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Some remarks on sign-balanced and maj-balanced posets

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1. Introduction

Let P be an n-element poset (partially ordered set), and let $\omega : P \to [n] = \{1, 2, \ldots, n\}$ be a bijection, called a *labeling* of P. We call the pair (P, ω) a *labeled poset*. A *linear extension* of P is an order-preserving bijection $f : P \to [n]$. We can regard f as defining a permutation $\pi = \pi(f)$ of the set $[n]$ given by $\pi(i) = j$ if $f(\omega^{-1}(j)) = i$. We write π in the customary way as a word $a_1 a_2 \cdots a_n$, where $\pi(i) = a_i = \omega(f^{-1}(i))$. We will say for instance that f is an *even linear extension* of (P, ω) if π is an even permutation (i.e., an element of the alternating group $\mathfrak{A}_n$). Let $\mathcal{E}_P$ denote the set of linear extensions of P, and set $\mathcal{L}_{P,\omega} = \{\pi(f)\colon f \in \mathcal{E}_P\}$.

We say that (P, ω) is *sign-balanced* if $\mathcal{L}_{P,\omega}$ contains the same number of even permutations as odd permutations. Note that the parity of a linear extension f depends on the labeling ω. However, the notion of sign-balanced depends only on P, since changing the labeling of P simply multiplies the elements of $\mathcal{L}_{P,\omega}$ by a fixed permutation in $\mathfrak{S}_n$, the symmetric group of all permutations of $[n]$. Thus we can simply say that P is sign-balanced without specifying ω.

We say that a function $\vartheta : \mathcal{E}_P \to \mathcal{E}_P$ is *parity-reversing* (respectively, *parity-preserving*) if for all $f \in \mathcal{E}_P$, the permutations $\pi(f)$ and $\pi(\vartheta(f))$ have opposite parity (respectively, the same parity). Note that the properties of parity-reversing and parity-preserving do not depend on ω; indeed, ϑ is parity-reversing (respectively, parity-preserving) if and only if for all $f \in \mathcal{E}_P$, the permutation $\vartheta f \circ f^{-1} \in \mathfrak{S}_n$ is odd (respectively, even).

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Sign-balanced posets were first considered by Ruskey [20]. He established the following result, which shows that many combinatorially occurring classes of posets, such as geometric lattices and Eulerian posets, are sign-balanced.

Theorem 1.1. *Suppose $\#P \geqslant 2$. If every nonminimal element of the poset P is greater than at least two minimal elements, then P is sign-balanced.*

Proof. Let $\pi = a_1a_2a_3\cdots a_n \in \mathcal{L}_{P,\omega}$. Let $\pi' = \pi(1,2) = a_2a_1a_3\cdots a_n \in \mathfrak{S}_n$. (We always multiply permutations from right to left.) By the hypothesis on P, we also have $\pi' \in \mathcal{L}_{P,\omega}$. The map $\pi \mapsto \pi'$ is a parity-reversing involution (i.e., exactly one of π and π' is an even permutation) on $\mathcal{L}_{P,\omega}$, and the proof follows. $\square$

The above proof illustrates what will be our basic technique for showing that a poset P is sign-balanced, viz., giving a bijection $\sigma : \mathcal{L}_{P,\omega} \to \mathcal{L}_{P,\omega}$ such that π and $\sigma(\pi)$ have opposite parity for all $\pi \in \mathcal{L}_{P,\omega}$. Equivalently, we are giving a parity-reversing bijection $\vartheta : \mathcal{E}_P \to \mathcal{E}_P$.

In 1992 Ruskey [21, Section 5, item 6] conjectured as to when the product $\boldsymbol{m} \times \boldsymbol{n}$ of two chains of cardinalities m and n is sign-balanced, viz., $m, n > 1$ and $m \equiv n \pmod 2$. Ruskey proved this when m and n are both even by giving a simple parity-reversing involution, which we generalize in Proposition 4.1 and Corollary 4.2. Ruskey's conjecture for m and n odd was proved by D. White [32], who also computed the "imbalance" between even and odd linear extensions in the case when exactly one of m and n is even (stated here as Theorem 3.5). None of our theorems below apply to the case when m and n are both odd. Ruskey [21, Section 5, item 5] also asked what order ideals I (defined below) of $\boldsymbol{m} \times \boldsymbol{n}$ are sign-balanced. Such order ideals correspond to integer partitions λ and will be denoted P_λ; the linear extensions of P_λ are equivalent to standard Young tableaux (SYT) of shape λ. White [32] also determined some additional λ for which P_λ is sign-balanced, and our results below will give some further examples. In Sections 5 and 6 we consider some analogous questions for the parity of the major index of a linear extension of a poset P.

Given $\pi = a_1a_2\cdots a_n \in \mathcal{L}_{P,\omega}$, let $\mathrm{inv}(f)$ denote the number of *inversions* of π, i.e.,

$$\mathrm{inv}(\pi) = \#\{(i,j)\colon\ i < j,\ a_i > a_j\}.$$

Let

$$I_{P,\omega}(q) = \sum_{\pi \in \mathcal{L}_{P,\omega}} q^{\mathrm{inv}(f)}, \tag{1}$$

the generating function for linear extensions of (P,ω) by number of inversions. Since f is an even linear extension if and only if $\mathrm{inv}(f)$ is an even integer, we see that P is sign-balanced if and only if $I_{P,\omega}(-1) = 0$. In general $I_{P,\omega}(q)$ seems difficult to understand, even when P is known to be sign-balanced.

2. Promotion and evacuation

Promotion and evacuation are certain bijections on the set $\mathcal{E}_P$ of linear extensions of a finite poset P. They were originally defined by M.-P. Schützenberger [22] and have subsequently arisen is many different situations (e.g., [6, Section 5], [10, Section 8], [11, Section 4], [16, Section 3]). To be precise, the original definitions of promotion and evacuation require an insignificant reindexing to become bijections. We will incorporate this reindexing into our definition. Let $f : P \to [n]$ be a linear extension of the poset P. Define a maximal chain $u_0 < u_1 < \cdots < u_\ell$ of P, called the *promotion chain* of f, as follows. Let $u_0 = f^{-1}(1)$. Once u_i is defined let u_{i+1} be that element u covering u_i (i.e., $u_i < u_{i+1}$ and no $s \in P$ satisfies $u_i < s < u_{i+1}$) for which $f(u)$ is minimal. Continue until reaching a maximal element u_ℓ of P. Now define the *promotion* $g = \partial f$ of f as follows. If $t \neq u_i$ for any i, then set $g(t) = f(t) - 1$. If $1 \leqslant i \leqslant k - 1$, then set $g(u_i) = f(u_{i+1}) - 1$. Finally, set $g(u_\ell) = n$. Figure 1 gives an example, with the elements in the promotion chain of f circled. (The vertex labels in Fig. 1 are the values of a linear extension and are unrelated to the (irrelevant) labeling ω.) It is easy to see that $\partial f \in \mathcal{E}_P$ and that the map $\partial : \mathcal{E}_P \to \mathcal{E}_P$ is a bijection.

Lemma 2.1. *Let P be an n-element poset. Then the promotion operator $\partial : \mathcal{E}_P \to \mathcal{E}_P$ is parity-reversing if and only if the length ℓ (or cardinality $\ell + 1$) of every maximal chain of P satisfies $n \equiv \ell \pmod 2$. Similarly, ∂ is parity-preserving if and only if the length ℓ of every maximal chain of P satisfies $n \equiv \ell + 1 \pmod 2$.*

Proof. Let $f \in \mathcal{E}_P$, and let $u_0 < u_1 < \cdots < u_\ell$ be the promotion chain of f. Then $(\partial f) f^{-1}$ is a product of two cycles, viz.,

$$(\partial f) f^{-1} = (n, n-1, \ldots, 1)(b_0, b_1, \ldots, b_\ell),$$

where $b_i = f(u_i)$. This permutation is odd if and only if $n \equiv \ell \pmod 2$, and the proof follows since every maximal chain of P is the promotion chain of some linear extension. $\square$

Corollary 2.2. *Let P be an n-element poset, and suppose that the length ℓ of every maximal chain of P satisfies $n \equiv \ell \pmod 2$. Then P is sign-balanced.*

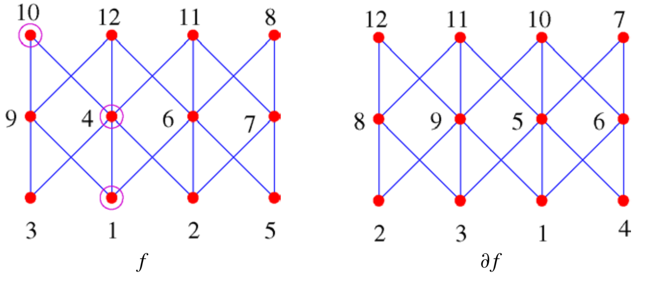

Fig. 1. The promotion operator ∂.

Proof. By the previous lemma, ∂ is parity-reversing. Since it is also a bijection, $\mathcal{E}_P$ must contain the same number of even linear extensions as odd linear extensions. $\square$

We now consider a variant of promotion known as evacuation. For any linear extension g of an m-element poset Q, let $u_0 < u_1 < \cdots < u_\ell$ be the promotion chain of g, so $\partial g(u_\ell) = m$. Define $\rho_g(Q) = Q - \{u_\ell\}$. The restriction of ∂g to $\rho_g(Q)$, which we also denote by ∂g, is a linear extension of $\rho_g(Q)$. Let

$$\mu_{g,k}(Q) = \rho_{\partial^k g}\, \rho_{\partial^{k-1} g} \cdots \rho_{\partial g}\, \rho_g(Q).$$

Now let $\#P = n$ and define the *evacuation* $\operatorname{evac}(f)$ of f to be the linear extension of P whose value at the unique element of $\mu_{g,k-1}(P) - \mu_{g,k}(P)$ is $n - k + 1$, for $1 \leqslant k \leqslant n$. Figure 2 gives an example of $\operatorname{evac}(f)$, where we circle the values of $\operatorname{evac}(f)$ as soon as they are determined. A remarkable theorem of Schützenberger [22] asserts that evac is an involution (and hence a bijection $\mathcal{E}_P \to \mathcal{E}_P$).

We say that the poset P is *consistent* if for all $t \in P$, the lengths of all maximal chains of the principal order ideal $\Lambda_t := \{s \in P\colon\ s \leqslant t\}$ have the same parity. Let $\nu(t)$ denote the length of the longest chain of Λ_t, and set

$$\Gamma(P) = \sum_{t \in P} \nu(t).$$

We also say that a permutation σ of a finite set has *parity* $k \in \mathbb{Z}$ if either σ and k are both even or σ and k are both odd. Equivalently, $\operatorname{inv}(\sigma) \equiv k \pmod 2$.

Proposition 2.3. *Suppose that P is consistent. Then* $\operatorname{evac}\colon \mathcal{E}_P \to \mathcal{E}_P$ *is parity-preserving if $\binom{n}{2} - \Gamma(P)$ is even, and parity-reversing if $\binom{n}{2} - \Gamma(P)$ is odd.*

Proof. The evacuation of a linear extension f of an n-element poset P consists of n promotions $\delta_1, \ldots, \delta_n$, where δ_i is applied to a certain subposet P_{i-1} of P with $n - i + 1$ elements. Let f_i be the linear extension of P whose restriction to P_i agrees with $\delta_i \delta_{i-1} \cdots \delta_1$, and whose value at the unique element of $P_{j-1} - P_j$ for $j \leqslant i$ is $n - i + 1$. Thus $f_0 = f$ and $f_n = \operatorname{evac}(f)$. (Figure 2 gives an example of the sequence $f_0, \ldots, f_5$.) Let u_i be the end (top) of the promotion chain for the promotion δ_i. Thus $\{u_1, u_2, \ldots, u_n\} = P$. Lemma 2.1

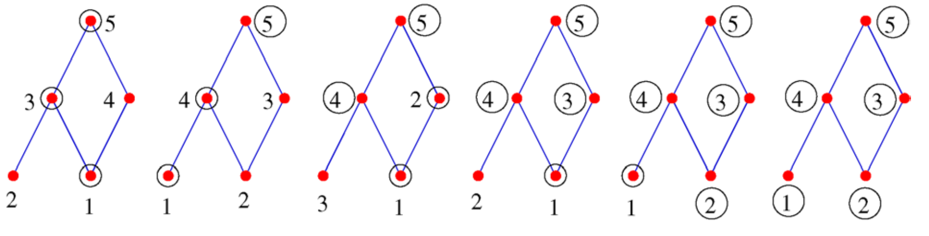

Fig. 2. The evacuation operator evac.

shows that if P is consistent, then $f_i f_{i-1}^{-1}$ has parity $n-i+1-(\nu(u_i)+1)$. Hence the parity of $\mathrm{evac}(f)f^{-1}$ is given by

$$\sum_{i=1}^{n}\bigl(n-i-\nu(u_i)\bigr)=\binom{n}{2}-\sum_{t\in P}\nu(P)=\binom{n}{2}-\Gamma(P),$$

from which the proof follows. □

Corollary 2.4. *Suppose that P is consistent and $\binom{n}{2}-\Gamma(P)$ is odd. Then P is sign-balanced.*

Note. In [25, pp. 50–51], [26, Corollary 19.5] it was shown using the theory of P-partitions that the number $e(P)$ of linear extensions of P is even if P is graded of rank ℓ (i.e., every maximal chain of P has length ℓ) and $n-\ell$ is even, and it was stated that it would be interesting to give a direct proof. Our Corollary 2.2 gives a direct proof of a stronger result. Similarly in [25, Corollary 4.6], [26, Corollary 19.6] it was stated (in dual form) that if for all $t\in P$ all maximal chains of Λ_t have the same length, and if $\binom{n}{2}-\Gamma(P)$ is odd, then $e(P)$ is even. Corollary 2.4 gives a direct proof of a stronger result.

3. Partitions

In this section we apply our previous results and obtain some new results for certain posets corresponding to (integer) partitions. We first review some notation and terminology concerning partitions. Further details may be found in [29, Chapter 7]. Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition of n, denoted $\lambda\vdash n$ or $|\lambda|=n$. Thus $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant 0$ and $\sum\lambda_i=n$. We can identify λ with its *diagram* $\{(i,j)\in\mathbb{P}\times\mathbb{P}\colon\ 1\leqslant j\leqslant\lambda_i\}$. Let μ be another partition such that $\mu\subseteq\lambda$, i.e., $\mu_i\leqslant\lambda_i$ for all i. Define the *skew partition* or *skew diagram* λ/μ by

$$\lambda/\mu=\bigl\{(i,j)\in\mathbb{P}\times\mathbb{P}\colon\ \mu_i+1\leqslant j\leqslant\lambda_i\bigr\}.$$

Write $|\lambda/\mu|=n$ to denote that $|\lambda|-|\mu|=n$, i.e., n is the number of squares in the shape λ/μ, drawn as a Young diagram [27, p. 29]. We can regard λ/μ as a subposet of $\mathbb{P}\times\mathbb{P}$ (with the usual coordinatewise ordering). We write $P_{\lambda/\mu}$ for this poset. As a set it is the same as λ/μ, but the notation $P_{\lambda/\mu}$ emphasizes that we are considering it to be a poset. In this section we will only be concerned with "ordinary" shapes λ, but in Section 5 skew shapes λ/μ will arise as a special case of Proposition 5.3.

The posets P_λ are consistent for any λ, so we can ask for which P_λ is evacuation parity-reversing, i.e., $\binom{n}{2}-\Gamma(P_\lambda)$ is odd. To this end, the *content* $c(i,j)$ of the cell (i,j) is defined by $c(i,j)=j-i$ [29, p. 373]. Also let $\mathcal{O}(\mu)$ denote the number of odd parts of the partition μ. An *order ideal* of a poset P is a subset $K\subseteq P$ such that if $t\in K$ and $s<t$, then $s\in K$. Similarly a *dual order ideal* or *filter* of P is a subset $F\subseteq P$ such that if $s\in F$ and $t>s$, then $t\in F$. If we successively remove two-element chains from P_λ which are dual order ideals of the poset from which they are removed, then eventually we reach a poset $\mathrm{core}_2(P_\lambda)$, called the 2-core of P_λ, that contains no dual order ideals which are two-element chains. The 2-core is *unique*, i.e., independent of the order in which the dual order ideals are removed, and is given by P_{δ_k} for some $k \geqslant 1$, where δ_k denotes the "staircase shape" $(k-1, k-2, \dots, 1)$. For further information see [29, Exercise 7.59].

Proposition 3.1. *Let $\lambda \vdash n$. The following numbers all have the same parity.*

(a) $\Gamma(P_\lambda)$.
(b) $\sum_{t \in P_\lambda} c(t)$.
(c) $\frac{1}{2}(\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))$.
(d) $\frac{1}{2}\left(n - \binom{k}{2}\right)$, *where* $\binom{k}{2} = \#\mathrm{core}_2(P_\lambda)$.

Hence if a_λ denotes any of the above four numbers, then evacuation is parity-reversing on P_λ if and only if $\binom{n}{2} - a_\lambda$ is odd.

Proof. It is easy to see that if $t \in P_\lambda$, then $\nu(t) \equiv c(t) \pmod 2$. Hence (a) and (b) have the same parity. It is well known and easy to see [17, Example 3, p. 11] that

$$\sum_{t \in P_\lambda} c(t) = \sum \binom{\lambda_i}{2} - \sum \binom{\lambda_i'}{2}.$$

Since $\sum \lambda_i = \sum \lambda_i'$, we have

$$\sum_{t \in P_\lambda} c(t) = \frac{1}{2}\left(\sum \lambda_i^2 - \sum (\lambda_i')^2\right).$$

Since $a^2 \equiv 0, 1 \pmod 4$ depending on whether a is even or odd, we see that (b) and (c) have the same parity. If we remove from P_λ a 2-element dual order ideal which is also a chain, then we remove exactly one element with an odd content. A 2-core is self-conjugate and hence has an even content sum. Hence the number of odd contents of P_λ is equal to the number of dominos that must be removed from P_λ in order to reach $\mathrm{core}_2(P_\lambda)$. It follows that (b) and (c) have the same parity, completing the proof. $\square$

It can be shown [30] that if $t(n)$ denotes the number of partitions $\lambda \vdash n$ for which a_λ is even, then $t(n) = \frac{1}{2}(p(n) + f(n))$, where $p(n)$ denotes the total number of partitions of n and

$$\sum_{n \geqslant 0} f(n) x^n = \prod_{i \geqslant 1} \frac{1 + x^{2i-1}}{(1 - x^{4i})(1 + x^{4i-2})^2}.$$

Hence the number $g(n)$ of partitions $\lambda \vdash n$ for which evac is parity-reversing on P_λ is given by

$$g(n) = \begin{cases} \frac{1}{2}\big(p(n) + f(n)\big), & \text{if } \binom{n}{2} \text{ is odd,} \\ \frac{1}{2}\big(p(n) - f(n)\big), & \text{if } \binom{n}{2} \text{ is even.} \end{cases}$$

We conclude this section with some applications of the theory of domino tableaux. A *standard domino tableau* (SDT) of shape $\lambda \vdash 2n$ is a sequence

$$\emptyset = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^n = \lambda$$

of partitions such that each skew shape λ^i/λ^{i-1} is a *domino*, i.e., two squares with an edge in common. Each of these dominos is either horizontal (two squares in the same row) or vertical (two squares in the same column). Let Dom_λ denote the set of all SDT of shape λ. Given $D \in \mathrm{Dom}_\lambda$, define $\mathrm{ev}(D)$ to be the number of vertical dominos in even columns of D, where an *even column* means the $2i$th column for some $i \in \mathbb{P}$. For the remainder of this section, fix the labeling ω of P_λ to be the usual "reading order," i.e., the first row of λ is labeled $1, 2, \ldots, \lambda_1$; the second row is labeled $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$, etc. We write $I_\lambda(q)$ for $I_{P_\lambda,\omega}(q)$ and set $I_\lambda = I_\lambda(-1)$, the *imbalance* of the partition λ. It is shown in [32, Theorem 12] (by analyzing the formula that results from setting $q = -1$ in (13)) that

$$I_\lambda = \sum_{D \in \mathrm{Dom}_\lambda} (-1)^{\mathrm{ev}(D)}.$$

Let $\lambda \vdash n$. Lascoux, Leclerc, and Thibon [14, (27)] define a certain class of symmetric functions $\tilde{G}_\lambda^{(k)}(x; q)$ (defined earlier by Carré and Leclerc [4] for the special case $k = 2$ and $\lambda = 2\mu$). We will only be concerned with the case $k = 2$ and $q = -1$, for which we write $G_\lambda = \tilde{G}_\lambda^{(2)}(x; -1)$. The symmetric function G_λ vanishes unless $\mathrm{core}_2(\lambda) = \emptyset$, so we may assume $n = 2m$. If $\mathrm{core}_2(\lambda) = \emptyset$, then G_λ is homogeneous of degree $m = n/2$. We will not define it here but only recall the properties relevant to us. The connection with the imbalance I_λ is provided by the formula (immediate from the definition of G_λ in [14] together with [32, Theorem 12])

$$[x_1 \cdots x_m] G_\lambda = (-1)^{r(\lambda)} I_\lambda, \tag{2}$$

where $[x_1 \cdots x_m]F$ denotes the coefficient of $x_1 \cdots x_m$ in the symmetric function F, and $r(\lambda)$ is the maximum number of vertical dominos that can appear in even columns of a domino tableau of shape λ. Also define $d(\lambda)$ to be the maximum number of disjoint vertical dominos that can appear in the diagram of λ, i.e.,

$$d(\lambda) = \sum_i \left\lfloor \frac{1}{2} \lambda'_{2i} \right\rfloor.$$

Note that $d(\lambda) \geqslant r(\lambda)$, but equality need not hold in general. For instance, $d(4,3,1) = 1$, $r(4,3,1) = 0$. However, we do have $d(2\mu) = r(2\mu)$ for any partition μ. Let us also note that our $r(\lambda)$ is denoted $d(\lambda)$ in [32] and is defined only for λ with an empty 2-core.

Theorem 3.2. (a) *We have*

$$\sum_{\mu \vdash m} I_{2\mu} = 1$$

for all $m \geqslant 1$.

(b) *Let $v(\lambda)$ denote the maximum number of disjoint vertical dominos that fit in the shape λ. Equivalently,*

$$v(\lambda) = \sum_{i \geqslant 1} \left\lfloor \frac{1}{2} \lambda_i' \right\rfloor.$$

Then

$$\sum_{\lambda \vdash 2m} (-1)^{v(\lambda)} I_\lambda^2 = 0.$$

Proof. (a) Barbasch and Vogan [2] and Garfinkle [9] define a bijection between elements π of the hyperoctahedral group B_m, regarded as signed permutations of $1, 2, \ldots, m$, and pairs (P, Q) of SDT of the same shape $\lambda \vdash 2m$. (See [15, p. 25] for further information.) A crucial property of this bijection, stated implicitly without proof in [12] and proved by Shimozono and White [23, Theorem 30], asserts that

$$\mathrm{tc}(\pi) = \frac{1}{2}\big(v(P) + v(Q)\big), \tag{3}$$

where $\mathrm{tc}(\pi)$ denotes the number of minus signs in π and $v(R)$ denotes the number of vertical dominos in the SDT R.

Carré and Leclerc, [4, Definition 9.1], define a symmetric function $H_\mu(x;q)$ which satisfies $H_\mu(x,-1) = (-1)^{v(\mu)} G_{2\mu}$. In [12, Theorem 1] is stated the identity

$$\sum_\mu H_\mu(x;q) = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} \prod_{i \geqslant j} \frac{1}{1 - q x_i x_j}. \tag{4}$$

The proof of (4) in [12] is incomplete, since it depends on a semistandard version of the $P = Q$ case of (3) (easily deduced from (3)), which had not yet been proved. The proof of (3) in [23] therefore completes the proof of (4). A generalization of (4) was later given by Lam [13, Theorem 28].

Setting $q = -1$ in (4) gives

$$\sum_\mu (-1)^{v(\mu)} G_{2\mu} = \prod_i \frac{1}{(1 - x_i)(1 + x_i^2)} \prod_{i<j} \frac{1}{1 - x_i^2 x_j^2}.$$

Taking the coefficient of $x_1 \cdots x_m$ on both sides and using (2) together with $v(\mu) = d(2\mu) = r(2\mu)$ completes the proof.

(b) It is easy to see that for any SDT D we have

$$v(D) = v(\lambda) - 2d(\lambda) + 2\,\mathrm{ev}(D).$$

Thus by (3) we have

$$0 = \sum_{\pi \in B_m} (-1)^{\mathrm{tc}(\pi)} = \sum_{P,Q} (-1)^{\frac{1}{2}(v(P)+v(Q))} = \sum_{\lambda \vdash 2m} \Bigl(\sum_{D \in \mathrm{Dom}_\lambda} (-1)^{\frac{1}{2}v(D)} \Bigr)^2$$
$$= \sum_{\lambda \vdash 2m} (-1)^{v(\lambda)} \Bigl(\sum_{D \in \mathrm{Dom}_\lambda} (-1)^{\mathrm{ev}(D)} \Bigr)^2 = \sum_{\lambda \vdash 2m} (-1)^{v(\lambda)} I_\lambda^2. \qquad \square$$

In the same spirit as Theorem 3.2 we have the following conjecture.

Conjecture 3.3. [2] (a) *For all* $n \geqslant 0$ *we have*

$$\sum_{\lambda \vdash n} q^{v(\lambda)} t^{d(\lambda)} x^{v(\lambda')} y^{d(\lambda')} I_\lambda = (q+x)^{\lfloor n/2 \rfloor}. \tag{5}$$

(b) *If* $n \not\equiv 1 \pmod 4$, *then*

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} t^{d(\lambda)} I_\lambda^2 = 0.$$

It is easy to see that $d(\lambda) = d(\lambda')$ for all λ. (E.g., consider the horizontal and vertical line segments in Fig. 3.) Hence the variable y is superfluous in Eq. (5), but we have included it for the sake of symmetry. In particular, if $F_n(q,t,x,y)$ denotes the left-hand side of (5) then

$$F_n(q,0,x,y) = F_n(q,t,x,0) = F_n(q,0,x,0).$$

Note also that $d(\lambda) = 0$ if and only λ is a *hook*, i.e., a partition of the form $(n-k, 1^k)$.

The case $t = 0$ (or $y = 0$, or $t = y = 0$) of Eq. (5) follows from the following proposition, which in a sense "explains" where the right-hand side $(q+x)^{\lfloor n/2 \rfloor}$ comes from.

Proposition 3.4. *For all* $n \geqslant 0$ *we have*

$$\sum_{\lambda = (n-k, 1^k)} q^{v(\lambda)} x^{v(\lambda')} I_\lambda = (q+x)^{\lfloor n/2 \rfloor}, \tag{6}$$

where λ *ranges over all hooks* $(n-k, 1^k)$, $0 \leqslant k \leqslant n-1$.

[2] A combinatorial proof of (a) was found by Thomas Lam [13] after this paper was written. Later a combinatorial proof of both (a) and (b) was given by Jonas Sjöstrand [24]. Sjöstrand's main result [24, Theorem 2.3] leads to further identities, such as $\sum_{\mu \vdash n} q^{v(\mu)} I_{2\mu} = 1$, thereby generalizing our Theorem 3.2(a).

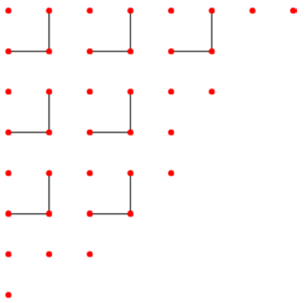

Fig. 3. $d(86655431) = d(86655431')$.

First proof. Let $\lambda = (n-k, 1^k)$. Let ω denote the "reading order" labeling of P_λ as above. The set $\mathcal{L}_{P,\omega}$ consists of all permutations $1, a_2, \ldots, a_m$, where $a_2, \ldots, a_m$ is a *shuffle* of the permutations $2, 3, \ldots, n-k$ and $n-k+1, n-k+2, \ldots, n$. It follows, e.g., from [27, Proposition 1.3.17] that

$$I_\lambda(q) = \begin{bmatrix} n-1 \\ k \end{bmatrix},$$

a q-binomial coefficient.

Suppose first that $n = 2m+1$. By [27, Exercize 3.45(b)],

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_{q=-1} = \begin{cases} \binom{m}{j}, & k = 2j, \\ 0, & k = 2j+1. \end{cases}$$

Note that if $\lambda = (n-2j, 1^{2j})$, then $v(\lambda) = j$ and $v(\lambda') = m-j$. Hence

$$\sum_{\lambda=(n-k,1^k)} q^{v(\lambda)} x^{v(\lambda')} I_\lambda = \sum_{j=0}^{m} q^j x^{m-j} \binom{m}{j} = (q+x)^m,$$

as desired. The proof for n even is similar and will be omitted. $\square$

Second proof. Assume first that $n = 2m$. We use an involution argument analogous to the proof of Theorem 1.1 or to arguments in [32, Section 5] and Section 4 of this paper. Let T be an SYT of shape $\lambda = (n-k, 1^k)$, which can be regarded as an element of $\mathcal{L}_{P_\lambda,\omega}$. Let i be the least positive integer (if it exists) such that $2i-1$ and $2i$ appear in different rows and in different columns of T. Let T' denote the SYT obtained from T by transposing $2i-1$ and $2i$. Since multiplying by a transposition changes the sign of a permutation, we have $(-1)^{\mathrm{inv}(T)} + (-1)^{\mathrm{inv}(T')} = 0$. The surviving SYT are obtained by first placing $1, 2$ in the same row or column, then $3, 4$ in the same row or column, etc. If $k = 2j$ or $2j+1$, then the number of survivors is easily seen to be $\binom{m-1}{j}$. Because the entries of T come in pairs $2i-1, 2i$, the number of inversions of each surviving SYT is even. Moreover, if $k = 2j$ then $v(\lambda) = j$ and $v(\lambda') = m-j$, while if $k = 2j+1$ then $v(\lambda) = j+1$ and $v(\lambda') = m-1-j$. Hence

$$\sum_{\lambda=(n-k,1^k)} q^{v(\lambda)} x^{v(\lambda')} I_\lambda = \sum_{j=0}^{m-1} (q+x)\binom{m-1}{j} q^j x^{m-1-j} = (q+x)^m,$$

as desired.

The proof is similar for $n = 2m+1$. Let i be the least positive integer (if it exists) such that $2i$ and $2i+1$ (rather than $2i-1$ and $2i$) appear in different rows and in different columns of T. There are now no survivors when $k = 2j+1$ and $\binom{m}{j}$ survivors when $k = 2j$. Other details of the proof remain the same, so we get

$$\sum_{\lambda=(n-k,1^k)} q^{v(\lambda)} x^{v(\lambda')} I_\lambda = \sum_{j=0}^{m} \binom{m-1}{j} q^j x^{m-j} = (q+x)^m,$$

completing the proof. $\square$

There are some additional properties of the symmetric functions G_λ that yield information about I_λ. For instance, there is a product formula in [12, Theorem 2] for $\sum_\mu G_{2\mu\cup 2\mu}$, where μ ranges over all partitions and

$$2\mu \cup 2\mu = (2\mu_1, 2\mu_1, 2\mu_2, 2\mu_2, \ldots),$$

which implies that $\sum_{\mu\vdash n} I_{2\mu\cup 2\mu} = 0$. In fact, in [4, Corollary 9.2] it is shown that $G_{2\mu\cup 2\mu}(x) = \pm s_\mu(x_1^2, x_2^2, \ldots)$, from which it follows easily that in fact $I_{2\mu\cup 2\mu} = 0$. However, this result is just a special case of Corollary 2.2 and of Proposition 2.3, so we obtain nothing new.

Also relevant to us is an expansion of G_λ into Schur functions due to Shimozono (see [32, Theorem 18]) for certain shapes λ, namely, those whose 2-quotient (in the sense, e.g., of [17, Example I.1.8]) is a pair of rectangles. This expansion was used by White [32, Corollary 20] to evaluate I_λ for such shapes. White [32, Section 8] also gives a combinatorial proof, based on a sign-reversing involution, in the special case that λ itself is a rectangle. We simply state here White's result for rectangles.

Theorem 3.5. *Let λ be an $m \times n$ rectangle. Then*

$$I_\lambda = \begin{cases} 1, & \textit{if } m = 1, \textit{ or } n = 1, \\ 0, & \textit{if } m \equiv n \pmod 2 \textit{ and } m, n > 1, \\ \pm g^\mu, & m \not\equiv n \pmod 2, \end{cases}$$

where g^μ denotes the number of shifted standard tableaux (*as defined, e.g., in* [17, Example III.8.12]) *of shape*

$$\mu = \left(\frac{m+n-1}{2}, \frac{m+n-3}{2}, \ldots, \frac{|n-m|+3}{2}, \frac{|n-m|+1}{2}\right).$$

(*An explicit "hook length formula" for any g^μ appears, e.g., in the reference just cited.*)

It is natural to ask whether Theorem 3.5 can be generalized to other partitions λ. In this regard, A. Eremenko and A. Gabrielov (private communication) have made a remarkable conjecture. Namely, if we fix the number ℓ of parts and parity of each part of λ, then there are integers $c_1, \dots, c_k$ and integer vectors $\gamma_1, \dots, \gamma_k \in \mathbb{Z}^\ell$ such that

$$I_\lambda = \sum_{i=1}^{k} c_i g^{\frac{1}{2}(\lambda+\gamma_i)}.$$

One defect of this conjecture is that the expression for I_λ is not unique. We can insure uniqueness, however, by the additional condition that all the vectors γ_i have coordinate sum 0 when $|\lambda|$ is even and -1 when $|\lambda|$ is odd (where $|\lambda| = \sum \lambda_i$). In this case, however, we need to define properly g^μ when μ is not a strictly decreasing sequence of nonnegative integers. See the discussion preceding Conjecture 3.6. For instance, we have

$$\begin{aligned}
&I_{(2a,2b,2c)} = g^{(a,b,c)} - g^{(a+1,b,c-1)},\\
&I_{(2a+1,2b,2c)} = g^{(a,b,c)} + g^{(a+1,b-1,c)},\\
&I_{(2a,2b+1,c)} = 0,\\
&I_{(2a,2b,2c+1)} = -g^{(a+1,b-1,c)} - g^{(a+1,b,c-1)},\\
&I_{(2a+1,2b+1,2c)} = g^{(a+1,b,c)} + g^{(a+1,b+1,c-1)},\\
&I_{(2a+1,2b,2c+1)} = 0,\\
&I_{(2a,2b+1,2c+1)} = g^{(a+1,b,c)} + g^{(a,b+1,c)},\\
&I_{(2a+1,2b+1,2c+1)} = g^{(a,b+1,c)} + g^{(a+1,b+1,c-1)},\\
&I_{(2a,2b,2c,2d)} = g^{(a,b,c,d)} - g^{(a+1,b,c-1,d)} - g^{(a+1,b+1,c-1,d-1)} - 2g^{(a+1,b,c,d-1)}.
\end{aligned}$$

It is easy to see that $I_{(2a,2b+1,c)} = I_{(2a+1,2b,2c+1)} = 0$, viz., the 2-cores of the partitions $(2a, 2b+1, c)$ and $(2a+1, b, 2c+1)$ have more than one square. More generally, we have verified by induction the formulas for I_μ when $\ell(\mu) \leqslant 3$.

We have found a (conjectured) symmetric function generalization of the Eremenko–Gabrielov conjecture. If $f(x)$ is any symmetric function, define

$$f(x/x) = f(p_{2i-1} \to 2p_{2i-1},\ p_{2i} \to 0).$$

In other words, write $f(x)$ as a polynomial in the power sums p_j and substitute $2p_{2i-1}$ for p_{2i-1} and 0 for p_{2i}. In λ-ring notation, $f(x/x) = f(X - X)$. Let Q_μ denote Schur's shifted Q-function [17, Section 3.8]. The Q_μ's form a basis for the ring $\mathbb{Q}[p_1, p_3, p_5, \dots]$. Hence $f(x/x)$ can be written uniquely as a linear combination of Q_μ's.

We mentioned above that the symmetric function G_λ was originally defined only when $\mathrm{core}_2(\lambda) = \emptyset$. We can extend the definition to any λ as follows. The original definition has the form

$$G_\lambda(x) = \sum_D (-1)^{\mathrm{cospin}(D)} x^D, \tag{7}$$

summed over all semistandard domino tableaux of shape λ, where $\mathrm{cospin}(\lambda)$ is a certain integer and x^D a certain monomial depending on λ. If $\#\mathrm{core}_2(\lambda) = 1$, then define G_λ exactly as in (7), except that we sum over all semistandard domino tableaux of the skew shape $\lambda/1$. If $\#\mathrm{core}_2(\lambda) > 1$, then define $G_\lambda = 0$. (In certain contexts it would be better to define G_λ by (7), summed over all semistandard domino tableaux of the skew shape $\lambda/\mathrm{core}_2(\lambda)$, but this is not suitable for our purposes.) Equation (2) then continues to hold for any $\lambda \vdash n$, where $m = \lfloor n/2 \rfloor$.

We also need to define $G_\mu(x/x)$ properly when μ is not a strictly decreasing sequence of positive integers. The following definition seems to be correct, but perhaps some modification is necessary. Let $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k$. Trailing 0's are irrelevant and can be ignored, so we may assume $\mu_k > 0$. If μ is not a sequence of distinct nonnegative integers, then $G_\mu(x/x) = 0$. Otherwise $G_\mu(x/x) = \varepsilon_\mu G_\lambda(x/x)$, where λ is the decreasing rearrangement of μ and ε_μ is the sign of the permutation that converts μ to λ.

Conjecture 3.6. *Fix the number ℓ of parts and parity of each part of the partition λ. Then there are integers $c_1, \ldots, c_k$ and integer vectors $\gamma_1, \ldots, \gamma_k \in \mathbb{Z}^\ell$ such that*

$$(-1)^{r(\lambda)} G_\lambda(x/x) = \sum_{i=1}^{k} c_i Q_{\frac{1}{2}(\lambda+\gamma_i)}(x). \tag{8}$$

Let $\lambda \vdash 2n$ or $\lambda \vdash 2n+1$. Take the coefficient of $x_1 x_2 \cdots x_n$ on both sides of (8). By (2) the left-hand side becomes $2^n I_\lambda$. Moreover, if $\mu \vdash m$ then the coefficient of $x_1 \cdots x_m$ in Q_μ is $2^m g^\mu$ [17, (8.16)]. Hence Conjecture 3.6 specializes to the Eremenko–Gabrielov conjecture. At present we have no conjecture for the values of the coefficients c_i. Here is a short table (due to Eremenko and Gabrielov for I_λ; they have extended this table to the case of four and five rows) of the three-row case of Conjecture 3.6. For simplicity we write $\pm$ for $(-1)^{r(\lambda)}$.

$$\pm G_{(2a,2b,2c)}(x/x) = Q_{(a,b,c)}(x) - Q_{(a+1,b,c-1)}(x),$$

$$\pm G_{(2a+1,2b,2c)}(x/x) = Q_{(a,b,c)}(x) + Q_{(a+1,b-1,c)}(x),$$

$$\pm G_{(2a,2b+1,2c)}(x/x) = 0,$$

$$\pm G_{(2a,2b,2c+1)}(x/x) = -Q_{(a+1,b-1,c)}(x) - Q_{(a+1,b,c-1)}(x),$$

$$\pm G_{(2a+1,2b+1,2c)}(x/x) = Q_{(a+1,b,c)}(x) + Q_{(a+1,b+1,c-1)}(x),$$

$$\pm G_{(2a+1,2b,2c+1)}(x/x) = 0,$$

$$\pm G_{(2a,2b+1,2c+1)}(x/x) = Q_{(a+1,b,c)}(x) + Q_{(a,b+1,c)}(x),$$

$$\pm G_{(2a+1,2b+1,2c+1)}(x/x) = Q_{(a,b+1,c)}(x) + Q_{(a+1,b+1,c-1)}(x).$$

We now discuss some general properties of the polynomial $I_\lambda(q)$ and its value $I_\lambda(-1)$. Let $C(\lambda)$ denote the set of *corner squares* of λ, i.e., those squares of the Young diagram of λ whose removal still gives a Young diagram. Equivalently, Pieri's formula [29, Theorem 7.15.7] implies that

$$s_{\lambda/1} = \sum_{t\in C(\lambda)} s_{\lambda-t}. \tag{9}$$

Let f^λ denote the number of SYT of shape λ [29, Proposition 7.10.3], so

$$f^\lambda = \sum_{t\in C(\lambda)} f^{\lambda-t}. \tag{10}$$

Note that $I_\lambda(1) = f^\lambda$, so $I_\lambda(q)$ is a (nonstandard) q-analogue of f^λ. The q-analogue of Eq. (10) is the following result.

Proposition 3.7. *We have*

$$I_\lambda(q) = \sum_{t\in C(\lambda)} q^{b_\lambda(t)} I_{\lambda-t}(q),$$

where $b_\lambda(t)$ denotes the number of squares in the diagram of λ in a lower row than that of t.

Proof. We have by definition

$$I_\lambda(q) = \sum_T q^{\mathrm{inv}(\pi(T))},$$

where T ranges over all SYT of shape λ and $\pi(T)$ is the permutation obtained by reading the entries of T in the usual reading order, i.e., left-to-right and top-to-bottom when T is written in "English notation" as in [17,27,29]. Suppose $\lambda \vdash n$. If T is an SYT of shape λ, then the square t occupied by n is a corner square. The number of inversions (i, j) of $\pi(T) = a_1 \cdots a_m$ such that $a_i = n$ is equal to $b_\lambda(t)$, and the proof follows. $\square$

Now let D_1 denote the linear operator on symmetric functions defined by $D_1(s_\lambda) = s_{\lambda/1}$. We then have the commutation relation [29, Exercise 7.24(a)]

$$D_1 s_1 - s_1 D_1 = I, \tag{11}$$

the identity operator. This leads to many enumerative consequences, discussed in [28]. There is an analogue of (11) related to I_λ, though we do not know of any applications. Define a linear operator $D(q)$ on symmetric functions by

$$D(q)s_\lambda = \sum_{t\in C(\lambda)} q^{b_\lambda(t)} s_{\lambda-t}.$$

Let $U(q)$ denote the adjoint of $D(q)$ with respect to the basis $\{s_\lambda\}$ of Schur functions, so

$$U(q)s_\mu = \sum_t q^{b_{\mu+t}(t)} s_{\mu+t},$$

where t ranges over all boxes that we can add to the diagram of μ to get the diagram of a partition $\mu + t$ (for which necessarily $t \in C(\mu + t)$). Note that $U(1) = s_1$ (i.e., multiplication by s_1) and $D(1) = D_1$ as defined above. It follows from Proposition 3.7 that

$$U(q)^n \cdot 1 = \sum_{\lambda \vdash n} I_\lambda(q) s_\lambda,$$

where $U(q)^n \cdot 1$ denotes $U(q)^n$ acting on the symmetric function $1 = s_\emptyset$. Write $U = U(-1)$ and $D = D(-1)$. Let A be the linear operator on symmetric functions given by $As_\lambda = (2k(\lambda) + 1)s_\lambda$, where $k(\lambda) = \#C(\lambda)$, the number of corner boxes of λ.

Proposition 3.8. *We have $DU + UD = A$.*

Proof. The proof is basically a brute force computation. Write $\bar\lambda_i = \lambda_i + \lambda_{i+1} + \cdots$. Suppose μ is obtained from λ by adding a box in row $r - 1$ and deleting a box in row $s - 1$, where $r < s$. Then the coefficient of s_μ in $(D(q)U(q) + U(q)D(q))s_\lambda$ is given by

$$\bigl\langle s_\mu, \bigl(D(q)U(q) + U(q)D(q)\bigr)s_\lambda\bigr\rangle = q^{\bar\lambda_r} q^{\bar\lambda_s} + q^{\bar\lambda_s} q^{\bar\lambda_r - 1},$$

which vanishes when $q = -1$. Similarly if $r > s$ we get

$$\bigl\langle s_\mu, \bigl(D(q)U(q) + U(q)D(q)\bigr)s_\lambda\bigr\rangle = q^{\bar\lambda_s} q^{\bar\lambda_r + 1} + q^{\bar\lambda_r} q^{\bar\lambda_s},$$

which again vanishes when $q = -1$. On the other hand, if $\lambda = \mu$ we have

$$\bigl\langle s_\lambda, \bigl(D(q)U(q) + U(q)D(q)\bigr)s_\lambda\bigr\rangle = \bigl(c(\lambda) + 1\bigr)q^{2\bar\lambda_r} + c(\lambda)q^{2\bar\lambda_r} = \bigl(2c(\lambda) + 1\bigr)q^{2\bar\lambda_r}.$$

When $q = -1$ the right-hand side become $2c(\lambda) + 1$, completing the proof. $\square$

4. Chains of order ideals

Suppose that P is an n-element poset, and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n, i.e., $\alpha_i \in \mathbb{P} = \{1, 2, \ldots\}$ and $\sum \alpha_i = n$. Define an *α-chain* of order ideals of P to be a chain

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_k = P \tag{12}$$

of order ideals satisfying $\#(K_i - K_{i-1}) = \alpha_i$ for $1 \leqslant i \leqslant k$. The following result is quite simple but has a number of consequences.

Proposition 4.1. *Let P be an n-element poset and α a fixed composition of n. Suppose that for every α-chain* (12) *of order ideals of P, at least one subposet $K_i - K_{i-1}$ is sign-balanced. Then P is sign-balanced.*

Proof. Let $\mathcal{C}$ be the α-chain (12). We say that a linear extension f is *$\mathcal{C}$-compatible* if

$$K_1 = f^{-1}\big(\{1,\ldots,\alpha_1\}\big), \qquad K_2 - K_1 = f^{-1}\big(\{\alpha_1+1,\ldots,\alpha_1+\alpha_2\}\big),$$

etc. Let $\mathrm{inv}(\mathcal{C})$ be the minimum number of inversions of a $\mathcal{C}$-compatible linear extension. Clearly

$$\sum_f q^{\mathrm{inv}(f)} = q^{\mathrm{inv}(\mathcal{C})} \prod_{i=1}^k I_{K_i - K_{i-1}}(q),$$

where the sum is over all $\mathcal{C}$-compatible f. Since every linear extension is compatible with a unique α-chain, there follows

$$I_{P,\omega}(q) = \sum_{\mathcal{C}} q^{\mathrm{inv}(\mathcal{C})} \prod_{i=1}^k I_{K_i - K_{i-1}}(q), \tag{13}$$

where $\mathcal{C}$ ranges over all α-chains of order ideals of P. The proof now follows by setting $q = -1$. $\square$

Define a finite poset P with $2m$ elements to be *tilable by dominos* if there is a chain $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_m = P$ of order ideals such that each subposet $K_i - K_{i-1}$ is a two-element chain. Similarly, if $\#P = 2m+1$ and $1 \leqslant j \leqslant m+1$ then we say that P is *j-tilable by dominos* if there is a chain $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_{m+1} = P$ of order ideals such that $\#(K_i - K_{i-1}) = 2$ if $1 \leqslant i \leqslant m+1$ and $i \neq j$ (so $\#(K_j - K_{j-1}) = 1$). Note that being tilable by dominos is stronger than the existence of a partition of P into cover relations (or two element saturated chains). For instance, the poset P with cover relations $a < c, b < c, a < d, b < d$ can be partitioned into the two cover relations $a < c$ and $b < d$, but P is not tilable by dominos. When $n = 2m$, we define a *P-domino tableau* to be a chain $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_m = P$ of order ideals such that $K_i - K_{i-1}$ is a two-element chain for $1 \leqslant i \leqslant m$. Similarly, when $n = 2m+1$, we define a (standard) *P-domino tableau* to be a chain $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_{m+1} = P$ of order ideals such that $K_i - K_{i-1}$ is a two-element chain for $1 \leqslant i \leqslant m$ (so that $K_{m+1} - K_m$ consists of a single point). Thus for $\lambda \vdash 2n$, a P_λ-domino tableau coincides with our earlier definition of an SDT of shape λ.

Corollary 4.2. *Let $\#P = 2m$, and assume that P is not tilable by dominos. Then P is sign-balanced. Similarly if $\#P = 2m+1 \geqslant 3$ and P is not j-tilable by dominos for some j, then P is sign-balanced.*

Proof. Let $\alpha = (2, 2, \ldots, 2)$ (m 2's). If $\#P = 2m$ and P is not tilable by dominos, then for any α-chain (12) there is an i for which $K_i - K_{i-1}$ consists of two disjoint points. Since a poset consisting of two disjoint points is sign-balanced, it follows from Proposition 4.1 that P is sign-balanced. The argument is similar for $\#P = 2m+1$. $\square$

Corollary 4.2 was proved in a special case (the product of two chains with an even number of elements, with the $\hat{0}$ and $\hat{1}$ removed), using essentially the same proof as we have given, by Ruskey [21, Section 5, item 6].

Corollary 4.2 is particularly useful for the posets P_λ. From this corollary and the definition of $\mathrm{core}_2(\lambda)$ we conclude the following.

Corollary 4.3. *If* $\mathrm{core}_2(P_\lambda)$ *consists of more than one element, then* P_λ *is sign-balanced.*

It follows from [29, Exercise 7.59(e)] that if $f(n)$ denotes the number of partitions $\lambda \vdash n$ such that $\#\mathrm{core}_2(\lambda) \leqslant 1$, then

$$\sum_{n\geqslant 0} f(n)x^n = \frac{1+x}{\prod_{i\geqslant 1}(1-x^{2i})^2}.$$

Standard partition asymptotics (e.g., [1, Theorem 6.2]) shows that

$$f(n) \sim \frac{C}{n^{5/4}} \exp\bigl(\pi\sqrt{2n/3}\bigr)$$

for some $C > 0$. Since the total number $p(n)$ of partitions of n satisfies

$$p(n) \sim \frac{C'}{n} \exp\bigl(\pi\sqrt{2n/3}\bigr),$$

it follows that $\lim_{n\geqslant 0} f(n)/p(n) = 0$. Hence as $n \to \infty$, P_λ is sign-balanced for almost all $\lambda \vdash n$.

5. Maj-balanced posets

If $\pi = a_1 a_2 \cdots a_m$ is a permutation of $[n]$, then the *descent set* $D(\pi)$ of π is defined as

$$D(\pi) = \{i\colon\ a_i > a_{i+1}\}.$$

An element of $D(\pi)$ is called a *descent* of π, and *major index* $\mathrm{maj}(\pi)$ is defined as

$$\mathrm{maj}(\pi) = \sum_{i\in D(\pi)} i.$$

The major index has many properties analogous to the number of inversions, e.g., a classic theorem of MacMahon states that inv and maj are equidistributed on the symmetric group $\mathfrak{S}_n$ [7,8]. Thus it is natural to try to find “maj analogues” of the results of the preceding sections. In general, the major index of a linear extension of a poset can be

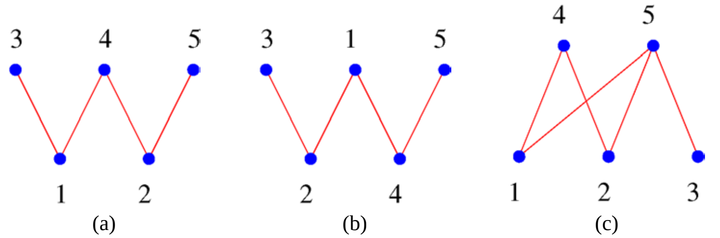

Fig. 4. Some counterexamples.

more tractable or less tractable than the number of inversions. Thus, for example, in Theorem 5.1 we are able to completely characterize naturally labeled maj-balanced posets. An analogous result for sign-balanced partitions seems very difficult. On the other hand, since multiplying a permutation by a fixed permutation has no definite effect on the parity of the major index, many of the results for sign-balanced posets are false (Theorem 1.1, Lemma 2.1, Proposition 2.3).

Let f be a linear extension of the labeled poset (P, ω), and let $\pi = \pi(f)$ be the associated permutation of $[n]$. In analogy to our definition of $\mathrm{inv}(f)$, define $\mathrm{maj}(f) = \mathrm{maj}(\pi)$ and

$$W_{P,\omega}(q) = \sum_{f \in \mathcal{E}_P} q^{\mathrm{maj}(f)} = \sum_{\pi \in \mathcal{L}_{P,\omega}} q^{\mathrm{maj}(\pi)}.$$

We say that (P, ω) is *maj-balanced* if $W_{P,\omega}(-1) = 0$, i.e., if the number of linear extensions of P with even major index equals the number with odd major index. Unlike the situation for sign-balanced posets, the property of being maj-balanced can depend on the labeling ω. Thus an interesting special case is that of *natural labelings*, for which $\omega(s) < \omega(t)$ whenever $s < t$ in P. We write $W_P(q)$ for $W_{P,\omega}(q)$ when ω is natural. It is a basic consequence of the theory of P-partitions [27, Theorem 4.5.8] that $W_P(q)$ does not depend on the choice of natural labeling of P.

Figures 4(a) and 4(b) show two different labelings of a poset P. The first labeling (which is natural) is not maj-balanced, while the second one is. Moreover, the dual poset P^* to the poset P in Fig. 4(b), whether naturally labeled or labeled the same as P, is maj-balanced. Contrast that with the trivial fact that the dual of a sign-balanced poset is sign-balanced. As a further example of the contrast between sign- and maj-balanced posets, Fig. 4(c) shows a naturally labeled maj-balanced poset Q. However, if we adjoin an element $\hat{0}$ below every element of Q and label it 0 (thus keeping the labeling natural) then we get a poset which is no longer maj-balanced. On the other hand, it is clear that such an operation has no effect on whether a poset is sign-balanced. (In fact, it leaves $I_{Q,\omega}(q)$ unchanged.)

Corollary 4.2 carries over to the major index in the following way.

Theorem 5.1. (a) *Let P be naturally labeled. Then $W_P(-1)$ is equal to the number of P-domino tableaux. In particular, P is maj-balanced if and only if there does not exist a P-domino tableau.*

(b) *A labeled poset (P, ω) is maj-balanced if there does not exist a P-domino tableau.*

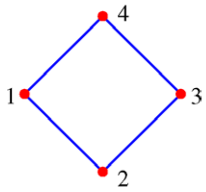

Fig. 5. A maj-balanced labeled poset tilable by dominos.

Proof. (a) Let $\pi = a_1 \cdots a_m \in \mathcal{L}_{P,\omega}$. Let i be the least number (if it exists) for which $\pi' = a_1 \cdots a_{2i} a_{2i+2} a_{2i+1} a_{2i+3} \cdots a_m \in \mathcal{L}_{P,\omega}$. Note that $(\pi')' = \pi$. Now exactly one of π and π' has a descent at $2i+1$. The only other differences in the descent sets of π and π' occur (possibly) for the even numbers $2i$ and $2i+2$. Hence $(-1)^{\mathrm{maj}(\pi)} + (-1)^{\mathrm{maj}(\pi')} = 0$. The surviving permutations $\sigma = b_1 \cdots b_m$ in $\mathcal{L}_{P,\omega}$ are exactly those for which $\emptyset \subset \{b_1, b_2\} \subset \{b_1, \ldots, b_4\} \subset \cdots$ is a P-domino tableau with $\omega^{-1}(b_{2i-1}) < \omega^{-1}(b_{2i})$ in P. (If n is even, then the P-domino tableau ends as $\{b_1, \ldots, b_{n-2}\} \subset P$, while if n is odd it ends as $\{b_1, \ldots, b_{n-1}\} \subset P$.) Since ω is natural we have $b_{2i-1} < b_{2i}$ for all i, so $\mathrm{maj}(\sigma)$ is even. Hence $W_P(-1)$ is equal exactly to the number of P-domino tableaux.

(b) Regardless of the labeling ω, if there does not exist a P-domino tableau then there will be no survivors in the argument of (a), so $W_P(-1) = 0$. $\square$

The converse to Theorem 5.1(b) is false. The labeled poset (P, ω) of Fig. 5 is tilable by dominos and is maj-balanced.

Given an n-element poset P with dual P^*, set $\Delta(P) = \Gamma(P^*)$. In [25, Theorem 4.4], [26, Proposition 18.4], [27, Theorem 4.5.2] it is shown that the following two conditions are equivalent:

(i) For all $t \in P$, all maximal chains of the principal dual order ideal $V_t = \{s \in P\colon s \geqslant t\}$ have the same length.
(ii) $q^{\binom{n}{2} - \Delta(P)} W_P(1/q) = W_P(q)$.

It follows by setting $q = -1$ that if (i) holds and $\binom{n}{2} - \Delta(P)$ is odd, then P is maj-balanced. Corollary 2.4 suggests in fact the following stronger result.

Corollary 5.2. *Suppose that P is naturally labeled and dual consistent* (*i.e., P^* is consistent*). *If $\binom{n}{2} - \Delta(P)$ is odd, then P is maj-balanced.*

Proof. By Theorem 5.1 we need to show that there does not exist a P-domino tableau. Given $t \in P$, let $\delta(t)$ denote the length of the longest chain of V_t, so $\Delta(P) = \sum_{t \in P} \delta(t)$. First suppose that $n = 2m$, and assume to the contrary that $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_m = P$ is a P-domino tableau. If $s, t \in I_i - I_{i-1}$ then by dual consistency $\delta(s) + \delta(t) \equiv 1 \pmod 2$. Hence $\Delta(P) \equiv m \pmod 2$, so

$$\binom{n}{2} - \Delta(P) \equiv m(2m-1) - m \equiv 0 \pmod 2,$$

a contradiction.

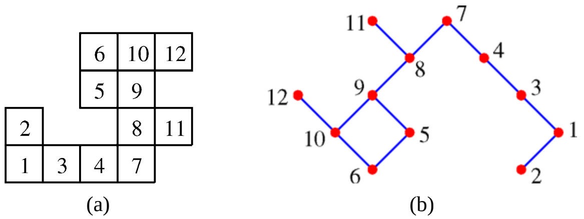

Fig. 6. A set $\mathcal{S}$ of squares and the Schur labeled poset $P_{\mathcal{S}}$.

Similarly if $n = 2m + 1$, then the existence of a P-domino tableau implies $\Delta(P) \equiv m \pmod 2$, so

$$\binom{n}{2} - \Delta(P) \equiv m(2m+1) - m \equiv 0 \pmod 2,$$

again a contradiction. □

Now let $\mathcal{S}$ be a finite subset of solid unit squares with integer vertices in $\mathbb{R} \times \mathbb{R}$ such that the set $|\mathcal{S}| = \bigcup_{S \in \mathcal{S}}$ is simply-connected. For $S, T \in \mathcal{S}$, define $S < T$ if the center vertices (s_1, s_2) of S and (t_1, t_2) of T satisfy either (a) $t_1 = s_1$ and $t_2 = s_2 + 1$ or (b) $t_1 = s_1 + 1$ and $t_2 = s_2$. Regard $\mathcal{S}$ as a poset, denoted $P_{\mathcal{S}}$, under the transitive (and reflexive) closure of the relation $<$. Figure 6 gives an example, where (a) shows $\mathcal{S}$ as a set of squares and (b) as a poset. Note that the posets $P_{\lambda/\mu}$ are a special case.

A *Schur labeling* ω of $P_{\mathcal{S}}$ is a labeling that increases along rows and decreases along columns, as illustrated in Fig. 6. For the special case $P_{\lambda/\mu}$, Schur labelings play an important role in the expansion of skew Schur functions $s_{\lambda/\mu}$ in terms of quasisymmetric functions [29, pp. 360–361]. Suppose that $\#P_{\mathcal{S}}$ is even and that $P_{\mathcal{S}}$ is tilable by dominos. Then $\mathcal{S}$ itself is tilable by dominos in the usual sense. It is known (implicit, for instance, in [31], and more explicit in [5]) that any two domino tilings of $\mathcal{S}$ can be obtained from each other by "2×2 flips," i.e., replacing two horizontal dominos in a 2×2 square by two vertical dominos or vice versa. It follows that if D is a domino tiling of $\mathcal{S}$ with $v(D)$ vertical dominos, then $(-1)^{v(D)}$ depends only on $\mathcal{S}$. Set $\operatorname{sgn}(\mathcal{S}) = (-1)^{v(D)}$ for any domino tiling of $\mathcal{S}$.

Proposition 5.3. *Let $\mathcal{S}$ be as above, and let ω be a Schur labeling of $P_{\mathcal{S}}$, where $\#P_{\mathcal{S}}$ is even, say $\#P_{\mathcal{S}} = n$. Then $\operatorname{sgn}(\mathcal{S}) W_{P_{\mathcal{S}}}(-1)$ is the number of $P_{\mathcal{S}}$-domino tableaux.*

Proof. The proof parallels that of Theorem 5.1. Define the involution $\pi \mapsto \pi'$ as in the proof of Theorem 5.1. Each survivor $\sigma = b_1 \cdots b_m$ corresponds to a $P_{\mathcal{S}}$-domino tableau D. We have $b_{2i-1} > b_{2i}$ if and only if the domino labeled with b_{2i-1} and b_{2i} is vertical. As noted above, $(-1)^{v(D)} = \operatorname{sgn}(\mathcal{S})$, independent of D. Hence $(-1)^{\operatorname{maj}(\sigma)} = \operatorname{sgn}(\sigma)$, and the proof follows as in Theorem 5.1(a). □

A result analogous to Proposition 5.3 holds for $\#P_{\mathcal{S}}$ odd (with essentially the same proof) provided $P_{\mathcal{S}}$ has a $\hat{0}$ or $\hat{1}$. The special case $P_{\lambda/\mu}$ of Proposition 5.3 (and its analogue for $\#P_S$ odd) can also be proved using the theory of symmetric functions, notably, [29, Proposition 7.19.11] and the Murnaghan–Nakayama rule [29, Corollary 7.17.5].

6. Hook lengths

In this section we briefly discuss a class of posets P for which $W_P(q)$, and sometimes even $I_{P,\omega}(q)$, can be explicitly computed. For this class of posets we get a simple criterion for being maj-balanced and, if applicable, sign-balanced.

Following [26, p. 84], an n-element poset P is called a *hook length poset* if there exist positive integers $h_1, \dots, h_n$, the *hook lengths* of P, such that

$$W_P(q) = \frac{[n]!}{(1-q^{h_1})\cdots(1-q^{h_n})}, \tag{14}$$

where $[n]! = (1-q)(1-q^2)\cdots(1-q^n)$. It is easy to see that if P is a hook length poset, then the multiset of hook lengths is unique. In general, if P is an “interesting” hook length poset, then each element of P should have a hook length associated to it in a “natural” combinatorial way.

Note. We could just as easily have extended our definition to *labeled* posets (P,ω), where now

$$W_{P,\omega}(q) = \frac{q^c[n]!}{(1-q^{h_1})\cdots(1-q^{h_n})}$$

for some $c \in \mathbb{N}$. However, little is known about the labeled situation except when we can reduce it to the case of natural labelings by subtracting certain constants from the values of σ.

The following result is an immediate consequence of Eq. (14).

Proposition 6.1. *Suppose that P is a hook length poset with hook lengths $h_1, \dots, h_n$. Then P is maj-balanced if and only if the number of even hook lengths is less than $\lfloor n/2 \rfloor$. If P is not maj-balanced, then the maj imbalance is given by*

$$W_P(-1) = \frac{\lfloor n/2 \rfloor!}{\prod_{h_i\,\mathrm{even}}(h_i/2)}.$$

It is natural to ask at this point what are the known hook length posets. The strongest work in this area is due to Proctor [18,19]. We will not state his remarkable results here, but let us note that his d-*complete* posets encompass all known “interesting” examples of hook length posets. These include forests (i.e., posets for which every element is covered by at most one element) and the duals P_λ^* of the posets P_λ of Section 3.

Björner and Wachs [3, Theorem 1.1] settle the question of what naturally labeled posets (P,ω) satisfy

$$I_{P,\omega}(q) = W_{P,\omega}(q). \tag{15}$$

Namely, P is a forest and ω is a postorder labeling. Hence for postorder labeled forests, Proposition 6.1 holds also for $I_{P,\omega}(-1)$. Björner and Wachs also obtain less definitive results for arbitrary labelings, whose relevance to sign and maj imbalance we omit.

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