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Note

The notion and basic properties of M-transversals

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Abstract

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Let I be a finite index set and let $\mathscr{A}$ denote the family $(A_i: i \in I)$ of finite subsets of S. Let M be a matroid without loops on I. A family $(x_i: i \in I)$ of elements of S is an M-system of representatives of $\mathscr{A}$ if $x_i \in A_i$, for any $i \in I$, and the set $\{i \in I: x_i = s\}$ is independent in M, for any $s \in S$. Let $(x_i: i \in I)$ be an M-system of representatives of $\mathscr{A}$; then the set $X = \{x_i: i \in I\}$ (i.e., the set of distinct elements of the system $(x_i: i \in I)$) is called the M-transversal of $\mathscr{A}$. (If U_k is the k-uniform matroid of rank k, then the U_k-transversal is usually described as k-transversal, or as system of representatives with repetition.) The aim of this note is to prove an M-transversal version of Rado's and Perfect's Theorem and to give a short proof of a similar result known for k-transversals.

1. Introduction

Throughout this paper I denotes a finite index set and $\mathscr{A}$ denotes the family $(A_i: i \in I)$ of subsets of a finite set S. Let M be a matroid without loops on I.

We expect the reader to be familiar with matroid theory. All terminology and notation related to matroids are essentially the same as that of Welsh [9].

We say that a family $(x_i: i \in I)$ of elements of S is a *system of representatives* (SR) of $\mathscr{A}$ if $x_i \in A_i$, for any $i \in I$. If, in addition, $x_i \neq x_j$, for any $i \neq j$, then $(x_i: i \in I)$ is called the *system of distinct representatives* (SDR) of $\mathscr{A}$.

The subset X of S is called a *transversal* of $\mathscr{A}$ if there exists an SDR $(x_i: i \in I)$ of $\mathscr{A}$ such that $X = \{x_i: i \in I\}$.

A system of representatives $(x_i: i \in I)$ of $\mathscr{A}$ we call an *M-system of representatives* (M-SR) of $\mathscr{A}$ if the set $\{i \in I: x_i = s\}$ is independent in M, for any $s \in S$.

A subset X of S is called an *M-transversal* of $\mathscr{A}$ if there exists an M-system of representatives $(x_i: i \in I)$ of $\mathscr{A}$ such that $X = \{x_i: i \in I\}$ (i.e., X is the set of the distinct elements of the family $(x_i: i \in I)$).

If $J \subseteq I$, then denote by $\mathscr{A}_J$ the subfamily $(A_i: i \in J)$ of $\mathscr{A}$.

Let $J \subseteq I$, and let $X \subseteq S$ be a transversal of $\mathscr{A}_J$. Then X is called a *partial transversal* of $\mathscr{A}$. Furthermore, $|J|$ and $|I - J|$ are called the *length* and *defect* of the partial transversal X, respectively.

Similarly, let $J \subseteq I$, and let $X \subseteq S$ be an M-transversal of $\mathscr{A}_J$. Then X is called a *partial M-transversal* of $\mathscr{A}$. Suppose K to be the maximal set (with respect to the cardinality) such that X is an M-transversal of $\mathscr{A}_K$. Then $|K|$ and $|I - K|$ are called the *length* and *defect* of the partial M-transversal X, respectively.

Let U_k be the uniform matroid of rank k (i.e., $J \subseteq I$ is independent in U_k iff $|J| \leq k$). Then an SR $(y_i: i \in I)$ of $\mathscr{A}$ is a U_k-SR of $\mathscr{A}$ iff $|\{i \in I: y_i = s\}| \leq k$, for any $s \in S$. Let us note, that U_k-transversals were usually denoted as k-transversals (see [7, 9]), or were described as systems of representatives with repetition (see [1, 3]). Furthermore, U_1-transversal is a transversal in the usual sense.

We shall, for brevity, write $A(J) = \bigcup \{A_i: i \in J\}$ and use an analogous notation for families denoted by other letters.

Rado's Theorem [5, 6] is perhaps the most fundamental result in transversal theory.

Theorem 1. *Let M_2 be a matroid, with rank function r_2, on the set S. The finite family of subsets $\mathscr{A} = (A_i: i \in I)$ of S has a transversal which is independent in the matroid M_2 if and only if $\mathscr{A}$ satisfies the following condition: for any $J \subseteq I$,*

$$r_2(A(J)) \geq |J|.$$

2. Main results

We shall need some additional notations. Set $S' = S \times I$. Let $X \subseteq S'$. Set

$$X/s = \{i \in I: (s, i) \in X\}, \quad \text{for any } s \in S,$$

$$X/i = \{s \in S: (s, i) \in X\}, \quad \text{for any } i \in I,$$

$$X/I = \bigcup_{i \in I} X/i \ (\subseteq S).$$

Next, for the matroid M, with rank function r, on the set I, we denote by M' the matroid on S', with rank function r', such that

$$r'(X) = \sum_{s \in S} r(X/s), \tag{1}$$

for any $X \subseteq S'$. (M' is in fact a direct sum of matroids M'_s on $\{s\} \times I$ such that M'_s is induced from M by bijection $i \mapsto (s, i)$, $i \in I$.)

Next, considering again a family $\mathscr{A} = (A_i : i \in I)$ of subsets of S, let us denote the elements of A_i, for $i \in I$, by $a_{i,1}, \ldots, a_{i,n_i}$. Then set, for any $i \in I$, $A'_i = \{(a_{i,1}, i), \ldots, (a_{i,n_i}, i)\} \subseteq S'$. Thus $\mathscr{A}$ determines exactly one family $\mathscr{A}' = (A'_i : i \in I)$ of subsets of S'.

Let M_2 be a matroid, with rank function r_2, on the set S. The matroid M'', with rank function r'', on the set S' is defined by the requirement that, for any $X \subseteq S'$,

$$r''(X) = r_2(X/I). \tag{2}$$

(The set $\{(x_1, i_1), \ldots, (x_n, i_n)\}$ is independent in M'' iff the set $\{x_1, \ldots, x_n\}$ is an n-element set, independent in M_2, and $i_1, \ldots, i_n$ are arbitrary, not necessarily distinct elements of I.)

The following lemma is easy to check.

Lemma 1. *Let $\mathscr{A}$, $\mathscr{A}'$, M, M', M_2 and M'' have the same meaning as above. Then the following conditions are equivalent:*

(a) *There exists $X \subseteq S$ such that $r_2(X) \geq t$ and X is a partial M-transversal of $\mathscr{A}$ with defect d.*

(b) *There exists $X' \subseteq S'$ such that X' is independent in M', $r''(X') \geq t$, and X' is a partial transversal of $\mathscr{A}'$ with defect d.*

We shall also need another auxiliary lemma.

Lemma 2. *Let X, $Y \subseteq S'$ and r', r'' have the same meaning as above. Then*

$$r'(X) + r''(Y) \geq r'(X \cap Y) + r''(X \cup Y).$$

Proof. Let X, $Y \subseteq S'$. Suppose $u \in X/I - Y/I$. Then $X/u \neq \emptyset$ and $Y/u = \emptyset$. Since M is a matroid without loops, then

$$r(X/u) \geq 1 \quad \text{and} \quad r(X/u \cap Y/u) = 0.$$

Suppose $u \notin X/I - Y/I$. Then $r(X/u) \geq r(X/u \cap Y/u)$. Hence

$$\begin{aligned}\sum_{s \in S} r(X/s) &= \sum_{s \in S} r(X/s \cap Y/s) + |X/I - Y/I| \\ &\geq \sum_{s \in S} r(X/s \cap Y/s) + r_2(X/I \cup Y/I) - r_2(Y/I).\end{aligned}$$

Thus $r'(X) + r''(Y) \geq r'(X \cap Y) + r''(X \cup Y)$.

If $\mathscr{A} = (A_i : i \in I)$ is a family of sets, we write for any $s \in S$, $J \subseteq I$,

$$A(s, J) = \{i \in J : s \in A_i\}.$$

(Let us note that $A(s, J) \subseteq I$.) Then, for any $s \in S$, $J \subseteq I$,

$$A'(J)/s = A(s, J). \tag{3}$$

We now prove a general theorem which is a slight modification of Welsh's Theorem [7, 9].

Theorem 2. *Let $\{f_q : q \in Q\}$ be a set of submodular and nondecreasing functions from 2^I to $\mathbb{Z}^+$, i.e., for any $q \in Q$; and any $X, Y \subseteq S$, f_q satisfies the following conditions:*

if $X \subseteq Y \subseteq S$ then $f_q(X) \leqslant f_q(Y)$, and

$f_q(X) + f_q(Y) \geqslant f_q(X \cup Y) + f_q(X \cap Y)$.

Moreover, let them be mutually submodular, i.e., for any q, $p \in Q$, and any $X, Y \subseteq S$,

$$f_q(X) + f_q(Y) \geqslant \min\{f_q(X \cup Y) + f_p(X \cap Y), f_p(X \cup Y) + f_q(X \cap Y)\}$$

holds. Let $\mathcal{A}$ denote a finite family $(A_i : i \in I)$ of subsets of S. Then $\mathcal{A}$ has a system of representatives $(x_i : i \in I)$ such that

$$f_q\{x_i : i \in J\} \geqslant |J|, \quad \text{for any } q \in Q, J \subseteq I,$$

if and only if

$$f_q A(J) \geqslant |J|, \quad \text{for any } q \in Q, J \subseteq I.$$

Proof. It is easy to see that

$$f(X) = \min_{q \in Q} \{f_q(X)\}$$

is a submodular and nondescending function from 2^I to $\mathbb{Z}^+$. Hence, it is enough to prove this theorem in the case $|Q| = 1$. But this has been done in Welsh [7, 9], concluding the proof. □

In general, the collection of k-transversals of a family $\mathcal{A}$ of subsets of S does not form the basis of a matroid (see [9]). Thus also the collection of M-transversals does not. However, we generalize the Rado's and Perfect's Theorems to get the following.

Theorem 3. *Let $\mathcal{A} = (A_i : i \in I)$ be a family of subsets of S, and let M be a matroid without loops, with rank function r, on the set I. Let M_2 be a matroid, with rank function r_2, on the set S. Let d be a nonnegative integer, $d \leqslant |I|$. Then $\mathcal{A}$ has partial M-transversal X with defect d and such that $r_2(X) \geqslant t$ if and only if, for all $J \subseteq I$,*

$$\sum_{s \in S} rA(s, J) \geqslant |J| - d,$$

$$r_2(A(J)) \geqslant |J| - |I| + t.$$

Proof. Take $Q = \{1, 2\}$, $f_1 = r' + d$, $f_2 = r'' + |I| - t$ in Theorem 2. f_1, f_2 are nondecreasing, submodular and, by Lemma 2, mutually submodular. By Lemma

1 and Theorem 2, $\mathscr{A}$ has the required partial M-transversal if and only if, for all $J \subseteq I$, $r'(A'(J)) + d \geq |J|$, and $r''(A'(J)) + |I| - t \geq |J|$ hold. But from (1), (2), (3) follows:

$$r'(A'(J)) = \sum_{s\in S} r(A'(J)/s) = \sum_{s\in S} rA(s, J),$$

and

$$r''(A'(J)) = r_2(A'(J)/I) = r_2(A(J)),$$

concluding the proof. □

As an application of this theorem we can give a shorter proof of the following result (see Welsh [8, 9]).

Corollary 1. *Let $\mathscr{A} = (A_i: i \in I)$ be a family of subsets of S, $k \geq 1$, and M_2 be a matroid, with rank function r_2, on the set S. Then $\mathscr{A}$ has a k-transversal (i.e. U_k-transversal) with rank not less than t if and only if, for any $J \subseteq I$,*

$$k\ |A(J)| \geq |J|, \tag{4}$$

$$r_2(A(J)) \geq |J| - |I| + t. \tag{5}$$

Proof. The necessity of (4) and (5) is obvious. Conversely, let (4) and (5) be true for any $J \subseteq I$. We show that the conditions of Theorem 2 are satisfied replacing d by 0 and $r(X)$ by $\min\{|X|, k\}$, the rank of U_k. For if not, take J the smallest subset of I such that

$$\sum_{s\in S} \min\{|A(s, J)|, k\} = \sum_{s\in A(J)} \min\{|A(s, J)|, k\} < |J|. \tag{6}$$

From minimality of J, it follows that, for any $j \in J$,

$$\sum_{s\in A(J)} \min\{|A(s, J - \{j\})|, k\} \geq |J| - 1.$$

If there exist $j \in J$ and $s \in A_j$ such that

$$\min\{|A(s, J)|, k\} > \min\{|A(s, J - \{j\})|, k\}$$

then (6) fails to hold. Hence, for any $j \in J$, and any $s \in A_j$, $\min\{|A(s, J - \{j\})|, k\} = k$, and, thus, for any $s \in A(J)$, $\min\{|A(s, J)|, k\} = k$. Then

$$\sum_{s\in A(J)} \min\{|A(s, J)|, k\} = k\ |A(J)|.$$

Comparing this with (4) and (6) we get a contradiction. Thus $\mathscr{A}$ satisfies the conditions of Theorem 2 and $\mathscr{A}$ has a k-transversal with rank not less than t, completing the proof. □

3. $\mathcal{M}$-transversals

It is of some interest to note, that we can extend our results in the following way. Let $\mathcal{M}$ denote the family $(M_s: s \in S)$ of matroids without loops on the set I. The system of representatives $(x_i: i \in I)$ of $\mathcal{A}$ we call *$\mathcal{M}$-system of representatives* ($\mathcal{M}$-SR) of $\mathcal{A}$ if, for any $s \in S$, $\{i \in I: x_i = s\}$ is independent in M_s.

The subset X of S we will call an *$\mathcal{M}$-transversal* of $\mathcal{A}$ if there exists an $\mathcal{M}$-system of representatives $(x_i: i \in I)$ of $\mathcal{A}$ such that $X = \{x_i: i \in I\}$. Partial $\mathcal{M}$-transversals of $\mathcal{A}$, their length and defect, can be defined similarly as for M-transversals.

For any $s \in S$, let r_s denote the rank function of the matroid M_s. Let us denote by $M'_{\mathcal{M}}$ the matroid on S' with rank function $r'_{\mathcal{M}}$ such that, for any $X \subseteq S'$,

$$r'_{\mathcal{M}}(X) = \sum_{s \in S} r_s(X/s),$$

hold. The following theorem can be proved in the same way as Theorem 3.

Theorem 4. *Let $\mathcal{A} = (A_i: i \in I)$ be a family of subsets of S and let $\mathcal{M}$ be a family $(M_s: s \in S)$ of matroids without loops on I. For any $s \in S$, let r_s be the rank function of M_s. Let M_2 be a matroid, with rank function r_2, on the set S. Then $\mathcal{A}$ has a partial $\mathcal{M}$-transversal X with defect d such that $r_2(X) \geqslant t$ if and only if, for all $J \subseteq I$,*

$$\sum_{s \in S} r_s A(s, J) \geqslant |J| - d,$$

$$r_2 A(J) \geqslant |J| - |I| + t.$$

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