Cartan models and cellular decompositions of symmetric Riemannian spaces

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\textbf{A B S T R A C T}

In this paper, by making use of the Cartan models, we will construct cellular decompositions of some symmetric Riemannian spaces such as $\text{Sp}(n)/\text{U}(n)$, $\text{U}(n)/\text{O}(n)$, $\text{U}(2n)/\text{Sp}(n)$, $\text{O}(2n)/\text{U}(n)$, $\text{SU}(n)/\text{SO}(n)$, $\text{SU}(2n)/\text{Sp}(n)$, $\text{SO}(2n)/\text{U}(n)$.

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1. Introduction

We consider a symmetric space $M$ which is homeomorphic with $G/G^\sigma$ for some Lie group $G$ and its invariant subgroup $G^\sigma = \{ g \in G \mid \sigma(g) = g \}$ of an involution, where the space $M$ can be embedded into $G$ by the Cartan model. In particular, the simply connected, compact, irreducible symmetric Riemannian spaces, other than Lie groups, of classical type are classified as follows:

- A I $\text{SU}(n)/\text{SO}(n)$ for $n \geq 3$,
- A II $\text{SU}(2n)/\text{Sp}(n)$ for $n \geq 2$,
- A III $\text{U}(p+q)/\text{U}(p) \times \text{U}(q)$ for $p \geq q \geq 1$,
- BD I $\text{SO}(p+q)/\text{SO}(p) \times \text{SO}(q)$ for $p \geq q \geq 2$, $p+q \neq 4$,
- BD II $\text{SO}(n+1)/\text{SO}(n)$ for $n \geq 2$,
- D III $\text{SO}(2n)/\text{U}(n)$ for $n \geq 4$,
- C I $\text{Sp}(n)/\text{U}(n)$ for $n \geq 3$,
- C II $\text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q)$ for $p \geq q \geq 1$.

Among the simply connected, compact, (irreducible) symmetric Riemannian spaces of classical type, cellular decompositions are known for the following cases:

- A III the complex Grassmann manifold $\mathbb{G}_{p,q}(\mathbb{C}) = \text{U}(p+q)/\text{U}(p) \times \text{U}(q)$ is done by Griffiths and Harris [3, Chapter 5], Milnor and Stasheff [7, Chapter 6], Schwartz [9, Chapter III].

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BD I the oriented real Grassmann manifold \(\tilde{G}_{p,q}(\mathbb{R}) = \text{SO}(p+q)/\text{SO}(p) \times \text{SO}(q)\) is a double covering space of \(G_{p,q}(\mathbb{R})\) by the projection which ignores the orientations of bases, one obtains cellular decompositions of the space \(G_{p,q}(\mathbb{R})\) from that of \(G_{p,q}(\mathbb{R})\).

BD II the sphere \(S^n = \text{SO}(n+1)/\text{SO}(n)\) is trivial.

BD III the symmetric space \(\text{SO}(2n)/U(n)\) is given in Miller [6, Section 10] and also is given in Duan [1] as generalized flag manifold.

C I the symmetric space \(\text{Sp}(n)/U(n)\) is given in Duan [1] as generalized flag manifold.

C II the quaternionic Grassmann manifold \(G_{p,q}(\mathbb{H}) = \text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q)\) is constructed in a way similar to the complex case.

In this paper, we will give cellular decompositions of symmetric Riemannian spaces by making use of the Cartan models of them and the cellular decompositions of the classical Lie groups \(O(n), U(n), \text{Sp}(n), \text{SO}(n),\) and \(SU(n)\) given in [6,10–12]. In other words, taking these results into consideration, one obtains, due to the Cartan models, that homology and cohomology groups, since the cardinal number of the cells of the cellular decomposition be minimal. Yokota principle.

In other words, taking these results into consideration, one obtains, due to the Cartan models, that

\[ M = \bigcup_{e \in \text{a cell of } G} (e \cap M). \]

We will construct characteristic maps of the cells and prove that the above union gives a cellular decomposition of \(M\). More concretely, we will give cellular decompositions of the following symmetric Riemannian spaces:

\(\text{Sp}(n)/U(n), \ U(n)/O(n), \ U(2n)/\text{Sp}(n), \ O(2n)/U(n), \ SU(n)/\text{SO}(n), \ SU(2n)/\text{Sp}(n), \ \text{SO}(2n)/U(n)\).

Remark that the homogeneous space \(O(2n)/U(n)\) is not a symmetric Riemannian space in the strict sense, since it is not connected. However, we will deal with it quite similarly to the other symmetric Riemannian spaces in this paper.

Our results in this paper cover the remaining two cases in the above list, those of types A I and A II, so that we completely give explicit cellular decompositions of all the simply connected, compact, irreducible symmetric Riemannian spaces, other than Lie groups, of classical type. Our cellular decompositions of \(\text{Sp}(n)/U(n)\) and \(\text{SO}(2n)/U(n)\), however, are entirely different from those of Duan [1]. Furthermore, it is interesting that our constructions, which are purely topological, show the strong similarities between \(\text{Sp}(n)/U(n)\) and \(U(n)/O(n)\), between \(U(2n)/\text{Sp}(n)\) and \(O(2n)/U(n)\) and between \(SU(2n)/\text{Sp}(n)\) and \(\text{SO}(2n)/U(n)\) in their topological structures. Note that our cellular decomposition of \(\text{SO}(2n)/U(n)\) is much simpler than the one in Miller [6, Section 10].

The paper is organized as follows. In Section 2, we prepare the notations used in this paper. In Section 3, we study the Cartan models and the spectral resolutions or the normal form of matrices concerned with symmetric Riemannian spaces. In Section 4, we recall basic facts on the cellular decompositions of Lie groups for later use. We give cellular decompositions of \(\text{Sp}(n)/U(n)\) and \(U(n)/O(n)\) in Section 5, \(SU(n)/\text{SO}(n)\) in Section 6, \(U(2n)/\text{Sp}(n)\) and \(O(2n)/U(n)\) in Section 7, \(SU(2n)/\text{Sp}(n)\) and \(\text{SO}(2n)/U(n)\) in Section 8.

Notice that all of these cellular decompositions, except for \(\text{SO}(p+q)/\text{SO}(p) \times \text{SO}(q)\), are very useful to calculate the homology and cohomology groups, since the cardinal number of the cells of the cellular decomposition be minimal. Yokota constructs a cellular decomposition of \(U(n)\) in [12] so as to satisfy this property; we hereafter call it the Yokota principle after him.

There are several applications of the results of this paper. For example, some of them are the following: the estimation of the Lusternik–Schnirelmann category of the symmetric Riemannian spaces mentioned in the above [4], and stable splittings of the symmetric Riemannian spaces [5].

2. Notations

The field of the real numbers \(\mathbb{R}\), the field of the complex numbers \(\mathbb{C}\), and the skew-field of the quaternions \(\mathbb{H}\) have structures of real vector spaces. The field \(\mathbb{H}\) has three imaginary units \(i, j, k\). We use the symbol \(i\) for the imaginary unit of \(\mathbb{C} = \mathbb{R} + j\mathbb{R}\) whose imaginary unit is \(j\).

For the rest of the paper, we fix a positive integer \(n\). We regard the \(n\)-dimensional vector space \(\mathbb{K}^n\) over the field \(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{R} + j\mathbb{R}, \mathbb{H}\) as a right module over \(\mathbb{K}\), that is, the scalar product of a (column) vector \(x \in \mathbb{K}^n\) and a scalar \(\lambda \in \mathbb{K}\) is defined by \(x\lambda\). For an integer \(m\) satisfying \(1 \leq m < n\), the \(m\)-dimensional vector space \(\mathbb{K}^m\) is embedded in \(\mathbb{K}^n\) by a map

\[ \mathbb{K}^m \ni x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{K}^n. \]

For each \(i = 1, \ldots, n\), we denote by \(e_i\) the unit vector whose \(i\)-th component is 1 and the others are 0. Then the set \(\{e_1, \ldots, e_n\}\) forms the canonical basis of \(\mathbb{K}^n\).

For a matrix \(X\) with entries in \(\mathbb{K}\), the transposed matrix of \(X\) is denoted by \(X^T\), the conjugate matrix of \(X\) by \(\overline{X}\) and the transposed conjugate matrix of \(X\) by \(X^*\). The inner product of two vectors \(x, y \in \mathbb{K}^n\) is defined by \(x^*y\).

We define a Lie group \(U(n, \mathbb{K})\) by

\[ U(n, \mathbb{K}) = \{U \text{ is an } n \times n \text{ matrix whose entries belong to } \mathbb{K} \mid U^*U = I_n\}. \]
where \( I_n \) denotes the unit matrix. When \( \mathbb{K} \) is equal to \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{R} + i\mathbb{R} \), the determinant \( \det : U(n, \mathbb{K}) \to \mathbb{K} \) is defined, which allows us to define a Lie group \( SU(n, \mathbb{K}) \) by

\[
SU(n, \mathbb{K}) = \{ U \in U(n, \mathbb{K}) \mid \det U = 1 \}.
\]

**Example.** \( O(n) = U(n, \mathbb{R}), U(n) = U(n, \mathbb{C}) \approx U(n, \mathbb{R} + j\mathbb{R}), Sp(n) = U(n, \mathbb{H}), SO(n) = SU(n, \mathbb{R}), SU(n) = SU(n, \mathbb{C}) \approx SU(n, \mathbb{R} + j\mathbb{R}) \), where \( \approx \) denotes a homeomorphism.

If a pair \( (K, \mathbb{F}) \) is equal to \( (\mathbb{C}, \mathbb{H}), (\mathbb{R}, \mathbb{R} + j\mathbb{R}) \), the canonical embedding maps an \( m \times n \) matrix \( (a_{ij} + jb_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) with entries \( a_{ij}, b_{ij} \) in the field \( \mathbb{K} \) \((1 \leq i \leq m, 1 \leq j \leq n)\) to a \( 2m \times 2n \) matrix

\[
\begin{pmatrix}
  a_{11} & -b_{11} & \cdots & a_{1n} & -b_{1n} \\
  b_{11} & a_{11} & \cdots & b_{1n} & a_{1n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & -b_{m1} & \cdots & a_{mn} & -b_{mn} \\
  b_{m1} & a_{m1} & \cdots & b_{mn} & a_{mn}
\end{pmatrix}.
\]

We consider a scalar as a \( 1 \times 1 \) matrix, a vector \( x \in \mathbb{F}^n \) as an \( n \times 1 \) matrix. Consequently \( \mathbb{F} \), \( \mathbb{F}^n \) and \( U(n, \mathbb{F}) \) are embedded into \((\mathbb{K}^2)^n, (\mathbb{K}^{2n})^2\) and \( U(2n, \mathbb{K}) \) respectively.

### 3. Cartan models

In general, a symmetric Riemannian space \( G/G^\sigma \) can be embedded into some Lie group. The embedding is called **Cartan model** as is explained below.

We define two subsets \( M, N \) of \( G \) by

\[
M = \{ g\sigma(g^{-1}) \in G \mid g \in G \}, \quad N = \{ g \in G \mid \sigma(g^{-1}) = g \}.
\]

Then we have \( G/G^\sigma \approx M \subset N \); the inclusion is the obvious one, and the homeomorphism is proved by using an action

\[
G \times M \ni (g, x) \mapsto gx\sigma(g^{-1}) \in M.
\]

We identify the symmetric Riemannian space \( G/G^\sigma \) with the subset \( M \) of \( G \) by this homeomorphism.

Remark that the subspace \( M \) is closed in the group \( G \), since \( N \) is closed and \( M \) is the connected component including the identity element of \( N \) (see Fomenko [2, Chapter 4, Theorem 15.1]).

When a symmetric space \( M \) is one of the following:

\[
U(n)/O(n), \quad Sp(n)/U(n), \quad U(2n)/Sp(n), \quad O(2n)/U(n),
\]

we study a spectral resolution or a normal form of a matrix \( X \) which belongs to \( N \); this property will be used later to construct cellular decompositions of symmetric Riemannian spaces.

Observe that, for any orthonormal system \( \{x_1, \ldots, x_n\} \subset \mathbb{K}^n \) and scalars \( \xi_1, \ldots, \xi_n \) of \( U(1, \mathbb{K}) \) we have

\[
(X_1 \quad \cdots \quad X_n)
\begin{pmatrix}
  \xi_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & \xi_n
\end{pmatrix}
\begin{pmatrix}
  X_1^* \\
  \vdots \\
  X_n^*
\end{pmatrix} = \sum_{i=1}^n \xi_i x_i^*.
\]

1. We study a spectral resolution of a matrix \( X \in N \) in the case that \( M = U(n)/O(n) \). The Lie group \( G \) is equal to \( U(n) \) and the involution \( \sigma \) maps a matrix \( Y \) to \( \overline{Y} \). Consequently the subspace \( N \) of \( G \) is equal to \( \{ X \in U(n) \mid X^T = X \} \).

**Theorem 3.1.** A matrix \( X \) of \( U(n) \) satisfies that \( X^T = X \) if and only if there exist an orthonormal basis \( \{x_1, \ldots, x_n\} \subset \mathbb{R}^n \) and scalars \( \xi_1, \ldots, \xi_n \in U(1) \) such that they form a spectral resolution of \( X \) as

\[
X = \sum_{i=1}^n \xi_i x_i^*.
\]

**Proof.** That the condition is sufficient is clear; therefore, we need only show it is necessary. Suppose that the matrix \( X \) satisfies \( X^T = X \). Then there exist an orthonormal basis \( \{y_1, \ldots, y_n\} \subset \mathbb{C}^n \) and scalars \( \xi_1, \ldots, \xi_n \in U(1) \) such that

\[
X = \sum_{i=1}^n \xi_i y_i^*.
\]

For each scalar \( \xi \in U(1) \), we define an orthogonal projection \( P(X, \xi) \) by

\[
P(X, \xi) = \sum_{\xi_i = \xi} y_i y_i^*.
\]
whose image is the eigenspace with eigenvalue \( \xi \). Then we obtain that

\[ P(X, \xi) = P(X^T, \xi) = P(X, \xi)^T = P(X, \xi). \]

Thus the orthogonal projection \( P(X, \xi) \) is a real matrix. We can form an orthonormal basis \( \{x_1, \ldots, x_n\} \subset \mathbb{R}^n \) from real eigenvectors \( x_1, \ldots, x_n \) with eigenvalues \( \xi_1, \ldots, \xi_n \) respectively. Therefore we obtain a spectral resolution \( X = \sum_{i=1}^n x_i \xi_i x_i^* \).

(2) We study a spectral resolution of a matrix \( X \in \mathbb{N} \) in the case that \( M = \text{Sp}(n)/U(n) \). The Lie group \( G \) is equal to \( \text{Sp}(n) \), and the involution \( \sigma \) maps a matrix \( Y \) to \( -iY \). Consequently the subspace \( N \) of \( G \) is equal to \( \{X \in \text{Sp}(n) | -iX^{-1}I = X\} \). A spectral resolution of a matrix \( X \in \text{Sp}(n) \) is expressed by

\[ X = \sum_{i=1}^n x_i \xi_i x_i^* \]

for some quaternionic orthonormal basis \( \{x_1, \ldots, x_n\} \subset \mathbb{H}^n \) and some quaternions \( \xi_1, \ldots, \xi_n \in \text{Sp}(1) \). However, the set of all the eigenvectors of \( X \) with eigenvalue \( \xi \in \text{Sp}(1) \) is not necessarily a vector subspace over \( \mathbb{H} \) but is always a vector subspace over \( \mathbb{R} + \xi \mathbb{R} \).

For each quaternion \( \eta \), the real part of \( \eta \) is denoted by \( \Re(\eta) \). We need a lemma.

**Lemma 3.2.** Let \( X \) belong to \( \mathbb{N} \). Suppose that the real parts of all the eigenvalues except for 1 of \( X \) are identical. Then there exist an orthonormal system \( \{x_1, \ldots, x_k\} \subset \mathbb{C}^n \) and a scalar \( \xi \in \text{Sp}(1) \cap (\mathbb{R} + \mathbf{j} \mathbb{R}) \) such that

\[ X = I_n + \sum_{i=1}^k x_i (\xi - 1) x_i^* \]

for some \( k = 0, 1, \ldots, n \).

**Proof.** The matrix \( X \) has a spectral resolution, which is expressed as follows. There exist an orthonormal system \( \{z_1, \ldots, z_k\} \subset \mathbb{H}^n \) and scalars \( \xi_1, \ldots, \xi_k \in \text{Sp}(1) \) such that

\[ X = I_n + \sum_{i=1}^k z_i (\xi_i - 1) z_i^* \]

for some \( k = 0, \ldots, n \). The orthonormal projection \( I_n - \sum_{i=1}^k z_i z_i^* \) is the eigenspace of \( X \) with eigenvalue 1. Two quaternions \(!\lambda \) and \( \mu \) of length 1 satisfy \( \Re(\lambda) = \Re(\mu) \) if and only if there exists a quaternion \( v \) of length 1 such that \( \lambda = v \mu \). Hence there exist a real number \( \theta \in [0, \pi] \) and quaternions \( v_1, \ldots, v_k \in \text{Sp}(1) \) such that \( \cos \theta + j \sin \theta = \sum_{i=1}^k v_i v_i^* \) for each \( i = 1, \ldots, k \). We define a vector \( y_i = z_i v_i \) and a scalar \( \xi = \cos \theta + j \sin \theta \) for each \( i = 1, \ldots, k \). Then we have

\[ X = I_n + \sum_{i=1}^k z_i y_i (\xi - 1) y_i^* = I_n + \sum_{i=1}^k y_i (\xi - 1) y_i^*. \]

Since \( X = -iX^{-1}I \), we have

\[ X = -i \left( I_n + \sum_{i=1}^k y_i (\xi - 1) y_i^* \right) I = I_n + \sum_{i=1}^k -iy_i (\xi - 1) (-iy_i)^*. \]

Hence the vectors \( y_1 \) and \( -iy_1i \) are eigenvectors of \( X \) with eigenvalue \( \xi \). Choose two complex vectors \( p \) and \( q \) so as to satisfy \( p + jq = y_1 \). Then the vectors \( p \) and \( q \) are eigenvectors of \( X \) with eigenvalue \( \xi \), since \( 2p = y_1 - iy_1i \) and \( 2q = -(y_1 + iy_1i)j \).

Define a complex vector \( x_1 \) by

\[ x_1 = \begin{cases} \frac{1}{|p|}p & \text{if } p \neq 0, \\ \frac{1}{|q|}q & \text{if } p = 0. \end{cases} \]

The vector \( x_1 \) is well defined, since \( |p|^2 + |q|^2 = |y_1|^2 = 1 \). The vector \( x_1 \) is a complex eigenvector of \( X \) with eigenvalue \( \xi \). Consequently we can express the matrix \( X \) by

\[ X = I_n + x_1 (\xi - 1) x_1^* + \sum_{i=2}^k y_i' (\xi - 1) y_i'^* \]

with some vectors \( y_2', \ldots, y_k' \in \mathbb{H}^n \). By applying the same argument to the vector \( y_2' \), one can obtain a complex eigenvector of \( X \) with eigenvalue \( \xi \) which is perpendicular to \( x_1 \). So we obtain that

\[ X = I_n + \sum_{i=1}^2 x_i (\xi - 1) x_i^* + \sum_{i=3}^k y_i'' (\xi - 1) y_i''^* \]
with some vectors $y_1, \ldots, y_k \in \mathbb{H}^n$. By applying the same argument inductively, one can construct an orthonormal system
\[ \{x_1, \ldots, x_k\} \subset \mathbb{C}^n \] such that $X = I_n + \sum_{i=1}^{k} x_i (\xi - 1)x_i^*$. \qed

Now we study a spectral resolution of a matrix $X \in N$.

**Theorem 3.3.** A matrix $X$ of $\text{Sp}(n)$ satisfies that $-iX^*i = X$ if and only if there exist an orthonormal basis $\{x_1, \ldots, x_n\} \subset \mathbb{C}^n$ and scalars $\xi_1, \ldots, \xi_n \in \text{Sp}(1) \cap (\mathbb{R} + j\mathbb{R} + k\mathbb{R})$ such that they form a spectral resolution of $X$ as
\[ X = \sum_{i=1}^{n} x_i \xi_i x_i^*. \]

**Proof.** That the condition is sufficient is clear; therefore, we need only show it is necessary. Suppose that a matrix $X \in \text{Sp}(n)$ satisfies $-iX^*i = X$. Then there exist an orthonormal basis $\{z_1, \ldots, z_n\} \subset \mathbb{H}^n$ and scalars $\xi_1, \ldots, \xi_n \in \text{Sp}(1)$ such that
\[ X = \sum_{i=1}^{n} z_i\xi_i z_i^*. \]

For each number $\theta \in [0, \pi]$, we define a matrix $R(X, \theta) \in \text{Sp}(n)$ by
\[ R(X, \theta) = I_n + \sum_{\Re(\xi_i) = \cos \theta} z_i (\xi_i - 1)z_i^*. \]

Observe that the definition of the matrix $R(X, \theta)$ does not depend on a representative of the spectral resolution. The equations
\[ -iX^*i = \sum_{i=1}^{n} iz_i\xi_i z_i^* (-i) = \sum_{i=1}^{n} iz_i\xi_i (iz_i)^* \]

imply that
\[ R(X, \theta) = R(-iX^*i, \theta) = I_n + \sum_{\Re(\xi_i) = \cos \theta} iz_i (\xi_i - 1)(iz_i)^* = -i(R(X, \theta)^{-1})i. \]

By Lemma 3.2 there exist an orthonormal system $\{y_{(\theta,1)}, \ldots, y_{(\theta,k_0)}\} \subset \mathbb{C}^n$ and a scalar $\eta_\theta \in \text{Sp}(1) \cap (\mathbb{R} + j\mathbb{R} + k\mathbb{R})$ such that
\[ R(X, \theta) = I_n + \sum_{j=1}^{k_0} y_{(\theta,j)} (\eta_\theta - 1)y_{(\theta,j)^*} \]

for some $k_0 = 0, 1, \ldots, n$. Then we have
\[ X = \sum_{\theta \in [0, \pi]} \sum_{j=1}^{k_0} y_{(\theta,j)} \eta_\theta y_{(\theta,j)^*}. \]

We can choose vectors $x_1, \ldots, x_n$ so as to satisfy
\[ \{x_1, \ldots, x_n\} = \{y_{(\theta,j)} \mid \theta \in [0, \pi], j = 1, \ldots, k_0\}. \]

Therefore we obtain that
\[ X = \sum_{i=1}^{n} x_i \xi_i x_i^*, \]
where $\xi_i$ denotes the eigenvalue of eigenvector $x_i$ for each $i = 1, \ldots, n$. \qed

(3) We study a spectral resolution of $X \in N$ in the case that $M = U(2n)/\text{Sp}(n)$. The Lie group $G$ is equal to $U(2n)$ and the involution $\sigma$ maps a matrix $Y$ to $JYJ^*$, where $J$ denotes the matrix
\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Consequently the subspace $N$ of $G$ is equal to $\{X \in U(2n) \mid JX^T J^* = X\}$. 
Theorem 3.4. A matrix $X$ of $U(2n)$ satisfies that $JX^TJ^* = X$ if and only if there exist an orthonormal basis $\{x_1, Jx_1, \ldots, x_n, Jx_n\} \subset \mathbb{C}^{2n}$ and scalars $\xi_1, \ldots, \xi_n \in U(1)$ such that they form a spectral resolution of $X$ as

$$X = \sum_{i=1}^{n} (x_i^* \xi_i^* + Jx_i^* J \xi_i^*).$$

Proof. That the condition is sufficient is clear; therefore, we need only show it is necessary. Suppose that a matrix $X \in U(2n)$ satisfies $JX^TJ^* = X$. If a vector $x$ is an eigenvector of $X$ with eigenvalue $\xi$, then we have that

$$X(Jx) = JX^TJ^*x = JX^TJx = Jx \xi = Jx \xi,$$

which implies that the vector $Jx$ is also an eigenvector of $X$ with eigenvalue $\xi$. The vector $Jx$ is perpendicular to $x$, since the inner product $x^*(Jx)$ is equal to 0. Hence there exist an orthonormal basis $\{x_1, Jx_1, \ldots, x_n, Jx_n\} \subset \mathbb{C}^{2n}$ and scalars $\xi_1, \ldots, \xi_n \in U(1)$ such that

$$X = \sum_{i=1}^{n} (x_i^* \xi_i^* + Jx_i^* J \xi_i^*).$$

(4) We study a normal form of $X \in N$ in the case that $M = O(2n)/U(n)$. The Lie group $G$ is equal to $O(2n)$ and the involution $\sigma$ maps a matrix $X$ to $FY^T$. Consequently the subspace $N$ of $G$ is equal to $\{X \in O(2n) | JX^TJ^* = X\}$.

Theorem 3.5. A matrix $X$ of $O(2n)$ satisfies that $JX^TJ^* = X$ if and only if there exist an orthonormal basis $\{x_1, Jx_1, \ldots, x_n, Jx_n\} \subset \mathbb{R}^{2n}$ and numbers $\theta_1, \ldots, \theta_k \in [0, 2\pi)$ such that they form the following as

$$X = \sum_{i=1}^{k} (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i) (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i)^T + \sum_{i=1}^{l} (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i) (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i)^T$$

for some integers $k$ and $l$ satisfying $0 \leq 2k \leq l \leq n$.

Proof. That the condition is sufficient is clear; therefore, we need only show it is necessary. Suppose that a matrix $X \in O(2n)$ satisfies $JX^TJ^* = X$. It follows from Theorem 3.4 that there exist an orthonormal basis $\{y_1, Jy_1, \ldots, y_n, Jy_n\} \subset \mathbb{C}^{2n}$ and scalars $\eta_1, \ldots, \eta_n \in U(1)$ such that

$$X = \sum_{i=1}^{n} (y_i^* \eta_i^* + Jy_i^* J \eta_i^*).$$

For each complex eigenvalue $\eta$ of $X$ and its complex eigenvector $y$, we have that $\overline{X}y = \overline{\eta}y$, since $X$ is a real matrix. Consequently, by replacing indices, we can express

$$X = \sum_{i=1}^{k} (y_i^* \eta_i^* + Jy_i^* J \eta_i^*)$$

for some integers $k$ and $l$ satisfying $0 \leq 2k \leq l \leq n$, and hence the eigenvectors $y_{2k+1}, \ldots, y_n$ are real vectors. Define real vectors $x_1, \ldots, x_k$ by

$$x_i = \frac{1}{\sqrt{2}} y_i + \frac{1}{\sqrt{2}} \overline{y}_i, \quad x_{k+1} = -\frac{i}{\sqrt{2}} y_i + \frac{i}{\sqrt{2}} \overline{y}_i$$

for each $i = 1, \ldots, k$ and real vectors $x_{2k+1}, \ldots, x_n$ by $x_i = y_i$ for each $i = 2k + 1, \ldots, n$. Define numbers $\theta_1, \ldots, \theta_k \in [0, 2\pi)$ by

$$\cos \theta_i + i \sin \theta_i = \eta_i$$

for each $i = 1, \ldots, k$. Therefore the set $\{x_1, Jx_1, \ldots, x_n, Jx_n\}$ forms an orthonormal basis, and hence we obtain that

$$X = \sum_{i=1}^{k} (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i) (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i)^T + \sum_{i=1}^{l} (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i) (x_i^* \cos \theta_i - \sin \theta_i \sin \theta_i \cos \theta_i)^T$$

for some integers $k$ and $l$ satisfying $0 \leq 2k \leq l \leq n$. □
4. Recollection of results on cellular decompositions of Lie groups

In this section, we recall from [6,10–12] the basic facts of the cellular decompositions of the classical Lie groups and prepare some spaces needed for cellular decompositions of symmetric Riemannian spaces.

We use here a symbol $\mathbb{F}$ for fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and a symbol $\mathcal{V}$ for real vector spaces $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R} + j\mathbb{R}$, $\mathbb{R} + j\mathbb{R} + k\mathbb{R}$. Because of the theorems in Section 3, one can choose all the eigenvalues of $X$ in $U(n, \mathbb{K})/U(n, \mathbb{F})$ or $U(2n, \mathbb{K})/U(n, \mathbb{F})$ from $\mathcal{V}$ and eigenvectors of $X$ with them from $\mathbb{F}^n$. We assume that the space $\mathcal{V} \cap U(1, \mathbb{K})$ is a set of eigenvalues and the space $\mathbb{F}^n$ a set of eigenvectors.

The real dimension $\dim_{\mathbb{R}} \mathcal{V}$ is denoted by $\nu$ and $\dim_{\mathbb{F}} \mathcal{F}$ by $f$. For each $m = 1, \ldots, n$, the unit sphere $S_{\nu-1}^{\mathcal{V}}$ is defined by

$$S_{\nu-1}^{\mathcal{V}} = \{ x \in \mathcal{V} \mid |x| = 1 \}.$$ 

An $f(m-1)$-dimensional disk $D_{\mathcal{F}}^{(m-1)}$ and an $f(m-1)$-dimensional ball $B_{\mathcal{F}}^{(m-1)}$ are defined respectively by

$$D_{\mathcal{F}}^{(m-1)} = \{ x \in S_{\mathcal{F}}^{f(m-1)} \mid \text{the } m\text{th component of } x \text{ is real and } \geq 0 \},$$

$$B_{\mathcal{F}}^{(m-1)} = \{ x \in S_{\mathcal{F}}^{f(m-1)} \mid \text{the } m\text{th component of } x \text{ is real and } > 0 \}.$$ 

A $(\nu - 1)$-dimensional disk $D_{\nu-1}^\mathcal{V}$ and a $(\nu - 1)$-dimensional ball $B_{\nu-1}^\mathcal{V}$ are defined respectively by

$$D_{\nu-1}^\mathcal{V} = \{ \xi \in S_{\mathcal{V}}^{\nu-1} \mid |\mathcal{V}(\xi)| \geq 0 \},$$

$$B_{\nu-1}^\mathcal{V} = \{ \xi \in S_{\mathcal{V}}^{\nu-1} \mid |\mathcal{V}(\xi)| > 0 \}.$$ 

We use for indices a Schubert symbol $\omega = (m_1, \ldots, m_k)$ with integers $m_1, \ldots, m_k$ satisfying that $n \geq m_k > m_{k-1} > \cdots > m_1 > 0$.

For each Schubert symbol $\omega = (m_1, \ldots, m_1)$, we define spaces $V_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1}$, $E_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1}$, and $\partial V_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1}$ respectively by

$$V_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1} = \prod_{i=1}^k (D_{\mathcal{F}}^{(m_i-1)} \times D_{\mathcal{V}}^{\nu-1}),$$

$$E_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1} = \prod_{i=1}^k (B_{\mathcal{F}}^{(m_i-1)} \times B_{\mathcal{V}}^{\nu-1}),$$

$$\partial V_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1} = \bigcup_{i=1}^k (D_{\mathcal{F}}^{(m_i-1)} \times E_{\mathcal{V}}^{\nu-1}).$$

By omitting subscripts, the spaces $V_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1}$ and $E_{\mathcal{F}, \mathcal{V}}^{f\omega-f+v-1}$ are denoted by $V_{f\omega-f+v-1}$ and $E_{f\omega-f+v-1}$ respectively. Observe that the dimensions of the spaces $V_{f\omega-f+v-1}$ and $E_{f\omega-f+v-1}$ are both equal to $\sum_{i=1}^k (m_i - f + v - 1)$.

In the case that $k = 0$, we can extend the definitions of a Schubert symbol $\omega$ and the spaces $V_{f\omega-f+v-1}$, $E_{f\omega-f+v-1}$, and $\partial V_{f\omega-f+v-1}$ as follows: a Schubert symbol $\omega$ is empty, $V_{f\omega-f+v-1}$, $E_{f\omega-f+v-1}$ are one point spaces, and $\partial V_{f\omega-f+v-1}$ is empty.

We recall from [6,10–12] the cellular decompositions of the Lie groups $U(n, \mathbb{K})$. For the rest of the section we assume that $\mathcal{V} = \mathbb{F} = \mathbb{K}$, so that we have $v = f$. For each Schubert symbol $\omega$, we define a continuous map

$$\varphi_{U(n, \mathbb{K})}^{f\omega-1} : V_{f\omega-1} \to U(n, \mathbb{K})$$

by

$$\varphi_{U(n, \mathbb{K})}^{f\omega-1}(x, \xi_1)_{j=1}^k = (I_n + x_k(-\xi_k^2 - 1)x_k^*) \cdots (I_n + x_1(-\xi_1^2 - 1)x_1^*),$$

for each $(x_i, \xi_i)_{j=1}^k \in V_{f\omega-1}$. Under this definition, if a Schubert symbol $\omega$ is empty, then $\varphi_{U(n, \mathbb{K})}^{f\omega-1}$ maps $V_{f\omega-1}$, which is the one point space, to the unit matrix $I_n$. We define a cell $e_{U(n, \mathbb{K})}^{f\omega-1} \subset U(n, \mathbb{K})$ by

$$e_{U(n, \mathbb{K})}^{f\omega-1} = \varphi_{U(n, \mathbb{K})}^{f\omega-1}(E_{f\omega-1}).$$

We need the following two lemmas in order to construct the cellular decomposition of $U(n, \mathbb{K})$.

**Lemma 4.1.** The map of pair

$$\varphi_{U(n, \mathbb{K})}^{f\omega-1} : (V_{f\omega-1}, \partial V_{f\omega-1}) \to (e_{U(n, \mathbb{K})}^{f\omega-1}, \partial e_{U(n, \mathbb{K})}^{f\omega-1})$$

is a relative homeomorphism.

**Lemma 4.2.** Let $m_1$ and $m_2$ be two integers satisfying $0 < m_1 < m_2 \leq n$. Then we have

$$e_{U(n, \mathbb{K})}^{f(m_2-1)} e_{U(n, \mathbb{K})}^{f(m_1-1)} = e_{U(n, \mathbb{K})}^{f(m_2-m_1)},$$

For each integer $0 < m \leq n$, we have

$$e_{U(n, \mathbb{K})}^{f(m-1)} e_{U(n, \mathbb{K})}^{f(m-1)} \subset e_{U(n, \mathbb{K})}^{f(m-1)-1}.$$
Then a cellular decomposition of $U(n, \mathbb{K})$ is given by the following.

**Theorem 4.3.** The Lie group $U(n, \mathbb{K})$ has a cellular decomposition

$$\bigcup_{k=0}^{n} \bigcup_{n \geq m_k > m_{k-1} > \ldots > m_1 > 0} e_{U(n, \mathbb{K})}^{(m_n, \ldots, m_1)-1}.$$ 

We consider the canonical projection $p : U(n, \mathbb{K}) \to SU(n, \mathbb{K})$ defined by

$$p(X) = X \Delta(\det X^{-1}, 1, \ldots, 1)$$

for each $X \in U(n, \mathbb{K})$, where $\Delta(\lambda_1, \ldots, \lambda_m)$ denotes the diagonal matrix whose $(i, i)$-entry is equal to $\lambda_i$. Then a cellular decomposition of $SU(n, \mathbb{K})$ is given by the following.

**Theorem 4.4.** The Lie group $SU(n, \mathbb{K})$ has a cellular decomposition

$$\bigcup_{k=0}^{n-1} \bigcup_{n \geq m_k > m_{k-1} > \ldots > m_1 > 1} e_{SU(n, \mathbb{K})}^{(m_n, \ldots, m_1)-1},$$

where $e_{SU(n, \mathbb{K})}^{(m_n, \ldots, m_1)-1} = p(e_{U(n, \mathbb{K})}^{(m_n, \ldots, m_1)-1})$ for each Schubert symbol $\omega = (m_k, \ldots, m_1)$.

For proofs of these lemmas and theorems, the reader is referred to, for example, [6,10–12]. We study some more facts on the cellular decomposition of $U(n, \mathbb{K})$; Lemma 4.2 induces the following.

**Proposition 4.5.** Let $\omega = (m_k, \ldots, m_1)$ be a Schubert symbol. Suppose that matrices $X_i, Y_i \in e_{U(n, \mathbb{K})}^{(m_i)-1}$ $(i = 1, \ldots, k)$ satisfy $X_k X_{k-1} \cdots X_1 = Y_1 \cdots Y_{k-1} Y_k$. Then $X_i = Y_1 \cdots Y_{i-1} Y_i Y_{i-1}^{-1} \cdots Y_1^{-1}$ for all $i = 1, \ldots, k$.

**Proof.** It is clear that

$$Y_1 \cdots Y_{k-1} Y_k = (Y_1 \cdots Y_{k-1} Y_k Y_{k-1}^{-1} \cdots Y_1^{-1})(Y_1 \cdots Y_{k-1} Y_k Y_{k-1}^{-1} \cdots Y_1^{-1}) \cdots Y_1.$$ 

The product $Y_1 \cdots Y_{i-1} Y_i Y_{i-1}^{-1} \cdots Y_1^{-1}$ belongs to $e_{U(n, \mathbb{K})}^{(m_i)-1}$ for each $i = 1, \ldots, k$, while the product $Y_1 \cdots Y_{i-1} Y_i Y_{i-1}^{-1} \cdots Y_1^{-1}$ is equal to $X_i$ for each $i = 1, \ldots, k$ by Lemma 4.1. □

The following proposition is clear by the definition of the characteristic map $\varphi_{U(n, \mathbb{K})}^{(m_n, \ldots, m_1)-1}$.

**Proposition 4.6.** Let $(x_i, \xi_i)_{i=1}^k$ belong to the space $V_{(m_n, \ldots, m_1)-1}$ and define a matrix $X$ by $X = \varphi_{U(n, \mathbb{K})}^{(m_n, \ldots, m_1)-1}(x_i, \xi_i)_{i=1}^k$. Then the orthogonal complement of the subspace $\langle x_1, \ldots, x_k \rangle$ spanned by $\{x_1, \ldots, x_k\}$ is a vector subspace of the eigenspace of $X$ with eigenvalue 1.

Moreover, if the point $(x_i, \xi_i)_{i=1}^k$ belongs to the space $E_{(m_n, \ldots, m_1)-1}$, then the orthogonal complement of the subspace $\{x_1, \ldots, x_k\}$ spanned by $\{x_1, \ldots, x_k\}$ is equal to the eigenspace of $X$ with eigenvalue 1.

5. **Cellular decompositions of $Sp(n)/U(n)$ and $U(n)/O(n)$**

In this section, we consider the cases that

$$(\mathbb{K}, \mathbb{F}, V) = (\mathbb{H}, \mathbb{C}, \mathbb{R} + j\mathbb{R} + k\mathbb{R}), (\mathbb{R} + j\mathbb{R}, \mathbb{R}, \mathbb{R} + j\mathbb{R})$$

and give a cellular decomposition of the symmetric Riemannian space $M = U(n, \mathbb{K})/U(n, \mathbb{F})$, that is, $M = Sp(n)/U(n), U(n)/O(n)$. Then we have $v = f + 1$ and $\dim \mathbb{K} = 2f$. The Lie group $G$ is equal to $U(n, \mathbb{K})$, the involution $\sigma : G \to G$ maps $X$ to $-iX$, and $G^\sigma = U(n, \mathbb{F})$. For each Schubert symbol $\omega$, we define a subset $e_{N}^{\omega} \subset N$ by

$$e_{N}^{\omega} = e_{G}^{\omega} \cap N.$$ 

We will define characteristic maps into cells of $N$ as follows. A continuous map $\varphi_{N}^{\omega} : V_{\omega} \to G$ is defined by

$$\varphi_{N}^{\omega}(x_i, \xi_i)_{i=1}^k = X_1 \cdots X_k \sigma(X_k^*) \cdots \sigma(X_1^*)$$

for each $(x_i, \xi_i)_{i=1}^k \in V^{\omega}$, where $X_i = I_2 + x_i(\xi_i j - 1)x_i^*$. The purpose of this section is to prove that the map $\varphi_{N}^{\omega} : (V, \omega \triangleright V) \to (e_{N}^{\omega}, \omega e_{N}^{\omega})$ is well defined and that it is a relative homeomorphism for each Schubert symbol $\omega$. 

Lemma 5.1. Let $m$ be an integer satisfying $1 \leq m \leq n$. For each matrix $X \in e^m_\mathbb{N}$, there exists a point $(x, \xi) \in E^m$ such that $X = \phi^m_\mathbb{N} (x, \xi)$.

Proof. By Theorems 3.1 and 3.3, there exist an eigenvector $x \in B^f_\mathbb{P} (m-1)$ and its eigenvalue $\eta \in S^1_{\mathbb{P}}$ of $X$ such that $X = I_n + x(\eta - 1)x$. Since the matrix $X$ belongs to $e^m_\mathbb{N} = e^2_{G} \cap \mathbb{N}$. Consequently there exists a scalar $\xi \in B^f_{\mathbb{P}}$ such that $\xi^2 = -\eta$, since $-\eta \neq -1$. Thus we have

$$X = I_n + x(-\xi^2 - 1)x^* = (I_n + x(\xi - 1)x^*)i(I_n + x(\xi - 1)x^*)(-i).$$

Therefore the point $(x, \xi)$ belongs to $E^m$ and hence we have $X = \phi^m_\mathbb{N} (x, \xi)$. □

Lemma 5.2. Let $\omega = (m_1, \ldots, m_k)$ be a Schubert symbol. Suppose that a matrix $X_i$ belongs to $e^2_{G} \cap \mathbb{N}$ for each $i = 1, \ldots, k$ and that the product $X_k \cdots X_1$ belongs to $\mathbb{N}$. Then the matrix $X_i$ belongs to $\mathbb{N}$.

Proof. We have equations

$$X_k \cdots X_1 = \sigma (X_k \cdots X_1^*) = \sigma (X_1^*) \cdots \sigma (X_k^*),$$

since $X_k \cdots X_1 \in \mathbb{N}$. For each $i = 1, \ldots, k$, the matrix $\sigma (X_i^*)$ belongs to $e^2_{G} \cap \mathbb{N}$. Hence the matrix $X_1$ is equal to $\sigma (X_1^*)$ by Proposition 4.5. Therefore the matrix $X_i$ belongs to $\mathbb{N}$. □

The following lemmas are concerned with the relative homeomorphism $\phi^f_\mathbb{N}$.

Lemma 5.3. The image $\phi^f_\mathbb{N} (E^f_\mathbb{N})$ is a subset of the space $e^f_\mathbb{N}$.

Proof. Take a point $(x_i, \xi_i)_{i=1}^k \in E^f_\mathbb{N}$. It is clear that the image point $\phi^f_\mathbb{N} (x_i, \xi_i)_{i=1}^k$ belongs to the subspace $M \subset \mathbb{N}$ by the definition of $\phi^f_\mathbb{N}$. Put $X_i = I_n + x_i(\xi - 1)x_i^*$ for each $i = 1, \ldots, k$. We have

$$X_1 \cdots X_{i-1}X_i = \sigma (X_1^*) \cdots \sigma (X_i^*),$$

since $X_i(\sigma (X_i^*)) = I_n + x_i(-\xi_i^2 - 1)x_i^* \in e^2_{G} \cap \mathbb{N}$ for each $i = 1, \ldots, k$. The image point $\phi^f_\mathbb{N} (x_i, \xi_i)_{i=1}^k$ belongs to $e^2_{G} \cap \mathbb{N}$, since we have

$$\phi^f_\mathbb{N} (x_i, \xi_i)_{i=1}^k = X_1 \cdots X_{i-1}X_i(\sigma (X_i^*)) \cdots \sigma (X_1^*)$$

Therefore the image $\phi^f_\mathbb{N} (E^f_\mathbb{N})$ is a subset of $e^2_{G} \cap \mathbb{N} = e^f_\mathbb{N}$.

Lemma 5.4. The image $\phi^f_\mathbb{N} (\partial V^f_\mathbb{N})$ is a subset of the boundary $\partial e^f_\mathbb{N}$.

Proof. Take a point $(x_i, \xi_i)_{i=1}^k \in \partial V^f_\mathbb{N}$. The image point $\phi^f_\mathbb{N} (x_i, \xi_i)_{i=1}^k$ belongs to $e^f_\mathbb{N}$ by Lemma 5.3, since the map $\phi^f_\mathbb{N} : V^f_\mathbb{N} \rightarrow G$ is continuous. Put $X_i = I_n + x_i(\xi - 1)x_i^*$ for each $i = 1, \ldots, k$. If the point $(x_i, \xi_i)$ belongs to $\partial V^f_\mathbb{N}$, we have that

$$X_i(\sigma (X_i^*)) = I_n + x_i(-\xi_i^2 - 1)x_i^* \in e^2_{G} \cap \mathbb{N}$$

and that

$$X_1 \cdots X_{i-1}X_i(\sigma (X_i^*)) \cdots X_1^* \in \partial e^2_{G} \cap \mathbb{N}$$

for each $i = 1, \ldots, k$. The image point $\phi^f_\mathbb{N} (x_i, \xi_i)_{i=1}^k$ belongs to $\partial e^2_{G} \cap \mathbb{N}$, since we have

$$\phi^f_\mathbb{N} (x_i, \xi_i)_{i=1}^k = X_1 \cdots X_{i-1}X_i(\sigma (X_i^*)) \cdots \sigma (X_1^*)$$

Therefore the image $\phi^f_\mathbb{N} (\partial V^f_\mathbb{N})$ is a subset of $e^f_\mathbb{N} \cap \partial e^2_{G} \cap \mathbb{N} = \partial e^f_\mathbb{N}$. □
Lemma 5.5. The image $\varphi_N^{f_0}(E^{f_0})$ contains the space $e_N^{f_0}$.

Proof. Take a matrix $X \in e_N^{f_0}$. The matrix $X$ belongs also to the space $e_G^{2f_0-1}$. Hence there exists a point $(X_k, \ldots, X_1) \in e_G^{2f_0-1} \times \cdots \times e_G^{2f_0-1}$ such that

$$X = X_k \cdots X_1$$

by Lemma 4.1. The matrix $X$ belongs also to $N$, which implies that the matrix $X_1$ belongs to $N$ by Lemma 5.2. Consequently there exists a point $(x_1, \xi_1) \in V^{f_0}$ such that $\varphi_N^{f_0}(x_1, \xi_1) = X_1$ by Lemma 5.1. Put $Y_1 = I_n + x_1(\xi_1 j - 1)x_1^*$, and then the matrix $Y_1 \sigma(Y_1^*)$ is equal to $X_1$. We have equations

$$Y_1^* X \sigma(Y_1) = Y_1^* x_k \cdots x_2 Y_1 \sigma(Y_1^*) \sigma(Y_1) = (Y_1^* x_k Y_1) (Y_1^* x_{k-1} Y_1) \cdots (Y_1^* x_2 Y_1)$$

and so the matrix $Y_1^* X_1 Y_1$ belongs to $e_G^{2f_0-1}$ for each $i = 2, \ldots, k$. Hence the matrix $Y_1^* X \sigma(Y_1)$ belongs to $e_G^{2f_0-1}$. The matrix $Y_1^* X_1 \sigma(Y_1)$ belongs also to the space $N$, since

$$\sigma((Y_1^* X \sigma(Y_1))^*) = \sigma(\sigma(Y_1)^* X^* Y_1) = Y_1^* \sigma(X^*) \sigma(Y_1) = Y_1^* X_1 \sigma(Y_1).$$

Therefore the matrix $Y_1^* X_1 \sigma(Y_1)$ belongs to $e_N^{(m_1, \ldots, m_2)}$. By using Lemmas 5.1 and 5.2 inductively, we see that there exists a point $(x_i, \xi_i) \in V^{f_0}$ satisfying

$$\varphi_N^{f_0}(x_i, \xi_i) = Y_{i-1}^* \cdots Y_1^* x_i Y_1 \cdots Y_{i-1}$$

for each $i = 2, \ldots, k$. Put $Y_i = I_n + x_i(\xi_i j - 1)x_i^*$, and then we obtain an equation

$$Y_1 \cdots Y_{i-1} Y_i \sigma(Y_i^*) Y_{i-1}^* \cdots Y_1^* = X_i.$$

Finally we have that

$$\varphi_N^{f_0}(x_i, \xi_i)^k_{i=1} = Y_1 \cdots Y_k \sigma(Y_k^*) \cdots \sigma(Y_1^*)$$

$$= (Y_1 \cdots Y_k Y_k^* Y_{k-1} Y_{k-1}^* \cdots Y_1 Y_1^*)$$

$$= X_k \cdots X_1$$

$$= X.$$

Therefore the image $\varphi_N^{f_0}(E^{f_0})$ contains the space $e_N^{f_0}$. □

Lemma 5.6. The restriction $\varphi_N^{f_0}|_{E^{f_0}}$ is injective.

Proof. Take two points $(x_i, \xi_i)^k_{i=1}$, $(y_i, \eta_i)^k_{i=1} \in E^{f_0}$ and suppose that

$$\varphi_N^{f_0}(x_i, \xi_i)^k_{i=1} = \varphi_N^{f_0}(y_i, \eta_i)^k_{i=1},$$

that is, if we define two matrices $X_i$ and $Y_i$ respectively by

$$X_i = I_n + x_i(\xi_i j - 1)x_i^*, \quad Y_i = I_n + y_i(\eta_i j - 1)y_i^*$$

for each $i = 1, \ldots, k$, then we obtain an equation

$$X_1 \cdots X_k \sigma(X_k^*) \cdots \sigma(X_1^*) = Y_1 \cdots Y_k \sigma(Y_k^*) \cdots \sigma(Y_1^*).$$

By Lemma 4.1 we have an equation

$$X_1 \cdots X_{i-1} Y_i \sigma(X_i^*) X_{i-1}^* \cdots X_1^* = Y_1 \cdots Y_{i-1} Y_i \sigma(Y_i^*) Y_{i-1}^* \cdots Y_1^*$$

for each $i = 1, \ldots, k$. We will show by induction that

$$x_i = y_i, \quad \xi_i = \eta_i, \quad X_i = Y_i$$

for all $i = 1, \ldots, k$ as follows: we have equations

$$I_n + x_i(\xi_i^2 - 1)x_i^* = X_i \sigma(X_i^*) = Y_1 \sigma(Y_1^*) = I_n + y_i(\eta_i^2 - 1)y_i^*$$

which imply that $x_i = y_1$ and $\xi_i^2 = -\eta_i^2$. Since the numbers $\xi_1$ and $\eta_1$ belong to $B^{-1}_y$, we have the equations $\xi_1 = \eta_1$ and $X_1 = Y_1$. Suppose that $X_j = Y_j$ for all $j = 1, \ldots, i - 1$. Then we obtain equations

$$I_n + x_i(\xi_i^2 - 1)x_i^* = X_i \sigma(X_i^*) = Y_1 \sigma(Y_1^*) = I_n + y_i(\eta_i^2 - 1)y_i^*.$$
which imply that $x_i = y_i$ and $-\xi_i^2 = -\eta_i^2$. Since the numbers $\xi_i$ and $\eta_i$ belong to $B^1_0$, we have the equations $\xi_i = \eta_i$ and $X_i = Y_i$. Therefore the restriction $\psi^f_\omega | E^f_\omega$ is injective. □

Now it follows from Lemmas 5.3–5.6 that

**Theorem 5.7.** The map 

$$\psi^f_\omega : (V^f_\omega, \partial V^f_\omega) \to (e^f_M, \partial e^f_M)$$

is a relative homeomorphism.

By Theorem 5.7 and the definition of the characteristic map $\psi^f_\omega$, we obtain the following.

**Corollary 5.8.** The subset $M$ is equal to $N$.

Moreover we obtain the following.

**Theorem 5.9.** The symmetric Riemannian space $M = U(n, \mathbb{K})/U(n, \mathbb{F})$ has a cellular decomposition

$$\bigcup_{k=0}^{n} \bigcup_{n \geq m_k > m_{k-1} > \ldots > m_1 > 0} e^f_M(m_k, m_{k-1}, \ldots, m_1).$$

6. A cellular decomposition of $SU(n)/SO(n)$

In this section, we consider the case that $(\mathbb{K}, \mathbb{F}, \mathbb{V}) = (\mathbb{R} + j\mathbb{R}, \mathbb{R} + j\mathbb{R})$ and give a cellular decomposition of the symmetric Riemannian space $M = SU(n, \mathbb{K})/SU(n, \mathbb{F})$, that is, $M = SU(n)/SO(n)$. Then we have $\nu = f + 1$ and $\dim_\mathbb{R} \mathbb{K} = 2f$. The Lie group $G$ is equal to $SU(n, \mathbb{K})$, the involution $\sigma : G \to G$ maps $X$ to $\overline{X} = -IX$, and $\mathbb{G}^0 = SU(n, \mathbb{F})$. For each Schubert symbol $\omega = (m_k, \ldots, m_1)$ with $m_1 > 1$, we define a subset $e^f_{\omega} \subset N$ by

$$e^f_{\omega} = e^f_G \cap N.$$

We will define characteristic maps into cells of $N$ as follows. A continuous map $\psi^f_{\omega} : V^f_\omega \to G$ is defined by

$$\psi^f_{\omega}(x_i, \xi_i)_{i=1}^k = X_0 \cdots X_k \sigma (X_k^*) \cdots \sigma (X_0^*)$$

for each $(x_i, \xi_i)_{i=1}^k \in V^f_\omega$, where $X_i = I_n + x_i(\xi_i(j-1)x_i^* + \xi_i^2(-j)^k - 1)e_1^*$.

The purpose in this section is to prove that the map

$$\psi^f_{\omega} : (V^f_\omega, \partial V^f_\omega) \to (e^f_M, \partial e^f_M)$$

is well defined and that it is a relative homeomorphism for each Schubert symbol $\omega = (m_k, \ldots, m_1)$ with $m_1 > 1$. The following four lemmas are concerned with the relative homeomorphism $\psi^f_{\omega}$.

**Lemma 6.1.** The image $\psi^f_{\omega}(E^f_\omega)$ is a subset of the space $e^f_{\omega}$.

**Proof.** Take a point $(x_i, \xi_i)_{i=1}^k \in E^f_\omega$ and define a pair $(x_0, \xi_0)$ by $(x_0, \xi_0) = (e_1, \xi_1 \cdots \xi_k(-j)^k)$. The point $\psi^f_{\omega}(x_i, \xi_i)_{i=1}^k$ belongs to the subspace $M \subset N$, since $\det \psi^f_{\omega}(x_i, \xi_i)_{i=1}^k = 1$.

Put $X_0 = I_n + x_i(\xi_i(j-1)x_i^*)$ for each $i = 0, \ldots, k$. Then we have

$$X_0 \cdots X_{i-1}X_i \sigma (X_i^*)X_{i-1}^* \cdots X_0^* \in e^f_{U(n, \mathbb{K})},$$

since $X_i \sigma (X_i^*) = I_n + x_i((-\xi_i^2 - 1)x_i^* \in e^f_{U(n, \mathbb{K})}$ for all $i = 1, \ldots, k$. We have equations

$$\psi^f_{\omega}(x_i, \xi_i)_{i=1}^k = X_0 \cdots X_k \sigma (X_k^*) \cdots \sigma (X_0^*)$$

$$= (X_0 \cdots X_{k-1}X_k \sigma (X_k^*) \cdots X_0^*) \cdots (X_0X_1 \sigma (X_1^*)X_0^*) \cdots (X_0 \sigma (X_0^*).$$
By Lemma 5.3 we obtain that
\[
\varphi_{N}^{f^\omega}(x, \xi)^k_{i=1} \in \begin{cases} 
    e_{U(n,K)/U(n,F)}^{f^\omega} & \text{if } X_0 = I_n, \\
    e_{U(n,K)/U(n,F)}^{f_{(o,1)}} & \text{if } X_0 \neq I_n.
\end{cases}
\]

The matrix \(\varphi_{N}^{f^\omega}(x, \xi)^k_{i=1}\) belongs to \(e_{G}^{2f^\omega-1}\), since \(\det \varphi_{N}^{f^\omega}(x, \xi)^k_{i=1} = 1\). Therefore the image \(\varphi_{N}^{f^\omega}(E^f)^{\omega}\) is a subset of \(e_{G}^{2f^\omega-1} \cap N = e_{N}^{f^\omega}\).

**Lemma 6.2.** The image \(\varphi_{N}^{f^\omega}(\partial V^{f^\omega})\) is a subset of the boundary \(\partial e_{N}^{f^\omega}\).

The proof of Lemma 6.2 is the same as that of Lemma 5.4; in fact, one can prove it by replacing Lemma 5.3 in Lemma 5.4 with Lemma 6.1.

**Lemma 6.3.** The image \(\varphi_{N}^{f^\omega}(E^f)^{\omega}\) contains the space \(e_{N}^{f^\omega}\).

**Proof.** Take a matrix \(X \in e_{N}^{f^\omega}\). The space \(e_{N}^{f^\omega}\) is a subset of \(e_{U(n,K)/U(n,F)}^{f^\omega} \cup e_{U(n,K)/U(n,F)}^{f_{(o,1)}}\). If the matrix \(X\) belongs to the space \(e_{U(n,K)/U(n,F)}^{f^\omega}\), we see by Lemma 5.5 that there exists a point \((y_i, \eta_i)^k_{i=1} \in V^{f^\omega}\) satisfying
\[
\varphi_{U(n,K)/U(n,F)}^{f^\omega}(y_i, \eta_i)^k_{i=1} = X.
\]

Define a matrix \(Y_0\) by
\[
Y_0 = I_n + e_1(\gamma_1 \cdots \gamma_k (-j)^{k} - 1)e_1^*.
\]

Then the number \((\gamma_1 \cdots \gamma_k (-j)^{k})^2\) is equal to 1, since \(\det X = 1\). Define a pair \((x_i, \xi_i)\) by \((x_i, \xi_i) = (Y_0y_i, \eta_i)\) for each \(i = 1, \ldots, k\). Consequently we obtain that
\[
\varphi_{N}^{f^\omega}(x, \xi)^k_{i=1} = \varphi_{U(n,K)/U(n,F)}^{f^\omega}(y_i, \eta_i)^k_{i=1} = X.
\]

If the matrix \(X\) belongs to the space \(e_{U(n,K)/U(n,F)}^{f_{(o,1)}}\), we see by Lemma 5.5 that there exists a point \((y_i, \eta_i)^k_{i=1} \in V^{f_{(o,1)}}\) satisfying
\[
\varphi_{U(n,K)/U(n,F)}^{f_{(o,1)}}(y_i, \eta_i)^k_{i=1} = X.
\]

Define a matrix \(Y_0\) by
\[
Y_0 = I_n + e_1(\gamma_1 \cdots \gamma_{k+1} (-j)^{k+1} - 1)e_1^*.
\]

Then the number \((\gamma_1 \cdots \gamma_{k+1} (-j)^{k+1})^2\) is equal to 1, since \(\det X = 1\). Define a pair \((x_i, \xi_i)\) by \((x_i, \xi_i) = (Y_0y_{i+1}, \eta_{i+1})\) for each \(i = 1, \ldots, k\). Consequently we obtain that
\[
\varphi_{N}^{f^\omega}(x, \xi)^k_{i=1} = \varphi_{U(n,K)/U(n,F)}^{f_{(o,1)}}(y_i, \eta_i)^k_{i=1} = X.
\]

Therefore the image \(\varphi_{N}^{f^\omega}(E^f)^{\omega}\) contains the space \(e_{N}^{f^\omega}\). □

**Lemma 6.4.** The restriction \(\varphi_{N}^{f^\omega}|E^f\) is injective.

**Proof.** Take two points \((x_i, \xi_i)^k_{i=1}, (y_i, \eta_i)^k_{i=1} \in E^f\) and suppose that
\[
\varphi_{N}^{f^\omega}(x_i, \xi_i)^k_{i=1} = \varphi_{N}^{f^\omega}(y_i, \eta_i)^k_{i=1},
\]

that is, if we define two matrices \(X_i\) and \(Y_i\) respectively by
\[
X_i = I_n + x_i(\xi_i(-j) - 1)x_i^*, \quad Y_i = I_n + y_i(\eta_i(-j) - 1)y_i^*
\]

for each \(i = 1, \ldots, k\) and two matrices \(X_0\) and \(Y_0\) respectively by
\[
X_0 = I_n + e_1(\xi_1 \cdots \xi_k (-j)^{k} - 1)e_1^*, \quad Y_0 = I_n + e_1(\eta_1 \cdots \eta_k (-j)^{k} - 1)e_1^*,
\]

then we have equations
\[
X_0 \cdots X_k \sigma (X_0) \cdots \sigma (X_0^*) = Y_0 \cdots Y_k \sigma (Y_k) \cdots \sigma (Y_0^*).
\]
\[
X_0 \cdots X_k \sigma (X_0^*) \cdots \sigma (X_0) = X_0^* Y_0 \cdots Y_k \sigma (Y_k^*) \cdots \sigma (Y_0^*) \sigma (X_0).
\]
Define a vector $z_i$ by $z_i = X_0^* Y_0 y_i$ for each $i = 1, \ldots, k$. We obtain by Lemma 5.6 that $(x_i, \xi_i)^k_{i=1} = (z_i, \eta_i)^k_{i=1}$, which implies that $X_0 = Y_0$ and $(x_i, \xi_i)^k_{i=1} = (z_i, \eta_i)^k_{i=1}$. Therefore the restriction $\varphi_N^{f, \omega} | E f, \omega$ is injective. □

Now it follows from Lemmas 6.1–6.4 that

**Theorem 6.5.** The map

$$\varphi_N^{f, \omega} : (V f, \omega, \partial V f, \omega) \rightarrow (e_N^{f, \omega}, \partial e_N^{f, \omega})$$

is a relative homeomorphism.

By Theorem 6.5 and the definition of the characteristic map $\varphi_N^{f, \omega}$, we obtain the following.

**Corollary 6.6.** The subset $M$ is equal to $N$.

Moreover we obtain the following.

**Theorem 6.7.** The symmetric Riemannian space $M = SU(n, \mathbb{K})/SU(n, \mathbb{F})$ has a cellular decomposition

$$\bigcup_{k=0}^{n-1} \bigcup_{n \geq m_k > m_{k-1} > \ldots > m_1 > 1} e_M^{f(m_k, m_{k-1}, \ldots, m_1)}.$$

7. **Cellular decompositions of $U(2n)/Sp(n)$ and $O(2n)/U(n)$**

In this section, we consider the cases that

$$(\mathbb{K}, \mathbb{F}, \mathbb{V}) = (\mathbb{C}, \mathbb{H}, \mathbb{C}), (\mathbb{R}, \mathbb{R} + j\mathbb{R}, \mathbb{R})$$

and give a cellular decomposition of the symmetric Riemannian space $M = U(2n, \mathbb{K})/U(n, \mathbb{F})$, that is, $M = U(2n)/Sp(n)$, $O(2n)/U(n)$. Then we have $2v = f$. The Lie group $G$ is equal to $U(2n, \mathbb{K})$, the involution $\sigma : G \rightarrow G$ maps $X$ to $f \overline{X} f^*$, and $G^\sigma = U(n, \mathbb{F})$. For each Schubert symbol $\omega = (m_k, \ldots, m_1)$ we define a subset $e_N^{2\omega v, v-1} \subset N$ by

$$e_N^{2\omega v, v-1} = e_G^{(2\omega, 2\omega - 1)} \cap N,$$

where $(2\omega, 2\omega - 1)$ denotes a Schubert symbol $(2m_k, 2m_k - 1, \ldots, 2m_1, 2m_1 - 1)$. We will define characteristic maps into cells of $N$ as follows. A continuous map $\varphi_N^{2\omega v, v-1} : V^{2\omega v, v-1} \rightarrow G$ is defined by

$$\varphi_M^{2\omega v, v-1} (x_i, J x_i, \xi_i)^k_{i=1} = X_1 \cdot \ldots \cdot X_k \sigma(X_k^{-1}) \cdot \ldots \cdot \sigma(X_1^{-1})$$

for each $(x_i, J x_i, \xi_i)^k_{i=1} \in V^{2\omega v, v-1}$, where $X_i = I_n + x_i(\xi_i^2 - 1)x_i^*$. The purpose in this section is to prove that the map

$$\varphi_N^{2\omega v, v-1} : (V^{2\omega v, v-1}, \partial V^{2\omega v, v-1}) \rightarrow (e_N^{2\omega v, v-1}, \partial e_N^{2\omega v, v-1})$$

is well defined and that it is a relative homeomorphism for each Schubert symbol $\omega$.

**Lemma 7.1.** Let $m$ be an integer satisfying $1 \leq m \leq n$. For each matrix $X \in e_N^{2vm, v-1}$, there exists a point $(x, j \tilde{x}, \tilde{\xi}) \in e_G^{2vm, v-1}$ such that $X = \varphi_N^{2vm, v-1} (x, j \tilde{x}, \tilde{\xi})$.

**Proof.** The matrix $X$ belongs to $e_N^{2vm, v-1} = e_G^{(2m, 2m - 1)} \cap N$. We consider the case that $M = U(2n)/Sp(n)$. Then by Theorem 3.4 there exist a scalar $\eta \in U(1)$ and a vector $x \in S_C^{4m-3}$ such that

$$X = I_n + x(\eta - 1)x^* + j \tilde{x}(\eta - 1)(j \tilde{x})^*.$$

Then there exists a scalar $\xi \in B_1^1$ such that $-\eta = \xi^2$, since $-\eta$ is not equal to $-1$.

We consider the case that $M = O(2n)/U(n)$. By Theorem 3.5 there exists a vector $x \in S_R^{2m-1}$ such that

$$X = I_n - 2x x^T - 2j x (j x)^T.$$

Define a scalar $\xi$ by $\xi = 1$. Observe that the integer 1 belongs to $B_R^0$ and $(-1)^2 = 1$. 

In both of these cases, we have
\[ X = I_n + x(-\xi^2 - 1)x^* + Jx(-\xi^2 - 1)(Jx)^* \]
\[ = (I_n + x(-\xi^2 - 1)x^*)(I_n + Jx(-\xi^2 - 1)(Jx)^*). \]

Therefore the point \((x, Jx, \xi)\) belongs to \(E^{2v_{m-1}}\), and hence we have \(X = \varphi_N^{2v_{m-1}}(x, Jx, \xi)\).

**Lemma 7.2.** Let \(\omega = (m_1, \ldots, m_1)\) be a Schubert symbol. Suppose that a matrix \(X_i\) belongs to \(e_G^{v_{m_1} - 1}\) for each \(i = 1, \ldots, k\) and that the product \(X_k \cdots X_1\) belongs to \(N\). Then

1. The integer \(k\) is even;
2. The integer \(m_1\) is odd;
3. The integer \(m_1 + 1\) is equal to \(m_2\);
4. The matrix \(X_i^*\) is equal to \(\sigma(X_i)^*\).

**Proof.** Before proving, we prepare the following. We put the matrix \(X = X_k \cdots X_1\) and \(W_1\) to be the eigenspace of \(X\) with eigenvalue 1. Observe that, for each \(i = 1, \ldots, k\), there exists a point \((x_i, \xi_i) \in B_{v_1}^{v_{m_1} - 1} \times B_{v_2}^{v_{m_2} - 1}\) such that
\[ I_n + x_i(-\xi_i^2 - 1)x_i^* = x_i. \]

Then the vector subspace \(W_1^1 \cap K^{m_1}\) is equal to \((x_1)\).

1. By Proposition 4.6, Theorems 3.4 and 3.5 we see that the integer \(k\) is even.
2. We will prove it by reduction ad absurdum; suppose that the integer \(m_1\) is even. Then
\[ JX_1 \in W_1^1 \cap K^{m_1} = (x_1), \]
which contradicts that the vector \(JX_1\) is perpendicular to \(x_1\). Therefore the integer \(m_1\) is odd.
3. It is clear that \(m_1 + 1 \leq m_2\). We have
\[ (x_1, JX_1) \subset W_1^1 \cap K^{m_1+1} \subset W_1^1 \cap K^{m_2} = (x_1, x_2). \]

The space \((x_1, JX_1)\) is equal to \((x_1, x_2)\), since the vector \(JX_1\) is perpendicular to \(x_1\). Consequently the vector \(x_2\) belongs to the space \((x_1, JX_1) \subset K^{m_1+1}\) and \(m_1 + 1 \geq m_2\). Therefore the integer \(m_1 + 1\) is equal to \(m_2\).

4. Since the matrix \(X\) belongs to \(N\), we have
\[ (JX_2^T J^*) \cdots (JX_k^T J^*) = JX^T J^* = X \in e_G^{v_{m_1} - 1}. \]

The vectors \(JX_1, JX_2, \ldots, JX_k\) belong to \(K^{m_2}\). Consequently we have
\[ (JX_1, JX_2) \subset (JX_1, \ldots, JX_k) \cap K^{m_2} = W_1^1 \cap K^{m_2} = (x_1, x_2). \]

Hence we obtain that \((JX_1, JX_2) = (x_1, x_2)\), since \((JX_1, JX_2)\) is linearly independent. The space \((JX_1, \ldots, JX_k) \cap K^{m_2}\) is equal to \(0\), since the set \((JX_1, \ldots, JX_k)\) is linearly independent. By Lemma 4.2 and Proposition 4.6 we obtain that
\[ (JX_1^T J^*) \cdots (JX_k^T J^*) \in e_G^{v_{m_1} - 1}. \]

By Lemma 4.1 and 4.2, we have
\[ X_2 X_1 = (JX_1^T J^*) (JX_2^T J^*) = J(X_2 X_1)^T J^*. \]

Hence the matrix \(X_2 X_1\) belongs to \(N\) as well as \(e_N^{2v_{m_1} - 1}\). Therefore the matrix \(X_2\) is equal to \(\sigma(X_1)^*\) by Lemma 7.1.

The following four lemmas are concerned with the relative homeomorphism \(\varphi_N^{2v_\omega - 1}\).

**Lemma 7.3.** The image \(\varphi_N^{2v_\omega - 1}(E^{2v_\omega - 1})\) is a subset of the space \(e_N^{2v_\omega - 1}\).

The proof of Lemma 7.3 is the same as that of Lemma 5.3.

**Lemma 7.4.** The image \(\varphi_N^{2v_\omega - 1}(\partial V^{2v_\omega - 1})\) is a subset of the boundary \(\partial e_N^{2v_\omega - 1}\).

The proof of Lemma 7.4 is the same as that of Lemma 5.4; in fact, one can prove it by replacing Lemma 5.3 in Lemma 5.4 with Lemma 7.3.

**Lemma 7.5.** The image \(\varphi_N^{2v_\omega - 1}(E^{2v_\omega - 1})\) contains the space \(e_N^{2v_\omega - 1}\).
The proof of Lemma 7.5 is the same as that of Lemma 5.5; in fact, one can prove it by replacing Lemma 5.1 with Lemma 7.1 and Lemma 5.2 with Lemma 7.2 in Lemma 5.5 respectively.

**Lemma 7.6.** The restriction \( \psi^{|e^{2v_0-v-1}} \) is injective.

The proof of Lemma 7.6 is the same as that of Lemma 5.6. Now it follows from Lemmas 7.3–7.6 that

**Theorem 7.7.** The map
\[
\psi^{|e^{2v_0-v-1}} : (V^{2v_0-v-1}, \delta V^{2v_0-v-1}) \rightarrow (e^{2v_0-v-1}_N, \delta e^{2v_0-v-1}_N)
\]
is a relative homeomorphism.

To give a cellular decomposition, we need the following.

**Theorem 7.8.** Let \( \omega \) be a Schubert symbol and suppose that \( e^{2v_0-1}_G \cap N \) is not empty. Then there exist integers \( m_1, \ldots, m_k \) such that \( \omega = (2m_{k}, 2m_k - 1, \ldots, 2m_1, 2m_1 - 1) \).

For, similarly to the proof of Lemma 7.5, one can prove Theorem 7.8 by using Lemma 7.2 inductively.

By Theorems 7.7, 7.8, and the definition of the characteristic map \( \psi^{|e^{2v_0-v-1}}_N \), we obtain the following.

**Corollary 7.9.** The subset \( M \) is equal to \( N \).

Moreover we obtain the following.

**Theorem 7.10.** The symmetric Riemannian space \( M = U(2n, K)/U(n, F) \) has a cellular decomposition
\[
\bigcup_{k=0}^{n} \left( \bigcup_{n \geq m_k > m_{k-1} > \cdots > m_1 > 0} e^{2v(m_k, m_{k-1}, \ldots, m_1)-v-1}_M \right).
\]

8. **Cellular decompositions of \( SU(2n)/Sp(n) \) and \( SO(2n)/U(n) \)**

In this section, we consider the cases that
\[
(\mathbb{K}, F, V) = (\mathbb{C}, \mathbb{H}, \mathbb{C}), (\mathbb{R}, \mathbb{R} + j\mathbb{R}, \mathbb{R})
\]
and give a cellular decomposition of the symmetric Riemannian space \( M = SU(2n, \mathbb{K})/U(n, F) \), that is, \( M = SU(2n)/Sp(n), SO(2n)/U(n) \). Then we have \( 2v = f \). The Lie group \( G \) is equal to \( SU(2n, \mathbb{K}) \), the involution \( \sigma : G \rightarrow G \) maps \( X \) to \( JX^tJ \), and \( G^\sigma = U(n, F) \).

We define two subsets \( N_+ \) and \( N_- \) respectively by
\[
N_+ = \{ X \sigma(X^*) \in N \mid X \in U(2n, K), \text{det} X = 1 \},
\]
\[
N_- = \{ X \sigma(X^*) \in N \mid X \in U(2n, K), \text{det} X = -1 \}.
\]

It is clear that \( N_+ = M \).

**Lemma 8.1.** The space \( N \) is composed of the two connected components \( N_+ \) and \( N_- \).

**Proof.** It is clear that the spaces \( N_+ \) and \( N_- \) are connected. The spaces \( N_+ \) and \( N_- \) do not intersect each other, since \( \text{det} Y = 1 \) for any matrix \( Y \in U(n, F) \). Take a matrix \( Y \in N \). By Corollary 7.9, there exists a matrix \( X \in U(2n, K) \) such that \( X \sigma(X^*) = Y \). By the definition of \( \sigma \), we obtain that \( \text{det} \sigma(X^*) = \text{det} X \). By calculating the determinants, we have \( (\text{det} X)^2 = \text{det} Y = 1 \). Consequently the matrix \( Y \) belongs to \( N_+ \cup N_- \). Therefore the set \( N \) is equal to the disjoint union of \( N_+ \) and \( N_- \) as sets. Furthermore, the space \( N \) is equal to the disjoint union of \( N_+ \) and \( N_- \) as topological spaces, since the spaces \( N_+ \) and \( N_- \) are compact. Thus the space \( N \) is composed of the two connected components \( N_+ \) and \( N_- \). \( \square \)

For each Schubert symbol \( \omega = (m_k, \ldots, m_1) \) with \( m_1 > 1 \), we define a subset \( e^{2v_0-v-1}_M \) by
\[
e^{2v_0-v-1}_M = e^{(2v_0, 2v_0-1)}_G \cap M.
\]
We will define characteristic maps into cells of $M$ as follows. A continuous map $\varphi_{M}^{2v_0-v-1} : V^{2v_0-v-1} \to G$ is defined by

$$\varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} = x_0 \cdots x_i \sigma(x_i^*) \cdots \sigma(x_0^*)$$

for each $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} \in V^{2v_0-v-1}$, where $x_i = I_0 + x_i(-\xi_i^2 - 1)x_i^*$ and $x_0 = I_0 + e_1((-1)^{k+1} \xi_1^2 \cdots \xi_k^2 - 1)e_1^*$. The purpose in this section is to prove that the map

$$\varphi_{M}^{2v_0-v-1} : (V^{2v_0-v-1}, \partial V^{2v_0-v-1}) \to (e_{M}^{2v_0-v-1}, \partial e_{M}^{2v_0-v-1})$$

is well defined and that it is a relative homeomorphism for each Schubert symbol $\omega = (m_1, \ldots, m_1)$ with $m_1 > 1$. The following four lemmas are concerned with the relative homeomorphism $\varphi_{M}^{2v_0-v-1}$.

**Lemma 8.2.** The image $\varphi_{M}^{2v_0-v-1}(E_{v_0-v-1})$ is a subset of the space $e_{M}^{2v_0-v-1}$.

**Proof.** Take a point $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} \in E_{v_0-v-1}$ and define a matrix $X$ by $X = \varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k}$. The matrix $X$ belongs to $M$ by the definition of the map $\varphi_{M}^{2v_0-v-1}$. The matrix $X$ belongs to $e_{U(2n, K)}^{v_0} \cup e_{U(2n, K)}^{v_0} \cup e_{U(2n, K)}^{v_0}$ by Lemma 7.3. The matrix $X$ belongs also to $e_{G}^{v_0}$, since the determinant of $X$ is equal to 1. Therefore the image $\varphi_{M}^{2v_0-v-1}(E_{v_0-v-1})$ is a subset of $e_{M}^{2v_0-v-1}$.

**Lemma 8.3.** The image $\varphi_{M}^{2v_0-v-1}(\partial V^{2v_0-v-1})$ is a subset of the boundary $\partial e_{M}^{2v_0-v-1}$.

The proof of Lemma 8.3 is the same as that of Lemma 5.4; in fact, one can prove it by replacing Lemma 5.3 in Lemma 5.4 with Lemma 8.2.

**Lemma 8.4.** The image $\varphi_{M}^{2v_0-v-1}(E_{v_0-v-1})$ contains the space $e_{M}^{2v_0-v-1}$.

**Proof.** Take a matrix $X \in e_{M}^{2v_0-v-1}$, where the space $e_{M}^{2v_0-v-1}$ is a subset of the union $e_{U(2n, K)}^{v_0} \cup e_{U(2n, K)}^{v_0} \cup e_{U(2n, K)}^{v_0}$. Then by Lemma 7.5 there exists a point $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} \in E_{v_0-v-1}$ such that $X = \varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k}$. The product $(-1)^{k-1} \xi_1^2 \cdots \xi_k^2$ is equal to 1, since $M = N_i$. Hence the matrix $\varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k}$ is equal to $X$.

We assume that the matrix $X$ belongs to $e_{U(2n, K)}^{v_0} \cup e_{U(2n, K)}^{v_0} \cup e_{U(2n, K)}^{v_0}$. Then by Lemma 7.5 there exists a point $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} \in E_{v_0-v-1}$ such that $X = \varphi_{U(2n, K)}^{v_0}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k+1}$. The product $(-1)^{k+1} \xi_1^2 \cdots \xi_k^2$ is equal to 1, since $M = N_i$. Moreover the vector $x_i$ is equal to $e_1$. Therefore the point $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k+1}$ belongs to $e_{v_0-v-1}$ and $\varphi_{M}^{2v_0-v-1}(x_i+1, jI \xi+1, \xi_i^{k+1})_{i=1}^{k+1}$ is equal to $X$.

**Lemma 8.5.** The restriction $\varphi_{M}^{2v_0-v-1}|_{E_{v_0-v-1}}$ is injective.

**Proof.** Take two points $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} \in E_{v_0-v-1}$ and suppose that $\varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} = \varphi_{M}^{2v_0-v-1}(y_i, jI \eta, \eta_i^{k})_{i=1}^{k}$.

If the image point $\varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k}$ belongs to $e_{U(2n, K)}^{v_0}$, then we have $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} = (y_i, jI \eta, \eta_i^{k})_{i=1}^{k}$ by Lemma 7.6.

If the image point $\varphi_{M}^{2v_0-v-1}(x_i, jI \xi, \xi_i^{k})_{i=1}^{k}$ belongs to $U_{U(2n, K)}^{v_0}$, then there exist numbers $\xi_0, \eta_0 \in B_{v_0}$ such that

$$-\xi_0^2 = (-1)^{k} \xi_1^2 \cdots \xi_k^2, \quad -\eta_0^2 = (-1)^{k} \eta_1^2 \cdots \eta_k^2,$$

since $(-1)^{k} \xi_1^2 \cdots \xi_k^2 \neq 1$ and $(-1)^{k} \eta_1^2 \cdots \eta_k^2 \neq 1$. Define two vectors $x_0, y_0$ by $x_0 = y_0 = e_1$. Then we have $(x_i, jI \xi, \xi_i^{k})_{i=0}^{k} = (y_i, jI \eta, \eta_i^{k})_{i=0}^{k}$ by Lemma 7.6. Hence we obtain that $(x_i, jI \xi, \xi_i^{k})_{i=1}^{k} = (y_i, jI \eta, \eta_i^{k})_{i=1}^{k}$. Therefore the restriction $\varphi_{M}^{2v_0-v-1}|_{E_{v_0-v-1}}$ is injective.

Now it follows from Lemmas 8.2–8.5 that

**Theorem 8.6.** The map

$$\varphi_{M}^{2v_0-v-1} : (V^{2v_0-v-1}, \partial V^{2v_0-v-1}) \to (e_{M}^{2v_0-v-1}, \partial e_{M}^{2v_0-v-1})$$

is a relative homeomorphism.
By Theorems 7.8 and 8.6, and the definition of the characteristic map $\varphi^{2v_\omega - v - 1}_M$, we obtain the following.

**Theorem 8.7.** The symmetric Riemannian space $M = SU(2n, K)/U(n, F)$ has a cellular decomposition

$$
\bigcup_{k=0}^{n-1} \bigcup_{n \geq m_k \gg m_{k-1} \gg \ldots \gg m_1 > 1} \mathbb{Z}^{2v(m_k, m_{k-1}, \ldots, m_1) - v - 1}.
$$

**Concluding remark.** It follows easily from the cellular decompositions of symmetric Riemannian spaces stated in this section as well as in the earlier ones that the homology groups of them are given as follows:

$$
H_*(Sp(n)/U(n); \mathbb{Z}) \cong H_*(S^2 \times S^4 \times \cdots \times S^{2n}; \mathbb{Z}),
H_*(U(n)/O(n); \mathbb{Z}_2) \cong H_*(S^1 \times S^2 \times \cdots \times S^n; \mathbb{Z}_2),
H_*(SU(n)/SO(n); \mathbb{Z}_2) \cong H_*(S^1 \times S^3 \times \cdots \times S^n; \mathbb{Z}_2),
H_*(SU(2n)/Sp(n); \mathbb{Z}) \cong H_*(S^1 \times S^5 \times \cdots \times S^{4n-3}; \mathbb{Z}),
H_*(SU(2n)/Sp(n); \mathbb{Z}) \cong H_*(S^5 \times S^9 \times \cdots \times S^{4n-3}; \mathbb{Z}),
H_*(O(2n)/U(n); \mathbb{Z}) \cong H_*(S^0 \times S^2 \times \cdots \times S^{2n-2}; \mathbb{Z}),
H_*(SO(2n)/U(n); \mathbb{Z}) \cong H_*(S^2 \times S^4 \times \cdots \times S^{2n-2}; \mathbb{Z}).
$$

and also that the cohomology groups of them are given as follows:

$$
H^*(Sp(n)/U(n); \mathbb{Z}) \cong H^*(S^2 \times S^4 \times \cdots \times S^{2n}; \mathbb{Z}),
H^*(U(n)/O(n); \mathbb{Z}_2) \cong H^*(S^1 \times S^2 \times \cdots \times S^n; \mathbb{Z}_2),
H^*(SU(n)/SO(n); \mathbb{Z}_2) \cong H^*(S^2 \times S^3 \times \cdots \times S^n; \mathbb{Z}_2),
H^*(SU(2n)/Sp(n); \mathbb{Z}) \cong H^*(S^1 \times S^5 \times \cdots \times S^{4n-3}; \mathbb{Z}),
H^*(SU(2n)/Sp(n); \mathbb{Z}) \cong H^*(S^5 \times S^9 \times \cdots \times S^{4n-3}; \mathbb{Z}),
H^*(O(2n)/U(n); \mathbb{Z}) \cong H^*(S^0 \times S^2 \times \cdots \times S^{2n-2}; \mathbb{Z}),
H^*(SO(2n)/U(n); \mathbb{Z}) \cong H^*(S^2 \times S^4 \times \cdots \times S^{2n-2}; \mathbb{Z}).
$$

These are only algebraic isomorphisms. As for the ring structure of cohomology, the reader is referred to Mimura and Toda [8].

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**References**