MATHEMATICS

ON THE EXISTENCE OF ENTIRE FUNCTIONS MAPPING COUNTABLE DENSE SETS ONTO EACH OTHER

BY

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Between two countable everywhere dense subsets of $\mathbb{R}$, there exists an order preserving map ([3], 2.2. problem 1). The extension to $\mathbb{R}$ of such a map is almost everywhere differentiable. Do there, in general, also exist nicer maps? The answer is yes.

This has been known since 1925 [4]. However, the existing literature is not easy to read, nor very general. Franklin [4] finds a function which is analytic on $\mathbb{R}$, but not entire, due to the fact that his prime interest is not $\mathbb{R}$ but [0, 1]. In [1] the standard Bagemihl-Seidel argument together with Mergelyan's theorem is used. In [6], both dense sets equal $\mathbb{Q}$ and the field structure of $\mathbb{Q}$ is explicitly used.

We feel that the theorem presented below and its proof show that the solution of the problem is basically quite simple.

Let EM be the space of entire functions on $\mathbb{C}$, whose restriction to $\mathbb{R}$ is a real monotonically nondecreasing function.

THEOREM. Let S and T be countable everywhere dense subsets of $\mathbb{R}$, let p be a continuous positive real function such that $\lim_{t\to\infty} t^{-n}p(t)=\infty$ for all $n \in \mathbb{N}$ and let $f_0 \in EM$.

Then there exists a function $f \in EM$ such that

i) f is strictly increasing on $\mathbb{R}$ and $f(S)=T$

ii) $|f(z)-f_0(z)|<p(|z|)$ for all $z \in \mathbb{C}$.

PROOF. The reader may find it convenient to think of the following construction as an effort to construct the graph of f, by starting with f_0 and wiggling it in such a way that the whole of S is mapped onto T, thereby using progressively smaller wiggles, that do not disturb points of the graph constructed earlier.

Let S and T have been enumerated and let $x_1 \in S$, $x_1 \neq 0$ and $\delta>0$ be such that

$$f_0(x_1)+\delta x_1 \in T \text{ and } |\delta z|<\tfrac{1}{2}p(|z|) \text{ for all } z \in \mathbb{C}.$$

We define

$$f_1(z) := f_0(z) + \delta z, \ S_1 := \{x_1\} \text{ and } T_1 := \{f_1(x_1)\}.$$

We now construct, starting with f_1, S_1 and T_1, a sequence $\{f_n\}$ in EM and sequences $\{S_n\}$ and $\{T_n\}$ of finite subsets of S and T respectively such that

$$f_n'(x) \geqslant (2^{-1} + 2^{-n})\delta \text{ and } f_n(S_n) = T_n \text{ for all } n \in \mathbf{N}, \ x \in \mathbf{R}.$$

Suppose that f_n, S_n and T_n have been constructed and choose a polynomial g with real coefficients such that

1) $$g(z) = 0 \Leftrightarrow z \in S_n \ (z \in \mathbf{C})$$

2) $$|g(z)| < 2^{-n-1} p(|z|) \ (z \in \mathbf{C})$$

3) $$g'(x) \geqslant -2^{-n-1}\delta \ (x \in \mathbf{R}).$$

(Any polynomial of odd degree with positive leading coefficient for which 1) is valid will also obey 2) and 3) after it has been multiplied by a small enough positive constant. The degree can be chosen odd, by adjusting the multiplicity of one of the zeros.)

For each $M \in [0, 1]$ we have

$$(f_n + Mg)'(x) = f_n'(x) + Mg'(x) \geqslant (2^{-1} + 2^{-n-1})\delta \ (x \in \mathbf{R}),$$

so $f_n + Mg$ is strictly monotonic on $\mathbf{R}$.

Moreover if we let M vary in $[0, 1]$, then for $x \notin S_n$, $(f_n + Mg)(x)$ varies in an interval of $\mathbf{R}$ that contains points of $T \backslash T_n$ and for $y \notin T_n$, $(f_n + Mg)^{-1}(y)$ varies in an interval of $\mathbf{R}$ that contains points of $S \backslash S_n$.

Now for n odd, let x be the point of $S \backslash S_n$ with smallest index and let $M \in [0, 1]$ be such that $(f_n + Mg)(x) \in T$. We define

$$f_{n+1} := f_n + Mg, \ S_{n+1} := S_n \cup \{x\} \text{ and } T_n \cup \{f_{n+1}(x)\}.$$

For n even, let y be the point of $T \backslash T_n$ with smallest index and let $M \in [0, 1]$ be such that $(f_n + Mg)^{-1}(y) \in S$. We define

$$f_{n+1} := f_n + Mg, \ T_{n+1} := T_n \cup \{y\} \text{ and } S_{n+1} := S_n \cup \{f_{n+1}^{-1}(y)\}.$$

The following properties of the constructed sequences are easily verified:

a) $$|f_n(z) - f_{n-1}(z)| < 2^{-n} p(|z|) \ (n \in \mathbf{N}, \ z \in \mathbf{C})$$

b) $$\bigcup_{n=1}^{\infty} S_n = S, \ \bigcup_{n=1}^{\infty} T_n = T \text{ and } f_m(S_n) = T_n \ (m, n \in \mathbf{N}, \ m \geqslant n).$$

From a) it follows that $\{f_n\}$ converges pointwise to a function f for which

$$|f(z)-f_0(z)| < p(|z|) \quad (z \in \mathbb{C}).$$

Now $p(|z|)$ is a function of z which is bounded on compact subsets of $\mathbb{C}$, so the convergence of $\{f_n\}$ is uniform on such sets. From this we conclude that f is an entire function.

For each $n \in \mathbb{N}$ we have $f_n'(x) \geqslant \frac{1}{2}\delta$ $(x \in \mathbb{R})$, so the same is true for f. Hence f is strictly increasing on $\mathbb{R}$. From b) it follows that $f(S_n) = T_n$ for each n and so $f(S) = T$.

Remark 1. By similar procedures one may construct a real analytic homemorphism mapping one dense countable subset of $\mathbb{R}^n$ onto another one. This latter may be done by starting with the identity map for f_0 and taking each time for $g(\mathbf{x})$ the product of a polynomial vanishing precisely on S_n and $\exp(-|\mathbf{x}|^2)$; instead of a constant $M \in [0, 1]$, $\mathbf{M}$ is now a vector chosen from an ε-ball around $\mathbf{o}$, ε so small that at every stage $|f_n(\mathbf{x}) - f_n(\mathbf{y})| > \frac{1}{2}|\mathbf{x}-\mathbf{y}|$ is ensured.

This result is mainly negative: countable dense subsets cannot be distinguished topologically, and also not diffeomorphically nor by any other feasible class of homeomorphisms.

Remark 2. If we start with $f_0 \equiv 0$, each f_n is polynomial. If moreover, for each n, we give the zeros of $f_{n-1} - f_n$ multiplicities higher than the degree of f_n, then the Taylor expansion of f around any point of S has all its coefficients in the vectorspace over $F(S)$ generated by T, where $F(S)$ is the field generated by S.

Remark 3. Generalization to dense subsets T and S of $\mathbb{C}$ is not so elegant: the class of entire functions is not a feasible class of homeomorphisms, mainly because it is not a class of homeomorphisms.

One still can obtain $f(S) = T$ and $f|_S$ injective, if one starts with $f_0 = 0$. Indeed, all f_n are then polynomials hence surjections from $\mathbb{C}$ onto $\mathbb{C}$, which makes the "even" step possible. If one starts with an arbitrary entire function f_0 the existence of exceptional values may complicate the even step.

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