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Existence and multiplicity of solutions for semilinear elliptic equations with Hardy terms and Hardy–Sobolev critical exponents☆

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Abstract

Some existence and multiplicity results are obtained for solutions of semilinear elliptic equations with Hardy terms, Hardy–Sobolev critical exponents and superlinear nonlinearity by the variational methods and some analysis techniques.

Keywords: Hardy terms; Hardy–Sobolev critical exponents; $(PS)_c$ condition; Mountain Pass Lemma; Semilinear elliptic equations

1. Introduction and main results

Consider the following semilinear elliptic problem

$$\begin{cases} -\triangle u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s} u + f(x, u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain in R^N $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $0 \leq \mu < \overline{\mu} \triangleq \frac{(N-2)^2}{4}$, $0 \leq s < 2$, $f \in C(\overline{\Omega} \times R, R)$, $2^*(s) = \frac{2(N-s)}{N-2}$ is the Hardy–Sobolev critical exponent and $2^* = 2^*(0) = \frac{2N}{N-2}$ is the Sobolev critical exponent. $F(x, t)$ is a primitive function of $f(x, t)$ defined by $F(x, t) = \int_0^t f(x, s)\mathrm{d}s$ for $x \in \Omega$, $t \in R$. $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is the Sobolev space with the norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) \mathrm{d}x\right)^{\frac{1}{2}},$$

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which is equivalent to the usual norm of $H_0^1(\Omega)$ due to the Hardy inequality (see Lemma 3.1 in [7]) and

$$A_{\mu,s}(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2}\right) \mathrm{d}x}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \mathrm{d}x\right)^{\frac{2}{2^*(s)}}} \tag{2}$$

is the best Hardy–Sobolev constant which is independent of Ω by Theorem 3.1 in [7], so we denote $A_{\mu,s}$ instead of $A_{\mu,s}(\Omega)$.

Problem (1) in the case $s = 0$ and $\mu = 0$ has been widely studied since Brezis and Nirenberg (see [1,4,10] and the references here). In recent years, people have paid much attention to the existence of solutions for singular problems concerning the operator $-\Delta - \frac{\mu}{|x|^2}$ $(0 \le \mu < \overline{\mu})$ with Sobolev critical exponents (the case that $s = 0$) (see [2,5,6, 9,15] and their references). Some authors also studied the singular problems with Hardy–Sobolev critical exponents (the case that $s \neq 0$) (see [7,8,11,12,14]). But there are few results dealing with the case $0 \le \mu < \overline{\mu}$, $0 \le s < 2$ and the general form $f(x,t)$. In [11,12,14] and so on, the authors only studied the special cases of $f(x,t)$. For example, in [11], $f(x,t) = \lambda |t|^{q-1} t$ with suitable q. In the present paper, we use a variational method to deal with problem (1) with general form and generalize the results in [11].

Due to the lack of compactness of the embeddings in $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2}\mathrm{d}x)$ and $H_0^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |x|^{-s}\mathrm{d}x)$, we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais–Smale ((PS) for short) condition in $H_0^1(\Omega)$. However, a local (PS) condition can be established in a suitable range. Then the existence result is obtained via constructing a minimax level within this range and the Mountain Pass Lemma due to A. Ambrosetti and P.H. Rabinowitz (see also [13]).

Here are the main results of this paper:

Theorem 1. *Suppose that $N \ge 3$, $0 \le \mu < \overline{\mu}$, $0 \le s < 2$,*

(f_1) *$f \in C(\overline{\Omega} \times R^+, R)$, and $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$, $\lim_{t\to+\infty} \frac{f(x,t)}{t^{2^*-1}} = 0$ uniformly for $x \in \overline{\Omega}$, and*

(f_2) *There exists a constant ρ, $\rho > 2$, such that $0 < \rho F(x,t) \le f(x,t)t$ for all $x \in \overline{\Omega}$, $t \in R^+ \setminus \{0\}$.*

Assume that

$$\rho > \max\left\{2, \frac{N}{\sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}}, \frac{N - 2\sqrt{\overline{\mu} - \mu}}{\sqrt{\overline{\mu}}}\right\} \triangleq r_0. \tag{3}$$

Then problem (1) has at least a positive solution.

Corollary 1. *Suppose that $N \ge 4$, $0 \le \mu \le \overline{\mu} - 1$, $0 \le s < 2$. Assume that* (f_1) *and* (f_2) *hold. Then problem (1) has at least a positive solution.*

Theorem 2. *Suppose that $N \ge 3$, $0 \le \mu < \overline{\mu}$, $0 \le s < 2$,*

(f_3) *$f \in C(\overline{\Omega} \times R, R)$, and $\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0$, $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{2^*-1}} = 0$ uniformly for $x \in \overline{\Omega}$, and*

(f_4) *There exists a constant ρ, $\rho > 2$, such that $0 < \rho F(x,t) \le f(x,t)t$ for all $x \in \overline{\Omega}$ and $t \in R \setminus \{0\}$.*

Assume that (3) holds. Then problem (1) has at least two distinct nontrivial solutions.

Corollary 2. *Suppose that $N \ge 4$, $0 \le \mu \le \overline{\mu} - 1$, $0 \le s < 2$. Assume that* (f_3) *and* (f_4) *hold. Then problem (1) has at least two distinct nontrivial solutions.*

Remark 1. Theorem 1 generalizes Theorem 1.1 in [11] where the author only studied the special situation that $f(x,t) = \lambda |t|^{q-2} t$ with $r_0 < q < 2^*$. There are functions f satisfying the assumptions of our Theorem 1 and not satisfying those in [6,7,11,12]. For example, let

$$f(x,t) = g(x)|t|^{k-2}t + \alpha |t|^{l-2}t$$

for $(x,t) \in \overline{\Omega} \times R$, where $g(x) > 0$, $g \in L^\infty(\Omega)$, $\alpha > 0$ and $r_0 < k < l < 2^*$. Then f satisfies the conditions of Theorem 1, while it doesn't satisfy the conditions of Theorem 1.1 in [11] and others.

2. Proof of theorems

It is obvious that the values of $f(x,t)$ for $t<0$ are irrelevant in Theorem 1, and we may define

$$f(x,t)=0 \quad \text{for } x\in\Omega,\ t\le 0.$$

In order to study the existence of positive solutions for (1) we shall firstly consider the existence of nontrivial solutions to the problem

$$-\triangle u-\mu\frac{u}{|x|^2}=\frac{(u^+)^{2^*(s)-1}}{|x|^s}+f(x,u),\quad x\in\Omega\setminus\{0\},\ u=0,\ x\in\partial\Omega. \tag{4}$$

The energy functional corresponding to (4) is given by

$$I(u)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)\mathrm{d}x-\frac{1}{2^*(s)}\int_\Omega\frac{(u^+)^{2^*(s)}}{|x|^s}\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x,\quad u\in H_0^1(\Omega).$$

By the Hardy and Hardy–Sobolev inequalities (see Lemma 3.2 in [7]) and (f_1), $I\in C^1(H_0^1(\Omega),R)$. Now it is well known that there exists a one to one correspondence between the weak solutions of problem (4) and the critical points of I on $H_0^1(\Omega)$. More precisely we say that $u\in H_0^1(\Omega)$ is a weak solution of problem (4), if for any $v\in H_0^1(\Omega)$, there holds

$$\langle I'(u),v\rangle=\int_\Omega\left(\nabla u\nabla v-\mu\frac{uv}{|x|^2}\right)\mathrm{d}x-\int_\Omega\frac{(u^+)^{2^*(s)-1}}{|x|^s}v\mathrm{d}x-\int_\Omega f(x,u)v\mathrm{d}x=0.$$

Let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ and $c\in R$. $\{u_n\}$ is called to be a $(\mathrm{PS})_c$ sequence in $H_0^1(\Omega)$ if $I(u_n)\to c$, $I'(u_n)\to 0$ in $(H_0^1(\Omega))^*$ as $n\to\infty$. We say I satisfies $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence $\{u_n\}\subset H_0^1(\Omega)$ has a convergent subsequence.

Lemma 1. *Assume* (f_1) *and* (f_2) *hold. Suppose* $c\in(0,\frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}})$, *then* I *satisfies* $(\mathrm{PS})_c$ *condition.*

Proof. Suppose that $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence in $H_0^1(\Omega)$. By (f_2), we have

$$\begin{aligned}c+1+o(1)\|u_n\| &\ge I(u_n)-\frac{1}{\theta}\langle I'(u_n),u_n\rangle\\ &=\left(\frac12-\frac1\theta\right)\|u_n\|^2+\left(\frac1\theta-\frac{1}{2^*(s)}\right)\int_\Omega\frac{(u_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x-\int_\Omega\left(F(x,u_n)-\frac1\theta f(x,u_n)u_n\right)\mathrm{d}x\\ &\ge\left(\frac12-\frac1\theta\right)\|u_n\|^2,\end{aligned}$$

where $\theta=\min\{\rho,2^*(s)\}$. Hence we conclude $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega)$. By the continuity of embedding, we have $\|u_n\|_{2^*}^{2^*}\le C_1<\infty$. Going if necessary to a subsequence, one can get that

$$\begin{cases}u_n\to u & \text{weakly in } H_0^1(\Omega),\\ u_n\to u & \text{in } L^\gamma(\Omega),\ 1<\gamma<2^*,\\ u_n\to u & \text{a.e. in } \Omega,\end{cases}$$

as $n\to\infty$. By (f_1), for any $\varepsilon>0$ there exists $a(\varepsilon)>0$ such that

$$|f(x,t)t|\le\frac{1}{2C_1}\varepsilon|t|^{2^*}+a(\varepsilon)\quad \text{for } (x,t)\in\overline{\Omega}\times(0,+\infty).$$

Set $\delta=\frac{\varepsilon}{2a(\varepsilon)}>0$. When $E\subset\Omega$, mes $E<\delta$, we get

$$\begin{aligned}\left|\int_E f(x,u_n)u_n\mathrm{d}x\right| &\le\int_E|f(x,u_n)u_n|\mathrm{d}x\\ &\le\int_E a(\varepsilon)\mathrm{d}x+\frac{1}{2C_1}\varepsilon\int_E|u_n|^{2^*}\mathrm{d}x\end{aligned}$$

$$\leq a(\varepsilon)\text{mes } E + \frac{1}{2C_1}\varepsilon C_1$$
$$< \varepsilon.$$

Hence $\left\{\int_\Omega f(x,u_n)u_n \mathrm{d}x, n \in N\right\}$ is equi-absolutely-continuous. It follows easily from Vitali Convergence Theorem that

$$\int_\Omega f(x,u_n)u_n \mathrm{d}x \to \int_\Omega f(x,u)u\mathrm{d}x, \tag{5}$$

as $n \to \infty$. Using the same method, we can prove that

$$\int_\Omega F(x,u_n)\mathrm{d}x \to \int_\Omega F(x,u)\mathrm{d}x, \tag{6}$$

as $n \to \infty$. Let $v_n = u_n - u$, since $I'(u_n) \to 0$ in $\left(H_0^1(\Omega)\right)^*$, we obtain

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega f(x,u_n)u_n\mathrm{d}x = o(1).$$

From the Brezis–Lieb Lemma in [3] and (5), we have

$$\|v_n\|^2 + \|u\|^2 - \int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega f(x,u)u\mathrm{d}x = o(1), \tag{7}$$

and

$$\lim_{n\to\infty}\langle I'(u_n), u\rangle = \|u\|^2 - \int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega f(x,u)u\mathrm{d}x = 0. \tag{8}$$

It follows from (8) that

$$I(u) = \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\mathrm{d}x + \frac{1}{2}\int_\Omega f(x,u)u\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x.$$

From (f_2), we conclude that

$$I(u) \geq 0. \tag{9}$$

Since $I(u_n) \to c(n \to \infty)$, together with the Brezis–Lieb Lemma and (6), we obtain

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x + o(1) \\ &= I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x + o(1) \\ &= c + o(1). \end{aligned}$$

Therefore, one gets that

$$I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x = c + o(1). \tag{10}$$

From (7) and (8), we have

$$\|v_n\|^2 - \int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x = o(1),$$

then $\|v_n\|^2 \to 0$ as $n \to \infty$. Otherwise, there exists a subsequence (still denoted by v_n) such that

$$\lim_{n\to\infty}\|v_n\|^2 = k, \qquad \lim_{n\to\infty}\int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x = k, \tag{11}$$

where k is a positive constant. By (2), we deduce that

$$\|v_n\|^2 \geq A_{\mu,s}\left(\int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\right)^{\frac{2}{2^*(s)}} \quad \text{for all } n \in N,$$

then $k \geq A_{\mu,s}k^{\frac{2}{2^*(s)}}$, i.e., $k \geq (A_{\mu,s})^{\frac{N-s}{2-s}}$, which, together with (10) and (11), shows that

$$I(u) = c - \frac{1}{2}k + \frac{1}{2^*(s)}k \leq c - \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}} < 0,$$

which contradicts (9). Therefore, we get

$$\|v_n\|^2 \to 0 \quad \text{as } n \to \infty.$$

This proves $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$.

From the discussion above, I satisfies $(\mathrm{PS})_c$ condition. □

From Lemma 2.2 in [11], we know that $A_{\mu,s}$ is attained when $\Omega = R^N$ by the functions

$$y_\varepsilon(x) = \frac{\left[\frac{2\varepsilon(N-s)(\overline{\mu}-\mu)}{\sqrt{\overline{\mu}}}\right]^{\frac{\sqrt{\overline{\mu}}}{2-s}}}{|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}\left(\varepsilon + |x|^{\frac{(2-s)\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}}}}\right)^{\frac{N-2}{2-s}}}$$

for all $\varepsilon > 0$. Moreover, the functions $y_\varepsilon(x)$ solve the equation

$$-\Delta u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u, \quad \text{in } R^N \setminus \{0\}.$$

Let

$$C_\varepsilon = \left(\frac{2\varepsilon(N-s)(\overline{\mu}-\mu)}{\sqrt{\overline{\mu}}}\right)^{\frac{N-2}{2(2-s)}}, \qquad U_\varepsilon(x) = \frac{y_\varepsilon(x)}{C_\varepsilon}.$$

Define a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leq R$, $\varphi(x) = 0$ for $|x| \geq 2R$, $0 \leq \varphi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, $v_\varepsilon(x) = u_\varepsilon(x)/(\int_\Omega |u_\varepsilon|^{2^*(s)}|x|^{-s}\mathrm{d}x)^{1/2^*(s)}$, so that $\int_\Omega |v_\varepsilon|^{2^*(s)}|x|^{-s}\mathrm{d}x = 1$. Then we can get the following results by the method used in [7]:

$$A_{\mu,s} + C_2\varepsilon^{\frac{N-2}{2-s}} \leq \|v_\varepsilon\|^2 \leq A_{\mu,s} + C_3\varepsilon^{\frac{N-2}{2-s}}, \tag{12}$$

and

$$\begin{cases} C_4\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q} \leq \int_\Omega |v_\varepsilon|^q\mathrm{d}x \leq C_5\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q}, & 1 \leq q < \dfrac{N}{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \\ C_4\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q}|\ln\varepsilon| \leq \int_\Omega |v_\varepsilon|^q\mathrm{d}x \leq C_5\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q}|\ln\varepsilon|, & q = \dfrac{N}{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \\ C_4\varepsilon^{\frac{\sqrt{\overline{\mu}}(N-q\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}} \leq \int_\Omega |v_\varepsilon|^q\mathrm{d}x \leq C_5\varepsilon^{\frac{\sqrt{\overline{\mu}}(N-q\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}}, & \dfrac{N}{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}} < q < 2^*. \end{cases} \tag{13}$$

Moreover, we can obtain

$$\int_\Omega |v_\varepsilon|^{2^*}\mathrm{d}x \leq C_6(2A_{\mu,s})^{\frac{N}{N-2}}, \quad \text{for } \varepsilon \to 0^+. \tag{14}$$

In fact, since $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and (12) holds, one can deduce

$$\int_\Omega |v_\varepsilon|^{2^*}\mathrm{d}x \leq C_7\left(\int_\Omega |\nabla v_\varepsilon|^2\mathrm{d}x\right)^{\frac{2^*}{2}}$$

$$= C_7 \left(A_{\mu,s} + C_8 \varepsilon^{\frac{N-2}{2-s}} \right)^{\frac{N}{N-2}}$$
$$\leq C_6 (2A_{\mu,s})^{\frac{N}{N-2}} \quad (\varepsilon \to 0^+).$$

Lemma 2. *Suppose that* $0 \leq \mu < \overline{\mu}$, $0 \leq s < 2$. *Assume that* (f_1), (f_2) *and* (3) *hold. Then there exists* $u_0 \in H_0^1(\Omega)$, $u_0 \not\equiv 0$, *such that*

$$\sup_{t \geq 0} I(tu_0) < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}.$$

Proof. We consider the functions

$$g(t) = I(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega F(x, tv_\varepsilon)\mathrm{d}x,$$
$$\widetilde{g}(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)}.$$

Note that $\lim_{t \to +\infty} g(t) = -\infty$, $g(0) = 0$, $g(t) > 0$ for $t \to 0^+$, so $\sup_{t \geq 0} g(t)$ is attained for some $t_\varepsilon > 0$. Since

$$0 = g'(t_\varepsilon) = t_\varepsilon \left(\|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-2} - \frac{1}{t_\varepsilon} \int_\Omega f(x, t_\varepsilon v_\varepsilon) v_\varepsilon \mathrm{d}x \right),$$

we have

$$\|v_\varepsilon\|^2 = t_\varepsilon^{2^*(s)-2} + \frac{1}{t_\varepsilon} \int_\Omega f(x, t_\varepsilon v_\varepsilon) v_\varepsilon \mathrm{d}x \geq t_\varepsilon^{2^*(s)-2}.$$

Therefore, one gets

$$t_\varepsilon \leq \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}} \triangleq t_\varepsilon^0.$$

By (f_1), it is easy to verify that

$$|f(x,t)| \leq \varepsilon t^{2^*-1} + d(\varepsilon)t, \quad d(\varepsilon) > 0.$$

Hence, we obtain

$$\|v_\varepsilon\|^2 \leq t_\varepsilon^{2^*(s)-2} + \varepsilon \int_\Omega |t_\varepsilon|^{2^*-2} |v_\varepsilon|^{2^*} \mathrm{d}x + d(\varepsilon) \int_\Omega |v_\varepsilon|^2 \mathrm{d}x.$$

By (12)–(14), when ε is small enough, we conclude that

$$t_\varepsilon^{2^*(s)-2} \geq \frac{A_{\mu,s}}{2}. \tag{15}$$

On the one hand, from (12) we claim that

$$\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} \leq A_{\mu,s}^{\frac{N-s}{2-s}} + C_9 \varepsilon^{\frac{N-2}{2-s}}. \tag{16}$$

In order to prove this, we first prove the following inequality:

$$(a+b)^\lambda \leq a^\lambda + \lambda(a+1)^{\lambda-1} b, \quad a > 0,\ 0 \leq b \leq 1,\ \lambda \geq 1. \tag{17}$$

In fact, set

$$h(x) = (a+x)^\lambda - a^\lambda - \lambda(a+1)^{\lambda-1} x, \quad a > 0,\ 0 \leq x \leq 1,\ \lambda \geq 1.$$

Clearly, $h'(x) < 0$, $x \in (0,1)$, so $h(b) \leq h(0) = 0$, then (17) holds. Let $a = A_{\mu,s}$, $b = C_3 \varepsilon^{\frac{N-2}{2-s}}$, $\lambda = \frac{N-s}{2-s}$, then (16) holds.

On the other hand, the function $\widetilde{g}(t)$ attains its maximum at t_ε^0 and is increasing in the interval $[0, t_\varepsilon^0]$, together with (12), (15) and (16) and $F(x,t) \ge C_{10}|t|^\rho$ which is directly got from (f_2), we deduce that

$$\begin{aligned} g(t_\varepsilon) &\le \widetilde{g}(t_\varepsilon^0) - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x \\ &= \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x \\ &\le \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + C_{11}\varepsilon^{\frac{N-2}{2-s}} - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x \\ &\le \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + C_{11}\varepsilon^{\frac{N-2}{2-s}} - C_{10}\int_\Omega t_\varepsilon^\rho |v_\varepsilon|^\rho \mathrm{d}x \\ &\le \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + C_{11}\varepsilon^{\frac{N-2}{2-s}} - C_{10}\left(\frac{A_{\mu,s}}{2}\right)^{\frac{\rho}{2^*(s)-2}} \int_\Omega |v_\varepsilon|^\rho \mathrm{d}x, \end{aligned}$$

where $C_{11} = C_9 \frac{2-s}{2(N-s)}$. Furthermore, from (13), we get

$$\int_\Omega |v_\varepsilon|^\rho \mathrm{d}x \ge C_4 \varepsilon^{\frac{\sqrt{\bar\mu}(N-\rho\sqrt{\bar\mu})}{(2-s)\sqrt{\bar\mu-\mu}}}.$$

By (3), we obtain that

$$\frac{N-2}{2-s} > \frac{\sqrt{\bar\mu}(N-\rho\sqrt{\bar\mu})}{(2-s)\sqrt{\bar\mu-\mu}}.$$

Choosing ε small enough, we have

$$\sup_{t\ge 0} I(tv_\varepsilon) = g(t_\varepsilon) < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}. \quad \square$$

Proof of Theorem 1. Let $X = H_0^1(\Omega)$. From the Hardy and Hardy–Sobolev inequalities, we can easily get:

$$\|u\|_{L^2}^2 \le C\|u\|^2; \qquad \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\mathrm{d}x \le C\|u\|^{2^*(s)}; \qquad \|u\|_{2^*}^{2^*} \le C\|u\|^{2^*} \quad \text{for } \forall u \in X. \tag{18}$$

It follows from (f_1) that

$$\begin{aligned} &\exists \delta_1 > 0 \quad \text{such that } |f(x,t)| < t^{2^*-1} \quad \text{for } t > \delta_1; \\ &\forall \varepsilon > 0,\ \exists 0 < \delta_2 < \delta_1, \quad \text{such that } |f(x,t)| < \varepsilon t \quad \text{for } 0 < t < \delta_2; \\ &\exists M_1 > 0, \qquad |f(x,t)| \le M_1 \quad \text{for all } t \in [\delta_2, \delta_1] \end{aligned}$$

for all $x \in \overline{\Omega}$. Therefore, we deduce that

$$|f(x,t)| \le \varepsilon t + t^{2^*-1} + M_1 \le \varepsilon t + (1 + M_1\delta_2^{1-2^*})t^{2^*-1}$$

for all $t \in R^+$ and for $x \in \overline{\Omega}$. Then one gets

$$|F(x,t)| \le \frac{1}{2}\varepsilon|t|^2 + C_{12}|t|^{2^*} \tag{19}$$

for all $t \in R$ and for $x \in \overline{\Omega}$, where $C_{12} = \frac{1}{2^*}(1 + M_1\delta_2^{1-2^*})$. By (18) and (19) we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|^2 - \frac{C}{2^*(s)}\|u^+\|^{2^*(s)} - \frac{\varepsilon}{2}\|u\|_{L^2}^2 - C_{12}\|u\|_{2^*}^{2^*} \\ &\ge \frac{1}{2}\|u\|^2 - \frac{C}{2^*(s)}\|u^+\|^{2^*(s)} - \frac{C\varepsilon}{2}\|u\|^2 - CC_{12}\|u\|^{2^*} \end{aligned}$$

for ε small enough. So there exists $\beta > 0$ such that $I(u) \geq \beta$ for all $u \in \partial B_r = \{u \in H_0^1(\Omega), \|u\| = r\}$, where $r > 0$ small enough. By Lemma 2 there exists $u_0 \in H_0^1(\Omega)$, $u_0 \not\equiv 0$, such that

$$\sup_{t\geq 0} I(tu_0) < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}.$$

It follows from the nonnegativity of $F(x,t)$ that

$$\begin{aligned} I(tu_0) &= \frac{1}{2}t^2\|u_0\|^2 - \frac{1}{2^*(s)}t^{2^*(s)}\int_\Omega \frac{(u_0^+)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega F(x,tu_0)\mathrm{d}x \\ &\leq \frac{1}{2}t^2\|u_0\|^2 - \frac{1}{2^*(s)}t^{2^*(s)}\int_\Omega \frac{(u_0^+)^{2^*(s)}}{|x|^s}\mathrm{d}x, \end{aligned}$$

$\lim_{t\to+\infty} I(tu_0) \to -\infty$. Hence we can choose $t_0 > 0$ such that $\|t_0u_0\| > r$ and $I(t_0u_0) \leq 0$. Applying the Mountain Pass Lemma in [13], there is a sequence $u_n \subset X$ satisfying

$$I(u_n) \to c \geq \beta \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$c = \inf_{h\in\tau} \max_{t\in[0,1]} I(h(t)),$$
$$\tau = \{h \in ([0,1], X) | h(0) = 0, h(1) = t_0u_0\}.$$

Note that

$$0 < \beta \leq c = \inf_{h\in\tau} \max_{t\in[0,1]} I(h(t)) \leq \max_{t\in[0,1]} I(tt_0u_0) \leq \sup_{t\geq 0} I(tu_0) < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}.$$

Now Lemma 1 suggests $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$. Assume that $\{u_n\}$ converges to $u \subset H_0^1(\Omega)$. From the continuity of I' we know that u is a weak solution of problem (4). Then $\langle I'(u), u^-\rangle = 0$ where $u^- = \min\{u, 0\}$; thus $u \geq 0$. Moreover, we can get that u is a nonnegative solution of (1). By the Strong Maximum Principle, we get that u is a positive solution of problem (1), so Theorem 1 holds. □

Proof of Theorem 2. By Theorem 1 problem (1) has a positive solution u_1. Set $g(x,t) = -f(x,-t)$ for $t \in R$. It follows from Theorem 1 that the equation

$$-\triangle u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u + g(x,u)$$

has at least a positive solution v. Let $u_2 = -v$, then u_2 is a solution of

$$-\triangle u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u + f(x,u).$$

It is obvious that $u_1 \neq 0$, $u_2 \neq 0$ and $u_1 \neq u_2$. So equation (1) has at least two nontrivial solutions. Therefore, Theorem 2 holds. □

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