Some congruences involving central $q$-binomial coefficients

Victor J.W. Guo$^{a,*}$, Jiang Zeng$^{b}$

$^a$ Department of Mathematics, East China Normal University, Shanghai 200062, People’s Republic of China
$^b$ Université de Lyon, Université Lyon 1, Institut Camille Jordan, UMR 5208 du CNRS, 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

**Abstract**

Motivated by recent works of Sun and Tauraso, we prove some variations on the Green–Krammer identity involving central $q$-binomial coefficients, such as

$$
\sum_{k=0}^{n-1} (-1)^k q^{-\left(\frac{k+1}{2}\right)} \left\lfloor \frac{2k}{k} \right\rfloor_q \equiv \left(\frac{n}{5}\right) q^{-\left\lfloor \frac{n}{5}/5\right\rfloor} \pmod{\Phi_n(q)},
$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol and $\Phi_n(q)$ is the $n$th cyclotomic polynomial. As consequences, we deduce that

$$
\sum_{k=0}^{3a-1} q^k \left\lfloor \frac{2k}{k} \right\rfloor_q \equiv 0 \pmod{1 - q^{3a}/(1 - q)},
$$

$$
\sum_{k=0}^{5a-1} (-1)^k q^{-\left(\frac{k+1}{2}\right)} \left\lfloor \frac{2k}{k} \right\rfloor_q \equiv 0 \pmod{1 - q^{5a}/(1 - q)},
$$

for $a, m \geq 1$, the first one being a partial $q$-analogue of the Strauss–Shallit–Zagier congruence modulo powers of 3. Several related conjectures are proposed.

© 2010 Elsevier Inc. All rights reserved.

* Corresponding author.

E-mail addresses: jwguo@math.ecnu.edu.cn (V.J.W. Guo), zeng@math.univ-lyon1.fr (J. Zeng).

1. Introduction

The p-adic order of several sums involving central binomial coefficients have attracted much attention. For example, among other things, Pan and Sun [13] and Sun and Tauraso [18,19] proved the following congruences modulo a prime \( p \):

\[
\sum_{k=0}^{p^a-1} \binom{2k}{k+d} \equiv \left( \frac{p^a - |d|}{3} \right) \pmod{p},
\]

\[
\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left( \frac{p^a}{5} \right) \pmod{p},
\]

and, for \( p \geq 3 \),

\[
\sum_{k=0}^{p^a-1} \binom{2k}{k+d} 2^{-k} \equiv \begin{cases} 
0 \pmod{p} & \text{if } p^a \equiv |d| \pmod{2}, \\
1 \pmod{p} & \text{if } p^a \equiv |d| + 1 \pmod{4}, \\
-1 \pmod{p} & \text{if } p^a \equiv |d| - 1 \pmod{4}.
\end{cases}
\]

where \( \left( \frac{n}{p} \right) \) is the Legendre symbol. It is well known that binomial identities or congruences usually have nice \( q \)-analogues (see [2]). Recently Tauraso [20] has noticed that an identity of Greene–Krammer [9] can be served as an inspiration for searching \( q \)-analogues of some identities in [18,19], and, in particular, he has proved the following generalization of (1.1):

\[
\sum_{k=0}^{n-1} q^k \left[ \frac{2k}{k+d} \right]_q \equiv \left( \frac{n - |d|}{3} \right) q^{3r(r+1)+|d|(2r+1)} \pmod{\Phi_n(q)},
\]

with \( r = \lfloor (2n - |d|)/3 \rfloor \). Here and in what follows \( \Phi_n(q) \) denotes the \( n \)th cyclotomic polynomial, and \( \left[ \frac{n}{k} \right]_q \) is the \( q \)-binomial coefficient defined by

\[
\left[ \frac{n}{k} \right]_q = \begin{cases} 
\frac{(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})} & \text{if } 0 \leq k \leq n, \\
0 & \text{otherwise},
\end{cases}
\]

where \( (z; q)_n = (1 - z)(1 - zq) \cdots (1 - zq^{n-1}) \) is the \( q \)-shifted factorial for \( n \geq 0 \).

The purpose of this paper is to study some \( q \)-versions of (1.1)–(1.3) as well as some variations of the same flavor as in [20]. For example, from a \( q \)-analogue of (1.2), we will deduce the following two congruences:

\[
\sum_{k=0}^{3^m-1} q^k \left[ \frac{2k}{k} \right]_q \equiv 0 \pmod{(1 - q^{2^m})/(1 - q)}.
\]

\[
\sum_{k=0}^{5^m-1} (-1)^k q^{-\binom{k+1}{2}} \left[ \frac{2k}{k} \right]_q \equiv 0 \pmod{(1 - q^{5^m})/(1 - q)}.
\]
Note that (1.5) may be deemed to be a partial $q$-analogue of the Strauss–Shallit–Zagier congruence [15]:

$$\sum_{k=0}^{3^m-1} \binom{2k}{k} \equiv 0 \pmod{3^m}.$$

The rest of the paper is organized as follows. In Section 2 we will give a $q$-analogue of (1.2) by using a finite Rogers–Ramanujan identity due to Schur. In Section 3 we will prove (1.5) and (1.6). Some different $q$-analogues of (1.3) will be given in Section 4 and some open problems will be proposed in the last section.

2. A $q$-analogue of (1.2)

It was conjectured by Krammer and proved by Greene [9] that

$$1 + 2 \sum_{k=1}^{n-1} (-1)^k q^{-\binom{k+1}{2}} \frac{2k - 1}{k} = \begin{cases} (\frac{n}{5}) \sqrt{5} & \text{if } n \equiv 0 \pmod{5}, \\ (\frac{n}{5}) & \text{otherwise}, \end{cases}$$

(2.1)

where $q = e^{2\pi i m/n}$ with $\gcd(m, n) = 1$ (see also [3,6] for some related results). If $n = p^a$, then the left-hand side of (2.1) is a $q$-analogue of that of (1.2). However, we cannot deduce the Sun–Tauraso congruence (1.2) from (2.1) in the case $n \equiv 0 \pmod{5}$. In this section we shall give a new $q$-series identity which is similar to (2.1) and will imply the Sun–Tauraso congruence (1.2) completely.

Theorem 2.1. For $n \geq 0$, there holds

$$\sum_{k=0}^{n-1} (-1)^k q^{-\binom{k+1}{2}} \frac{2k}{k} q^{-\lceil n^4/5 \rceil} \equiv (\frac{n}{5}) q^{-\lceil n^4/5 \rceil} \pmod{\Phi_n(q)}.$$  

(2.2)

In other words, letting $\omega = e^{2\pi i m/n}$ with $\gcd(m, n) = 1$, we have

$$\sum_{k=0}^{n-1} (-1)^k \omega^{-(\binom{k+1}{2})} \frac{2k}{k} \omega = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{5}, \\ -\omega^{-\lceil n/5 \rceil} & \text{if } n \equiv 1 \pmod{5}, \\ -\omega^{-\lceil 3n/5 \rceil} & \text{if } n \equiv 2 \pmod{5}, \\ -\omega^{-\lceil 2n/5 \rceil} & \text{if } n \equiv 3 \pmod{5}, \\ -\omega^{-\lceil 4n/5 \rceil} & \text{if } n \equiv 4 \pmod{5}. \end{cases}$$  

(2.3)

Proof. Since $\omega^k \neq 1$ for $1 \leq k \leq n - 1$ and $\omega^n = 1$, we can write

$$\omega^{-(\binom{k+1}{2})} \frac{2k}{k} \omega = \omega^{-(\binom{k+1}{2})} \prod_{j=1}^{k} \frac{1 - \omega^{-2k+1-j}}{1 - \omega^j}$$

$$= \omega^{-(\binom{k+1}{2})} (-1)^k \omega^{(3k^2+k)/2} \prod_{j=1}^{k} \frac{1 - \omega^{n-(2k+1-j)}}{1 - \omega^j}$$

$$= (-1)^k \omega^k \binom{n-k-1}{k} \omega.$$

(2.4)
Therefore, we derive from Schur’s identity (see, for example, [4, p. 50]) that
\[
\sum_{k=0}^{n-1} (-1)^k \omega^{-\binom{k+1}{2}} \binom{2k}{k}_\omega = \sum_{k=0}^{n-1} \omega^k \binom{n-k-1}{k}_\omega = \sum_{j=-\infty}^{\infty} (-1)^j \omega^{-\binom{j+1}{2}} \binom{n-1}{\frac{n-1-2j}{2}}_\omega.
\] (2.5)

Since \(\omega^n = 1\), we have
\[
\binom{n-1}{k}_\omega = \prod_{i=1}^{k} \frac{1-\omega^{n-i}}{1-\omega^i} = \prod_{i=1}^{k} \frac{1-\omega^{-i}}{1-\omega^i} = (-1)^k \omega^{-\binom{k+1}{2}}
\]
for \(0 \leq k \leq n-1\), and the identity (2.3) is easily deduced. For example, if \(n = 5m\), then there are \(2m\) non-zero terms in the right-hand side of (2.5). But the terms indexed \(j = -m + 2k\) and \(j = -m + 2k + 1\) cancel each other for \(k = 0, \ldots, m-1\). \(\square\)

Replacing \(q\) by \(q^{-1}\), one sees that (2.2) is equivalent to
\[
\sum_{k=0}^{n-1} (-1)^k q^{-\binom{k+1}{2}} \binom{2k}{k}_q = \left(\frac{n}{5}\right) q^{\left\lfloor \frac{\lfloor n/5 \rfloor}{2} \right\rfloor} \pmod{\Phi_n(q)}.
\]

If \(n = p^a\) is a prime power, letting \(q = 1\) in (2.2), one immediately gets the Sun–Tauraso congruence (1.2) by the formula
\[
\Phi_n(1) = \begin{cases} 
p & \text{if } n = p^a \text{ is a prime power}, \\
1 & \text{otherwise}. \end{cases} \quad (2.6)
\]
(Eq. (2.6) follows from the identity \(q^n - 1 = \prod_{d|n} \Phi_d(q)\) by induction.)

Remark. The first part of the proof of Theorem 2.1 can be generalized as follows. We define the \(q\)-Fibonacci polynomials (see [5]) by \(F_q^0(t) = 0, \ F_q^1(t) = 1, \) and
\[
F_q^n(t) = F_q^{n-1}(t) + q^{n-2} t F_q^{n-2}(t), \quad n \geq 2.
\]

The following is an explicit formula for the \(q\)-Fibonacci polynomials:
\[
F_q^n(t) = \sum_{k \geq 0} q^k \binom{n-k-1}{k}_q t^k. \quad (2.7)
\]

Let \(n > d \geq 0\) and let \(\omega\) be as in Theorem 2.1. Similarly to (2.4), we have
\[
\omega^{-\binom{k-d}{2}} \binom{2k}{k+d}_\omega = \omega^{-\binom{k-d}{2}} (-1)^{k-d} \omega^{(3k+d+1)(k-d)/2} \prod_{j=1}^{k-d} \frac{1 - \omega^{2n-(2k+1-j)}}{1-\omega^j} = (-1)^{k-d} \omega^{(k+d+1)(k-d)} \binom{2n-k-d-1}{k-d}_\omega.
\]
which yields the following congruence

\[
\sum_{k=0}^{n-1} q^{\frac{2k}{2^{e_d}}} \left[ \begin{array}{c} 2k \\ k + d \end{array} \right]_q \equiv t^d F_{2(n-d)}^q (-t q^{2d+1}) \pmod{\Phi_n(q)}
\]

by applying (2.7).

3. Congruences modulo \( \Phi_{3j}^j(q) \) and \( \Phi_{5j}^j(q) \)

In this section we give a proof of (1.5) and (1.6). It is well known that

\[
1 - q \equiv 1 - q^b = a \prod_{j=1}^{3j} \Phi_{p j}^j(q)
\]

for any prime \( p \). We need the following two lemmas.

**Lemma 3.1.** For \( n \geq 0 \), there holds

\[
\sum_{k=0}^{n} (-1)^k q^{2k} \left[ \begin{array}{c} n - k \\ k \end{array} \right]_q = (-1)^n \left( \frac{n+1}{3} \right) q^{\frac{n(n-1)}{6}}.
\]

**Lemma 3.2.** Let \( m, k, d \) be positive integers, and write \( m = ad + b \) and \( k = rd + s \), where \( 0 \leq b, s \leq d - 1 \). Let \( \omega \) be a primitive \( d \)th root of unity. Then

\[
\left[ \begin{array}{c} m \\ k \end{array} \right]_\omega = \left( \begin{array}{c} a \\ r \end{array} \right) \left[ \begin{array}{c} b \\ s \end{array} \right]_\omega.
\]

**Remark.** Lemma 3.1 has appeared in the literature from different origins (see [7]). A proof using mathematical induction is given in [20] and a multiple extension is proposed in [10]. Lemma 3.2 is equivalent to the \( q \)-Lucas theorem (see [12] and [8, Proposition 2.2]).

We first establish the following theorem.

**Theorem 3.3.** Let \( m, n \geq 1 \). Then

\[
\sum_{k=0}^{mn-1} q^k \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q \equiv \sum_{j=0}^{m-1} \left( \frac{2j}{n} \right) \sum_{k=0}^{n-1} q^k \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q \pmod{\Phi_n(q)}.
\]

\[
\sum_{k=0}^{mn-1} (-1)^k q^{-(k+1)} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q \equiv \sum_{j=0}^{m-1} (-1)^j \left( \frac{2j}{n} \right) \sum_{k=0}^{n-1} (-1)^k q^{-(k+1)} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q \pmod{\Phi_n(q)}.
\]

**Proof.** Let \( q = \omega \) be a primitive \( n \)th root of unity. Then \( \omega^n = 1 \) and

\[
\sum_{k=j_{n+n-1}}^{j_{n+k}} \omega^k \left[ \begin{array}{c} 2k \\ k \end{array} \right]_\omega = \sum_{k=0}^{n-1} \omega^k \left[ \begin{array}{c} 2jn + 2k \\ jn + k \end{array} \right]_\omega.
\]
\[
\sum_{k=jn}^{jn+n-1} (-1)^k \omega^{-\binom{k+1}{2}} \binom{2k}{k}_\omega = (-1)^{jn} \sum_{k=0}^{n-1} (-1)^k \omega^{-\binom{j+n+k+1}{2}} \binom{2jn+2k}{jn+k}_\omega. \tag{3.4}
\]

By Lemma 3.2 we have
\[
\binom{2jn+2k}{jn+k}_\omega = (2j) \binom{2k}{k}_\omega,
\]
which is equal to 0 if \(2k \geq n\). Noticing that
\[
\omega^{-\binom{j+n+k+1}{2}} = \omega^{-\binom{j+1}{2}} \cdot \omega^{-\binom{k+1}{2}}
\]
and
\[
(-1)^{jn} \omega^{-\binom{j+1}{2}} = (-1)^j,
\]
we can write Eqs. (3.3) and (3.4) as
\[
\sum_{k=jn}^{jn+n-1} \omega^k \binom{2k}{k}_\omega = \frac{(2j)}{j} \sum_{k=0}^{n-1} \omega^k \binom{2k}{k}_\omega, \tag{3.5}
\]
\[
\sum_{k=jn}^{jn+n-1} (-1)^k \omega^{-\binom{k+1}{2}} \binom{2k}{k}_\omega = (-1)^j \binom{2j}{j} \sum_{k=0}^{n-1} (-1)^k \omega^{-\binom{k+1}{2}} \binom{2k}{k}_\omega. \tag{3.6}
\]

Summing (3.5) and (3.6) over \(j\) from 0 to \(m-1\), we complete the proof. \(\square\)

We now state our main theorem in this section.

**Theorem 3.4.** Let \(a \geq 1\) and \(m \geq 1\). Then
\[
\sum_{k=0}^{3^m-1} q^k \binom{2k}{k}_q \equiv 0 \pmod{\prod_{j=1}^{a} \Phi_3(q)}. \tag{3.7}
\]
\[
\sum_{k=0}^{5^m-1} (-1)^k q^{-\binom{k+1}{2}} \binom{2k}{k}_q \equiv 0 \pmod{\prod_{j=1}^{a} \Phi_5(q)}. \tag{3.8}
\]

**Proof.** Let \(\omega\) be a primitive \(n\)th root of unity. Then
\[
\omega^k \binom{2k}{k}_\omega = \omega^{k^2+k} \binom{2k}{k}_\omega = \text{conj} \left( \omega^{-k^2-k} \binom{2k}{k}_\omega \right),
\]
where \(\text{conj}(z)\) denotes the complex conjugate of \(z \in \mathbb{C}\). From (2.4) we deduce that
\[
\omega^{-k^2-k} \binom{2k}{k}_\omega = (-1)^k \omega^{\binom{n}{2}} \binom{n-k-1}{k}. 
\]
Therefore, by Lemma 3.1, we have
\[ \sum_{k=0}^{n-1} \omega^k \left[ \frac{2k}{k} \right]_{\omega} = \text{con} \left( \sum_{k=0}^{n-1} (-1)^k \omega^k \left[ \frac{n-k-1}{k} \right]_{\omega} \right) = (-1)^{n-1} \left( \frac{n}{3} \right) \omega^{\frac{(n-1)(n-2)}{9}}. \]

This implies that
\[ \sum_{k=0}^{n-1} q^k \left[ \frac{2k}{k} \right]_{q} \equiv 0 \pmod{\Phi_n(q)} \quad \text{if } 3 | n, \quad (3.9) \]

which also follows directly from Tauraso’s congruence (1.4).

Now, letting \( n = 3^j \) with \( 1 \leq j \leq a \) in (3.9) and letting \( n = 5^j \) with \( 1 \leq j \leq a \) in (2.2), we get
\[ \sum_{k=0}^{3^j-1} q^k \left[ \frac{2k}{k} \right]_{q} \equiv 0 \pmod{\Phi_{3^j}(q)}, \]
\[ \sum_{k=0}^{5^j-1} (-1)^k q^{-\left(\frac{k+1}{2}\right)} k \left[ \frac{2k}{k} \right]_{q} \equiv 0 \pmod{\Phi_{5^j}(q)}. \]

Letting \( m \rightarrow 3^{a-j}m, n \rightarrow 3^j \) in (3.1) and \( m \rightarrow 5^{a-j}m, n \rightarrow 5^j \) in (3.2) respectively, we obtain
\[ \sum_{k=0}^{3^a m-1} q^k \left[ \frac{2k}{k} \right]_{q} \equiv 0 \pmod{\Phi_{3^j}(q)} \quad (1 \leq j \leq a), \]
\[ \sum_{k=0}^{5^a m-1} (-1)^k q^{-\left(\frac{k+1}{2}\right)} k \left[ \frac{2k}{k} \right]_{q} \equiv 0 \pmod{\Phi_{5^j}(q)} \quad (1 \leq j \leq a). \]

Since the cyclotomic polynomials are pairwise relatively prime, we complete the proof of (3.7) and (3.8). \( \square \)

We have the following conjecture.

**Conjecture 3.5.** Let \( a \geq 1 \) and \( m \geq 1 \). Then
\[ \sum_{k=0}^{3^a m-1} q^k \left[ \frac{2k}{k} \right]_{q} \equiv 0 \pmod{\prod_{j=1}^{a} \Phi_{3^j}^2(q)}, \]
\[ \sum_{k=0}^{5^a - 1} (-1)^k \left[ \frac{2k}{k} \right] \equiv 5^a \pmod{5^{a+1}}. \]

We now give a dual form of Theorem 2.1. The reader is encouraged to compare it with [20, Theorem 5.1].
Theorem 3.6. Let \( q = e^{2\pi mi/n} \) with \( \gcd(m, n) = 1 \). Then

\[
\sum_{k=0}^{n-1} q^{2k+1} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q = \begin{cases} \left( \frac{m}{n} \right) i \sqrt{3} & \text{if } 3 \mid n, \\ \left( \frac{q}{3} \right) & \text{otherwise}. \end{cases}
\]

Proof. First note that

\[
q^{2k+1} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q = q^{(k+1)^2} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_{q^{-1}} = \text{conj}\left(q^{-((k+1)^2)} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q \right)
\]

and \( \Phi_n(q) = 0 \). From (2.4) we deduce that

\[
q^{-(k+1)^2} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q = (-1)^k q^{(k+1)-2} \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right]_q.
\]

Therefore,

\[
\sum_{k=0}^{n-1} q^{2k+1} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q = \text{conj}\left(\sum_{k=0}^{n-1} (-1)^k q^{(k+1)-2} \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right]_q \right).
\]

Since

\[
q^{(k+1)-2} \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right]_q = q^{-n} \left( q^{(k+1)} \left[ \begin{array}{c} n-k \\ k+1 \end{array} \right]_q - q^{(k+1)} \left[ \begin{array}{c} n-k-1 \\ k+1 \end{array} \right]_q \right),
\]

by Lemma 3.1 we have

\[
\sum_{k=0}^{n-1} (-1)^k q^{(k+1)} \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right]_q = (-1)^n \left( \left( \frac{n+2}{3} \right) q^{\frac{n+5}{6}} + \left( \frac{n+1}{3} \right) q^{\frac{n-7}{6}} \right).
\]

The result then follows easily. \( \Box \)

Corollary 3.7. For any positive integer \( n \) with \( \gcd(n, 3) = 1 \), there holds

\[
\sum_{k=0}^{n-1} q^{2k+1} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q \equiv \left( \frac{n}{3} \right) \pmod{\Phi_n(q)}.
\]

For the following remarkable congruence of Sun and Tauraso [18, (1.1) with \( d = 0 \)]:

\[
\sum_{k=0}^{p^2-1} \left( \begin{array}{c} 2k \\ k \end{array} \right) \equiv \left( \frac{p^2}{3} \right) \pmod{p^2},
\]

we have two interesting \( q \)-versions to offer:
Conjecture 3.8. Let \( p \) be a prime and \( a \geq 1 \). Then

\[
\sum_{k=0}^{p^n-1} q^k \binom{2k}{k}_q \equiv \left( \frac{p^n}{3} \right) q^{\frac{p^n}{2} (-\frac{p^n}{2})} (p^n)^{\frac{p^n}{2} + \frac{p^n}{2}} (mod \ \Phi_{p^n}(q)),
\]

and, for \( p \neq 3 \),

\[
\sum_{k=0}^{p^n-1} q^{2k+1} \binom{2k}{k}_q \equiv \left( \frac{p^n}{3} \right) q^{\frac{p^n}{2} + \frac{p^n}{2}} (mod \ \Phi_{p^n}(q)).
\]

4. Some \( q \)-analogues of (1.3)

To give \( q \)-analogues of (1.3), we need to establish the following \( q \)-series identities:

Theorem 4.1. Let \( n \geq 1 \) and \( d = 0, 1, \ldots, n \). Then

\[
\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \binom{n}{k}_q \frac{2k}{k+d}_q \frac{(-q^{k+1}; q)_{n-k}}{(-q^{k}; q)_{n-k}} = \begin{cases} q^\frac{n^2-n^2}{2} \frac{\binom{n}{n^2}_q}{q^2} & \text{if } n-d \text{ is even}, \\ 0 & \text{if } n-d \text{ is odd}, \end{cases}
\]

(4.1)

\[
\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \binom{n}{k}_q \frac{2k}{k+d}_q \frac{(-q^{k}; q)_{n-k}}{(-q^{k}; q)_{n-k}} = \begin{cases} q^\frac{n^2-n^2}{2} \frac{\binom{n}{n^2}_q}{q^2} & \text{if } n-d \text{ is even}, \\ q^\frac{n^2-n^2}{2} (q^n-1) \frac{\binom{n}{n^2-1}_q}{q^2} & \text{if } n-d \text{ is odd}. \end{cases}
\]

(4.2)

Proof. The \( d = 0 \) case of (4.1) was found by Andrews [2, Theorem 5.5]. Both (4.1) and (4.2) can be proved similarly by using Andrews’s \( q \)-analogue of Gauss’s second theorem [1,2]:

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k q^{k+1}}{(q; q)_k (ab; q^2)_k} = \frac{(-q; q)_\infty (aq; q^2)_\infty (bq; q^2)_\infty}{(abq; q^2)_\infty},
\]

(4.3)

where \((z; q)_\infty = \lim_{n \to \infty} (z; q)_n\). We first sketch the proof of (4.1).

Recall that \((q; q)_{2n} = (q; q^n)(q^2; q^n)(a; q)_n(-a; q)_n = (a^2; q^n)_n\) and

\[
(a; q)_{n-k} = \frac{(a; q)_n}{(q^{1-n}/a; q)_k} \left( -\frac{q}{a} \right)^k q^{(k+1)/2}.
\]

Replacing \( k \) by \( n-k \), we can write the left-hand side of (4.1) as

\[
\sum_{k=0}^{n} (-1)^k q^{k+1} \binom{n}{k}_q \frac{2n-2k}{n-k+d}_q \frac{(-q^{n-k+1}; q)_k}{(-q^{n-k}; q)_k}
\]

\[= \begin{cases} 2n \sum_{k=0}^{n} \frac{(q^{-n-d}; q)_k (q^{-n-d}; q)_k q^{k(k+1)/2}}{(q; q)_k (q^{-2n+1}; q^2)_k} & \text{if } n-d \text{ is even}, \\ 2n \sum_{k=0}^{n} \frac{(q^{-n-d+1}; q^2)_\infty (q^{-n-d+1}; q^2)_\infty (q^{-2n+1}; q^2)_\infty}{(q^{-2n+1}; q^2)_\infty} & \text{if } n-d \text{ is odd}. \end{cases}
\]

(4.3)
we obtain the following result by substituting $n \in \mathbb{N}$ into (4.1) and (4.2).

This proves (4.1). Observing that

$$
\frac{(a; q)_k(b; q)_kq^{\frac{k}{2}}}{(q; q)_k(abq; q^2)_k} - \frac{(a; q)_k(b; q)_kq^{\frac{k+1}{2}}}{(q; q)_k(abq; q^2)_k} = \frac{(1 - a)(1 - b)(aq; q)_{k-1}(bq; q)_{k-1}q^{\frac{k}{2}}}{(1 - abq)(q; q)_{k-1}(abq^2; q^2)_{k-1}}.
$$

we derive the following $q$-series identity from (4.3):

$$
\sum_{k=0}^{\infty} \frac{(a; q)_k(b; q)_kq^{\frac{k}{2}}}{(q; q)_k(abq; q^2)_k} = \frac{(-q; q)_\infty(aq; q^2)_\infty(bq; q^2)_\infty}{(abq; q^2)_\infty} + \frac{(-q; q)_\infty(a; q^2)_\infty(b; q^2)_\infty}{(abq; q^2)_\infty}.
$$

Replacing $k$ by $n - k$, we can write the left-hand side of (4.2) as

$$
\sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} \left[ \sum_{q=2n-2k+1}^{n} \frac{2n - 2k}{2k} \right] (q^n - k; q)_k
\begin{align*}
&= \left[ \frac{2n}{n + d} \right] \sum_{q=2n-2k+1}^{n} \frac{(q^{n-d}; q)_k(q^{-n+d}; q)_kq^{\frac{k+1}{2}}}{(q; q)_k(q^{-2n+1}; q^2)_k} \left( 1 + q^{n-k} \right) \\
&= \left[ \frac{2n}{n + d} \right] \frac{(-q; q)_\infty(q^{-n-d+1}; q^2)_\infty(q^{-n+d+1}; q^2)_\infty}{(q^{-2n+1}; q^2)_\infty} \left( \frac{2n}{1 + q^n} \right) \\
&\quad + \frac{q^n}{1 + q^n} \left[ \frac{2n}{n + d} \right] \frac{(-q; q)_\infty(q^{-n-d}; q^2)_\infty(q^{-n+d}; q^2)_\infty}{(q^{-2n+1}; q^2)_\infty} \\
&\begin{cases} 
q^{\frac{n^2 - d^2}{2}} \left[ \frac{n - d}{2} \right] q^2 & \text{if } n - d \text{ is even}, \\
q^{\frac{n^2 - d^2 - 1}{2}}(q^n - 1) \left[ \frac{n - 1}{n - d - 1} \right] q^2 & \text{if } n - d \text{ is odd}.
\end{cases}
\end{align*}
$$

This proves (4.2). \hfill \Box

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$ and

$$
\left[ \begin{array}{c}
\frac{n - 1}{k} \\
\frac{n - 2}{k}
\end{array} \right] _q = \prod_{j=1}^{k} \frac{1 - q^{n-j}}{1 - q^j} \equiv (-1)^k q^{-k(k+1)} \pmod{\Phi_n(q)},
$$

$$
\left[ \begin{array}{c}
\frac{n - 1}{k} \\
\frac{n - 2}{k}
\end{array} \right] _{q^2} = \prod_{j=1}^{k} \frac{1 - q^{2n-2j-2}}{1 - q^{2j}} \equiv (-1)^k q^{-k(k+3)} \pmod{\Phi_n(q)},
$$

we obtain the following result by substituting $n$ with $n - 1$ in (4.1) and (4.2).
Corollary 4.2. Let $n \geq 1$ and $d = 0, 1, \ldots, n - 1$. Then

$$
\sum_{k=0}^{n-1} q^k \left[ \begin{array}{c} 2k \\ k+d \end{array} \right] q (-q^{k+1}; q)_{n-k-1}
\equiv \begin{cases} 
0 & \text{if } n - d \text{ is even,} \\
(-1)^{n+d-1} q \left( \frac{d(2n - 3d) - n^2 + 2d}{4} \right) \frac{1 - q^{n-d}}{1+q} & \text{if } n - d \text{ is odd}
\end{cases} \pmod{\Phi_n(q)}.
$$

Replacing $q$ by $q^{-1}$ in (4.5), we get

$$
\sum_{k=0}^{n-1} q^{-\left(\frac{k+1}{2}\right)} \left[ \begin{array}{c} 2k \\ k+d \end{array} \right] q (-q^{k+1}; q)_{n-k-1}
\equiv \begin{cases} 
0 & \text{if } n - d \text{ is even,} \\
(-1)^{n+d-1} q \frac{1 - q^{n-d}}{1+q} & \text{if } n - d \text{ is odd}
\end{cases} \pmod{\Phi_n(q)}.
$$

We also have the following variant of Theorem 4.1.

Theorem 4.3. Let $n \geq 1$ and $d = 0, 1, \ldots, n$. Then

\begin{align*}
\sum_{k=0}^{n} (-q)^{n-k} \left[ \begin{array}{c} n \\ k \end{array} \right] q \left[ \begin{array}{c} 2k \\ k+d \end{array} \right] q (-q^{k+1}; q)_{n-k} &= \begin{cases} 
\left[ \begin{array}{c} n-d \\ 2 \end{array} \right] q^2 & \text{if } n - d \text{ is even,} \\
(1 - q^{2n}) \left[ \begin{array}{c} n-1 \\ n-1-d \end{array} \right] q^2 & \text{if } n - d \text{ is odd}
\end{cases} \tag{4.7}
\end{align*}

\begin{align*}
\sum_{k=0}^{n} (-q)^{n-k} \left[ \begin{array}{c} n \\ k \end{array} \right] q \left[ \begin{array}{c} 2k \\ k+d \end{array} \right] q (-q^{k}; q)_{n-k} &= \begin{cases} 
\left[ \begin{array}{c} n-d \\ 2 \end{array} \right] q^2 & \text{if } n - d \text{ is even,} \\
(1 - q^{2n}) \left[ \begin{array}{c} n-1 \\ n-1-d \end{array} \right] q^2 & \text{if } n - d \text{ is odd.} \tag{4.8}
\end{cases}
\end{align*}

Proof. We would only prove (4.7), since the proof of (4.8) is similar. Replacing $q$ by $q^{-1}$ and multiplying by $q^{n^2-d^2+1}$, one sees that (4.7) is equivalent to the following identity:

\begin{align*}
\sum_{k=0}^{n} (-1)^{n-k} q^{\left(\frac{n-k-1}{2}\right)} \left[ \begin{array}{c} n \\ k \end{array} \right] q \left[ \begin{array}{c} 2k \\ k+d \end{array} \right] q (-q^{k+1}; q)_{n-k}
&= \begin{cases} 
q^{\frac{n^2-d^2+1}{2}} \left[ \begin{array}{c} n \\ 2 \end{array} \right] q^2 & \text{if } n - d \text{ is even,} \\
q^{\frac{(n-1)^2-d^2}{2}} (q^{2n} - 1) \left[ \begin{array}{c} n-1 \\ n-d-1 \end{array} \right] q^2 & \text{if } n - d \text{ is odd.} \tag{4.9}
\end{cases}
\end{align*}

Replacing $k$ by $n - k$, we can write the left-hand side of (4.9) as
\[ \sum_{k=0}^{n} \frac{(-1)^k q^{k+1}}{q} \binom{n}{k} \left( \binom{2n+1}{n} q^{n-k+1} : q \right) \]

\[ = \frac{2n}{n + d} q^{\frac{n}{4}} \sum_{k=0}^{n} \frac{(q^{-n-d} : q)_k (q^{n+d} : q)_k}{(q : q)_{k(q-2n+1)} : q^2} \]

\[ = \frac{q^{\frac{2n}{4}}}{n + d} \left( \frac{(q^{-n-d+1} : q^2)_{\infty} (q^{n+d+1} : q^2)_{\infty}}{(q : q)_{\infty} (q^{-2n+1} : q^2)_{\infty}} \right) (by \ (4.4)) ,\]

which is equal to the right-hand side of (4.9). □

**Remark.** Whenever they are discovered, both Theorem 4.1 and Theorem 4.3 can be proved by the $q$-Zeilberger algorithm (see, for example, [11, p. 113]).

As before, we have the following consequences.

**Corollary 4.4.** Let $n \geq 1$ and $d = 0, 1, \ldots, n - 1$. Then

\[ \sum_{k=0}^{n-1} q^{-k(k+3)/2} \binom{2k}{k+d} \left( \frac{-q^{k+1}}{q} \right)_{n-k-1} \]

\[ \equiv \begin{cases} 
(-1)^{n+d} q^{\frac{n(n-d+1)}{4}} (1 - q^{n-d}) & \text{if } n - d \text{ is even,} \\
(-1)^{n+d} q^{\frac{n(n-d+1)}{4}} & \text{if } n - d \text{ is odd}
\end{cases} \quad (\mod \Phi_n(q)) . \quad (4.10) \]

\[ \sum_{k=0}^{n-1} q^{-k(k+3)/2} \binom{2k}{k+d} \left( \frac{-q^k}{q} \right)_{n-k-1} \]

\[ \equiv \begin{cases} 
(-1)^{n+d} q^{\frac{n(n-d+1)}{4}} (1 - q^{n-d}) & \text{if } n - d \text{ is even,} \\
(-1)^{n+d} q^{\frac{n(n-d+1)}{4}} & \text{if } n - d \text{ is odd}
\end{cases} \quad (\mod \Phi_n(q)) . \quad (4.11) \]

If we change $q$ to $q^{-1}$, then the congruence (4.10) may be rewritten as

\[ \sum_{k=0}^{n-1} q^{2k} \binom{2k}{k+d} \left( \frac{-q^{k+1}}{q} \right)_{n-k-1} \]

\[ \equiv \begin{cases} 
(-1)^{n+d} q^{\frac{2(n-d+1)}{4} - 1} (1 - q^{n-d}) & \text{if } n - d \text{ is even,} \\
(-1)^{n+d} q^{\frac{2(n-d+1)}{4}} & \text{if } n - d \text{ is odd}
\end{cases} \quad (\mod \Phi_n(q)) . \]

while the congruences (4.6) and (4.11) exchange each other.

**5. Open problems**

Inspired by the $q = 1$ case of congruences (1.5)–(1.6) and the work of Sun [16], we would like to make the following conjectures:
Conjecture 5.1. Let $p$ be a prime factor of $4m - 1$ with $m \in \mathbb{Z}$ and let $a, n \geq 1$. Then
\[
\sum_{k=0}^{p^an-1} \binom{2k}{k} m^k \equiv 0 \pmod{p^a}.
\]

Conjecture 5.2. Let $m$ be a positive integer. Then
\[
\sum_{k=0}^{4m-2} \binom{2k}{k} m^k \equiv 0 \pmod{(4m - 1)},
\]
\[
\sum_{k=0}^{4m} \binom{2k}{k} (-m)^k \equiv 0 \pmod{(4m + 1)}.
\]

It is easy to see that Conjecture 5.1 implies Conjecture 5.2 but not vice versa.

Conjecture 5.3. Let $a$ be a positive integer. Then
\[
\sum_{k=0}^{3^a-1} (-2)^k \binom{2k}{k} \equiv 3^a \pmod{3^{a+1}},
\]
\[
\sum_{k=0}^{3^a-1} (-5)^k \binom{2k}{k} \equiv 2 \cdot 3^a \pmod{3^{a+1}},
\]
\[
\sum_{k=0}^{7^a-1} (-5)^k \binom{2k}{k} \equiv 7^a \pmod{7^{a+1}}.
\]

Conjecture 5.4. Let $m$ be a positive integer. If $4m - 1$ is a prime and $m \neq 1$, then
\[
\sum_{k=0}^{(4m-1)^a-1} \binom{2k}{k} m^k \equiv (4m - 1)^a \pmod{(4m - 1)^{a+1}}.
\]
If $4m + 1$ is a prime, then
\[
\sum_{k=0}^{(4m+1)^a-1} \binom{2k}{k} (-m)^k \equiv (4m + 1)^a \pmod{(4m + 1)^{a+1}}.
\]

Conversely, we make the following conjecture, which gives a sufficient condition for whether $4m - 1$ or $4m + 1$ is a prime. We have checked the cases $m \leq 1500$ via Maple, not finding any counterexamples.

Conjecture 5.5. Let $m$ be a positive integer. If $m \neq 30$ and
\[
\sum_{k=0}^{4m-2} \binom{2k}{k} m^k \equiv 4m - 1 \pmod{(4m - 1)^2},
\]
then $4m - 1$ is a prime. If
\[ \sum_{k=0}^{4m} \binom{2k}{k} (-m)^k \equiv 4m + 1 \pmod{(4m + 1)^2}, \]
then $4m + 1$ is a prime.

The following conjecture looks a little different but seems also very challenging.

**Conjecture 5.6.** Let $a$ and $n$ be positive integers. Then
\[ \sum_{k=0}^{5^n n - 1} \binom{4k}{2k} \binom{2k}{k}^2 \equiv 0 \pmod{5^a}, \]
\[ \sum_{k=0}^{5^n - 1} \binom{4k}{2k} \binom{2k}{k}^2 \equiv (-1)^a 5^a \pmod{5^{a+1}}. \]

**Remark.** Recently, Pan and Sun [14] have confirmed the first congruence in Conjecture 3.5 and Sun [17] has proved Conjectures 5.1–5.4 (naturally including the second congruence in Conjecture 3.5).

**Problem 5.7.** Are there any $q$-analogues of Conjectures 5.1–5.6?

**Acknowledgments**

The authors thank the anonymous referee for helpful comments on this paper. The first author was sponsored by Shanghai Educational Development Foundation under the Chenguang Project (#2007CG29), Shanghai Rising-Star Program (#09QA1401700), Shanghai Leading Academic Discipline Project (#B407), and the National Science Foundation of China (#10801054). The second author was supported by the project MIRA 2008 of Région Rhône-Alpes.

**References**