - q^{5n-4})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(q; q^2)_{n+1}}{(q; q)_n(q; q^2)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-2})(1 - q^{5n-3})}.$$

In light of our combinatorial interpretation of (9) and the Rogers–Ramanujan identities (1) and (2), the above identities can be proven by showing that the two series count the appropriate collection of Fibonacci tilings. In particular, the first series counts all Fibonacci tilings according to the number of dominoes in position 2 or more. More specifically, suppose that there are exactly $n$ dominoes that are in position 2 or more. First, construct all tilings that have a white square in position 1. To do so, place dominoes in positions $2, 4, 6, \ldots, 2n$ and then arbitrarily project these $n$ dominoes. In this context, to project a domino, simply increase its position by 1. Second, construct all tilings that have a domino in position 1. To do so, place dominoes in positions $1, 3, 5, \ldots, 2n + 1$ and then arbitrarily project the $n$ dominoes in positions $3, 5, 7, \ldots, 2n + 1$. Therefore, the generating function (with $z = 1$) for tilings that have exactly $n$ dominoes in position 2 or more is given by

$$\begin{aligned} \frac{q^{n^2+n}}{(q; q)_n} + \frac{q^{(n+1)^2}}{(q; q)_n} &= \frac{q^{n^2+n}(1 + q^{n+1})}{(q; q)_n} \\ &= \frac{q^{n^2+n}(-q; q)_{n+1}}{(q^2; q^2)_n}. \end{aligned}$$

The second series counts Fibonacci tilings that contain a square in position 1 by constructing all Fibonacci tilings and then removing the tilings where position 1 is covered by a domino. In particular, the difference between the generating functions (with $z = 1$) for Fibonacci tilings with $n$ dominoes and Fibonacci tilings with $n + 1$ dominoes where position 1 is covered by a domino (using the same construction as described above) is given by

$$\frac{q^{n^2}}{(q;q)_n} - \frac{q^{(n+1)^2}}{(q;q)_n} = \frac{q^{n^2}(1-q^{2n+1})}{(q;q)_n} = \frac{q^{n^2}(q;q^2)_{n+1}}{(q;q)_n(q;q^2)_n}.$$

A natural generalization of Fibonacci tilings is tilings that use only white squares and $k$-ominoes for a fixed value of $k \geq 1$. By analogy, the position of a $k$-omino refers to the left-most position covered by the tile and the weight of a $k$-omino in position $i$ would be given by $zq^i$. The parity of a $k$-omino refers to the residue class of the position of the tile mod $k$. In other words, a $k$-omino of parity $j$ is a $k$-omino in position $km + j$ for some $m \geq 0$ and $0 \leq j < k$.

To project a $k$-omino that is immediately followed by a white square, first find the $k$th white square that appears to its right. Suppose that the $k$-omino is in position $i$ and that the $k$th white square to its right is in position $j$. To project the $k$-omino in position $i$, first remove it from the board, decrease by $k$ the position of the tiles in positions $i+k$ through $j$, and finally reinsert the $k$-omino in position $j-k+1$. The overall effect on the weight of the tiling is to multiply by $q^k$.

The following theorem generalizes the results of this section by counting tilings using white squares and $k$-ominoes where the first $k$-omino appears in position $i$ or greater.

**Theorem 6.** *For* $1 \leq i \leq k$,

$$\sum_{n=0}^{\infty} \frac{z^n q^{k\binom{n}{2}+in}}{(q;q)_n} = (-zq^k;q^k)_\infty \sum_{n_1,n_2,\ldots,n_{k-1}\geq 0} \frac{z^{n_1+n_2+\cdots+n_{k-1}} q^N}{(q^k;q^k)_{n_1}(q^k;q^k)_{n_2}\cdots(q^k;q^k)_{n_{k-1}}(-zq^k;q^k)_{n_1+n_2+\cdots+n_{k-1}}} \tag{16}$$

*where*

$$N = k\binom{n_1+n_2+\cdots+n_{k-1}}{2} + n_1 + 2n_2 + \cdots + (k-1)n_{k-1} + k(n_1+\cdots+n_{i-1}).$$

**Proof.** We present an outline of the proof, as the constructions below mirror the proofs of Theorems 2 and 4. In particular, we will show that each side of (16) is the generating function for tilings using white squares and $k$-ominoes where the first $k$-omino appears in position $i$ or greater.

One method for constructing such tilings is to place $n \geq 0$ $k$-ominoes in positions $i, i+k, i+2k, \ldots, i+(n-1)k$ and then to increase the position of the $j$th $k$-omino by $m_j$ such that $0 \leq m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_n$. This construction corresponds to the left-hand side of (16).

Another method is to construct tilings based on the number of $k$-ominoes of each parity that appear in the tiling. In particular, let $n_i \geq 0$ represent the number of $k$-ominoes of parity $i$ for $1 \leq i \leq k-1$. Place $k$-ominoes in the first $n_1$ positions of parity 1, then place $n_2$ $k$-ominoes in the first available positions of parity 2, and so on. Now go through the remaining uncovered positions of parity 0 and decide whether or not to place a $k$-omino in that position. It remains to project the $k$-ominoes in all possible ways. This can be done by projecting the $k$-ominoes of parity 1, then projecting the $k$-ominoes of parity 2, and so on. In order to make sure that no $k$-omino appears in positions 1 through $i-1$, make sure to project all of the $k$-ominoes of parity 1 through $i-1$ at least once. This completes the construction associated with the right-hand side of (16). □

The $k = i = 1$ case of Eq. (16) simplifies to the usual series-product identity for partitions with distinct parts. The $k = 2$ case yields Theorems 2 and 4 for $i = 1$ and $i = 2$, respectively. Bressoud gave a bijective proof of a partition-theoretic variant of these cases. For example, Eq. (11) can be interpreted as the number of partitions of $n$ with minimal difference at least 2 between parts equals the number of partitions of $n$ into distinct parts wherein each even part is larger than twice the number of odd parts. Bressoud's bijection is described in Andrews [1, Theorem 6.2]. It should be noted that Fibonacci tilings are equivalent to partitions with minimal difference at least 2 between parts, where the position of a domino corresponds to a part in a partition. The construction used in the proof of Theorem 2 begins with the positions of each even domino being at least twice the number of odd dominoes placed at the beginning of the board. In other words, the tiling built after the first step of our construction is equivalent to a partition with $n$ odd parts $1, 3, 5, \ldots, 2n-1$ and each even part larger than twice the number of odd parts. However, instead of increasing the value of the odd parts without changing their parity or the value of the even parts as Bressoud did, we have simultaneously increased the positions of the odd dominoes and decreased the position of the even dominoes via the operation of projection to stay in the context of Fibonacci tilings or equivalently, partitions with minimal difference at least 2 between parts. The cumulative effect of projecting the odd tiles in this case is equivalent to rearranging the parts of the partitions as Bressoud did. Generalizations of Bressoud's work that correspond to the partition-theoretic interpretation of Theorem 6 can be found in [6].

## 3. Even weighted Fibonacci tilings

For the next two identities, we again consider Fibonacci tilings with the same projection operation from the previous section. However, the weight function is now given by

$$w(t) = \begin{cases} -zq^{2i} & \text{if } t \text{ is an even domino in position } 2i \\ zq^{2i} & \text{if } t \text{ is an odd domino in position } 2i-1 \\ 1 & \text{if } t \text{ is a square covering position } i. \end{cases}$$

Notice that projection still has the same effect on the weight of a tiling, even though the weight function has changed. In particular, increasing the position of an odd domino by one changes its weight by a factor of $-1$. Increasing the position of an even domino by one changes its weight by a factor of $-q^2$. It is easy to see that the projection described in the previous section involved moving one odd domino and one even domino and thus the cumulative effect of projection combined with this new weight function is to multiply by a factor of $q^2$.

The collection of Fibonacci tilings combined with the above weight function and projection will be referred to as *even weighted Fibonacci tilings*. The following theorem, which yields two additional Rogers identities, concerns even weighted Fibonacci tilings counted according to the number of odd dominoes.

**Theorem 7.**

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}. \tag{17}$$

**Proof.** Consider even weighted Fibonacci tilings that contain exactly $n$ odd dominoes. To construct such a tiling, first place $n$ odd dominoes in positions $1, 3, \ldots, 2n-1$, which accounts for a combined weight of

$$z^n q^{2+4+\cdots+2n} = z^n q^{n^2+n}.$$

Next, decide whether or not to place a domino in each of the remaining even positions. This accounts for a weight of

$$\prod_{j \geq n+1} (1 - zq^{2j}) = \frac{(zq^2;q^2)_\infty}{(zq^2;q^2)_n}.$$

And finally, project the odd dominoes. Applying Lemma 1 with $k = 2$ shows that

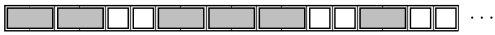
$$(zq^2;q^2)_\infty \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

is the generating function for even weighted Fibonacci tilings with exactly $n$ odd dominoes. Summing over $n \geq 0$ yields the right-hand side of (17).

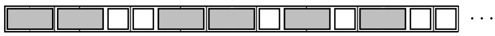
We now use a sign-reversing involution to simplify our construction. In particular, given an even weighted Fibonacci tiling, $T$, find the first occurrence (from left-to-right) of a sequence of consecutive odd dominoes of odd length or a sequence of consecutive odd dominoes of nonnegative even length followed by a square followed by an even domino. Note that if there is at least one even domino in $T$, then at least one of these sequences of tiles must appear in $T$.

If the consecutive odd dominoes of odd length appear first, then increase the position of the last of these odd dominoes by one, resulting in a sequence of consecutive odd dominoes of even length followed by a square followed by an even domino. If the consecutive odd dominoes of even length appear first, then decrease the position of the corresponding even domino by one, resulting in a sequence of consecutive odd dominoes of odd length.

For example, the tilings



and



are paired off under this involution. Note that the even positions covered by the dominoes remain the same under this involution, but the number of even dominoes changes by exactly one. In other words, the tilings have the same $z$ and $q$ weights, but opposite signs, and therefore cancel each other out in the right-hand side of (17).

The only tilings to which this involution cannot be applied are ones that do not contain any even dominoes or sequences of consecutive odd dominoes of odd length. Therefore, our above construction need only account for tilings that contain an even number of odd dominoes and each sequence of consecutive odd dominoes must contain an even number of dominoes. To construct such a tiling, place $2n$ odd dominoes in positions $1, 3, 5, \ldots, 4n-1$, which has a combined weight of $z^{2n}q^{4n^2+2n}$. Now project the odd dominoes *in pairs*. In other words, project the last two odd dominoes $p_n \geq 0$ times. Then project the next two odd dominoes (from right-to-left) $p_{n-1} \leq p_n$ times, and so on. Consequently, each projection (applied to two dominoes at a time) will increase the weight of the tiling by a factor of $q^4$. Applying Lemma 1 with $k = 4$ shows that

$$\frac{z^{2n}q^{4n^2+2n}}{(q^4;q^4)_n}$$

is the generating function for even weighted Fibonacci tilings that contain exactly $2n$ odd dominoes. Summing over $n \geq 0$ completes the proof. □

Theorem 7 unifies the following two identities of Rogers which appear as equations (4.9) and (4.10) in Andrews [1].

**Corollary 8.**

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^4;q^4)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n(q;q^2)_n} \tag{18}$$

$$\sum_{n=0}^{\infty} \frac{q^{4n^2+4n}}{(q^4;q^4)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(q;q^2)_{n+1}}. \tag{19}$$

**Proof.** The above identities follow by setting $z = 1/q$ and $z = 1$ in (17), respectively. □

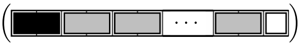
Eq. (18), after applying (1), appears in Slater's list [17, Equation (79)]. Eq. (19), after applying (2), also appears in Slater's list [17, Equation (96)].

## 4. Signed Jacobsthal tilings

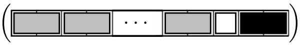
Now consider tilings of a $1 \times \infty$ board using white squares, black dominoes and gray dominoes. We will refer to such tilings as (infinite) Jacobsthal tilings since the number of ways to cover a $1 \times n$ board with one color of squares and two colors of dominoes is the $n$th Jacobsthal number, $J_n$, where $J_n = J_{n-1} + 2J_{n-2}, J_0 = 1$ and $J_1 = 1$. The weight of tile $t$ is defined as follows:

$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a black domino in position } i \\ -zq^i & \text{if } t \text{ is a gray domino in position } i \\ 1 & \text{if } t \text{ is a white square in position } i. \end{cases}$$

In this case, only black dominoes will be used as projectiles. Suppose that there is a black domino in position $i$, followed by $j \geq 0$ gray dominoes followed by a white square. To project the black domino, rearrange the tiles so that the gray dominoes start in position $i$, the white square is placed in position $i + 2j$ and the black domino is placed in position $i + 2j + 1$. Note that the effect of this operation is to increase the weight of the tiling by a factor of $q$, as illustrated below.

$w\Big($  $\Big)$

$$= zq^i(-zq^{i+2})(-zq^{i+4})\cdots(-zq^{i+2j}) = (-1)^j z^{j+1} q^{(i+j)(j+1)}$$

$w\Big($  $\Big)$

$$= (-zq^i)(-zq^{i+2})\cdots(-zq^{i+2j-2})zq^{i+2j+1} = (-1)^j z^{j+1} q^{(i+j)(j+1)+1}.$$

The collection of Jacobsthal tilings combined with the above weight function and projection will be referred to as *signed Jacobsthal tilings*.

**Theorem 9.**

$$\sum_{n=0}^{\infty} \frac{z^n q^{2n^2+n}}{(q^2; q^2)_n} = (zq^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{(3n^2+n)/2}}{(q; q)_n (zq^2; q^2)_n}. \tag{20}$$

**Proof.** Consider all signed Jacobsthal tilings that contain exactly $n$ black dominoes, each one of which is immediately preceded by at least one white square, and do not contain any odd gray dominoes. To construct such a tiling, first place $n$ black dominoes in positions $1, 3, 5, \ldots, 2n - 1$, which accounts for a weight of $z^n q^{n^2}$. Next, arbitrarily place even gray dominoes in positions $2j$ for $j \geq n + 1$. This accounts for a weight of

$$\prod_{j \geq n+1} (1 - zq^{2j}) = \frac{(zq^2; q^2)_\infty}{(zq^2; q^2)_n}.$$

Now project the $i$th black domino exactly $i$ times (starting with the right-most black domino and working right-to-left), for $i = 1, 2, \ldots, n$. This ensures that each black domino is immediately preceded by a white square and increases the weight of the tiling by a factor of

$$q^{1+2+3+\cdots+n} = q^{(n^2+n)/2}.$$

Lastly, project the black dominoes. Note that projection does not change the parity of any of the gray dominoes. Therefore, the generating function for all such tilings is given by

$$(zq^2; q^2)_\infty \frac{z^n q^{(3n^2+n)/2}}{(q; q)_n (zq^2; q^2)_n}.$$

Summing over $n \geq 0$ completes the construction and yields the right-hand side of (20).

We now use a sign-reversing involution to simplify our construction. In particular, find the first occurrence of an even domino. If the first even domino is black, convert it to a gray domino and vice versa. Notice that if the first even domino that appears is gray, then it must necessarily be preceded by a white square and thus converting it to a black domino results in a constructible tiling. Clearly, any two tilings paired off by this involution have the same $z$ and $q$ weights but opposite signs and, therefore, cancel each other out in the right-hand side of (20).

The only tilings to which this involution cannot be applied are ones that contain only squares and odd black dominoes. To construct such a tiling that has exactly $n \geq 0$ odd black dominoes, start by placing $n$ black dominoes in positions $3, 7, \ldots, 4n-1$ so that each domino has at least one white square preceding it, as required. This accounts for a weight of

$$z^n q^{3+7+\cdots+4n-1} = z^n q^{2n^2+n}.$$

Now project the black dominoes, making sure to project each one an even number of times to maintain its parity. In other words, think of projection as simply increasing the position of a domino by two, which increases the weight of a tiling by $q^2$. Therefore,

$$\frac{z^n q^{2n^2+n}}{(q^2;q^2)_n}$$

is the generating function for these tilings that contain exactly $n$ odd black dominoes. Summing over $n \geq 0$ completes the proof. $\square$

Theorem 9 unifies the following two identities of Rogers, which appear as equations (4.5) and (4.6) in Andrews [1].

**Corollary 10.**

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2-n)/2}}{(q;q)_n (q;q^2)_n} \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)/2}}{(q;q)_n (q;q^2)_{n+1}}. \tag{22}$$

**Proof.** The above identities follow by setting $z = 1/q$ and $z = 1$ in (20), respectively. $\square$

Eq. (21), after applying (1) with $q$ replaced by $q^2$, appears in Slater's list [17, Equation (46)]. Eq. (22), after applying (2) with $q$ replaced by $q^2$ also appears in Slater's list [17, Equation (44)].

Now suppose that instead of disallowing odd gray dominoes as we did in the previous proof, we disallow even gray dominoes. In order to use an analogous involution, we would have to insist that no gray domino is placed in position one. With this in mind, we can now prove the following theorem.

**Theorem 11.**

$$\sum_{n=0}^{\infty} \frac{z^n q^{2n^2}}{(q^2;q^2)_n} = (zq;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq;q^2)_{n+1}}. \tag{23}$$

**Proof.** Consider all signed Jacobsthal tilings that contain exactly $n$ black dominoes, each one of which is immediately preceded by at least one white square, and do not contain any even gray dominoes nor a gray domino in position one. To construct such a tiling, place $n$ black dominoes in positions $1, 3, 5, \ldots, 2n-1$. Next, arbitrarily place odd dominoes in positions $2j+1$ for $j \geq n+1$.

Now project the $i$th black domino exactly $i$ times, just as in the proof of the previous theorem. Note that since we did not place a gray domino in position $2j+1$ (i.e., there are at least two white squares before the first gray domino), there cannot be a gray domino in position one, as required.

Lastly, project the black dominoes. Therefore, the generating function for all such tilings is given by

$$(zq;q^2)_\infty \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq;q^2)_{n+1}}.$$

Summing over $n \geq 0$ completes the construction and yields the right-hand side of (23).

We now use a sign-reversing involution to simplify our construction. In particular, find the first occurrence of an odd domino. If the first odd domino is black, convert it to a gray domino and vice versa. This involution cancels out all tilings except those that contain only squares and even black dominoes. To construct such a tiling that has exactly $n \geq 0$ even black dominoes, place $n$ black dominoes in positions $2, 6, \ldots, 4n-2$ to make sure that each domino has at least one white square preceding it. Now project the black dominoes (i.e., increase position by two). Therefore,

$$\frac{z^n q^{2n^2}}{(q^2;q^2)_n}$$

is the generating function for those tilings that contain exactly $n$ odd black dominoes. Summing over $n \geq 0$ completes the proof. $\square$

**Corollary 12.**

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+n)/2}}{(q;q)_n (q;q^2)_{n+1}} \tag{24}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+5n)/2}}{(q;q)_n (q;q^2)_{n+2}}. \tag{25}$$

**Proof.** The above identities follow by setting $z = 1$ and $z = q^2$ in (23), respectively. $\square$

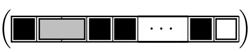
Eq. (24), after applying (1) with $q$ replaced by $q^2$ and dividing both sides by $(q;q^2)_\infty$, is due to Rogers [15, p. 330].

## 5. Signed Pell tilings

For the last pair of identities that we examine in this paper, consider tilings of a $1 \times \infty$ board using white squares, black squares and gray dominoes. We will refer to such tilings as (infinite) Pell tilings since the number of ways to tile a $1 \times n$ board with two colors of squares and one color of dominoes is given by the $n$th Pell number, $P_n$, where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 1$ and $P_1 = 2$. The weight of tile $t$ is defined as follows:

$$w(t) = \begin{cases} -zq^i & \text{if } t \text{ is a gray domino in position } i \\ zq^i & \text{if } t \text{ is a black square in position } i \\ 1 & \text{if } t \text{ is a white square in position } i. \end{cases}$$

In this case, only gray dominoes that are immediately preceded by a black square will be used as projectiles. Suppose that there is a gray domino in position $i$ which is immediately preceded by a black square and immediately followed by $j \geq 0$ black squares followed by a white square. To project the gray domino, rearrange the tiles so that the white square is in position $i-1$, followed by $j+1$ black squares followed by the gray domino. The effect of this operation is to increase the weight of the tiling by a factor of $q^2$, as illustrated below.

$w$ (  )
$$= zq^{i-1}(-zq^i)zq^{i+2} \cdot zq^{i+3} \cdots zq^{i+j+1}$$
$$= -z^{j+2} q^{i(j+2)+\binom{j+2}{2}-2}$$

$w$ (  )
$$= zq^i \cdot zq^{i+1} \cdots zq^{i+j}(-zq^{i+j+1})$$
$$= -z^{j+2} q^{i(j+2)+\binom{j+2}{2}}.$$

The collection of Pell tilings combined with the above weight function and projection will be referred to as *signed Pell tilings*.

**Theorem 13.**

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n(-zq;q)_{2n}}. \tag{26}$$

**Proof.** Consider signed Pell tilings with exactly $n$ dominoes where each domino is immediately preceded by at least one black square. To construct such a tiling, place $n$ black squares in positions $1, 4, 7, \dots, 3n-2$ and $n$ dominoes in positions $2, 5, 8, \dots, 3n-1$, which accounts for a weight of

$$(-1)^n z^{2n} q^{3+9+15+\cdots+6n-3} = (-1)^n z^{2n} q^{3n^2}.$$

Next, for each $j \geq 1$, determine whether or not to insert a black square immediately before the $j$th square (black or white) on the board, starting with $j = 1$. Suppose that you decide to insert a black square immediately before the $j$th square on the board, which has $0 \leq k \leq n$ dominoes appearing to its left. Thus the $j$th square appears in position $2k+j$ and has $n-k$ dominoes and $n-k$ black squares weakly to its right. Therefore, increasing the position of each of the $n-k$ dominoes and $n-k$ black squares by one and inserting a black square in position $2k+j$ increases the weight of the tiling by a factor of

$$zq^{2k+j}q^{2(n-k)} = zq^{2n+j}.$$

Thus the factor $(1+zq^{2n+j})$ represents the choice of whether or not to insert a black square immediately before the $j$th square. Therefore

$$\prod_{j\geq 1}(1+zq^{2n+j}) = \frac{(-zq;q)_\infty}{(-zq;q)_{2n}}$$

accounts for all possible choices of inserting black squares. At this stage, the tiling consists of a collection of dominoes and black squares mixed together followed by a collection of black and white squares mixed together. It remains to mix these two collections of tiles. We can accomplish this by projecting the dominoes. Therefore, the generating function for signed Pell tilings with exactly $n$ dominoes, each one of which is immediately preceded by at least one black square, is given by

$$(-zq;q)_\infty \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n(-zq;q)_{2n}}.$$

Summing over $n \geq 0$ completes the construction and yields the right-hand side of (26).

We now use a sign-reversing involution to simplify the right-hand side. In particular, find the first occurrence of a domino or two consecutive black squares followed by a white square. In the event that a domino appears first, simply replace this tile with a black square followed by a white square. In the event that two consecutive black squares followed by a white square appears first, replace the second of the two black squares and the white square with a single domino.

For example, the tilings

and

are paired off under this involution. Note that any two tilings paired by this involution clearly have the same $z$ and $q$ weights, but opposite signs, and therefore cancel each other out in the right-hand side of (26).

The fixed points of our involution are tilings that do not contain any dominoes or consecutive black squares. To construct such a tiling that contains $n \geq 0$ black squares, start by placing black squares in positions $1, 3, 5, \dots, 2n-1$, which accounts for a weight of $z^n q^{n^2}$. Now project the black squares. To project a black square, simply increase its position by one, which increases the weight of the tiling by a factor of $q$. Therefore,

$$\frac{z^n q^{n^2}}{(q;q)_n}$$

is the generating function for the remaining tilings. Summing over $n \geq 0$ completes the proof. □

**Corollary 14.**

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(q^2;q^2)_n(-q;q)_{2n}} \tag{27}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n+1}}. \tag{28}$$

**Proof.** The above identities follow by setting $z = 1$ and $z = q$ in (26), respectively. □

Eq. (27), after applying (1), appears in Slater's list [17, Equation (19)]. Eq. (28), after applying (2) and dividing both sides by $(-q;q)_\infty$, can be attributed to Ramanujan (see [2, Equation 11.2.7]). Proofs of these identities can also be found in [9, Chapter 5].

**Theorem 15.**

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}(1 - z^2 q^{2n+3})}{(q;q)_n} = (-zq;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n(-zq;q)_{2n+1}}. \tag{29}$$

**Proof.** Suppose that in the previous proof, we do not allow for a black square to be placed before the first square (black or white). In other words, for each $n \geq 0$, divide the $n$th term of the right-hand side of (26) by $(1 + zq^{2n+1})$, which results in the right-hand side of (29). Consequently, the right-hand side of (29) can be interpreted as the generating function for signed Pell tilings where the first domino has exactly one black square appearing to its left, or in the case there are no dominoes, the first position must be covered by a white square.

Since not all signed Pell tilings are allowed, we must also update our involution from the previous proof. In particular, if a black square appears in position one and a domino appears in position two, replacing the domino in position two with a black square followed by a white square results in a tiling which is no longer allowed. Therefore, in this case only, find the *second* occurrence of a domino or the first occurrence of two consecutive black square followed by a white square and then proceed as before.

Now the fixed points of our involution are tilings that start with a white square or start with a black square followed by a domino, with the rest of the board covered by white squares and nonconsecutive black squares. To construct a fixed point that starts with a white square and contains $n \geq 0$ nonconsecutive black squares, place black squares in positions $2, 4, \ldots, 2n$. Then project the black squares. In this case, to project a black square which is immediately followed by a white square, simply increase its position by one. Therefore, the corresponding generating function is given by

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q;q)_n}.$$

To construct a fixed point that starts with a black square followed by a domino and contains $n \geq 0$ additional black squares, place black squares in positions $1, 4, 6, 8, \ldots, 2n + 2$ and a gray domino in position two. Then project the last $n$ black squares. Therefore, the corresponding generating function is given by

$$-\sum_{n=0}^{\infty} \frac{z^{n+2} q^{n^2+3n+3}}{(q;q)_n}.$$

Summing these two generating functions completes the proof. □

**Corollary 16.**

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^2;q^2)_n(-q;q)_{2n}}. \tag{30}$$

**Proof.** Replacing $z$ with $1/q$ in the left-hand side of (29) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2}(1-q^{2n+1})}{(q;q)_n} &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} - \sum_{n=0}^{\infty} \frac{q^{n^2+2n+1}}{(q;q)_n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n} - \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_{n-1}} \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n} \big(1-(1-q^n)\big) \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}.
\end{aligned}$$

Making the same replacement in the right-hand side of (29) yields

$$(-1;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^2;q^2)_n(-1;q)_{2n+1}} = (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^2;q^2)_n(-q;q)_{2n}}$$

as claimed. □

Eq. (30), after applying (2), appears in Slater's list [17, Equation (15)].

## 6. Concluding thoughts

In this work, we have proven the collection of eight identities of Rogers (Corollaries 3, 5, 8 and 10) as presented by Andrews [1, Chapter 4], along with five additional related $q$-series identities (Corollaries 12, 14 and 16), in a very natural way using weighted tilings. While it is very satisfying to see this set of $q$-series identities proven in this manner, undoubtedly other identities can be given similar interpretations. Our goal for future work is to prove additional $q$-series identities via weighted tilings.

## References

[1] G.E. Andrews, $q$-series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra, in: Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, vol. 66, American Mathematical Society, Providence, R.I., 1986.
[2] G.E. Andrews, B. Berndt, Ramanujan's Lost Notebook, Part I, Springer, New York, 2005, pp. 437.
[3] R.J. Baxter, Hard hexagons: Exact solutions, J. Phys. A 13 (1980) L61–L70.
[4] R.J. Baxter, Rogers–Ramanujan identities in the hard hexagon model, J. Statist. Phys. 26 (1981) 427–452.
[5] D. Bowman, J. McLaughlin, A.V. Sills, Some more identities of the Rogers–Ramanujan type, Ramanujan J. 18 (2009) 307–325.
[6] D.M. Bressoud, J.P.O. Santos, P. Mondek, A Family of Partition Identities Proved Combinatorially, Ramanujan J. 4 (2000) 311–315.
[7] K.S. Briggs, D.P. Little, J.A. Sellers, Combinatorial proofs of various $q$-Pell identities via tilings, Ann. Combin. (in press).
[8] G.H. Hardy, P.V. Seshu Anjar, and B.M. Wilson (Eds.), Collected Papers of S. Ramanujan, Chelsea, 1962.
[9] M.D. Hirschhorn, Developments in the Theory of Partitions, Ph.D. Thesis, University of New South Wales, 1979.
[10] D.P. Little, J.A. Sellers, New Proofs of Identities of Lebesgue and Gollnitz via Tilings, J. Combin. Theory Ser. A 116 (2009) 223–231.

[11] J. McLaughlin, A.V. Sills, P. Zimmer, Rogers–Ramanujan–Slater type identities, Electron. J. Combin. 15 (2008) DS15.
[12] L.J. Rogers, On the expansion of some infinite products, Proc. London Math. Soc. 24 (1893) 337–352.
[13] L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25 (1894) 318–343.
[14] L.J. Rogers, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc. 26 (1895) 15–32.
[15] L.J. Rogers, On two theorems in combinatory analysis and some allied identities, Prc. LMS 16 (1917).
[16] L.J. Slater, A new proof of Rogers's tranformations of infinite series, Proc. London Math. Soc. 53 (1951) 460–475.
[17] L.J. Slater, Further identities of the Rogers–Ramanujan type, Proc. London Math. Soc. 54 (1952) 147–167.